Chapter 16

Greedy Algorithms

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16 Greedy Algorithms

- **Similar to dynamic programming. Used for optimization problems.**
- **Optimization problems typically go through a sequence of steps, with a set of choices at each step.**
- **For many optimization problems, using dynamic programming to determine the best choices is overkill.**
- **Greedy Algorithm: Simpler, more efficient**

16 Greedy Algorithms

Greedy algorithms (GA) do not always yield optimal solutions, but for many problems they do.

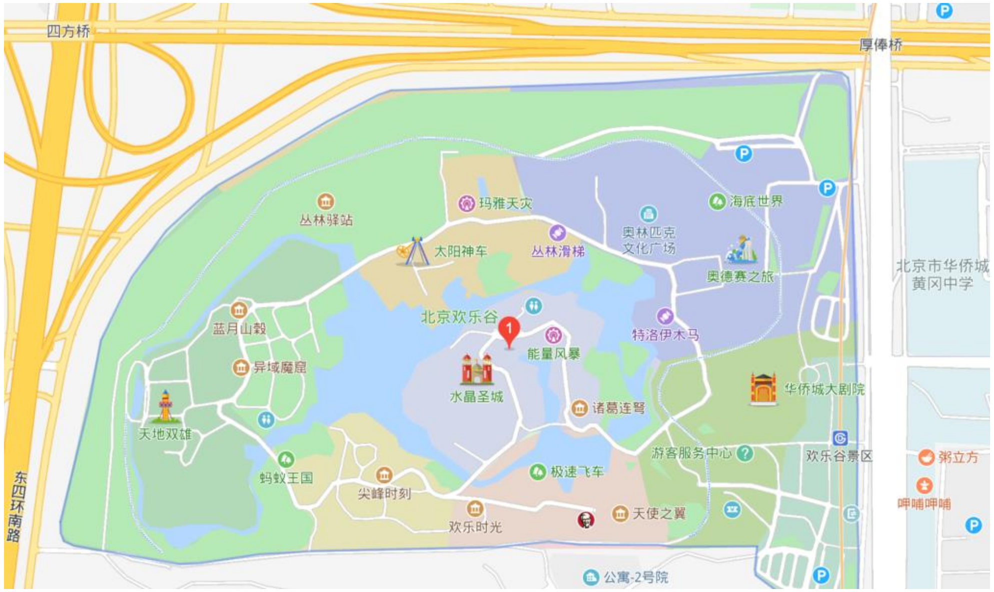
- **16.1, the activity-selection problem** （活动安排）
- **16.2, basic elements of the GA; knapsack prob.** （贪婪算法的基本特征；背包问题）
- **16.3, an important application: the design of data compression (Huffman) codes.** （哈夫曼编码）
- ***16.4 Matroids and greedy methods**
- ***16.5, A task-scheduling problem as matroid (unit-time tasks scheduling,** 有限期作业调度**)**

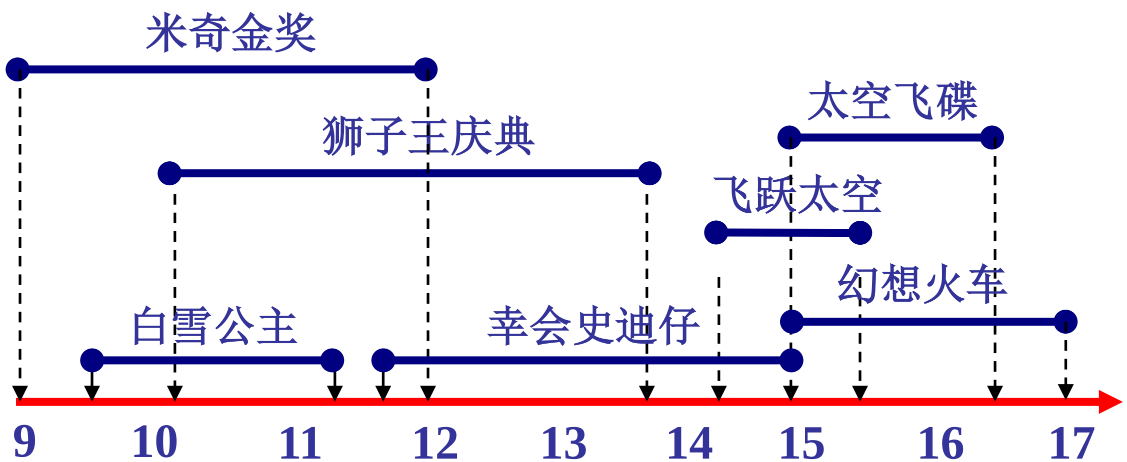
16 Greedy Algorithms

The greedy method is quite powerful and works well for a wide range of problems:

- **minimum-spanning-tree algorithms (Chap 23)**
  （最小生成树）
- **shortest paths from a single source (Chap 24)**
  （最短路径）
- **set-covering heuristic (Chap 35).**
  （集合覆盖）
- **…**

Example: 北京欢乐谷游玩的活动安排

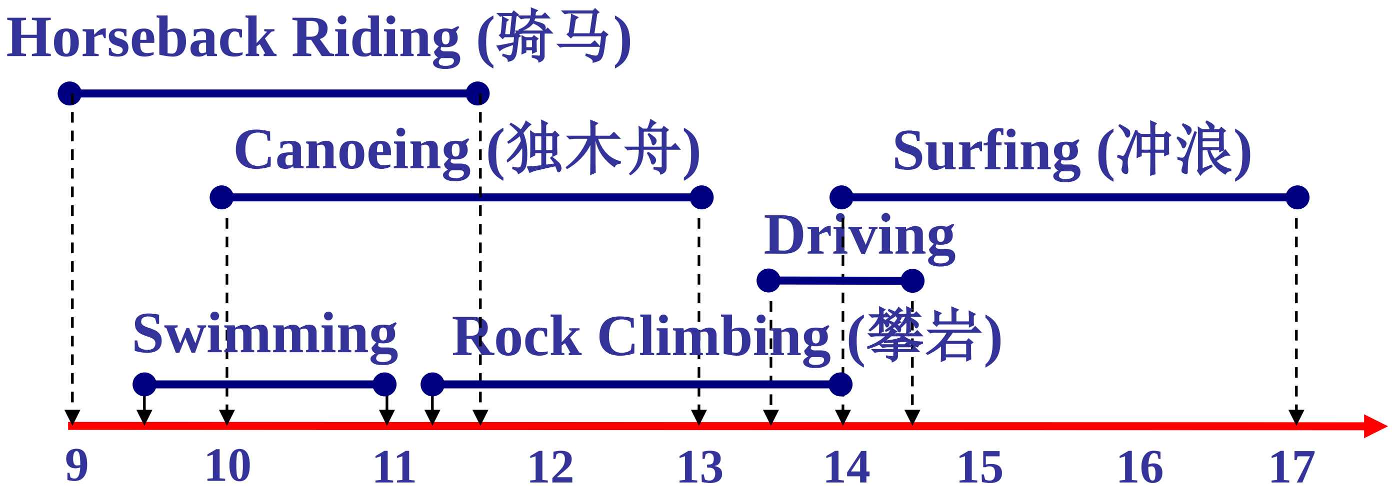




Activity Selection

- 欢乐谷
- Disneyland
- Universal Studio

Example: Activity Selection



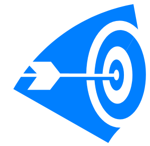
How to make an arrangement to have the more activities?

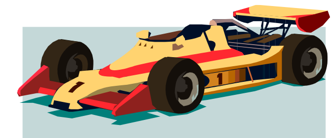
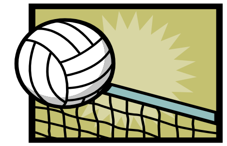
- **S1. Shortest activity first** （最短活动优先原则）
  Swimming , Driving
- **S2. First starting activity first** （最早开始活动优先原则）
  Horseback Riding , Driving
- **S3. First finishing activity first** （最早结束活动优先原则）
  Swimming , Rock Climbing , Surfing

16.1 An activity-selection problem

应用场景:

借体育馆、借会议室、借教室、…



n activities **require**

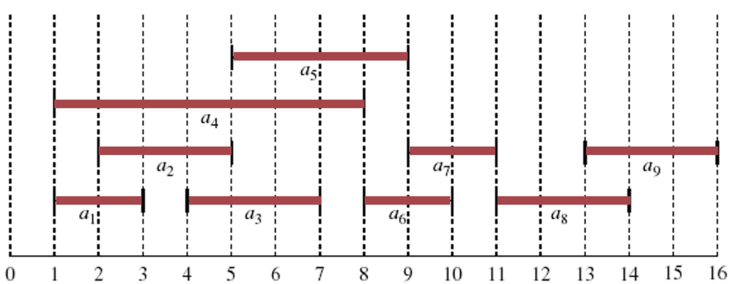
exclusive **use of a common resource.**

Example, scheduling the use of a classroom.

（*n* 个活动，1 项资源，任一活动进行时需唯一占用该资源）

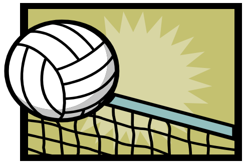
playground

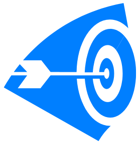
- **Set of activities $S = \{a_1, a_2, . . . , a_n\}$.**
- **a_i needs resource during period $[s_i , f_i)$, which is a half-open interval, where s_i is start time and f_i is finish time.**
- ***Goal:*** **Select the largest possible set of nonoverlapping (*mutually compatible*) activities.**
  安排一个活动计划，使得相容的活动数目最多
- **Other objectives:**
  Maximum duration time,
  Maximize income rental fees,
  …



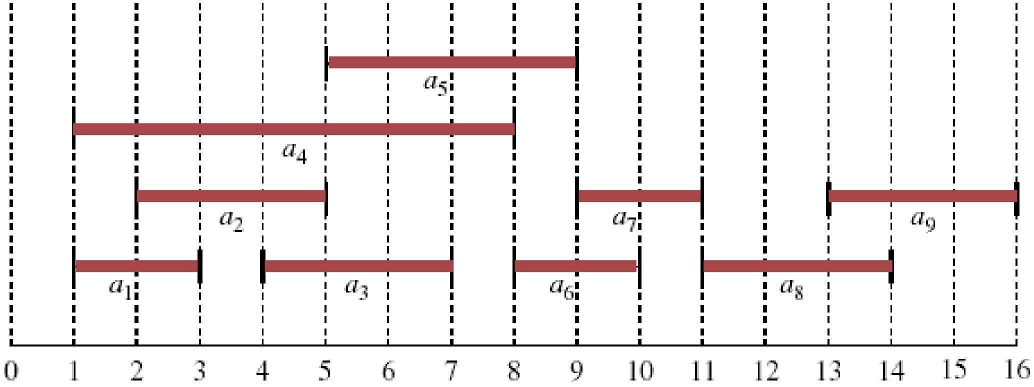
16.1 An activity-selection problem

- ***n activities*** **require** ***exclusive*** **use of a common resource.**
  - **Set of activities** $S = \{a_1, a_2, \ldots, a_n\}$
  - a_i **needs resource during period** $[s_i, f_i)$
- ***Example:*** S **sorted by finish time:**





i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16



Maximum-size mutually compatible set:

$\{a_1, a_3, a_6, a_8\}$.

