NOTES FOR INTRODUCTION TO COMPLEX ANALYSIS

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1. Sequences and limits

Definition 1. *A sequence in $\mathbb{C}$ is a function $\varphi : \mathbb{N} \to \mathbb{C}$. We use the notation $z_n = \varphi(n)$ and $\{z_n\}_{n=1}^{\infty}$ for the image $\varphi(\mathbb{N})$.*

Definition 2. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that the sequence converges to $z_0 \in \mathbb{C}$ if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ it holds that $|z_n - z_0| < \epsilon$. In this case z_0 is called the limit of the sequence $\{z_n\}_{n=1}^{\infty}$ and we write*

$$\lim_{n\to\infty} z_n = z_0.$$

Proposition 1. *If a sequence converges to a limit, this is unique.*

Proof. Supose that the sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ has two limits z_0 and w_0. Then for any $\epsilon > 0$ choose $n \in \mathbb{C}$ such that for any $n \geq n_0$, we have simultaneously

$$|z_n - z_0| < \frac{\epsilon}{2}, \text{ and } |z_n - w_0| < \frac{\epsilon}{2}.$$

Then

$$|z_0 - w_0| = |z_0 - z_n + z_n - w_0| \leq |z_n - z_0| + |z_n - w_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbtrary, this happens only if |z$_0$ - w$_0$| = 0. Then $z_0 = w_0$. □

Proposition 2. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$, with $z_n = x_n + iy_n$, and $z_0 = x_0 + iy_0$. Then the sequence converges to z_0 if and only if the sequences of real numbers $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge respectively to x_0 and y_0. Equivalently*

$$\lim_{n\to\infty} z_n = z_0 \iff \lim_{n\to\infty} x_n = x_0 \textit{ and } \lim_{n\to\infty} y_n = y_0.$$

Proof. Let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ the following holds:

$$|x_n - x_0| = |\mathrm{Re}(z_n - z_0)| \leq |z_n - z_0| < \epsilon.$$

This shows that $\lim_{n\to\infty} x_n = x_0$. The second part follows from a similar argument on the imaginary part of the sequence. □

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Proposition 3. *Every convergent sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is bounded, i.e., there exists a positive real number $M > 0$, such that $|z_n| \leq M$ for all $n \in \mathbb{N}$.*

Proof. Let $\epsilon = 1$. By convergence, there exists $n_0 \in \mathbb{N}$ such that whenever $k \geq n_0$ it holds that

$$|z_k - z_0| < 1.$$

Now consider $M = \max\{|z_1| + 1, |z_2| + 1, \ldots, |z_{k-1}| + 1, |z_0| + 1\}$. Then it is clear that

$$|z_n| \leq M, \text{ for all } n \in \mathbb{N}.$$

□

Remark The converse of last proposition is false. Discuss an example.

Theorem 1. *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{C}$ has a convergent sub-sequence.*

Proof. The proof relies on the same result known for sequences of Real numbers.

Let $\{z_n\}_{n=1}^{\infty}$ a bounded sequence where $z_n = x_n + iy_n$. Since the sequence is bounded we note that

$$|x_n| \leq |z_n| \leq M, \quad n \in \mathbb{N}.$$

Then the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is bounded. There exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit $x_0 \in \mathbb{R}$. Now consider the corresponding subsequence of complex numbers $\{z_{n_k}\}_{k=1}^{\infty}$, where $z_{n_k} = x_{n_k} + iy_{n_k}$. Since the sequence of complex numbers is bounded, we conclude that $\{y_{n_k}\}_{k=1}^{\infty}$ is also a bounded sequence of Real numbers. Then there exists a convergent subsequence $\left\{y_{n_{k_j}}\right\}_{j=1}^{\infty}$ wiht limit y_0. Recall that any subsequence of a convergent sequence is also convergent to the same limit, and conclude that $\left\{x_{n_{k_j}}\right\}_{j=1}^{\infty}$ has limit x_0. Then

$$\lim_{j \to \infty} z_{n_{k_j}} = \lim_{j \to \infty} \left(x_{n_{k_j}} + iy_{n_{k_j}}\right) = x_0 + iy$$

is a convergent subsequence of $\{z_n\}_{n=1}^{\infty}$. □

Definition 3. *A sequence $\{z_n\}_{n=1}^{\infty}$ is called Cauchy sequence if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, if $n, m \geq n_0$ then $|z_n - z_m| < \epsilon$.*

Theorem 2. *A sequence $\{z_n\}_{n=1}^{\infty}$ converges if and only if $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence.*

Proof. By writing $z_n = x_n + iy_n$ and from the inequalities

$$|x_n - x_m| \leq |z_n - z_m|,$$

and

$$|y_n - y_m| \leq |z_n - z_m|,$$

it follows that The sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is Cauchy if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences of Real numbers if and only if the sequences of real numbers $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent if and only if $\{z_n\}_{n=1}^{\infty}$ is a convergent sequence. □

Proposition 4. *Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ two convergent sequences, with $\lim_{n \to \infty} z_n = z_0$ and $\lim_{n \to \infty} w_n = w_0$. Then the following identities hold:*

(1) $\lim_{n \to \infty} (z_n + w_n) = z_0 + w_0$,

(2) $\lim_{n\to\infty} (z_n\, w_n) = z_0\, w_0$,
(3) $\lim_{n\to\infty} |z_n| = |z_0|$,
(4) *If* $w_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty} \left(\frac{z_n}{w_n}\right) = \frac{z_0}{w_0}$.

