NOTES FOR INTRODUCTION TO COMPLEX ANALYSIS

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Contents

1. Sequences and limits

Definition 1. *A sequence in $\mathbb{C}$ is a function $\varphi : \mathbb{N} \to \mathbb{C}$. We use the notation $z_n = \varphi(n)$ and $\{z_n\}_{n=1}^{\infty}$ for the image $\varphi(\mathbb{N})$.*

Definition 2. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that the sequence converges to $z_0 \in \mathbb{C}$ if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ it holds that $|z_n - z_0| < \epsilon$. In this case z_0 is called the limit of the sequence $\{z_n\}_{n=1}^{\infty}$ and we write*

$$\lim_{n\to\infty} z_n = z_0.$$

Proposition 1. *If a sequence converges to a limit, this is unique.*

Proof. Supose that the sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ has two limits z_0 and w_0. Then for any $\epsilon > 0$ choose $n \in \mathbb{C}$ such that for any $n \geq n_0$, we have simultaneously

$$|z_n - z_0| < \frac{\epsilon}{2}, \text{ and } |z_n - w_0| < \frac{\epsilon}{2}.$$

Then

$$|z_0 - w_0| = |z_0 - z_n + z_n - w_0| \leq |z_n - z_0| + |z_n - w_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this happens only if |z0 - w0| $= 0$. Then $z_0 = w_0$. □

Proposition 2. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$, with $z_n = x_n + iy_n$, and $z_0 = x_0 + iy_0$. Then the sequence converges to z_0 if and only if the sequences of real numbers $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge respectively to x_0 and y_0. Equivalently*

$$\lim_{n\to\infty} z_n = z_0 \iff \lim_{n\to\infty} x_n = x_0 \text{ and } \lim_{n\to\infty} y_n = y_0.$$

Date: *<2021-03-12 Fri>*.

Proof. Let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ the following holds:

$$|x_n - x_0| = |\mathrm{Re}(z_n - z_0)| \leq |z_n - z_0| < \epsilon.$$

This shows that $\lim_{n\to\infty} x_n = x_0$. The second part follows from a similar argument on the imaginary part of the sequence. □

Proposition 3. *Every convergent sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is bounded, i.e., there exists a positive real number $M > 0$, such that $|z_n| \leq M$ for all $n \in \mathbb{N}$.*

Proof. Let $\epsilon = 1$. By convergence, there exists $n_0 \in \mathbb{N}$ such that whenever $k \geq n_0$ it holds that

$$|z_k - z_0| < 1.$$

Now consider $M = \max\{|z_1| + 1, |z_2| + 1, \ldots, |z_{k-1}| + 1, |z_0| + 1\}$. Then it is clear that

$$|z_n| \leq M, \text{ for all } n \in \mathbb{N}.$$

□

Remark The converse of last proposition is false. Discuss an example.

Theorem 1. *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{C}$ has a convergent sub-sequence.*

Proof. The proof relies on the same result known for sequences of Real numbers.

Let $\{z_n\}_{n=1}^{\infty}$ a bounded sequence where $z_n = x_n + iy_n$. Since the sequence is bounded we note that

$$|x_n| \leq |z_n| \leq M, \quad n \in \mathbb{N}.$$

Then the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is bounded. There exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit $x_0 \in \mathbb{R}$. Now consider the corresponding subsequence of complex numbers $\{z_{n_k}\}_{k=1}^{\infty}$, where $z_{n_k} = x_{n_k} + iy_{n_k}$. Since the sequence of complex numbers is bounded, we conclude that $\{y_{n_k}\}_{k=1}^{\infty}$ is also a bounded sequence of Real numbers. Then there exists a convergent subsequence $\left\{y_{n_{k_j}}\right\}_{j=1}^{\infty}$ wiht limit y_0. Recall that any subsequence of a convergent sequence is also convergent to the same limit, and conclude that $\left\{x_{n_{k_j}}\right\}_{j=1}^{\infty}$ has limit x_0. Then

$$\lim_{j\to\infty} z_{n_{k_j}} = \lim_{j\to\infty} \left(x_{n_{k_j}} + iy_{n_{k_j}}\right) = x_0 + iy$$

is a convergent subsequence of $\{z_n\}_{n=1}^{\infty}$. □

Definition 3. *A sequence $\{z_n\}_{n=1}^{\infty}$ is called Cauchy sequence if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, if $n, m \geq n_0$ then $|z_n - z_m| < \epsilon$.*

Theorem 2. *A sequence $\{z_n\}_{n=1}^{\infty}$ converges if and only if $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence.*

Proof. By writing $z_n = x_n + iy_n$ and from the inequalities

$$|x_n - x_m| \leq |z_n - z_m|,$$

and

$$|y_n - y_m| \leq |z_n - z_m|,$$

it follows that The sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is Cauchy if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences of Real numbers if and only if the sequences of real numbers $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent if and only if $\{z_n\}_{n=1}^{\infty}$ is a convergent sequence. □

Proposition 4. *Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ two convergent sequences, with $\lim_{n\to\infty} z_n = z_0$ and $\lim_{n\to\infty} w_n = w_0$. Then the following identities hold:*

(1) $\lim_{n\to\infty} (z_n + w_n) = z_0 + w_0$,
(2) $\lim_{n\to\infty} (z_n\, w_n) = z_0\, w_0$,
(3) $\lim_{n\to\infty} |z_n| = |z_0|$,
(4) If $w_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \left(\dfrac{z_n}{w_n}\right) = \dfrac{z_0}{w_0}$.

Proof. Proof of 1. It follows from the triangle inequality and the definition. Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it simultaneously holds $|z_n - z_0| < \frac{\epsilon}{2}$ and $|w_n - w_0| < \frac{\epsilon}{2}$, then

$$|(z_n + w_n) - (z_0 + w_0)| = |(z_n - z_0) + (w_n - w_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

That is $\lim_{n\to\infty} (z_n + w_n) = z_0 + w_0$.

Proof of 2. It follows also from the definition, the triangle inequality and the fact that every convergent sequence is bounded. Since $\{z_n\}_{n=1}^{\infty}$ is convergent, there is $M_1 > 0$ such that $|z_n| < M_1$ for all $n \in \mathbb{N}$ and the same for $\{w_n\}_{n=1}^{\infty}$, there is $M_2 > 0$ such that $|w_n| < M_1$ for all $n \in \mathbb{N}$. Take $M = \max\{M_1, M_2\}$, and then $M > 0$ is a bound for both sequences. Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it simultaneously holds $|z_n - z_0| < \frac{\epsilon}{2M}$ and $|w_n - w_0| < \frac{\epsilon}{2M}$. Then

$$\begin{aligned} |z_n w_n - z_0 w_0| &= |z_n w_n - z_n w_0 + z_n w_0 - z_0 w_0| \\ &\leq |z_n w_n - z_n w_0| + |z_n w_0 - z_0 w_0| \\ &= |z_n||w_n - w_0| + |w_0||z_n - z_0| \\ &\leq M|w_n - w_0| + M|z_n - z_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

That is $\lim_{n\to\infty} (z_n\, w_n) = z_0\, w_0$.