Not unique: also

$\{a_2, a_5, a_7, a_9\}$.

16.1.1 Optimal substructure of activity selection

Space of subproblems

- $S_{ij} = \{a_k \in S : f_i \le s_k < f_k \le s_j\}$

  = activities that start after a_i finishes & finish before a_j starts

- Activities in S_{ij} are compatible with
  - all activities that finish by f_i（完成时间早于f_i 的活动）, and
  - all activities that start no earlier than s_j .
- To represent the entire problem, add fictitious activities:
  - $a_0 = [-\infty, 0)$; $a_{n+1} = [\infty, \text{“}\infty+1\text{”})$
  - We don't care about $-\infty$ in a_0 or “∞+1” in a_{n+1}.
- Then $S = S_{0,n+1}$. Range for S_{ij} is $0 \le i, j \le n + 1$.

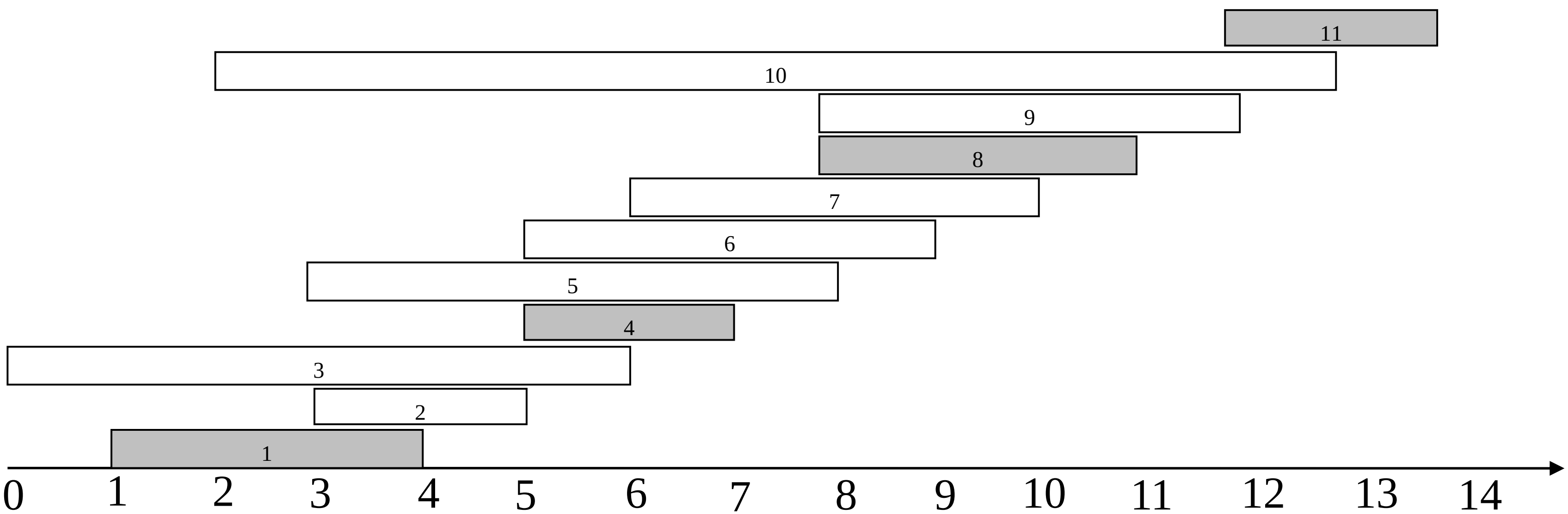
16.1.1 Optimal substructure of activity selection

Space of subproblems

- $S_{ij} = \{a_k \in S : f_i \le s_k < f_k \le s_j\}$

- **Assume that activities are sorted by monotonically increasing finish time** （以结束时间单调增的方式对活动进行排序）

$$f_0 \le f_1 \le f_2 \le \cdots \le f_n < f_{n+1} \quad (\text{if } i \le j, \text{ then } f_i \le f_j) \qquad (16.1)$$



16.1.1 Optimal substructure of activity selection

if $f_0 \le f_1 \le f_2 \le \cdot \cdot \cdot \le f_n < f_{n+1}$ **(if $i \le j$, then $f_i \le f_j$)** **(16.1)**

- **Then** $i \ge j \Rightarrow S_{ij} = \varnothing$

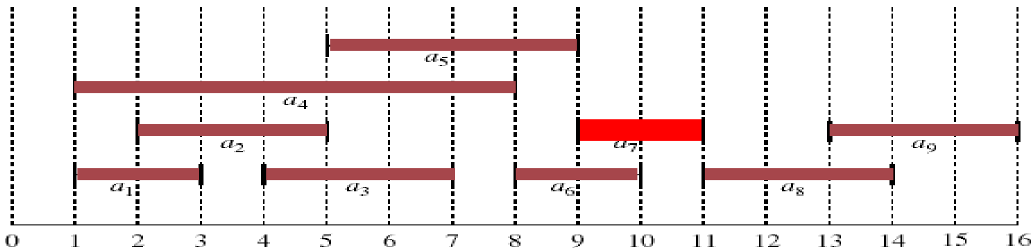
  Proof **If there exists $a_k \in S_{ij}$, then**

  $$f_i \le s_k < f_k \le s_j < f_j \Rightarrow f_i < f_j \,.$$

  But $i \ge j \Rightarrow f_i \ge f_j$. Contradiction.

- **So only need to worry about S_{ij} with $0 \le i < j \le n + 1$.**
  All other S_{ij} are $\varnothing$.

16.1.1 Optimal substructure of activity selection



- **Suppose that a solution to S_{ij} includes a_k . Have 2 sub-prob**
  - **S_{ik} (start after a_i finishes, finish before a_k starts)**
  - **S_{kj} (start after a_k finishes, finish before a_j starts)**
- **Solution to S_{ij} = (solution to S_{ik}) $\cup$ {a_k} $\cup$ (solution to S_{kj})**

  Since a_k is in neither of the subproblems, and the subproblems are disjoint, | solution to S | = | solution to S_{ik} | + 1 + | solution to S_{kj} | .
- **Optimal substructure: If an optimal solution to S_{ij} includes a_k , then the solutions to S_{ik} and S_{kj} used within this solution must be optimal as well. (use usual cut-and-paste argument).**
- **Let A_{ij} = optimal solution to S_{ij} ,**

  so $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$, (16.2)

  assuming: S_{ij} is nonempty; and we know a_k .

16.1.2 A recursive solution

- **Let $c[i, j]$ = size of maximum-size subset of mutually compatible activities in S_{ij} .**（$c[i, j]$ 表示 S_{ij} 相容的最大活动数）

  $i \geq j \Rightarrow S_{ij} = \varnothing \Rightarrow c[i, j] = 0$.
- **If $S_{ij} \neq \varnothing$, suppose that a_k is used in a maximum-size subsets of mutually S_{ij} . Then** $c[i, j] = c[i, k] + 1 + c[k, j]$.
- **But of course we don't know which k to use, and so**

$$c[i,j] = \begin{cases} 0 & , \quad \text{if } S_{ij} = \varnothing, \\ \max\limits_{i<k<j}\{c[i,k] + c[k,j] + 1\} & , \quad \text{if } S_{ij} \neq \varnothing. \end{cases} \qquad (16.3)$$

Why this range of k? Because $S_{ij} = \{a_k \in S: f_i \leq s_k < f_k \leq s_j\} \Rightarrow a_k$ can't be a_i or a_j (if $k = i$, we have $c[i, j] = c[i, j] + 1$).

核心要素：1. 递归；2. 遍历O(n)次，找出 k

16.1.3 Converting a DP solution to a greedy solution

$$c[i,j]=\begin{cases}0 & , \quad \text{if } S_{ij}=\varnothing, \\ \max\limits_{i<k<j}\{c[i,k]+c[k,j]+1\} & , \quad \text{if } S_{ij}\neq\varnothing.\end{cases} \qquad (16.3)$$

- **It may be easy to design an algorithm to the problem based on recurrence (16.3).**
  ① **Direct recursion algorithm (pseudo-code)? complexity?**
  ② **Dynamic programming algorithm (pseudo-code)? complexity?**
- **For (16.3):**
  ③ **How many choices?**
  ④ **How many subproblems for a choice?**
  ⑤ **How many subproblems totally?**

Exercise ?

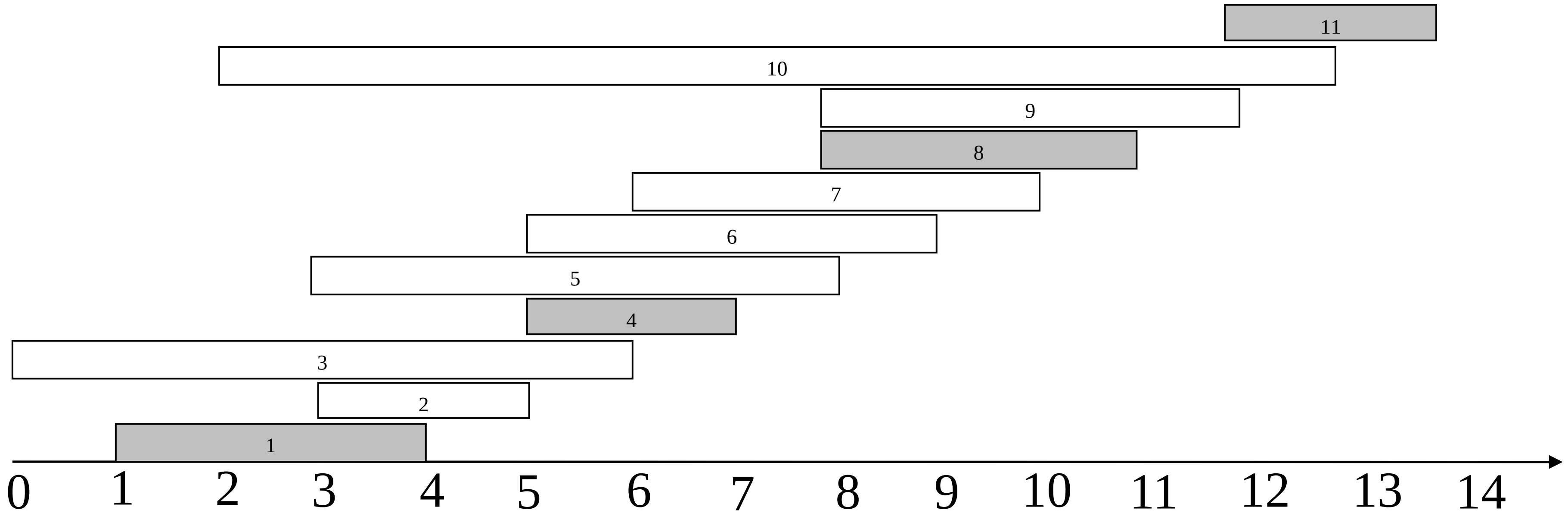
- **Can we simplify our solution?**

16.1.3 Converting a DP solution to a greedy solution

Theorem 16.1

Let $S_{ij} \neq \varnothing$, and let a_m be the activity in S_{ij} with the earliest finish time: $f_m = \min\{f_k : a_k \in S_{ij}\}$. Then