Proof. Proof of 1. It follows from the triangle inequality and the definition. Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it simultaneously holds $|z_n - z_0| < \frac{\epsilon}{2}$ and $|w_n - w_0| < \frac{\epsilon}{2}$, then

$$|(z_n + w_n) - (z_0 + w_0)| = |(z_n - z_0) + (w_n - w_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

That is $\lim_{n\to\infty} (z_n + w_n) = z_0 + w_0$.

Proof of 2. It follows also from the definition, the triangle inequality and the fact that every convergent sequence is bounded. Since $\{z_n\}_{n=1}^{\infty}$ is convergent, there is $M_1 > 0$ such that $|z_n| < M_1$ for all $n \in \mathbb{N}$ and the same for $\{w_n\}_{n=1}^{\infty}$, there is $M_2 > 0$ such that $|w_n| < M_1$ for all $n \in \mathbb{N}$. Take $M = \max\{M_1, M_2\}$, and then $M > 0$ is a bound for both sequences. Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it simultaneously holds $|z_n - z_0| < \frac{\epsilon}{2M}$ and $|w_n - w_0| < \frac{\epsilon}{2M}$. Then

$$\begin{aligned} |z_n w_n - z_0 w_0| &= |z_n w_n - z_n w_0 + z_n w_0 - z_0 w_0| \\ &\leq |z_n w_n - z_n w_0| + |z_n w_0 - z_0 w_0| \\ &= |z_n||w_n - w_0| + |w_0||z_n - z_0| \\ &\leq M|w_n - w_0| + M|z_n - z_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

That is $\lim_{n\to\infty} (z_n\, w_n) = z_0\, w_0$.

Proof of 3. It follows from a version of the triangle inequality. Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it holds $|z_n - z_0| < \epsilon$. Note then that

$$|\,|z_n| - |z_0|\,| \leq |z_n - z_0| < \epsilon.$$

That is $\lim_{n\to\infty} |z_n| = |z_0|$.

Proof of 4. Let $\epsilon_1 = \frac{|w_0|}{2}$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it holds $|w_n - w_0| < \frac{|w_0|}{2}$. First note that

$$\left|\frac{1}{w_n} - \frac{1}{w_0}\right| = |\frac{w_0 - w_n}{w_n w_0}| = \frac{|w_0 - w_n|}{|w_n|\,|w_0|}.$$

On the other hand by the triangle inequality

$$|w_0| - |w_n| \leq |\,|w_0| - |w_n|\,| \leq |w_0 - w_n| < \frac{|w_0|}{2},$$

this implies

$$\frac{|w_0|}{2} \leq |w_n|,$$

or equivalently

$$\frac{1}{|w_n|} \leq \frac{2}{|w_0|}.$$

Then we get

$$\left|\frac{1}{w_n} - \frac{1}{w_0}\right| \leq \frac{|w_0 - w_n|}{|w_n|\,|w_0|} \leq \frac{2|w_0 - w_n|}{|w_0|^2}.$$

For any $\epsilon > 0$ define $\epsilon_2 = \frac{|w_0|^2\epsilon}{2}$, then there exists $m_0 \in \mathbb{N}$ such that for any $n \geq m_0$ it holds $|w_n - w_0| < \frac{|w_0|^2\epsilon}{2}$. Take $N_0 = \max\{n_0, m_0\}$. Then we can improve our last inequality, for any $n \geq N_0$ it holds

$$\left|\frac{1}{w_n} - \frac{1}{w_0}\right| \leq \frac{|w_0 - w_n|}{|w_n|\,|w_0|} \leq \frac{2|w_0 - w_n|}{|w_0|^2} < \epsilon.$$

This shows that $\lim\limits_{n\to\infty} \dfrac{1}{w_n} = \dfrac{1}{w_0}$. Finally applying (2) of this proposition we have

$$\lim_{n\to\infty} \frac{z_0}{w_n} = \lim_{n\to\infty} z_n \frac{1}{w_n} = \lim_{n\to\infty} z_n \lim_{n\to\infty} \frac{1}{w_n} = \frac{z_0}{w_0}$$

□

Recall from real analysis, that every monotone and bounded sequence of real numbers is convergent.