Proof of 3. It follows from a version of the triangle inequality. Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it holds $|z_n - z_0| < \epsilon$. Note then that

$$\big|\,|z_n| - |z_0|\,\big| \leq |z_n - z_0| < \epsilon.$$

That is $\lim_{n\to\infty} |z_n| = |z_0|$.

Proof of 4. Let $\epsilon_1 = \frac{|w_0|}{2}$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it holds $|w_n - w_0| < \frac{|w_0|}{2}$. First note that

$$\left|\frac{1}{w_n} - \frac{1}{w_0}\right| = \left|\frac{w_0 - w_n}{w_n w_0}\right| = \frac{|w_0 - w_n|}{|w_n|\,|w_0|}.$$

On the other hand by the triangle inequality

$$|w_0| - |w_n| \leq \big|\,|w_0| - |w_n|\,\big| \leq |w_0 - w_n| < \frac{|w_0|}{2},$$

this implies
$$\frac{|w_0|}{2} \leq |w_n|,$$
or equivalently
$$\frac{1}{|w_n|} \leq \frac{2}{|w_0|}.$$
Then we get
$$\left|\frac{1}{w_n} - \frac{1}{w_0}\right| \leq \frac{|w_0 - w_n|}{|w_n|\,|w_0|} \leq \frac{2|w_0 - w_n|}{|w_0|^2}.$$

For any $\epsilon > 0$ define $\epsilon_2 = \frac{|w_0|^2\epsilon}{2}$, then there exists $m_0 \in \mathbb{N}$ such that for any $n \geq m_0$ it holds $|w_n - w_0| < \frac{|w_0|^2\epsilon}{2}$. Take $N_0 = \max\{n_0, m_0\}$. Then we can improve our last inequality, for any $n \geq N_0$ it holds
$$\left|\frac{1}{w_n} - \frac{1}{w_0}\right| \leq \frac{|w_0 - w_n|}{|w_n|\,|w_0|} \leq \frac{2|w_0 - w_n|}{|w_0|^2} < \epsilon.$$
This shows that $\lim\limits_{n\to\infty} \dfrac{1}{w_n} = \dfrac{1}{w_0}$. Finally applying (2) of this proposition we have
$$\lim_{n\to\infty} \frac{z_0}{w_n} = \lim_{n\to\infty} z_n \frac{1}{w_n} = \lim_{n\to\infty} z_n \lim_{n\to\infty} \frac{1}{w_n} = \frac{z_0}{w_0}$$
□

Recall from real analysis, that every monotone and bounded sequence of real numbers is convergent.

- If the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing then $\lim_{n\to\infty} a_n = \sup\{a_n | n \in \mathbb{N}\}$.
- If the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing then $\lim_{n\to\infty} a_n = \inf\{a_n | n \in \mathbb{N}\}$.

Definition 4. *We say that* $\lim_{n\to\infty} z_n = \infty$ *if* $\lim_{n\to\infty} |z_n| = \infty$. *In other words, if for every* $M > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *then* $|z_n| > M$.

Definition 5. *Let* $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ *be a sequence. A complex number* $w_0 \in \mathbb{C}$ *is called a limit point of the sequence* $\{z_n\}_{n=1}^{\infty}$ *if there exists a subsequence* $\{z_{n_k}\}_{k=1}^{\infty}$ *of* $\{z_n\}_{n=1}^{\infty}$ *such that*
$$\lim_{k\to\infty} z_{n_k} = w_0.$$

2. Continuity

Definition 6. *The function* $f(z)$ *is said to have the limit* $w_0 \in \mathbb{C}$ *as* z *tends to* $a \in \mathbb{C}$, *if for every* $\epsilon > 0$ *there exists a number* $\delta = \delta(\epsilon) > 0$ *with the property that if* $|z - a| < \delta$ *then it holds* $|f(z) - w_0| < \epsilon$, *and in that case we write*
$$\lim_{z\to a} f(z) = w_0.$$

Definition 7. *The function* $f(z)$ *is said to be continuous at* z_0 *if for if for every* $\epsilon > 0$ *there exists a number* $\delta = \delta(\epsilon) > 0$ *with the property that if* $|z - a| < \delta$ *then it holds* $|f(z) - f(a)| < \epsilon$, *and in that case we write*
$$\lim_{z\to a} f(z) = f(a).$$

Examples

- Let $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ the function given by
$$f(z) = \frac{z^2 - 4}{z - 2},$$
then $\lim_{z\to 2} f(z) = 4$.
- Let $f : \mathbb{C} \to \mathbb{C}$ be the conjugate function $f(z) = \bar{z}$. Then $f(z)$ is continuous at every $z_0 \in \mathbb{C}$. To show this, first note that $|z| = |\bar{z}|$. Let $\epsilon > 0$ and choose $0 < \delta = \epsilon$. Then, for any $z \in \mathbb{C}$ such that $|z - z_0|$ we can estimate
$$|f(z) - f(z_0)| = |\bar{z} - \bar{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon.$$
.

Proposition 5. *Let $U \subset \mathbb{C}$, $f : U \to \mathbb{C}$ and $a \in U$. Then the following three statements are equivalent:*

(1) f is continuous at a.
(2) For every $\epsilon > 0$ there exists a number $\delta > 0$ such that $f(B_\delta(a) \cap U) \subseteq B_\epsilon(f(a))$.
(3) For every sequence $\{z_n\}_{n=1}^\infty \subseteq U$ such that $\lim\limits_{n\to\infty} z_n = a$, we have
$$\lim_{n\to\infty} f(z_n) = f(a).$$

Proof. ($1 \Rightarrow 2$) Let $\epsilon > 0$, then there is a number $\delta > 0$ such that for every $z \in U$ with $|z - a| < \delta$ then $|f(z) - f(a)| < \epsilon$. This implies that if $z \in B_\delta(a) \cap U$ then $f(B_\delta(a) \cap U) \subseteq B_\epsilon(f(a))$.