1. **a_m is used in some maximum-size subset of mutually compatible activities of S_{ij} .**（a_m包含在某个最大相容活动子集中）
2. **$S_{im} = \varnothing$, so that choosing a_m leaves S_{mj} as the only nonempty subproblem.** （仅剩下一个非空子问题 S_{mj} ）



16.1.3 Converting a DP solution to a greedy solution

Theorem 16.1

Let $S_{ij} \neq \varnothing$, and let a_m be the activity in S_{ij} with the earliest finish time: $f_m = \min \{f_k : a_k \in S_{ij}\}$. Then

1. ……

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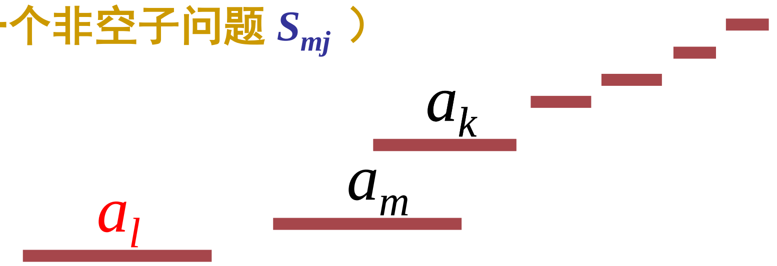


Proof

2. Suppose there is some $a_l \in S_{im}$. Then $f_i \leq s_l < f_l \leq s_m < f_m$ $\Rightarrow f_l < f_m$. Then $a_l \in S_{ij}$ and it has an earlier finish time than f_m, which contradicts our choice of a_m. Therefore, there is no $a_l \in S_{im} \Rightarrow S_{im} = \varnothing$.

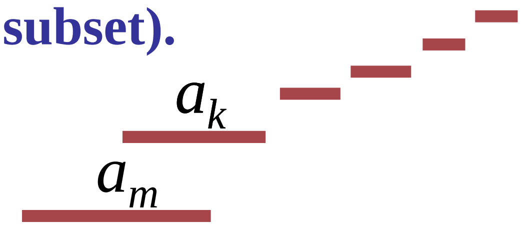
16.1.3 Converting a DP solution to a greedy solution

Theorem 16.1 Let $S_{ij} \neq \emptyset$, and let a_m be the activity in S_{ij} with the earliest finish time: $f_m = \min\ \{f_k : a_k \in S_{ij}\ \}$. Then

1. a_m is used in some maximum-size subset of mutually compatible activities of $S_{i\,j}$. （a_m包含在某个最大相容活动子集中）

Proof **1. Let A_{ij} be a maximum-size subset of mutually Compatible activities in S_{ij} . Order activities in A_{ij} in monotonically increasing order of finish time. Let a_k be the first activity in A_{ij} .**

- **If $a_k = a_m$, done (a_m is used in a maximum-size subset).**



- **Otherwise, construct B_{ij}=A_{ij} -$\{a_k\} \cup \{a_m\}$ (replace a_k by a_m).**
  Activities in B_{ij} are disjoint. (Activities in A_{ij} are disjoint, a_k is the first activity in A_{ij} to finish. $f_m \leq f_k \Rightarrow a_m$ doesn't overlap anything else in B_{ij}). Since $|B_{ij}| = |A_{ij}|$ and A_{ij} is a maximum-size subset, so is B_{ij} .

16.1.3 Converting a DP solution to a greedy solution

$$c[i,j] = \begin{cases} 0 & , \quad \text{if } S_{ij} = \varnothing, \\ \max\limits_{i<k<j}\{c[i,k] + c[k,j] + 1\} & , \quad \text{if } S_{ij} \neq \varnothing. \end{cases} \qquad (16.3)$$

Theorem 16.1 Let $S_{ij} \neq \varnothing$, and let a_m be the activity in S_{ij} with the earliest finish time: $f_m = \min\{f_k : a_k \in S_{ij}\}$. Then

1. a_m is used in some maximum-size subset of mutually compatible activities of S_{ij}. （a_m包含在某个最大相容活动子集中）
2. $S_{im} = \varnothing$, so that choosing a_m leaves S_{mj} as the only nonempty subproblem. （仅剩下一个非空子问题 S_{mj} ）

This theorem is great:

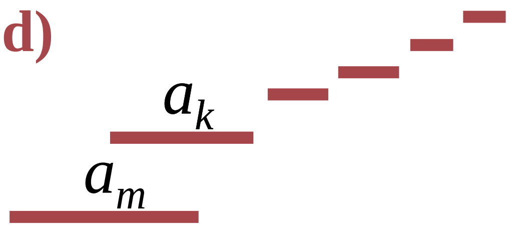
	before theorem	after theorem
# of sub-prob in optimal solution	2	1
# of choices to consider	$O(j-i-1)$	1

And, the choice is safe (a_m belongs to a solution)!

16.1.3 Converting a DP solution to a greedy solution

- **Theorem 16.1** Let $S_{ij} \neq \varnothing$, and let a_m be the activity in S_{ij} with the earliest finish time: $f_m = \min\{f_k : a_k \in S_{ij}\}$. Then
  1. a_m is used in some maximum-size subset of mutually compatible activities of S_{ij}.（a_m包含在某个最大相容活动子集中）
  2. $S_{im} = \varnothing$, so that choosing a_m leaves S_{mj} as the only nonempty subproblem. （仅剩下一个非空子问题 S_{mj} ）

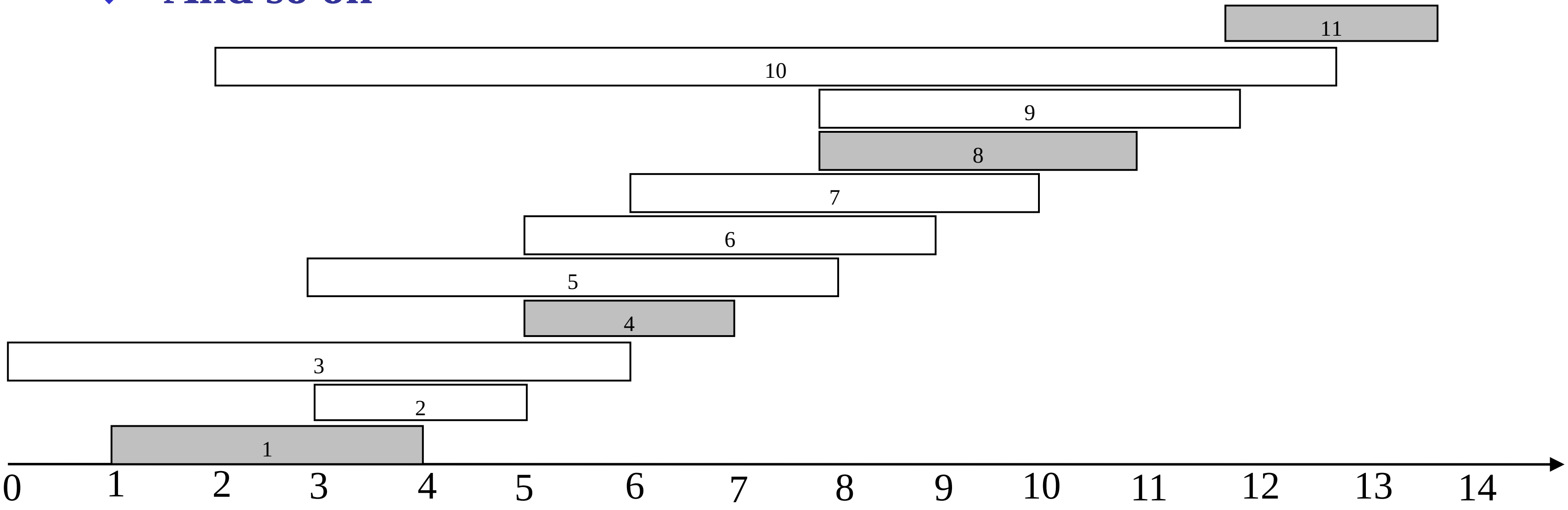
- **Now we can solve a problem S_{ij} in a top-down fashion (bottom-up for DP?)**
  - Choose $a_m \in S_{ij}$ with earliest finish time: the ***greedy choice***.
    (it leaves as much opportunity as possible
    for the remaining activities to be scheduled)
    留下尽可能多的时间来安排活动，贪心选择
  - Then solve S_{mj}



16.1.3 Converting a DP solution to a greedy solution

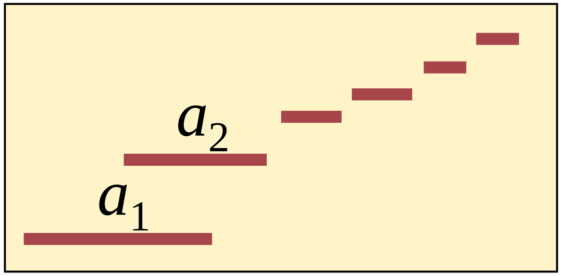
What are the subproblems?