- If the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing then $\lim_{n\to\infty} a_n = \sup\{a_n | n \in \mathbb{N}\}$.
- If the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing then $\lim_{n\to\infty} a_n = \inf\{a_n | n \in \mathbb{N}\}$.

Definition 4. *We say that* $\lim_{n\to\infty} z_n = \infty$ *if* $\lim_{n\to\infty} |z_n| = \infty$*. In other words, if for every* $M > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *then* $|z_n| > M$*.*

Definition 5. *Let* $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ *be a sequence. A complex number* $w_0 \in \mathbb{C}$ *is called a limit point of the sequence* $\{z_n\}_{n=1}^{\infty}$ *if there exists a subsequence* $\{z_{n_k}\}_{k=1}^{\infty}$ *of* $\{z_n\}_{n=1}^{\infty}$ *such that*

$$\lim_{k\to\infty} z_{n_k} = w_0.$$

2. Continuity

Definition 6. *The function* $f(z)$ *is said to have the limit* $w_0 \in \mathbb{C}$ *as* z *tends to* $a \in \mathbb{C}$*, if for every* $\epsilon > 0$ *there exists a number* $\delta = \delta(\epsilon) > 0$ *with the property that if* $|z - a| < \delta$ *then it holds* $|f(z) - w_0| < \epsilon$*, and in that case we write*

$$\lim_{z\to a} f(z) = w_0.$$

Definition 7. *The function* $f(z)$ *is said to be continuous at* z_0 *if for if for every* $\epsilon > 0$ *there exists a number* $\delta = \delta(\epsilon) > 0$ *with the property that if* $|z - a| < \delta$ *then it holds* $|f(z) - f(a)| < \epsilon$*, and in that case we write*

$$\lim_{z\to a} f(z) = f(a).$$

Examples

- Let $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ the function given by
$$f(z) = \frac{z^2 - 4}{z - 2},$$
then $\lim_{z\to 2} f(z) = 4$.
- Let $f : \mathbb{C} \to \mathbb{C}$ be the conjugate function $f(z) = \bar{z}$. Then $f(z)$ is continuous at every $z_0 \in \mathbb{C}$. To show this, first note that $|z| = |\bar{z}|$. Let $\epsilon > 0$ and choose $0 < \delta = \epsilon$. Then, for any $z \in \mathbb{C}$ such that $|z - z_0|$ we can estimate
$$|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon.$$
.

Proposition 5. *Let $U \subset \mathbb{C}$, $f : U \to \mathbb{C}$ and $a \in U$. Then the following three statements are equivalent:*

(1) f is continuous at a.
(2) For every $\epsilon > 0$ there exists a number $\delta > 0$ such that $f(B_\delta(a) \cap U) \subseteq B_\epsilon(f(a))$.
(3) For every sequence $\{z_n\}_{n=1}^\infty \subseteq U$ such that $\lim_{n\to\infty} z_n = a$, we have

$$\lim_{n\to\infty} f(z_n) = f(a).$$

Proof. $(1 \Rightarrow 2)$ Let $\epsilon > 0$, then there is a number $\delta > 0$ such that for every $z \in U$ with $|z - a| < \delta$ then $|f(z) - f(a)| < \epsilon$. This implies that if $z \in B_\delta(a) \cap U$ then $f(B_\delta(a) \cap U) \subseteq B_\epsilon(f(a))$.

$(2 \Rightarrow 1)$ Let $\epsilon > 0$, then there is a number $\delta > 0$ such that $f(B_\delta(a) \cap U) \subseteq B_\epsilon(f(a))$. This means that for every $z \in B_\delta(a) \cap U$ we have $f(z) \in B_\epsilon(f(a))$. This means that for every $z \in U$ with $|z - a| < \delta$ it holds $|f(z) - f(a)| < \epsilon$.

$(1 \Rightarrow 3)$ Let $f(z)$ be continuous at $a \in U$ and $\{z_n\}_{n=1}^\infty \subseteq U$ with limit a. For any $\epsilon > 0$ there exists a number $\delta > 0$ such that for every $z \in U$ with $|z - a| < \delta$ it holds $|f(z) - f(a)| < \epsilon$. For that $\delta > 0$ there is a $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $|z_n - a| \leq \delta$. Then by continuity this implies that $|f(z_n) - f(a)| < \epsilon$. This shows that $\lim_{n\to\infty} f(z_n) = f(a)$.