($2 \Rightarrow 1$) Let $\epsilon > 0$, then there is a number $\delta > 0$ such that $f(B_\delta(a) \cap U) \subseteq B_\epsilon(f(a))$. This means that for every $z \in B_\delta(a) \cap U$ we have $f(z) \in B_\epsilon(f(a))$. This means that for every $z \in U$ with $|z - a| < \delta$ it holds $|f(z) - f(a)| < \epsilon$.

($1 \Rightarrow 3$) Let $f(z)$ be continuous at $a \in U$ and $\{z_n\}_{n=1}^\infty \subseteq U$ with limit a. For any $\epsilon > 0$ there exists a number $\delta > 0$ such that for every $z \in U$ with $|z - a| < \delta$ it holds $|f(z) - f(a)| < \epsilon$. For that $\delta > 0$ there is a $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $|z_n - a| \leq \delta$. Then by continuity this implies that $|f(z_n) - f(a)| < \epsilon$. This shows that $\lim_{n\to\infty} f(z_n) = f(a)$.

($3 \Rightarrow 1$) Let $a \in U$ and assume that for any $\{z_n\}_{n=1}^\infty \subseteq U$ with limit a we have $\lim_{n\to\infty} f(z_n) = f(a)$. Suppose on the contrary, that f is not continuous at a. Then there exists a positive $\epsilon > 0$ such that for every $\delta > 0$ there is a point $z \in U$ with $|z - a| < \delta$ but $|f(z) - f(a)| \geq \epsilon$. Note that this implies that for any $n \in \mathbb{N}$ we can choose $\delta = \frac{1}{n}$ and $z_n \in U$ such that $|z_n - a| < \frac{1}{n}$, but $|f(z_n) - f(a)| \geq \epsilon$. In this way we have constructed a sequence $\{z_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} z_n = a$ but $\lim_{n\to\infty} f(z_n) \neq f(a)$ which is a contradiction to our initial assumptions.

($1 \Rightarrow 3$) Let $f(z)$ be continuous at a. Consider any sequence $\{z_n\}_{n=1}^\infty \subseteq U$ such that $\lim_{n\to\infty} z_n = a$. Then for $\epsilon > 0$ there is a number $\delta > 0$ such that for all $z \in U$ with $|z - a| < \delta$ we have $|f(z) - f(a)| < \epsilon$. Also for such $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $|z_n - a| < \delta$, which then implies $|f(z_n) - f(a)| < \epsilon$. □

Theorem 3. *Let $U \subseteq \mathbb{C}$ and $f, g : U \to \mathbb{C}$ be two continuous functions at $a \in U$. Then $f \pm g$, $f \cdot g$ are continuous in a. If $g(a) \neq 0$ then $\dfrac{f}{g}$ is also continuous at a.*

Theorem 4. *Let $U_1, U_2 \subseteq \mathbb{C}$, and $f : U_1 \to \mathbb{C}$, $g : U_2 \to \mathbb{C}$, such that $f(U_1) \subseteq U_2$. If f is continuous at $z_0 \in U_1$ and g is continuous at $w_0 = f(z_0) \in U_2$, then the composition $g \circ f$ is continuous at z_0.*

Proof. Consider any sequence $\{z_n\}_{n=1}^{\infty} \subseteq U_1$ such that $\lim_{n\to\infty} z_n = z_0$. Then define $w_n = f(z_n) \in U_2$. Since f is continuous at z_0, then $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} w_n = w_0 = f(z_0)$. Since g is continuous at w_0 then $\lim_{n\to\infty} g(w_n) = g(w_0)$. Note that on the left hand side $g(w_n) = g(f(z_n)) = (g\circ f)(z_n)$, and on the right hand side $g(w_0) = g(f(z_0)) = (g\circ f)(z_0)$. This shows that $\lim_{n\to\infty}(g\circ f)(z_n) = (g\circ f)(z_0)$. □

Definition 8. *Let $U \subseteq \mathbb{C}$ and consider a function $f : U \to \mathbb{C}$. f is called uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $z, w \in U$ such that $|z - w| < \delta$ then we have $|f(z) - f(w)| < \epsilon$.*

Example. Consider $f : A \to \mathbb{C}$, $f(z) = z^2$ in the following cases:

- $A = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$.

In this case f is uniformly continuous in A. Let $\epsilon > 0$ and take $\delta = \epsilon/2$. Then, for any $z, w \in A$ such that $|z - w| < \delta$ we have

$$|f(z) - f(w)| = |z^2 - w^2| = |z - w||z + w| \leq |z - w|(|z| + |w|) \leq 2|z - w| < \epsilon.$$

- $A = \mathbb{C}$.

In this case f is not uniformly continuous. Take $\epsilon = 1$. For every $\delta > 0$ there exists $n \in \mathbb{N}$ such that $n\delta > 1$. Now consider $z = n$ and $w = n + \delta/2$. Note that we have $|z - w| = \delta/2 < \delta$, but

$$|f(z) - f(w)| = |n^2 - \left(n + \frac{\delta}{2}\right)^2 | = n\delta + \frac{\delta^2}{4} > n\delta > 1 = \epsilon.$$

3. Basic topology of $\mathbb{C}$

The open and closed disks (balls) already defined, are basic subsets that may be used to build a topological structure of the complex plane. A topology allows us to define several notions of continuity.

$$B_r(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| < r\}.$$
$$\bar{B}_r(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| \leq r\}.$$

Definition 9. *A set $A \subseteq \mathbb{C}$ is called open if $\forall z \in A$ there exists a real number $r_z > 0$ such that $B_{r_z}(z) \subseteq A$.*

Theorem 5. *The following sentences are true*

(1) The sets $\mathbb{C}$, $\emptyset$, $B_r(z)$ (for any $r > 0$) and any $z \in \mathbb{C}$, are open sets.
(2) If $U_1, \ldots, U_n$ is a finite collection of open sets, then $\cap_{k=1}^n U_k$ is an open set.
(3) If $\{U_\alpha\}_{\alpha\in I}$ is a family of open sets, then $\cup_{\alpha\in I} U_\alpha$ is an open set.

Proof. $\mathbb{C}$ is open. For any $z \in \mathbb{C}$, take $r = 1$ and evidently $B_1(z) \subset \mathbb{C}$.

$\emptyset$ is open. If this wasn't true, there would exists $z \in \emptyset$ such that for any $r > 0$, $B_r(z) \not\subseteq \emptyset$. Then the statement is vacuously true.