- **Original problem is $S_{0,\, n+1}$** 〔 a_0 = [-∞, 0); a_{n+1} = [∞, "∞+1") 〕
- **Suppose our first choice is a_{m1} (in fact, it is a_1)**
- **Then next subproblem is $S_{m1,\, n+1}$**
- **Suppose next choice is a_{m2} (it must be a_2?)**
- **Next subproblem is $S_{m2,\, n+1}$**
- **And so on**



16.1.3 Converting a DP solution to a greedy solution

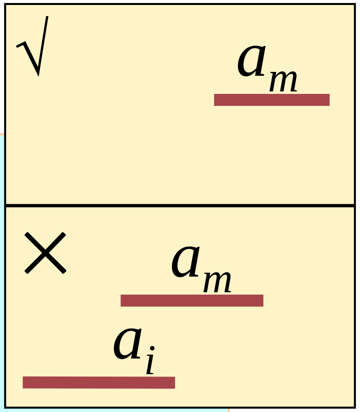
- **What are the subproblems?**
  - **Original problem is $S_{0,\ n+1}$**
  - **Suppose our first choice is a_{m1}**
  - **Then next subproblem is $S_{m1,\ n+1}$**
  - **Suppose next choice is a_{m2}**
  - **Next subproblem is $S_{m2,\ n+1}$**
  - **And so on**



- **Each subproblem is $S_{mi\ ,\ n+1}$.**
- **And the subproblems chosen have finish times that increase.** （所选的子问题，其完成时间是增序排列）
- **Therefore, we can consider each activity just once, in monotonically increasing order of finish time.**

16.1.4 A recursive greedy algorithm

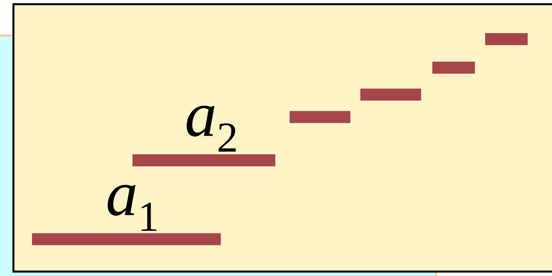
- **Original problem is $S_{0,\,n+1}$**
- **Each subproblem is $S_{mi\,,\,n+1}$**
- **Assumes activites already sorted by monotonically increasing finish time. (If not, then sort in $O(n\lg n)$ time.) Return an optimal solution for $S_{i,\,n+1}$:**



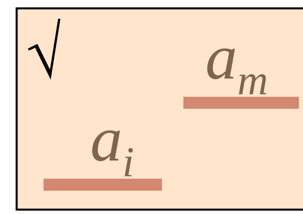

```
REC-ACTIVITY-SELECTOR(s, f, i, n)
   m ← i+1 // initially i = 0, m = 1
   while m ≤ n and s_m < f_i    // Find next activity in S_{i, n+1}.
         m ← m+1
   if m ≤ n
         return {a_m} ∪ REC-ACTIVITY-SELECTOR(s, f, m, n)
   else  return ∅
```

16.1.4 A recursive greedy algorithm

REC-ACTIVITY-SELECTOR(s, f, i, n)
$m \leftarrow i+1$
while $m \le n$ and $s_m < f_i$ // Find next activity in $S_{i,\, n+1}$
    $m \leftarrow m+1$
if $m \le n$
    return $\{a_m\} \cup$ REC-ACTIVITY-SELECTOR(s, f, m, n)
else **return** $\emptyset$



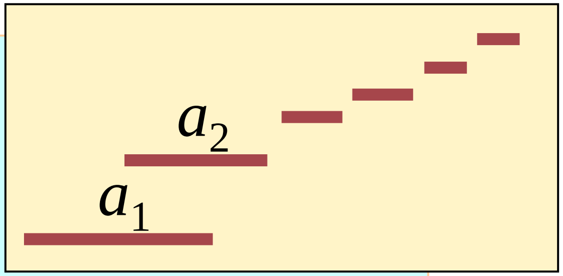
Initial call: REC-ACTIVITY-SELECTOR($s, f, 0, n$).



Idea: The **while** loop checks $a_{i+1}, a_{i+2}, \ldots, a_n$ until it finds an activity a_m that is compatible with a_i (need $s_m \ge f_i$).

- If the loop terminates because a_m is found ($m \le n$), then recursively solve $S_{m,\, n+1}$, and return this solution, along with a_m.
- If the loop never finds a compatible a_m ($m > n$), then just return empty set.

16.1.4 A recursive greedy algorithm



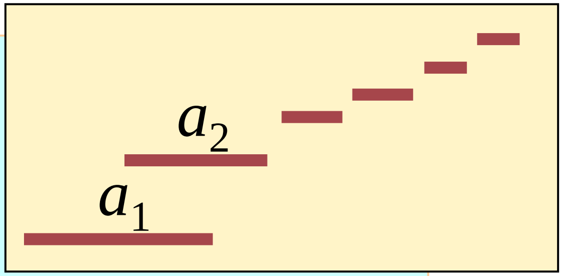
REC-ACTIVITY-SELECTOR(s, f, i, n)

$m \leftarrow i+1$
while $m \leq n$ and $s_m < f_i$ // Find next activity in $S_{i,\, n+1}$
  $m \leftarrow m+1$
if $m \leq n$
  return $\{a_m\} \cup$ REC-ACTIVITY-SELECTOR(s, f, m, n)
else **return** $\varnothing$

Running Time?

16.1.4 A recursive greedy algorithm

```
REC-ACTIVITY-SELECTOR(s, f, i, n)
m ← i+1
while m ≤ n and s_m < f_i        // Find next activity in S_{i, n+1}
        m ← m+1
if m ≤ n
        return {a_m} ∪ REC-ACTIVITY-SELECTOR(s, f, m, n)
else  return ∅
```



Running Time? $\Theta(n)$ — each activity examined exactly once.

$$T(n) = m_1 + T(n - m_1) = m_1 + m_2 + T(n - m_1 - m_2)$$
$$= m_1 + m_2 + m_3 + T(n - m_1 - m_2 - m_3) = \cdots$$
$$= \sum m_k + T(n - \sum m_k)$$

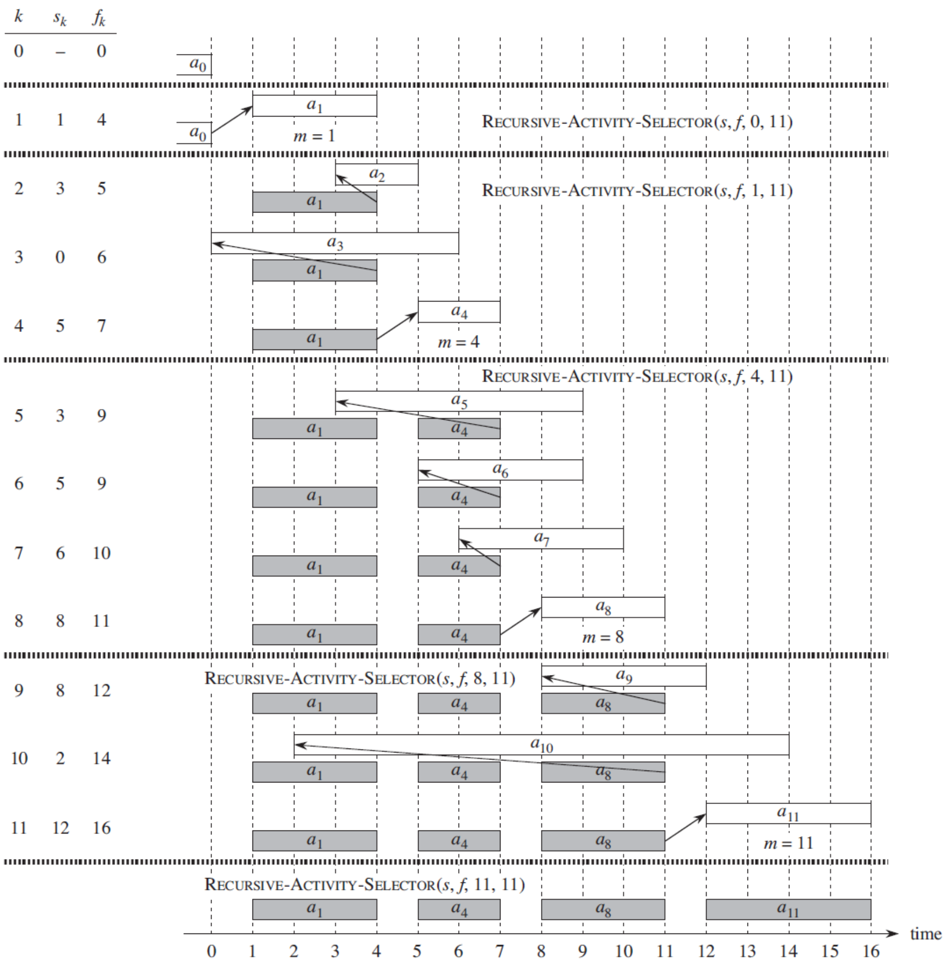
basecase: $n - \sum m_k = 1$, then $\sum m_k = n - 1$, $\sum m_k + T(1) = \Theta(n)$

16.1.4 A recursive greedy algorithm

***Initial call*:** REC-ACTIVITY-SELECTOR(s, f, 0, n).

Idea: The **while** loop checks $a_{i+1}, a_{i+2}, \ldots, a_n$ until it finds an activity a_m that is compatible with a_i (need $s_m \geq f_i$).

- If the loop terminates because a_m is found ($m \leq n$), then recursively solve $S_{m,n+1}$, and return this solution, along with a_m.
- If the loop never finds a compatible a_m ($m > n$), then just return empty set.