$(3 \Rightarrow 1)$ Let $a \in U$ and assume that for any $\{z_n\}_{n=1}^\infty \subseteq U$ with limit a we have $\lim_{n\to\infty} f(z_n) = f(a)$. Suppose on the contrary, that f is not continuous at a. Then there exists a positive $\epsilon > 0$ such that for every $\delta > 0$ there is a point $z \in U$ with $|z - a| < \delta$ but $|f(z) - f(a)| \geq \epsilon$. Note that this implies that for any $n \in \mathbb{N}$ we can choose $\delta = \frac{1}{n}$ and $z_n \in U$ such that $|z_n - a| < \frac{1}{n}$, but $|f(z_n) - f(a)| \geq \epsilon$. In this way we have constructed a sequence $\{z_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} z_n = a$ but $\lim_{n\to\infty} f(z_n) \neq f(a)$ which is a contradiction to our initial assumptions.

$(1 \Rightarrow 3)$ Let $f(z)$ be continuous at a. Consider any sequence $\{z_n\}_{n=1}^\infty \subseteq U$ such that $\lim_{n\to\infty} z_n = a$. Then for $\epsilon > 0$ there is a number $\delta > 0$ such that for all $z \in U$ with $|z - a| < \delta$ we have $|f(z) - f(a)| < \epsilon$. Also for such $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $|z_n - a| < \delta$, which then implies $|f(z_n) - f(a)| < \epsilon$. □

Theorem 3. *Let $U \subseteq \mathbb{C}$ and $f, g : U \to \mathbb{C}$ be two continuous functions at $a \in U$. Then $f \pm g$, $f \cdot g$ are continuous in a. If $g(a) \neq 0$ then $\dfrac{f}{g}$ is also continuous at a.*

Theorem 4. *Let $U_1, U_2 \subseteq \mathbb{C}$, and $f : U_1 \to \mathbb{C}$, $g : U_2 \to \mathbb{C}$, such that $f(U_1) \subseteq U_2$. If f is continuous at $z_0 \in U_1$ and g is continuous at $w_0 = f(z_0) \in U_2$, then the composition $g \circ f$ is continuous at z_0.*

Proof. Consider any sequence $\{z_n\}_{n=1}^\infty \subseteq U_1$ such that $\lim_{n\to\infty} z_n = z_0$. Then define $w_n = f(z_n) \in U_2$. Since f is continuous at z_0, then $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} w_n = w_0 = f(z_0)$. Since g is continuous at w_0 then $\lim_{n\to\infty} g(w_n) = g(w_0)$. Note that on the left hand side $g(w_n) = g(f(z_n)) = (g \circ f)(z_n)$, and on the right hand side $g(w_0) = g(f(z_0)) = (g \circ f)(z_0)$. This shows that $\lim_{n\to\infty}(g \circ f)(z_n) = (g \circ f)(z_0)$. □

Definition 8. *Let $U \subseteq \mathbb{C}$ and consider a function $f : U \to \mathbb{C}$. f is called uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $z, w \in U$ such that $|z - w| < \delta$ then we have $|f(z) - f(w)| < \epsilon$.*

Example. Consider $f : A \to \mathbb{C}$, $f(z) = z^2$ in the following cases:

- $A = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$.

In this case f is unifrmly continuous in A. Let $\epsilon > 0$ and take $\delta = \epsilon/2$. Then, for any $z, w \in A$ such that $|z - w| < \delta$ we have

$$|f(z) - f(w)| = |z^2 - w^2| = |z - w||z + w| \leq |z - w|(|z| + |w|) \leq 2|z - w| < \epsilon.$$

- $A = \mathbb{C}$.

In this case f is not uniformly continuous. Take $\epsilon = 1$. For every $\delta > 0$ there exists $n \in \mathbb{N}$ such that $n\delta > 1$. Now consider $z = n$ and $w = n + \delta/2$. Note that we have $|z - w| = \delta/2 < \delta$, but

$$|f(z) - f(w)| = |n^2 - \left(n + \frac{\delta}{2}\right)^2| = n\delta + \frac{\delta^2}{4} > n\delta > 1 = \epsilon.$$

3. Basic topology of $\mathbb{C}$

The open and closed disks (balls) already defined, are basic subsets that may be used to build a topological structure of the complex plane. A topology allows us to define several notions of continuity.

$$B_r(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| < r\}.$$
$$\bar{B}_r(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| \leq r\}.$$

Definition 9. *A set $A \subseteq \mathbb{C}$ is called open if $\forall z \in A$ there exists a real number $r_z > 0$ such that $B_{r_z}(z) \subseteq A$.*

Theorem 5. *The following sentences are true*

1. *The sets $\mathbb{C}$, $\emptyset$, $B_r(z)$ (for any $r > 0$) and any $z \in \mathbb{C}$, are open sets.*
2. *If $U_1, \dots, U_n$ is a finite collection of open sets, then $\cap_{k=1}^n U_k$ is an open set.*
3. *If $\{U_\alpha\}_{\alpha \in I}$ is a family of open sets, then $\cup_{\alpha \in I} U_\alpha$ is an open set.*

Proof. $\mathbb{C}$ is open. For any $z \in \mathbb{C}$, take $r = 1$ and evidently $B_1(z) \subset \mathbb{C}$.