$B_r(z)$ is open. Let $w \in B_r(z)$. Take $\delta = r - |w - z|$, and note that $\delta > 0$. To show that $B_\delta(w) \subset B_r(z)$ we need to show that any $\xi \in B_\delta(w)$ is also in $B_r(z)$. If $\xi \in B_\delta(w)$ then $|\xi - w| < \delta$. Since $|\xi - z| = |\xi - w + w - z| \leq |\xi - w| + |w - z| < \delta + |w - z| = r - |w - z| + |w - z| = r$. Then $\xi \in B_r(z)$.

$\cap_{k=1}^n U_k$ is open if each U_i is open. Let $z \in \cap_{k=1}^n U_k$. Then for each $1 \leq k \leq n$ there exists $r_k > 0$ such that $B_{r_k}(z) \subseteq U_k$. Take $r = \min\{r_1, \ldots, r_k\}$, and hence $B_r(z) \subset U_k$ for every $1 \leq k \leq n$. Hence $B_r(z) \subset \cap_{k=1}^n U_k$.

$\cup_{\alpha\in I}U_\alpha$ is open if each U_α is open. Let $z \in \cup_{\alpha\in I}U_\alpha$. Then there exists $\beta \in I$ such that $z \in U_\beta$. Since U_β is open, there is $r > 0$ such that $B_r(z) \subset U_\beta \subset \cup_{\alpha\in I}U_\alpha$. $\square$

Definition 10. *A set $G \subseteq \mathbb{C}$ is called closed if its complement $\mathbb{C} \setminus G$, is an open set.*

Proposition 6. *The following sentences are true*

(1) *The sets $\mathbb{C}$, $\emptyset$, are closed sets.*
(2) *If $G_1, \dots, G_n$ is a finite collection of closed sets, then $\cup_{k=1}^n G_k$ is a closed set.*
(3) *If $\{G_\alpha\}_{\alpha\in I}$ is a family of closed sets, then $\cap_{\alpha\in I}G_\alpha$ is a closed set.*

Definition 11. *Let $A \subset \mathbb{C}$. Then we define the following sets*

- *The interior of A:* $\mathit{Int}(A) = \cup\{U \mid U \subset A,\ \text{and } U \text{ is an open set}\}$.
- *The closure of A:* $\mathit{Cl}(A) = \cap\{G \mid A \subset G,\ \text{and } G \text{ is a closed set}\}$.
- *The boundary of A:* $\partial A = \mathit{Cl}(A) \cap \mathit{Cl}(\mathbb{C} \setminus A)$.

Proposition 7. *Let $A, B \subseteq \mathbb{C}$. Then following statements are true*

- $\mathit{Int}(A) \subset A$.
- $A \subset \mathit{Cl}(A)$.
- $\partial A \subseteq \mathit{Cl}(A)$.
- *A is open if and only if $A = \mathit{Int}(A)$.*
- *A is closed if and only if $A = \mathit{Cl}(A)$.*
- $\mathit{Cl}(A \cup B) = \mathit{Cl}(A) \cup \mathit{Cl}(B)$.
- *$\mathit{Cl}(A \cap B) \subseteq \mathit{Cl}(A) \cap \mathit{Cl}(B)$, but in general they are not equal.*
- $\mathit{Int}(A \cap B) = \mathit{Int}(A) \cap \mathit{Int}(B)$.
- *$\mathit{Int}(A \cup B) \supseteq \mathit{Int}(A) \cup \mathit{Int}(B)$, but in general they are not equal.*
- $z_0 \in \mathit{Int}(A) \Leftrightarrow \exists r > 0,$ *such that* $B_r(z_0) \subseteq A$.
- $z_0 \in \mathit{Cl}(A) \Leftrightarrow \forall r > 0,$ *it holds* $B_r(z_0) \cap A \neq \emptyset$.
- $z_0 \in \partial(A) \Leftrightarrow \forall r > 0,$ *it holds* $B_r(z_0) \cap A \neq \emptyset$ *and* $B_r(z_0) \cap (\mathbb{C} \setminus A) \neq \emptyset$.

The topological structure of $\mathbb{C}$ induces in a natural way a topological structure in any subset $A \subset \mathbb{C}$.

Definition 12. *Let $A \subset \mathbb{C}$. The set $B \subseteq A$ is called open (closed) in A, if there is an open (closed) set U of $\mathbb{C}$ such that $B = A \cap U$.*

Proposition 8. *Let $A \subseteq \mathbb{C}$. Then the following sentences are true*

(1) *The sets A, $\emptyset$, are open sets and closed sets in A.*
(2) *If the set $B \subseteq A$ is open in A, then $A \setminus B$ is closed in A.*
(3) *If the set $B \subseteq A$ is closed in A, then $A \setminus B$ is open in A.*
(4) *If $U_1, \dots, U_n$ is a finite collection of open sets in A, then $\cap_{k=1}^n U_k$ is an open set in A.*
(5) *If $\{U_\alpha\}_{\alpha\in I}$ is a family of open sets in A, then $\cup_{\alpha\in I}U_\alpha$ is an open set A.*
(6) *If $G_1, \dots, G_n$ is a finite collection of closed sets in A, then $\cup_{k=1}^n G_k$ is a closed set in A.*
(7) *If $\{G_\alpha\}_{\alpha\in I}$ is a family of closed sets in A, then $\cap_{\alpha\in I}G_\alpha$ is a closed set in A.*

Proposition 9. *Let $A \subset \mathbb{C}$. $z_0 \in \mathit{Cl}(A)$ if and only if there exists a sequence $\{z_n\}_n^\infty \subset A$ such that* $\lim_{n\to\infty} z_n = z_0$.

Proof. By the previous proposition, since $z_0 \in \mathrm{Cl}(A)$, we have that for every $n \in \mathbb{N}$ we can take $r_n = \frac{1}{n} > 0$, and it holds

$$B_{r_n}(z_0) \cap A \neq \emptyset.$$

Then for each n we can choose $z_n \in B_{r_n}(z_0) \cap A$. Clearly we have

$$|z_n - z_0| < \frac{1}{n},$$

for all $n \in \mathbb{N}$, which implies $\lim_{n\to\infty} z_n = z_0$. □

Definition 13. *(1) The set $A \subset \mathbb{C}$ is called **disconnected**, if there are two open sets U, V in $\mathbb{C}$ such that*
- *(a) $(A \cap U) \neq \emptyset$ and $(A \cap V) \neq \emptyset$.*
- *(b) U and V are disjoint in A: $(A \cap U) \cap (A \cap V) = \emptyset$*
- *(c) $A \subseteq U \cup V$.*

*(2) The set A is called **connected** if it is not disconnected.*

Proposition 10. *The set A is connected if and only if only the sets A and $\emptyset$ are the only sets that are both, open and closed in A.*

Proof. ($\Rightarrow$) Let A be connected and suppose on the contrary that there is a set $B \subset A$, that is both, open and closed, and different from A and $\emptyset$. Then B and $A \setminus B$ are open sets in A. Then, there are open sets $U, V \subseteq \mathbb{C}$ such that $B = A \cap U$ and $A \setminus B = A \cap V$. then we have:
1. $(A \cap U) \neq \emptyset$ and $(A \cap V) \neq \emptyset$.
2. $(A \cap U) \cap (A \cap V) = B \cap (A \setminus B) = \emptyset$.
3. $A = B \cup (A \setminus B) = (A \cap U) \cup (A \cap V) = A \cap (U \cup V)$ which implies that $A \subseteq U \cup V$.