16.1.5 An iterative greedy algorithm

- **REC-ACTIVITY-SELECTOR is almost "tail recursive".**
- **We easily can convert the recursive procedure to an iterative one. (Some compilers perform this task automatically)**

GREEDY-ACTIVITY-SELECTOR(s, f, n)

$A \leftarrow \{a_1\}$

$i \leftarrow 1$

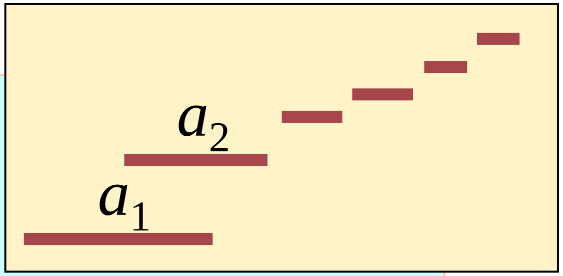
for $m \leftarrow 2$ **to** n

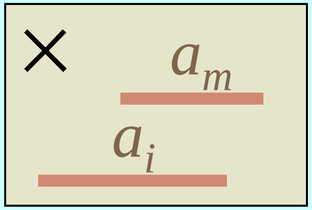
  if $s_m \geq f_i$

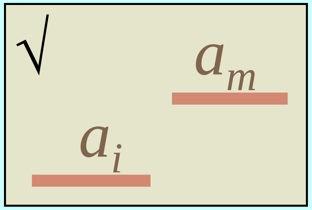
    $A \leftarrow A \cup \{a_m\}$

    $i \leftarrow m$ // a_i is most recent addition to A

return A







Review

- **Greedy Algorithm Idea: When we have a choice to make, make the one that looks best *right now*. Make a *locally optimal choice* in hope of getting a *globally optimal solution*.**
  希望当前选择是最好的，每一个局部最优选择能产生全局最优选择

- **Greedy Algorithm: Simpler, more efficient**

- **DP:** 每个子问题都必须求解
  T(n) = 子问题个数*求解每个子问题时所做的选择数
- **GA:** 甚至很多子问题都不需要求解

16 Greedy Algorithms

- **16.1, the activity-selection problem**

- **16.2**
  basic elements of the GA 贪婪算法的基本特征；背包问题
  knapsack prob 背包问题

- **16.3, an important application: the design of data compression (Huffman) codes**

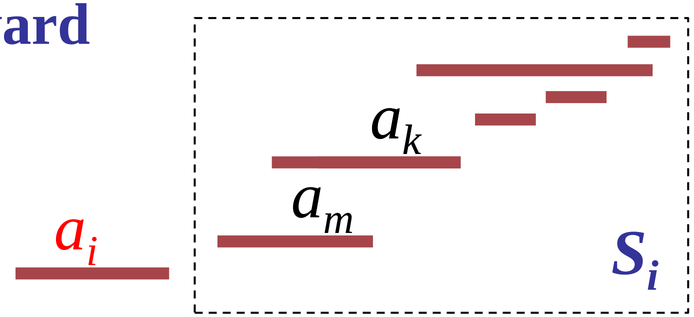
16.2 Elements of the greedy strategy

- **The choice that seems best at the moment is chosen**
  （每次决策时，当前所做的选择看起来是“最好”的）
- **What did we do for activity selection?**

  1. **Determine the optimal substructure.**
  2. **Develop a recursive solution.**
  3. **Prove that at any stage of recursion, one of the optimal choices is the greedy choice.**
  4. **Show that all but one of the subproblems resulting from the greedy choice are empty.** **（通过贪婪选择，只有一个子问题非空）**
  5. **Develop a recursive greedy algorithm.**
  6. **Convert it to an iterative algorithm.**

16.2 Elements of the greedy strategy

- **These steps looked like dynamic programming.**
- **Develop the substructure with an eye toward**
  - **making the greedy choice,**
  - **leaving just one subproblem.**



- **For activity selection, we showed that the greedy choice implied that in S_{ij} , only i varied, and j was fixed at n+1,**
- **So, we could have started out with a greedy algorithm in mind:**
  - **define $S_i = \{a_k \in S : f_i \leq s_k\}$,** （所有在 a_i 结束之后开始的活动）
  - **show the greedy choice, first a_m to finish in S_i combined with optimal solution to S_m** $\Rightarrow$ **optimal solution to S_i**

16.2 Elements of the greedy strategy

Typical streamlined steps

1. **Cast the optimization problem as one in which we make a choice and are left with one subproblem to solve.**
做选择，留下一个待求的子问题

2. **Prove that there's always an optimal solution that makes the greedy choice, so that the greedy choice is always safe.**
选择是解的一部分【贪婪】，因此贪婪选择是安全的

3. **Show that greedy choice and optimal solution to subproblem $\Rightarrow$ optimal solution to the problem.**
贪婪选择 + 子问题的最优解 $\Rightarrow$ 原问题的最优解

16.2 Elements of the greedy strategy

No general way to tell if a greedy algorithm is optimal, but two key ingredients are

没有一般化的规则来说明贪婪算法是否最优，但有两个基本要点

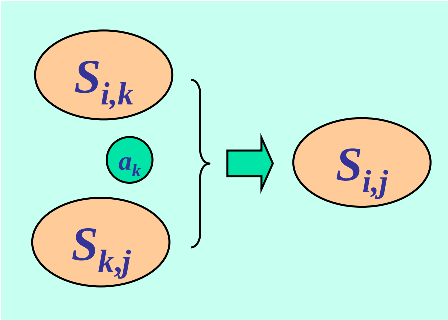
1. **greedy-choice property （贪婪选择属性）**

2. **optimal substructure**

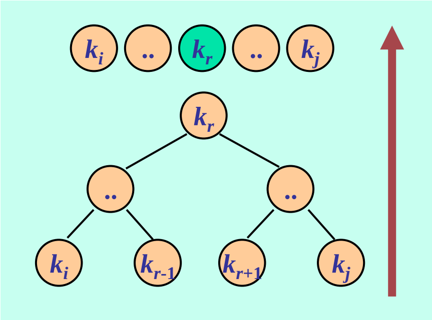
16.2.1 Greedy-choice property

Dynamic programming

- Choice depends on knowing optimal solutions to subproblems. Solve subproblems *first*.
  依赖于已知子问题的最优解再作出选择
- Solve *bottom-up*.



DP for Activity



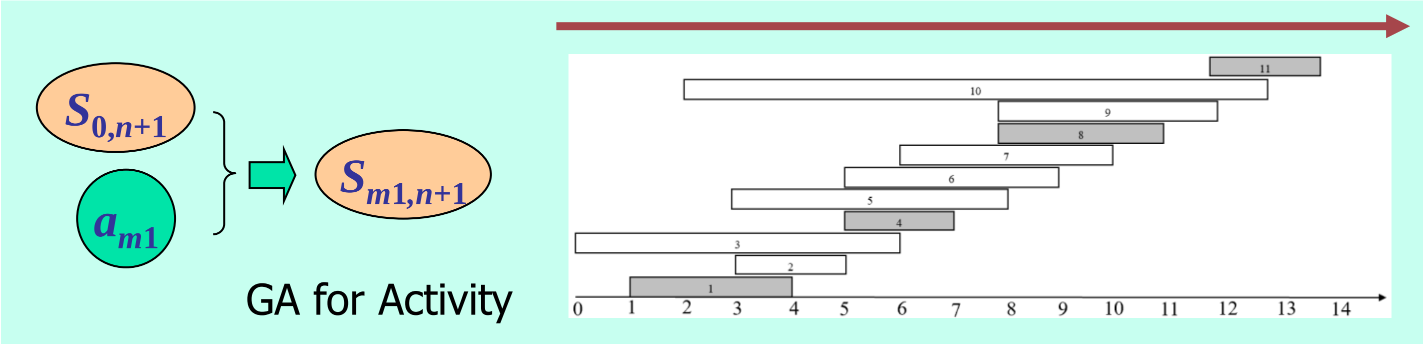
DP for OBST

Greedy

- Make the choice *before* solving the subproblems.
- Solve *top-down*.

GA for Activity

16.2.1 Greedy-choice property



- **We must prove that a greedy choice at each step yields a globally optimal solution. Sometimes it's Difficult! Cleverness may be required!**
- **Typically, Theorem 16.1, shows that the solution (A_{ij}) can be modified to use the greedy choice (a_m), resulting in one similar but smaller subproblem (A_{mj}).**
- **We can get efficiency gains from greedy-choice property. (For example, in activity-selection, examine each activity just once.)**

16.2.2 Optimal substructure

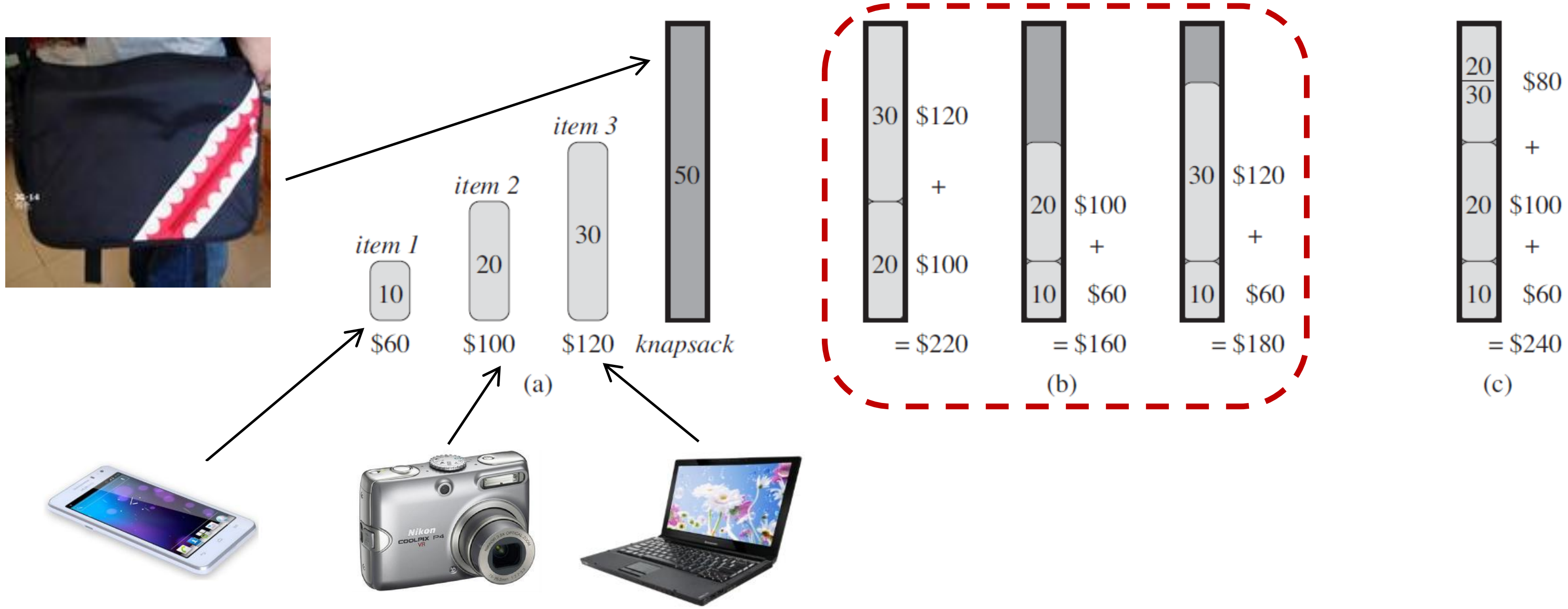
- ***optimal substructure*: an optimal solution to the problem contains within it optimal solutions to subproblems.**

- **Just show that optimal solution to subproblem and greedy choice $\Rightarrow$ optimal solution to problem.**