$\emptyset$ is open. If this wasn't true, there would exists $z \in \emptyset$ such that for any $r > 0$, $B_r(z) \not\subseteq \emptyset$. Then the statement is vacuously true.

$B_r(z)$ is open. Let $w \in B_r(z)$. Take $\delta = r - |w - z|$, and note that $\delta > 0$. To show that $B_\delta(w) \subset B_r(z)$ we need to show that any $\xi \in B_\delta(w)$ is also in $B_r(z)$. If $\xi \in B_\delta(w)$ then $|\xi - w| < \delta$. Since $|\xi - z| = |\xi - w + w - z| \leq |\xi - w| + |w - z| < \delta + |w - z| = r - |w - z| + |w - z| = r$. Then $\xi \in B_r(z)$.

$\cap_{k=1}^n U_k$ is open if each U_i is open. Let $z \in \cap_{k=1}^n U_k$. Then for each $1 \leq k \leq n$ there exists $r_k > 0$ such that $B_{r_k}(z) \subseteq U_k$. Take $r = \min\{r_1, \dots, r_k\}$, and hence $B_r(z) \subset U_k$ for every $1 \leq k \leq n$. Hence $B_r(z) \subset \cap_{k=1}^n U_k$.

$\cup_{\alpha \in I} U_\alpha$ is open if each U_α is open. Let $z \in \cup_{\alpha \in I} U_\alpha$. Then there exists $\beta \in I$ such that $z \in U_\beta$. Since U_β is open, there is $r > 0$ such that $B_r(z) \subset U_\beta \subset \cup_{\alpha \in I} U_\alpha$. □

Definition 10. *A set $G \subseteq \mathbb{C}$ is called closed if its complement $\mathbb{C} \setminus G$, is an open set.*

Proposition 6. *The following sentences are true*

1. *The sets $\mathbb{C}$, $\emptyset$, are closed sets.*
2. *If $G_1, \dots, G_n$ is a finite collection of closed sets, then $\cup_{k=1}^n G_k$ is a closed set.*
3. *If $\{G_\alpha\}_{\alpha \in I}$ is a family of closed sets, then $\cap_{\alpha \in I} G_\alpha$ is a closed set.*

Definition 11. *Let $A \subset \mathbb{C}$. Then we define the following sets*
- *The interior of A:* $\quad Int(A) = \cup\{U \,|\, U \subset A,$ *and U is an open set*$\}$.
- *The closure of A:* $\quad Cl(A) = \cap\{G \,|\, A \subset G,$ *and G is a closed set*$\}$.
- *The boundary of A:* $\quad \partial A = Cl(A) \cap Cl(\mathbb{C} \setminus A)$.

Proposition 7. *Let $A, B \subseteq \mathbb{C}$. Then following statements are true*
- $Int(A) \subset A$.
- $A \subset Cl(A)$.
- $\partial A \subseteq Cl(A)$.
- *A is open if and only if $A = Int(A)$.*
- *A is closed if and only if $A = Cl(A)$.*
- $Cl(A \cup B) = Cl(A) \cup Cl(B)$.
- $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$, *but in general they are not equal.*
- $Int(A \cap B) = Int(A) \cap Int(B)$.
- $Int(A \cup B) \supseteq Int(A) \cup Int(B)$, *but in general they are not equal.*
- $z_0 \in Int(A) \Leftrightarrow \exists r > 0,$ *such that* $B_r(z_0) \subseteq A$.
- $z_0 \in Cl(A) \Leftrightarrow \forall r > 0,$ *it holds* $B_r(z_0) \cap A \neq \emptyset$.
- $z_0 \in \partial(A) \Leftrightarrow \forall r > 0,$ *it holds* $B_r(z_0) \cap A \neq \emptyset$ *and* $B_r(z_0) \cap (\mathbb{C} \setminus A) \neq \emptyset$.

The topological structure of $\mathbb{C}$ induces in a natural way a topological structure in any subset $A \subset \mathbb{C}$.

Definition 12. *Let $A \subset \mathbb{C}$. The set $B \subseteq A$ is called open (closed) in A, if there is an open (closed) set U of $\mathbb{C}$ such that $B = A \cap U$.*

Proposition 8. *Let $A \subseteq \mathbb{C}$. Then the following sentences are true*
1. *The sets A, $\emptyset$, are open sets and closed sets in A.*
2. *If the set $B \subseteq A$ is open in A, then $A \setminus B$ is closed in A.*
3. *If the set $B \subseteq A$ is closed in A, then $A \setminus B$ is open in A.*
4. *If $U_1, \ldots, U_n$ is a finite collection of open sets in A, then $\cap_{k=1}^{n} U_k$ is an open set in A.*
5. *If $\{U_\alpha\}_{\alpha \in I}$ is a family of open sets in A, then $\cup_{\alpha \in I} U_\alpha$ is an open set A.*
6. *If $G_1, \ldots, G_n$ is a finite collection of closed sets in A, then $\cup_{k=1}^{n} G_k$ is a closed set in A.*
7. *If $\{G_\alpha\}_{\alpha \in I}$ is a family of closed sets in A, then $\cap_{\alpha \in I} G_\alpha$ is a closed set in A.*