These three points are a contradiction since A is assumed connected.

($\Leftarrow$) Let A and $\emptyset$ the only sets that are both, open and closed in A. Suppose on the contrary that A is disconnected. Then there are two open sets in $U, V \subset \mathbb{C}$ satisfying the definition above. Put $B_1 = A \cap U$ and $B_2 = A \cap V$. Then B_1 and B_2 are non-empty open sets in A. Since $B_1 \cap B_2 = \emptyset$, and $A = B_1 \cup B_2$. Then $A \setminus B_1 = B_2$, which implies that B_2 is also closed. Moreover, since $B_1 \neq \emptyset$, $B_2 \neq A$, which is a contradiction. Then A is connected. □

Definition 14. *Let $A \subset \mathbb{C}$. By a curve γ in A we mean a continuous map $\gamma : [0,1] \to A$. We say that A is called **arc-connected** or **path-connected** if for any two points $z_1, z_2 \in \mathbb{C}$ there exists a curve γ in A such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$.*

Theorem 6. *Let $A \subset \mathbb{C}$.*
1. *If A is arc-connected then A is connected.*
2. *If A is open and connected then A is arc-connected.*

Here is an example of a connected set which fails to be arc-connected Example.

A **domain** $\Omega \subseteq \mathbb{C}$ is an open connected set.

Definition 15. *Let $A \subseteq \mathbb{C}$. An open cover of A is a family $\{U_\alpha\}_{\alpha\in I}$ of open subsets of $\mathbb{C}$, such that $A \subset \cup_{\alpha\in I} U_\alpha$.*

A finite subcover is a collection $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ such that $A \subseteq \cup_{j=1}^{k} U_{\alpha_j}$.

Definition 16. *The set $K \subset \mathbb{C}$ is called **compact** if any open cover of K, has a finite subcover.*

Theorem 7. ***(Heine-Borel).*** *The set $K \subset \mathbb{C}$ is compact if and only K is closed and bounded.*

Proof. ($\Rightarrow$).

If K is compact then K is closed. Supose there is $z_0 \in \mathrm{cl}(K)$ such that $z_0 \notin K$, and consider the following sets: All balls $B_r(z_0)$ and for each $z \in K$ choose a ball $U_z := B_{r_z}(z)$ such that $r_z > 0$ is small enough to not intersect one of the $B_r(z_0)$. Note now that the collection $\{U_z\}_{z\in K}$ is an open cover of K but clearly any finite choice of U_z's fails to cover K.

If K is compact then K is bounded. For each \$z $\in K$ consider the set $U_z = B_1(z)$. Clearly $\{U_z\}_{z\in K}$ is an open cover of K, and then there is a finite open subcover $U_{z_1}, \dots, U_{z_n}$. Take $M = \max\{|z_1| + 1, \dots, |z_k| + 1\}$. Then $K \subset \cup_{k=1}^n U_{z_k} \subseteq B_M$, i.e. K is bounded.

($\Leftarrow$). Exercise. *Hint.* Show first that any closed subset of a compact set is also compact. Use the bounded property to show that K must be contained in a square of the form $G = [-r, r] \times [-r, r]$ and show that show that G is compact. □

Theorem 8. *$K \subset \mathbb{C}$ is compact if and only if for every sequence $\{z_n\}_{n=1}^\infty \subset K$ has a convergent subsequence, i.e., $\{z_{n_k}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} z_{n_k} = z_0$ for some $z_0 \in K$.*

Proof. ($\Rightarrow$) Let $K \subset \mathbb{C}$ compact. If $\{z_n\}_{n=1}^\infty \subset K$ is a bounded sequence since K is bounded. By Bolzano-Weierstrass' Theorem, the sequence has a convergent subsequece, say to a limit point $z_0 \in \mathrm{Cl}(K)$. Since K is closed then $z_0 \in K$.

($\Leftarrow$) Suppose that any $\{z_n\}_{n=1}^\infty \subset K$ has a convergent subsequece. IF K is not bounded, then we can construct a sequence $\{z_n\}_{n=1}^\infty$ such that $|z_n| > n$ for all $n \in \mathbb{N}$. But this sequence has no convergent subsequence, which is a contradiction. Then K should be bounded.

Now take any $z_0 \in \mathrm{Cl}(K)$. By a previous proposition, we can construct a sequence $\{z_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} z_n = z_0$. By our hypothesis, the sequence $\{z_{n_k}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} z_{n_k} = w_0$, for some $w_0 \in K$. By the uniqueness of the limit, $z_0 \in K$. We he have shown $\mathrm{Cl}(K) \subseteq K$, which implies $\mathrm{Cl}(K) = K$, i.e., K is closed. □

Proposition 11. *Let $f : A \to \mathbb{C}$ be a function. Then the following sentences are equivalent*

1. *f is continuous in A.*
2. *For every open set $V \subset \mathbb{C}$, the set $f^{-1}(V)$ is open in A.*
3. *For every closed set $G \subset \mathbb{C}$, the set $f^{-1}(G)$ is closed in A.*

Proof. ($1 \Rightarrow 2$) Assume f is continuous in A. Let $V \subset \mathbb{C}$ any open such that $f(A) \cap V \neq \emptyset$. We want to show that $f^{-1}(V)$ is an open subset of A. Let $z_0 \in f^{-1}(V)$, then $f(z_0) \in V$, and since V is open there exists an open ball $B_r(f(z_0)) \subset V$. Since f is assume to be continuous, if we let $\epsilon = r$, then there exists $\delta > 0$ such that for any $z \in A$ such that $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \epsilon$. This is the same as saying that there is a $\delta > 0$ such that $B_\delta(z_0) \cap A \subseteq f^{-1}(B_r(f(z_0))) \subset f^{-1}(V)$. Then \$U$_\delta =$ which implies z_0 is an interior point. Since z_0 is arbitary, $f^{-1}(V)$ is an open subset of A.