  （说明子问题的最优解和贪婪选择

  $\Rightarrow$ 原问题的最优解）

16.2.3 knapsack: Greedy vs. dynamic programming

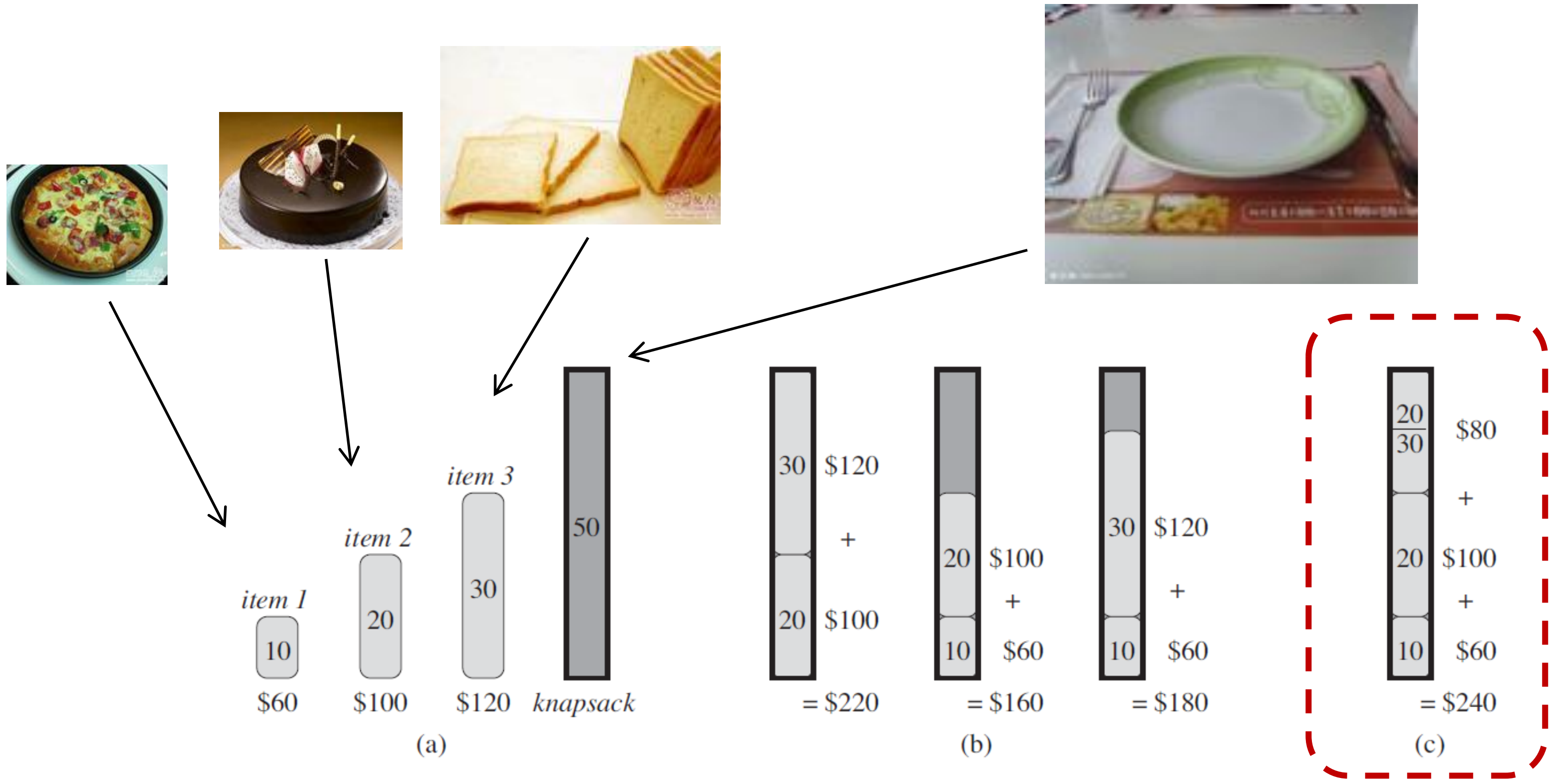


0-1 knapsack problem （0-1背包问题，小偷问题）

- *n* items
- Item *i* is worth $\$v_i$, weighs w_i P（物品*i* 价值v_i，重w_i）
- Find a most valuable subset of items with total weight $\leq W$.
- Have to either take an item or not take it—can't take part of it.

1 0

16.2.3 knapsack: Greedy vs. dynamic programming



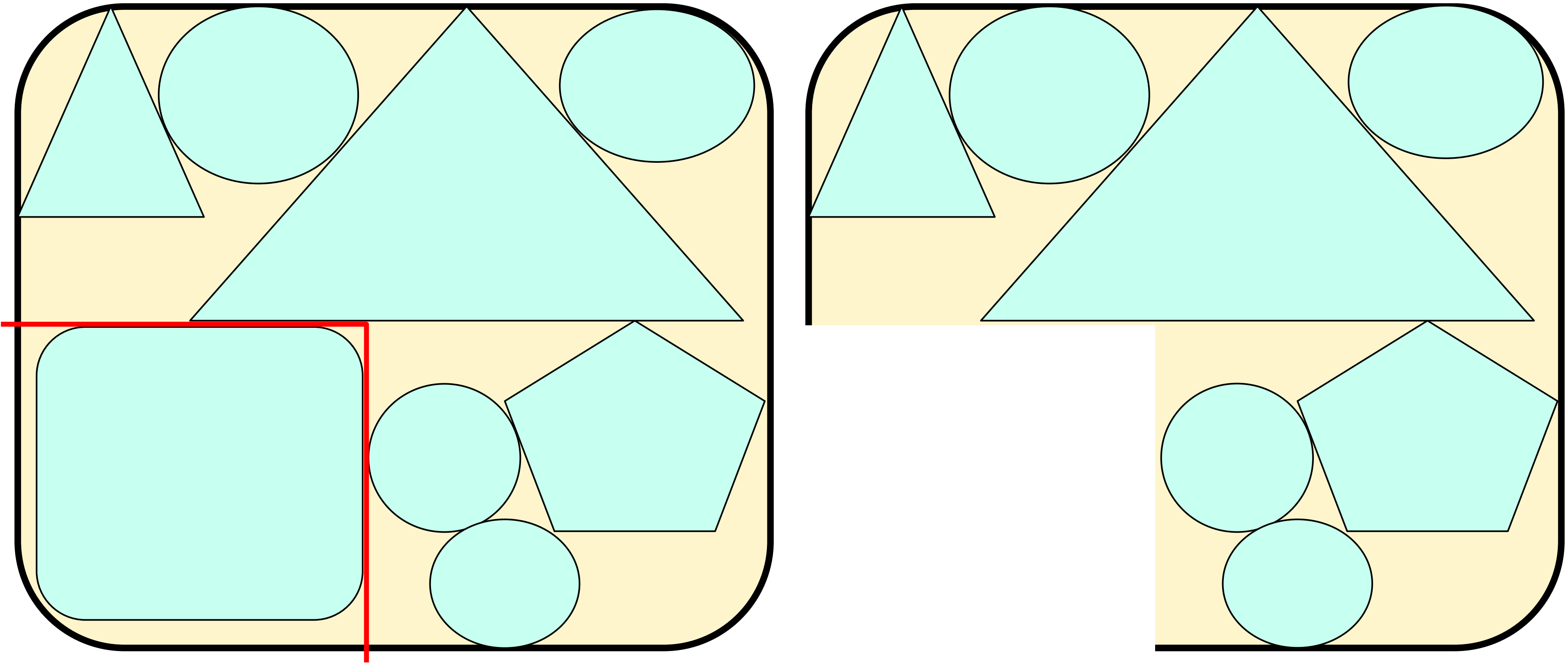
Fractional knapsack problem（分数背包问题，小偷问题）

- **Like the 0-1 knapsack problem, but can take fraction of an item.**

16.2.3 knapsack: Greedy vs. dynamic programming

- ***0-1 knapsack problem*** **（ 0-1背包问题，小偷问题）**
- ***Fractional knapsack problem*** **（分数背包问题，小偷问题）**

- **Both have optimal substructure property.**
  - **0-1 : ?**
  - **fractional: ?**



背包问题的最优子结构性质：

完整的圆角矩形框是一个最优背包。

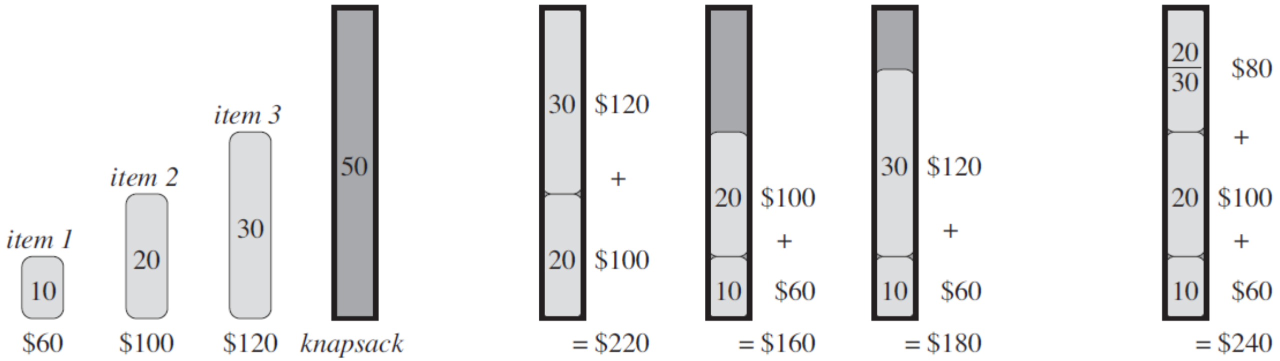
去掉右下角的红色部分剩下的部分是一个子背包，则该子背包也是一个最优背包。

16.2.3 knapsack: Greedy vs. dynamic programming

- ***0-1 knapsack problem*** **（0-1背包问题，小偷问题）**
- ***Fractional knapsack problem*** **（分数背包问题，小偷问题）**

- **But the fractional problem has the greedy-choice property, and the 0-1 problem does not.**

16.2.3 knapsack: Greedy vs. dynamic programming



(a) (b) (c)

- **Fractional knapsack problem has the greedy-choice property, and the <u>0-1 knapsack problem</u> does not.**
- **To solve the fractional problem, rank decreasingly items by v_i/w_i**
- **Let $v_i/w_i \geq v_{i+1}/w_{i+1}$ for all i**

```
FRACTIONAL-KNAPSACK(v,w,W)
load ← 0
i ← 1
while load < W and i ≤ n
      if w_i ≤ W - load
            take all of item i
      else  take W-load of w_i from item i
      add what was taken to load
      i ← i + 1
```