Proposition 9. *Let $A \subset \mathbb{C}$. $z_0 \in Cl(A)$ if and only if there exists a sequence $\{z_n\}_n^\infty \subset A$ such that $\lim_{n\to\infty} z_n = z_0$.*

Proof. By the previous proposition, since $z_0 \in \mathrm{Cl}(A)$, we have that for every $n \in \mathbb{N}$ we can take $r_n = \frac{1}{n} > 0$, and it holds

$$B_{r_n}(z_0) \cap A \neq \emptyset.$$

Then for each n we can choose $z_n \in B_{r_n}(z_0) \cap A$. Clearly we have

$$|z_n - z_0| < \frac{1}{n},$$

for all $n \in \mathbb{N}$, which implies $\lim_{n\to\infty} z_n = z_0$. □

Definition 13. (1) *The set $A \subset \mathbb{C}$ is called* ***disconnected****, if there are two open sets U, V in $\mathbb{C}$ such that*
   (a) $(A \cap U) \neq \emptyset$ *and* $(A \cap V) \neq \emptyset$.

   (b) U and V are disjoint in A: $(A \cap U) \cap (A \cap V) = \emptyset$
   (c) $A \subseteq U \cup V$.
(2) *The set A is called **connected** if it is not disconnected.*

Proposition 10. *The set A is connected if and only if only the sets A and $\emptyset$ are the only sets that are both, open and closed in A.*

Proof. ($\Rightarrow$) Let A be connected and suppose on the contrary that there is a set $B \subset A$, that is both, open and closed, and different from A and $\emptyset$. Then B and $A \setminus B$ are open sets in A. Then, there are open sets $U, V \subseteq \mathbb{C}$ such that $B = A \cap U$ and $A \setminus B = A \cap V$. then we have:

(1) $(A \cap U) \neq \emptyset$ and $(A \cap V) \neq \emptyset$.
(2) $(A \cap U) \cap (A \cap V) = B \cap (A \setminus B) = \emptyset$.
(3) $A = B \cup (A \setminus B) = (A \cap U) \cup (A \cap V) = A \cap (U \cup V)$ which implies that $A \subseteq U \cup V$.

These three points are a contradiction since A is assumed connected.

($\Leftarrow$) Let A and $\emptyset$ the only sets that are both, open and closed in A. Suppose on the contrary that A is disconnected. Then there are two open sets in $U, V \subset \mathbb{C}$ satisfying the definition above. Put $B_1 = A \cap U$ and $B_2 = A \cap V$. Then B_1 and B_2 are non-empty open sets in A. Since $B_1 \cap B_2 = \emptyset$, and $A = B_1 \cup B_2$. Then $A \setminus B_1 = B_2$, which implies that B_2 is also closed. Moreover, since $B_1 \neq \emptyset$, $B_2 \neq A$, which is a contradiction. Then A is connected. □

Definition 14. *Let $A \subset \mathbb{C}$. By a curve γ in A we mean a continuous map $\gamma : [0, 1] \to A$. We say that A is called **arc-connected** or **path-connected** if for any two points $z_1, z_2 \in \mathbb{C}$ there exists a curve γ in A such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$.*

Theorem 6. *Let $A \subset \mathbb{C}$.*

(1) *If A is arc-connected then A is connected.*
(2) *If A is open and connected then A is arc-connected.*

Here is an example of a connected set which fails to be arc-connected Example.
A **domain** $\Omega \subseteq \mathbb{C}$ is an open connected set.

Definition 15. *Let $A \subseteq \mathbb{C}$. An open cover of A is a family $\{U_\alpha\}_{\alpha \in I}$ of open subsets of $\mathbb{C}$, such that $A \subset \cup_{\alpha \in I} U_\alpha$.*

A finite subcover is a collection $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ such that $A \subseteq \cup_{j=1}^{k} U_{\alpha_j}$.

Definition 16. *The set $K \subset \mathbb{C}$ is called **compact** if any open cover of K, has a finite subcover.*

Theorem 7. ***(Heine-Borel)**. The set $K \subset \mathbb{C}$ is compact if and only K is closed and bounded.*

Proof. ($\Rightarrow$).

If K is compact then K is closed. Supose there is $z_0 \in \mathrm{cl}(K)$ such that $z_0 \notin K$, and consider the following sets: All balls $B_r(z_0)$ and for each $z \in K$ choose a ball $U_z := B_{r_z}(z)$ such that $r_z > 0$ is small enough to not intersect one of the $B_r(z_0)$. Note now that the collection $\{U_z\}_{z \in K}$ is an open cover of K but clearly any finite choice of U_z's fails to cover K.