($2 \Rightarrow 1$) Assume now that for any $V \subset \mathbb{C}$ we have that $f^{-1}(V)$ is an open set in A. Let $z_0 \in A$, then for $f(z_0) \in \mathbb{C}$, and for any $\epsilon > 0$ consider the open set $V = B_\epsilon(f(z_0))$. Then by our hypothesis, $f^{-1}(V)$ is an open set in A, which means that there is an open set $U \subset \mathbb{C}$ such that $U \cap A$ is completely contained in $f^{-1}(V) \cap A$. More over, since $z_0 \in U$ there exists $\delta > 0$ such that $B_\delta(z_0)$ is contained in U and also $f(B_\delta(z_0) \cap A) \subset B_\epsilon(f(z_0))$. The by one of our previous propositions, f is continuous at every point of A.

$(2 \Leftrightarrow 3)$ $f^{-1}(V)$ is open in A for any $V \in \mathbb{C}$ open $\Leftrightarrow A \setminus f^{-1}(V) = f^{-1}(\mathbb{C} \setminus V)$ is closed in A for any closed set $(\mathbb{C} \setminus V) \in \mathbb{C} \Leftrightarrow$ f-1(G) is closed for any $G \subset \mathbb{C}$ closed. □

Proposition 12. *Let $K \subset \mathbb{C}$ be compact and $f : K \to \mathbb{C}$ continuous. Then $f(K)$ is also compact.*

Proof. Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$, i.e., $f(K) \subseteq \cup_{\alpha \in I} V_\alpha$. By the properties of the inverse image we have $K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(\cup_{\alpha \in I} V_\alpha) = \cup_{\alpha \in I} f^{-1}(V_\alpha)$. Since f is continuous then $f^{-1}(V_\alpha)$ is open for every $\alpha \in I$. This means that $\{f^{-1}(V_\alpha)\}_{\alpha \in I}$ is an open cover of the compact set K. Then there exists a finite upen subcover $\{f^{-1}(V_{\alpha_i})\}_{i=1}^k$ such that $K \subset \cup_{i=1}^k f^{-1}(V_{\alpha_i})$. Now by properties of the direct image $f(K) \subset f^{-1}\left(\cup_{i=1}^k f^{-1}(V_{\alpha_i})\right) = \cup_{i=1}^k f\left(f^{-1}(V_{\alpha_i})\right) \subseteq \cup_{i=1}^k V_{\alpha_i}$. This implies that $\{V_{\alpha_i}\}_{i=1}^k$ is an open subcover for $f(K)$ and we can conclude that $f(K)$ is compact. □

Proposition 13. *Let $A \subset \mathbb{C}$ be connected and $f : A \to \mathbb{C}$ continuous. Then $f(A)$ is also connected.*

Proof. The proof goes by contradiction. Let A be connected and assume that $f(A)$ is not connected. Then, there are two open sets $U, V \subset \mathbb{C}$ such that:

(1) $(f(A) \cap U) \neq \emptyset$ and $(f(A) \cap V) \neq \emptyset$.
(2) U and V are disjoint in $\mathbb{C}$: $(f(A) \cap U) \cap (f(A) \cap V) = \emptyset$
(3) $f(A) \subseteq U \cup V$.

By the continuity of f we have that $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in A.

(1) By properties of the inverse image and using the point (1) we have $A \cap f^{-1}(U) \neq \emptyset$ and $A \cap f^{-1}(V) \neq \emptyset$.

(2) Again by properties of the inverse image and point (2) we have $\left(A \cap f^{-1}(U)\right) \cap \left(A \cap f^{-1}(V)\right) = \emptyset$.

(3) By properties of the inverse image and point (3) we also have $A \subseteq f^{-1}(U) \cup f^{-1}(V)$.

the last three points are in contradiction with A being connected. Hence $f(A)$ should also be connected. □

Proposition 14. *Let $K \subset \mathbb{C}$ be a compact set, $f : K \to \mathbb{C}$ continuous. Then f is uniformly continuous.*

Proof. For this proof, we use the characterisation of compact sets by sequences, and by contradiction.

Suppose then that f is continuous on the compact set K but f not uniformly continuous. Then there exists $\epsilon > 0$ such that for every $\delta > 0$ there exist $z_0, w_0 \in K$ such that on one side $|z_0 - w_0| < \delta$ and on the other $|f(z_0) - f(w_0)| > \epsilon$. Since this holds for any $\delta > 0$, take $\delta_n = \frac{1}{n}$. The for each $n \in \mathbb{N}$ there are $z_n, w_n \in K$ such that $|z_n - w_n| < \frac{1}{n}$ and $|f(z_n) - f(w_n)| > 0\epsilon$.

Now consider the sequence $\{z_n\}_{n=1}^{\infty} \subseteq K$ just constructed. By compactness we know that there exists a convergent subsequence, namely $\{z_{n_j}\}_{j=1}^{\infty}$, such that $\lim_{j\to\infty} z_{n_j} = a_0$.

The inequality $|z_{n_j} - w_{n_j}| < \frac{1}{n_j}$ implies that the subsequence $\{w_{n_j}\}_{j=1}^{\infty}$ converges also to the same limit, $\lim_{j\to\infty} w_{n_j} = a_0$.

By the continuity of f it holds

$$\lim_{j\to\infty} f(z_{n_j}) = f(w_{n_j}) = f(a_0).$$

Then, there exits $j_0 > 0$ such that for any $j > j_0$ it holds simultaneously

$$|f(z_{n_j}) - f(a_0)| < \frac{\epsilon}{2}, \quad \text{and} \quad |f(w_{n_j}) - f(a_0)| < \frac{\epsilon}{2}.$$

The then we have the following sequence of inequalities

$$\epsilon < |f(z_0) - f(w_0)| < |f(z_{n_j}) - f(a_0)| + |f(w_{n_j}) - f(a_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which its a contradiction itself. Consequently f is uniformly continuous in K. $\square$

4. Derivatives

The definition of the derivative of a complex function is very familiar from the one we know for the real valued functions. But also we should explore in this section how this definition relates with that of the derivative of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. The properties of complex differentiation have been extensively used and applied in other areas of mathematics, leading to astonishing consequences and results.

We will give the concept of derivative for functions defined on open subsets of the complex plane, although it can be generalised to subsets which are not necessarily open by writing the definitions using the induced topology. In order to keep the exposition and notation simple, we will be dealing only with functions defined on open subsets of the complex plane. Recall that a **domain** $\Omega \subseteq \mathbb{C}$ is an open connected set.