- ***Time:* $O(n\lg n)$ to sort, $O(n)$ to greedy choice thereafter.**

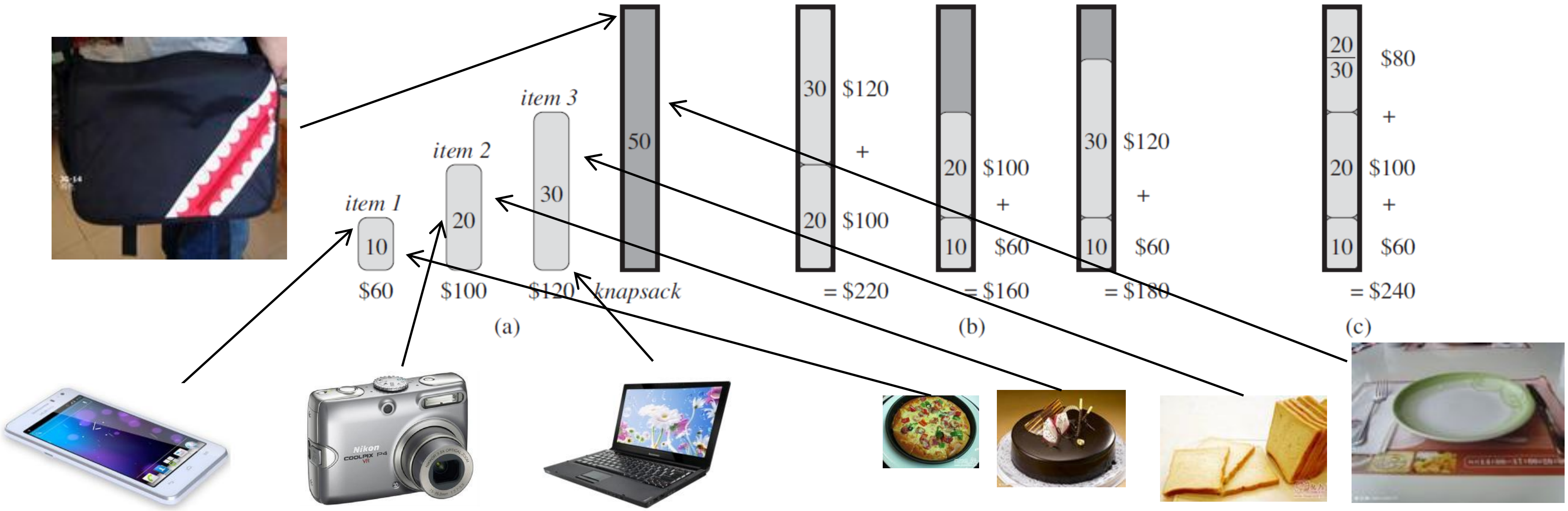
16.2.3 knapsack: Greedy vs. dynamic programming

i	1	2	3
v_i	60	100	120
w_i	10	20	30
v_i/w_i	6	5	4

- **0-1 knapsack problem has not the greedy-choice property.**
- **let *W* = 50 for the following example.**
  Greedy solution:
  - **take items 1 and 2**
  - **value = 160, weight = 30**

  20 pounds of capacity leftover.
- **Optimal solution:**
  - **Take items 2 and 3**
  - **value=220, weight=50**

  No leftover capacity. **(没有剩余空间)**



16 Greedy Algorithms

- **16.1, the activity-selection problem** （活动安排）
- **16.2, basic elements of the GA; knapsack prob**
- **16.3, an important application: the design of data compression (Huffman) codes** （哈夫曼编码）

1951年，MIT的老师Robert M. Fano给学生们布置了一个题目“寻找最有效的二进制编码”。1952年，David A. Huffman在MIT攻读博士时发表了《一种构建极小多余编码的方法》一文，它一般就叫做Huffman编码。

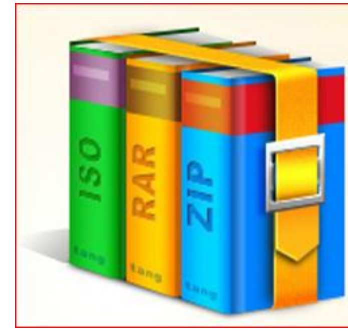
A method for the construction of minimum-redundancy codes

DA Huffman - Proceedings of the IRE, 1952 - ieeexplore.ieee.org

… Summary-An optimum method of coding an ensemble of messages consisting of a finite number of members is developed. A minimum-redundancy code is one constructed in such a way that the average number of coding digits per message is minimized. … Itwill be understood then that, in this paper, "optimum code"means "minimum-redundancy code." The following basic restrictions will be imposed on an ensemble …

16.3 Huffman codes

- **Huffman codes: widely used and very effective technique for <u>encoding file</u> or compressing data.**
  - **savings of 20% to 90%**



作业：每人写一个压缩软件

- **Consider the data to be a sequence of characters**

  abaaaabbbdcffeaaeaeec…abaadefe

- **Huffman's greedy algorithm:**

  uses a table of the frequencies of occurrence of the characters to build up an optimal way of representing each character as a binary string.

  依据字符出现的频率表，使用二进串来建立一种表示字符的最佳方法

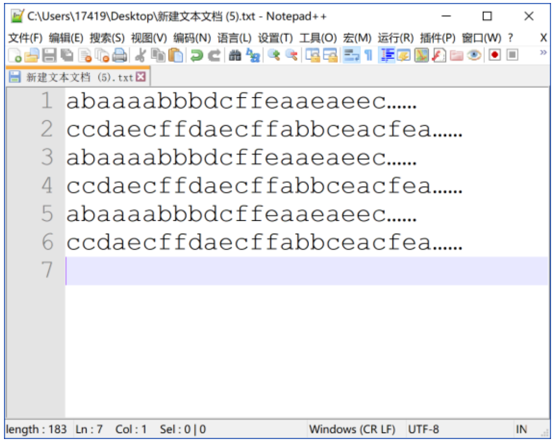
16.3 Huffman codes

- **Wish to store compactly 100,000-character data file.**

  Only six different characters appear.

  abaaaabbbdcffeaaeaeec……

  ccdaecffdaecffabbceacfea……



  Frequency table:

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

- **Many ways (encodes) to represent such a file of information**
- ***binary character code* (or *code* for short): each character is represented by a unique binary string.**
  - ***fixed-length code*: if use 3-bit codeword, the file can be encoded in 300,000 bits. Can we do better?**

16.3 Huffman codes

- 100,000-character data file

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

- *binary character code* (or *code* for short)
  - *variable-length code*: by giving frequent characters short codewords and infrequent characters long codewords, here the 1-bit string 0 represents a, and the 4-bit string 1100 represents f. （高频出现的字符以短字码表示；低频→长字码）

    $(45\cdot1 + 13\cdot3 + 12\cdot3 + 16\cdot3 + 9\cdot4 + 5\cdot4) \cdot 1{,}000 = 224{,}000$ bits

16.3 Huffman codes

- **100,000-character data file**

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

- ***binary character code*** **(or *code* for short)**
  - ***fixed-length code*: 300,000 bits**
  - ***variable-length code*: 224,000 bits, a savings of 25.3%. In fact, this is an optimal character code for this file.**

16.3.1 Prefix codes

- ***prefix codes*** **(prefix-free codes): no codeword is a prefix of some other codeword.**
  前缀码〔前缀无关码〕：没有字码是其他字码的前缀

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

- **Encoding is always simple for any binary character code**
  - **Concatenate（连接） the codewords representing each character. For example, "abc", with the variable-length prefix code as 0·101·100 = 0101100, where we use '·' to denote concatenation.**
- **Prefix codes simplify decoding**

16.3.1 Prefix codes

- *prefix codes* (prefix-free codes): no codeword is a prefix of some other codeword.
  前缀码〔前缀无关码〕：没有字码是其他字码的前缀

	a	b	c	d	e	f
Variable-length codeword	0	101	100	111	1101	1100

- Encoding is always simple for any binary character code
- Prefix codes simplify decoding
  - Since no codeword is a prefix of any other, the codeword that begins an encoded file is unambiguous（明确的）.
  - We can simply identify the initial codeword, translate it back to the original character, and repeat the decoding process on the remainder of the encoded file.
  - Exam: 001011101 uniquely as 0·0·101·1101, which decodes to “aabe”.

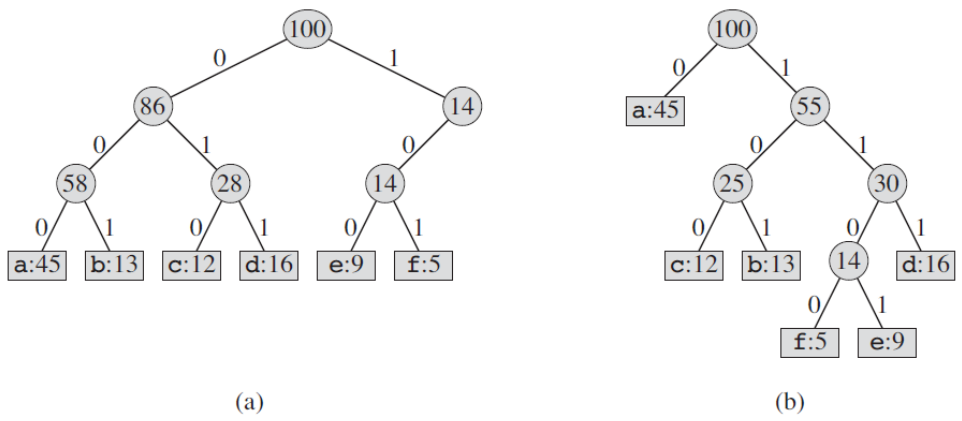
16.3.1 Prefix codes

a	b	c	d	e	f
0	101	100	111	1101	1100

001011101

uniquely as 0·0·101·1101,

which decodes to "aabe".