If K is compact then K is bounded. For each \$z $\in K$ consider the set $U_z = B_1(z)$. Clearly $\{U_z\}_{z \in K}$ is an open cover of K, and then there is a finite open subcover

$U_{z_1}, \dots, U_{z_n}$. Take $M = \max\{|z_1|+1, \dots, |z_k|+1\}$. Then $K \subset \cup_{k=1}^{n} U_{z_k} \subseteq B_M$, i.e. K is bounded.

($\Leftarrow$). Exercise. *Hint.* Show first that any closed subset of a compact set is also compact. Use the bounded property to show that K must be contained in a square of the form $G = [-r, r] \times [-r, r]$ and show that show that G is compact. $\square$

Theorem 8. *$K \subset \mathbb{C}$ is compact if and only if for every sequence $\{z_n\}_{n=1}^{\infty} \subset K$ has a convergent subsequence, i.e., $\{z_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} z_{n_k} = z_0$ for some $z_0 \in K$.*

Proof. ($\Rightarrow$) Let $K \subset \mathbb{C}$ compact. If $\{z_n\}_{n=1}^{\infty} \subset K$ is a bounded sequence since K is bounded. By Bolzano-Weierstrass' Theorem, the sequence has a convergent subsequece, say to a limit point $z_0 \in \mathrm{Cl}(K)$. Since K is closed then $z_0 \in K$.

($\Leftarrow$) Suppose that any $\{z_n\}_{n=1}^{\infty} \subset K$ has a convergent subsequece. IF K is not bounded, then we can construct a sequence $\{z_n\}_{n=1}^{\infty}$ such that $|z_n| > n$ for all $n \in \mathbb{N}$. But this sequence has no convergent subsequence, which is a contradiction. Then K should be bounded.

Now take any $z_0 \in \mathrm{Cl}(K)$. By a previous proposition, we can construct a sequence $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} z_n = z_0$. By our hypotesis, the sequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} z_{n_k} = w_0$, for some $w_0 \in K$. By the uniqueness of the limit, $z_0 \in K$. We he have shown $\mathrm{Cl}(K) \subseteq K$, which implies $\mathrm{Cl}(K) = K$, i.e., K is closed. $\square$

Proposition 11. *Let $f : A \to \mathbb{C}$ be a function. Then the following sentences are equivalent*

(1) f is continuous in A.
(2) For every open set $V \subset \mathbb{C}$, the set $f^{-1}(V)$ is open in A.
(3) For every closed set $G \subset \mathbb{C}$, the set $f^{-1}(G)$ is closed in A.

Proof. ($1 \Rightarrow 2$) Assume f is continuous in A. Let $V \subset \mathbb{C}$ any open such that $f(A) \cap V \neq \emptyset$. We want to show that $f^{-1}(V)$ is an open subset of A. Let $z_0 \in f^{-1}(V)$, then $f(z_0) \in V$, and since V is open there exists an open ball $B_r(f(z_0)) \subset V$. Since f is assume to be continuous, if we let $\epsilon = r$, then there exists $\delta > 0$ such that for any $z \in A$ such that $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \epsilon$. This is the same as saying that there is a $\delta > 0$ such that $B_\delta(z_0) \cap A \subseteq f^{-1}(B_r(f(z_0))) \subset f^{-1}(V)$. Then \$U$_\delta =$ which implies z_0 is an interior point. Since z_0 is arbitary, $f^{-1}(V)$ is an open subset of A.

($2 \Rightarrow 1$) Assume now that for any $V \subset \mathbb{C}$ we have that $f^{-1}(V)$ is an open set in A. Let $z_0 \in A$, then for $f(z_0) \in \mathbb{C}$, and for any $\epsilon > 0$ consider the open set $V = B_\epsilon(f(z_0))$. Then by our hypothesis, $f^{-1}(V)$ is an open set in A, which means that there is an open set $U \subset \mathbb{C}$ such that $U \cap A$ is completely contained in $f^{-1}(V) \cap A$. More over, since $z_0 \in U$ there exists $\delta > 0$ such that $B_\delta(z_0)$ is contained in U and also $f(B_\delta(z_0) \cap A) \subset B_\epsilon(f(z_0))$. The by one of our previous propositions, f is continuous at every point of A.