Definition 17. *Let $\Omega \subset \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$. We say that f is (complex) differentiable at $z_0 \in \Omega$ if the following limit exists*

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

*The limit is called the derivative of f at z_0 and will be denoted by $f'(z_0)$, or $\frac{df}{dz}(z_0)$. If f is differentiable at every point of a domain Ω, then we call f **holomorphic** in Ω.*

*We use the term **analytic** with the same meaning as holomorphic. If f is holomorphic on the whole complex plane $\mathbb{C}$, then we call f **entire**.*

Examples of holomorphic functions are:

- $f(z) = c$, (the constant function). With $f'(z) = 0$, for all $z \in \mathbb{C}$.
- $f(z) = z$, (the identity function). With $f'(z) = 1$, for all $z \in \mathbb{C}$.
- $f(z) = z^n$. With $f'(z) = nz^{n-1}$, for all $z \in \mathbb{C}$.

The function $f(z) = \bar{z}$ is not complex differentiable anywhere in $\mathbb{C}$. Take $z_0 \in \mathbb{C}$ and consider the following two sequences converging to the same complex number $z_0 = x_0 + iy_0$, $w_n = x_n + iy_0$ and $\xi_n = x_0 + iy_n$. Notice now that

$$\lim_{w_n\to z_0} \frac{f(w_n) - f(z_0)}{w_n - z_0} = \frac{\bar{w_n} - \bar{z_0}}{(x_n + iy_0) - (x_0 + iy_0)} = \frac{x_n - iy_0 - (x_0 - iy_0)}{x_n - x_0} = \frac{x_n - w_0}{x_n - x_0} = 1,$$

and a similar computation gives

$$\lim_{\xi_n \to z_0} \frac{f(\xi_n) - f(z_0)}{\xi_n - z_0} = \frac{x_n - iy_0 - (x_0 - iy_0)}{y_n - y_0} = \frac{-i(y_n - y_0)}{i(y_n - y_0)} = -1,$$

Proposition 15. *If $f : \Omega \to \mathbb{C}$ is complex differentiable at z_0 then it is continuous at z_0*

Proof. Note that since $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and $\lim_{z \to z_0} z - z_0 = 0$, then the limit of o the product exists and it is the product of the limits

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) = f'(z_0) \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0,$$

equivalently

$$\lim_{z \to z_0} f(z) = f(z_0),$$

then, f is continuous at z_0. □

The rules for differentiating complex functions and their proof are similar that the ones one encounter when dealing with real analysis:

Proposition 16. *Let $f, g : \Omega \to \mathbb{C}$ two complex differentiable functions at a point $z_0 \in \mathbb{C}$. Then*

(1) $(c\, f)$ is differentiable at $z_0 \in \mathbb{C}$ and $(c\, f)'(z_0) = c\, f'(z_0)$.
(2) $(f \pm g)$ is differentiable at $z_0 \in \mathbb{C}$ and $(f \pm g)'(z_0) = f'(z_0) + g'(z_0)$.
(3) $(f\, g)$ is differentiable at $z_0 \in \mathbb{C}$ and $(f\, g)'(z_0) = f'(z_0)\, g(z_0) + f(z_0)\, g'(z_0)$.
(4) $\left(\dfrac{f}{g}\right)$, $g(z_0) \neq 0$, is differentiable at $z_0 \in \mathbb{C}$ and $\left(\dfrac{f}{g}\right)'(z_0) = \frac{f'(z_0)\, g(z_0) - f(z_0)\, g'(z_0)}{g(z_0)^2}$.

Proposition 17. *Let $U, V \subseteq \mathbb{C}$ be two open sets and consider the functions $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ where $f(U) \subseteq V$. If f is complex differentiable at $z_0 \in U$, and g complex differentiable at $f(z_0) \in V$, then the composition $g \circ f : U \to \mathbb{C}$, given by $(g \circ f)(z) = g(f(z))$, is complex differentiable at $z_0 \in U$, and*

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0).$$

Proof. Case 1. Suppose $f'(z_0) \neq 0$. Then $f(z) - f(z_0) \neq 0$ holds for a small neighbourhood around z_0. We can write

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \frac{f(z)) - f(z_0)}{z - z_0}.$$

Since f is complex differentiable at z_0, then it is continuous and it holds

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Then, by taking the limit we obtain

$$\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \lim_{z \to z_0} \frac{f(z)) - f(z_0)}{z - z_0} = g'(f(z_0))\, f(z_0).$$

Case 2. Suppose now that $f'(z_0) = 0$. Then one can show that $g'(f(z_0)) = 0$ and then would finish the proof. Note that since $g'(f(z_0))$ exists, then there is a neighbourhood around $f(z_0)$ such that the quotient

$$\frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)},$$

is bounded, i.e. , there is a positive number $M > 0$, and $\delta > 0$ such that for every $0 < |f(z) - f(z_0)| < \delta$ we have

$$\left| \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \right| < M.$$

Equivalently

$$|g(f(z)) - g(f(z_0))| < M|f(z) - f(z_0)|.$$

Then

$$\left| \frac{g(f(z)) - g(f(z_0))}{z - z_0} \right| < M \left| \frac{f(z) - f(z_0)}{z - z_0} \right|.$$

This shows that $(g \circ f)'(z_0) = 0 = g'(f(z_0))f'(z_0) = g'(f(z_0)) \cdot 0$. □

Now, since $f : \mathbb{C} \to \mathbb{C}$ can also be viewed as a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, lets explore the relationship between the two definitions of derivatives. First recall

Definition 18. *A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ if there is a linear transformation $Df_a : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - Df_a(h)|}{|h|} = 0.$$

In order to represent this linear transformation as a matrix, we introduce partial derivatives for a function $f : \mathbb{R}^n \to \mathbb{R}$ at $a \in \mathbb{R}^n$ as the limit

$$\frac{\partial f(a)}{\partial x_i} = \lim_{h \to 0} \frac{f(a_1, a_2, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{h}.$$

Theorem 9. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $a \in \mathbb{R}^n$, then all the first order partial differential exist, and the matrix representing the linear transformation Df_a is*

$$(Df_a) = \begin{pmatrix} \frac{\partial f_1(a)}{\partial x_1} & \cdots & \frac{\partial f_1(a)}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m(a)}{\partial x_1} & \cdots & \frac{\partial f_m(a)}{\partial x_n} \end{pmatrix}.$$

Remark Note that (Df_a) is a matrix with m rows and n columns.