($2 \Leftrightarrow 3$) $f^{-1}(V)$ is open in A for any $V \in \mathbb{C}$ open $\Leftrightarrow A \setminus f^{-1}(V) = f^{-1}(\mathbb{C} \setminus V)$ is closed in A for any closed set $(\mathbb{C} \setminus V) \in \mathbb{C} \Leftrightarrow$ f-1(G) is closed for any $G \subset \mathbb{C}$ closed. $\square$

Proposition 12. *Let $K \subset \mathbb{C}$ be compact and $f : K \to \mathbb{C}$ continuous. Then $f(K)$ is also compact.*

Proof. Let $\{V_\alpha\}_{\alpha\in I}$ be an open cover of $f(K)$, i.e., $f(K) \subseteq \cup_{\alpha\in I} V_\alpha$. By the properties of the inverse image we have $K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(\cup_{\alpha\in I} V_\alpha) = \cup_{\alpha\in I} f^{-1}(V_\alpha)$. Since f is continuous then $f^{-1}(V_\alpha)$ is open for every $\alpha \in I$. This means that $\{f^{-1}(V_\alpha)\}_{\alpha\in I}$ is an open cover of the compact set K. Then there exists a finite upen subcover $\{f^{-1}(V_{\alpha_i})\}_{i=1}^k$ such that $K \subset \cup_{i=1}^k f^{-1}(V_{\alpha_i})$. Now by properties of the direct image $f(K) \subset f^{-1}\left(\cup_{i=1}^k f^{-1}(V_{\alpha_i})\right) = \cup_{i=1}^k f\left(f^{-1}(V_{\alpha_i})\right) \subseteq \cup_{i=1}^k V_{\alpha_i}$. This implies that $\{V_{\alpha_i}\}_{i=1}^k$ is an open subcover for $f(K)$ and we can conclude that $f(K)$ is compact. $\square$

Proposition 13. *Let $A \subset \mathbb{C}$ be connected and $f : A \to \mathbb{C}$ continuous. Then $f(A)$ is also connected.*

Proof. The proof goes by contradiction. Let A be connected and assume that $f(A)$ is not connected. Then, there are two open sets $U, V \subset \mathbb{C}$ such that:

(1) $(f(A) \cap U) \neq \emptyset$ and $(f(A) \cap V) \neq \emptyset$.
(2) U and V are disjoint in $\mathbb{C}$: $(f(A) \cap U) \cap (f(A) \cap V) = \emptyset$
(3) $f(A) \subseteq U \cup V$.

By the continuity of f we have that $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in A.

(1) By properties of the inverse image and using the point (1) we have $A \cap f^{-1}(U) \neq \emptyset$ and $A \cap f^{-1}(V) \neq \emptyset$.

(2) Again by properties of the inverse image and point (2) we have $\left(A \cap f^{-1}(U)\right) \cap \left(A \cap f^{-1}(V)\right) = \emptyset$.

(3) By properties of the inverse image and point (3) we also have $A \subseteq f^{-1}(U) \cup f^{-1}(V)$.

the last three points are in contradiction with A being connected. Hence $f(A)$ should also be connected. $\square$

Proposition 14. *Let $K \subset \mathbb{C}$ be a compact set, $f : K \to \mathbb{C}$ continuous. Then f is uniformly continuous.*

Proof. For this proof, we use the characterisation of compact sets by sequences, and by contradiction.

Suppose then that f is continuous on the compact set K but f not uniformly continuous. Then there exists $\epsilon > 0$ such that for every $\delta > 0$ there exist $z_0, w_0 \in K$ such that on one side $|z_0 - w_0| < \delta$ and on the other $|f(z_0) - f(w_0)| > \epsilon$. Since this holds for any $\delta > 0$, take $\delta_n = \frac{1}{n}$. The for each $n \in \mathbb{N}$ there are $z_n, w_n \in K$ such that $|z_n - w_n| < \frac{1}{n}$ and $|f(z_n) - f(w_n)| > 0\epsilon$.

Now consider the sequence $\{z_n\}_{n=1}^\infty \subseteq K$ just constructed. By compactness we know that there exists a convergent subsequence, namely $\{z_{n_j}\}_{j=1}^\infty$, such that $\lim_{j\to\infty} z_{n_j} = a_0$.

The inequality $|z_{n_j} - w_{n_j}| < \frac{1}{n_j}$ implies that the subsequence $\{w_{n_j}\}_{j=1}^\infty$ converges also to the same limit, $\lim_{j\to\infty} w_{n_j} = a_0$.

By the continuity of f it holds

$$\lim_{j\to\infty} f(z_{n_j}) = f(w_{n_j}) = f(a_0).$$

Then, there exits $j_0 > 0$ such that for any $j > j_0$ it holds simultaneously

$$|f(z_{n_j}) - f(a_0)| < \frac{\epsilon}{2}, \quad \text{and} \quad |f(w_{n_j}) - f(a_0)| < \frac{\epsilon}{2}.$$

The then we have the following sequence of inequalities

$$\epsilon < |f(z_0) - f(w_0)| < |f(z_{n_j}) - f(a_0)| + |f(w_{n_j}) - f(a_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which its a contradiction itself. Consequently f is uniformly continuous in K. $\square$