The converse is not true unless we add a stronger hypothesis

Theorem 10. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, with $f = (f_1, f_2, \dots, f_m)$. The derivative Df_a exists if all first order partial derivatives $\frac{\partial f_i}{\partial x_j}$, $1 \le j \le n$, and $1 \le i \le m$, exist in an open set containing a, and each of these partial derivatives are continuous at a.*

Then last statement says that Df exists if the functions $f_i \in C^1$ for $1 \le i \le m$.

> It is possible for a differentiable function to have discontinuous partial derivatives. That is, continuous partial derivatives imply differentiability but not vice-versa.

Example: Be aware that it may happen that a map has all partial derivative, and still not being differentiable at a given point. The following is an example of a map whose all partial derivatives at $(0,0)$ exist, but it is not differentiable at $(0,0)$.

$$f(x,\, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{for } (x,\, y) \ne (0,\, 0), \\ 0 & \text{for } (x,\, y) = (0,\, 0). \end{cases}$$

Remark. If $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at z_0, then its derivative is a complex number, $f'(z_0) = a_1 + ia_2$ for some $a_1, a_2 \in \mathbb{R}$. The corresponding linear transformation representing $f'(z_0)$ by complex multiplication is $T : \mathbb{R}^2 \to \mathbb{R}^2$ acting on to the basic elements $(1,0),(0,1)$ as $T(1,0) = (a_1, a_2)$, and $T(0,1) = (-a_2, a_1)$. Then the matrix representation of T is

$$T = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$$

5. Cauchy-Riemann Equations

Theorem 11. *Let $\Omega \subseteq \mathbb{C}$ an open set $f : \mathbb{C} \to \mathbb{C}$, and $z_0 = x_0 + iy_0 \in \Omega$. Write $f(z) = u(x,y) + iv(x,y)$, where $u, v : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. The function f is complex differentiable at $z_0 \Leftrightarrow u, v$ are real differentiable at (x_0, y_0) and the following identities hold*

$$\begin{aligned} &\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0), \\ &\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \end{aligned} \tag{1}$$

Proof. $(\Rightarrow)$

Suppose f is complex differentiable at z_0. Then by definition

$$\lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right] = 0,$$

Then taking module we also have

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| = \lim_{z \to z_0} \frac{|f(z) - f(z_0) - f'(z)(z - z_0)|}{|z - z_0|} = 0.$$

Now writing the last expression as mapping between $\mathbb{R}^2$, i.e, $z = x + iy = (x,y)$, $z_0 = x_0 + iy_0 = (x_0, y_0)$, and using the matrix representation of $f'(z_0) = a_1 + ia_2$, we can rewrite the last equaiton as follows

$$\lim_{z \to z_0} \frac{\left| f(x,y) - f(x_0,y_0) - \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right|}{|(x - x_0, y - y_0)|} = \lim_{(x,y) \to (x_0,y_0)} \frac{|f(x,y) - f(x_0,y_0) - T(x - x_0, y - y_0)|}{|(x - x_0, y - y_0)|} =$$

which is clearly equivalent to

$$\lim_{h \to 0} \frac{|f(x_0 + h_1, y_0 + h_2) - f(x_0,y_0) - T(h)|}{|h|} = 0,$$

hence $f : \mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable.

Then we know that if $f(x,y) = u(x,y) + iv(x,y)$ is real differentiable at $z_0 = (x_0, y_0)$. Then the partial derivatives of u and v exist and moreover we showed that $Df_{z_0} = T$. Looking at the matrix representation we have

$$Df_{z_0} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} = T$$

which implies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}(x_0,y_0) = a_1 = \frac{\partial v}{\partial y}(x_0,y_0), \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -a_2 = -\frac{\partial v}{\partial x}(x_0,y_0).$$

$(\Leftarrow)$

Let $u, v : \mathbb{R}^2 \to \mathbb{R}$, real differentiable functions at $z_0 = (x_0, y_0)$, satisfying the Cauchy-Riemann equations. Define a_1, a_2 by

$$\frac{\partial u}{\partial x}(x_0, y_0) = a_1 = \frac{\partial v}{\partial y}(x_0, y_0), \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -a_2 = -\frac{\partial v}{\partial x}(x_0, y_0),$$

and define the following linear transformation (in matrix representation)

$$T = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}.$$

Now, define the complex number $A = a_1 + ia_2$. It holds

$$\lim_{z\to z_0} \frac{|f(z) - f(z_0) - A(z - z_0)|}{|z - z_0|} = \lim_{(x,y)\to(x_0,y_0)} \frac{|f(x, y) - f(x_0, y_0) - T(x - x_0, y - y_0)|}{|(x, y) - (x_0, y_0)|} = 0.$$

We can conclude that

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = A,$$

f is complex diffetentiable at z_0. □

Corollary 1. *If $u, v : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ have **continuous first-order partial derivatives** at (x_0, y_0) that satisfy the **Cauchy-Riemann equations**, then $f(z) = u(x, y)+ iv(x, y)$ is **complex differentiable** at $z_0 = x_0 + iy_0$.*

Remark. A usual notation is for the Cauchy-Riemann equations is $u_x = v_y$ and $u_y = -v_x$.
Holomorphy is the property of a complex function of being differentiable at every point of a domain. Consequently, we can say that a complex function f, whose real and imaginary parts u and v are real-differentiable functions, is holomorphic if and only if, the Cauchy-Riemann equaitons are satisfied at every point of the domain.

Example. The following example will clarify the hypothesis of the previous theorem, that the Cauchy-Riemann equations are not sufficient to establish the complex differentiability of a function. Let $f : \mathbb{C} \to \mathbb{C}$ be defined as

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & \text{for} \quad z \neq 0, \\ 0, & \text{for} \quad z = 0. \end{cases}$$

The function f is not complex differentiable at $z = 0$. Let $z = x$, with $x \to 0$. Then

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - z_0} = \lim_{x\to 0} \frac{\frac{x^5}{|x|^4}}{x} = \lim_{x\to 0} \frac{x^4}{|x|^4} x = 1.$$

Now take $z = i\,y$, with $y \to 0$ and observe

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - z_0} = \lim_{y\to 0} \frac{\frac{(iy)^5}{|y|^4}}{y} = \lim_{y\to 0} \frac{-y^5}{|y|^4} = -1.$$

Then we conclude that

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0},$$

does not exists.

On the other hand

$$u(x, 0) = x, \quad u(0, y) = v(x, 0) = 0, \quad v(0, y) = y,$$

Then we have at $(0,0)$:

$$u_x = v_y = 1, \quad u_y = -v_x = 0.$$

What is happening here?

$$f(z) = \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2+y^2)^2} + i\frac{5x^4y - 10x^2y^3 + y^5}{(x^2+y^2)^2}$$