Decoding （编码的形式化表示能方便解码）

- **the process needs a convenient representation for the prefix code so that the initial codeword can be easily picked off.** （前置无关码方便解码）

- **A binary tree whose leaves are the given characters provides one such representation.** （二叉树是一种方便的表示方法，树叶为给定字符，从树根到树叶的过程就是解码过程）

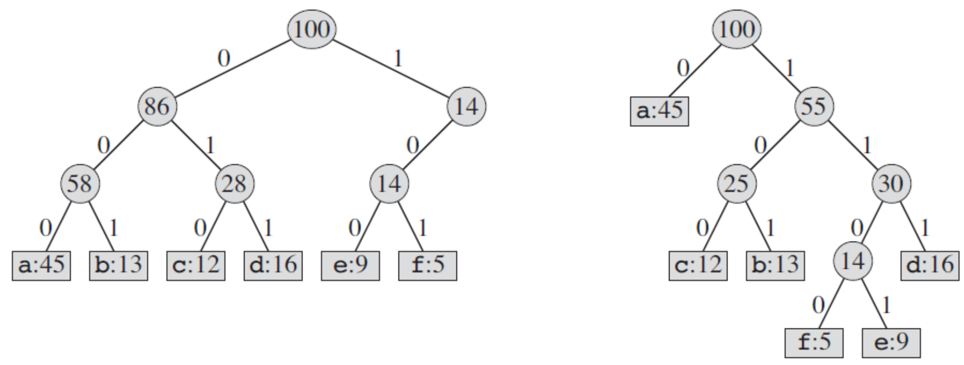
16.3.1 Prefix codes

a	b	c	d	e	f
0	101	100	111	1101	1100

001011101

uniquely as 0·0·101·1101,

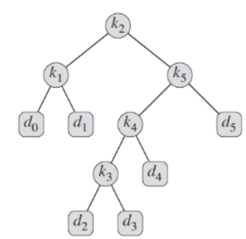
which decodes to “aabe”.



(a) (b)

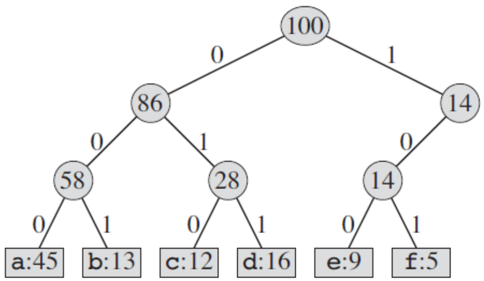
Decoding

- **We interpret the binary codeword for a character as the path from the root to that character.** **（字符的编码为一条从树根到树叶路径）**
- **It is not binary search trees, since the leaves need not appear in sorted order and internal nodes do not contain character keys.**

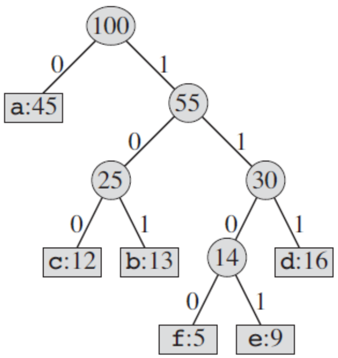


注意：不要混淆各种二叉树，如，最大（小）堆、二叉搜索树，哈夫曼树

16.3.1 Prefix codes



(a)



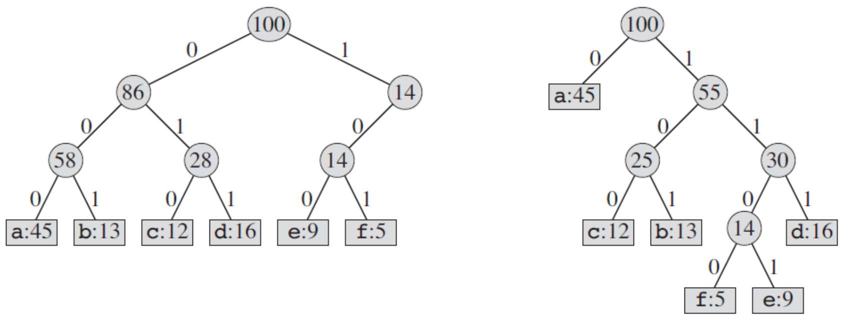
(b)

full binary tree 满二叉树

- 国际定义：除叶子节点外，所有节点都有两个孩子。
- 国内定义：除了满足如上定义，所有叶子节点还需要在同一层上。

- **An optimal code for a file is always represented by a *full* binary tree, every nonleaf node has two children (Ex16.3-1). The fixed-length code in our example is not optimal.**
- **We can restrict our attention to full binary trees**
  - ***C* is the alphabet,**
  - **all character frequencies >0**
  - **the tree for an optimal prefix code has |*C*| leaves, one for each letter of *C* , and exactly |*C*|-1 internal nodes.**

16.3.1 Prefix codes



(a) (b)

Compute # of bits required to encode a file:

Given a tree *T* corresponding to a prefix code, for each character *c* in the alphabet *C*,

- **$f(c)$: frequency of *c* in the file**
- **$d_T(c)$: depth of *c*'s leaf in the tree (length of the codeword for character *c*). Then, # of bits required to encode a file**

$$B(T) = \sum_{c \in C} f(c) d_T(c) \qquad (16.5)$$

which we define as the *cost* of the tree *T*.

16.3.2 Constructing a Huffman code

***Huffman code*: a greedy algorithm that constructs an optimal prefix code**

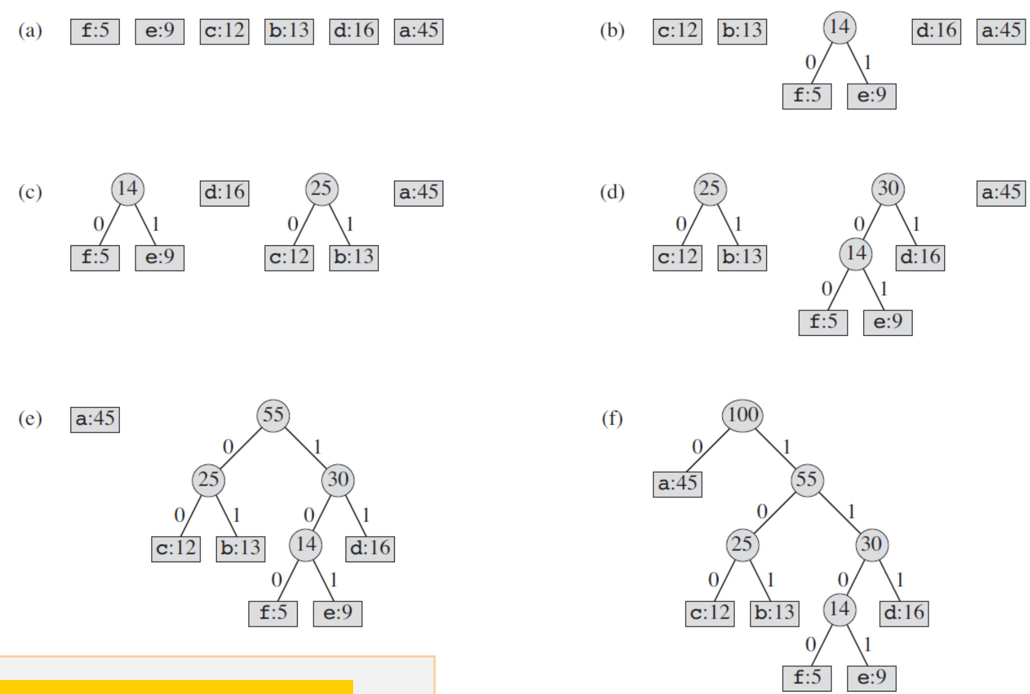
```
HUFFMAN(C)
n ← |C| ,  Q ← C
for i ← 1 to n - 1
        allocate(分配) a new node z
        left[z] ← x ← EXTRACT-MIN (Q)
        right[z] ← y ← EXTRACT-MIN (Q)
        f [z] ← f [x] + f [y]
        INSERT(Q, z)
return EXTRACT-MIN(Q)   //return the root of the tree.
```

***C*: set of *n* characters, $c \in C$: an object with frequency *f* [*c*].**

- **Build the tree *T* corresponding to the optimal code.**
- **Begin with |*C*| leaves, perform |*C*|-1 "merging" operations.**
- **A min-priority queue *Q*, keyed on *f* , is used to identify the two least-frequent objects to merge together. Result of the merger is a new object whose frequency is the sum of the frequencies of the two objects that were merged.**

16.3.2 Constructing a Huffman code

Example:
Huffman's algorithm proceeds.
6 letters, 5 merge steps.
The final tree represents the optimal prefix code.




```
HUFFMAN(C)
  n ← |C| ,  Q ← C
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      allocate(分配) a new node z
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      f [z] ← f [x] + f [y]
      INSERT(Q, z)
  return EXTRACT-MIN(Q)   //return the root of the tree.
```

Running time ?

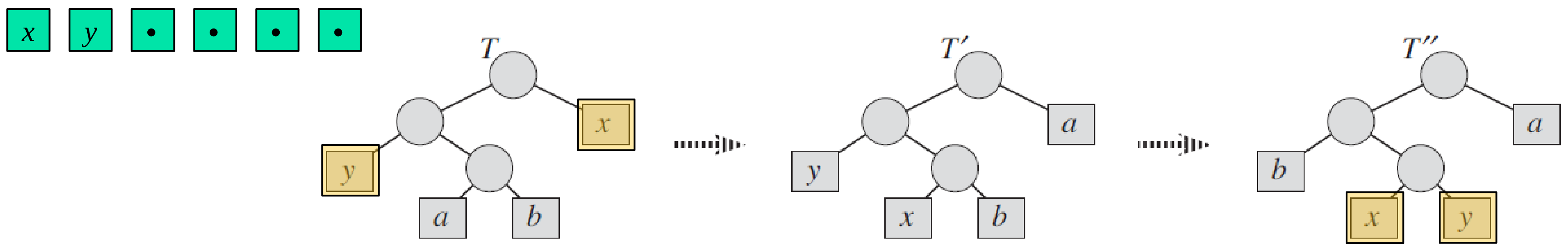
Each heap operation requires time O(lg*n*), the loop contributes O(*n*lg*n*).
We can reduce the running time to O(*n*lglg*n*) by replacing the binary min-heap with a van Emde Boas tree.

16.3.3 Correctness of Huffman's algorithm*

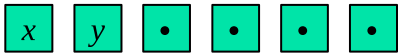
Problem of determining an optimal prefix code exhibits the greedy-choice and optimal-substructure properties.

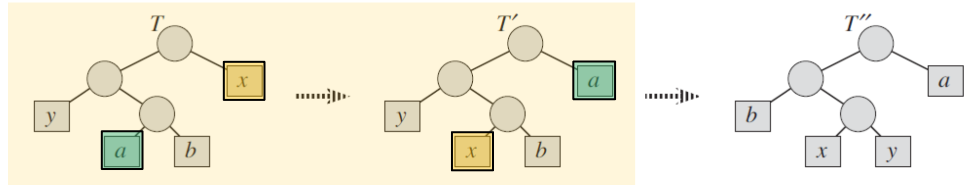
Lemma 16.2 (greedy-choice property): Let C be an alphabet, each character $c \in C$ has frequency $f[c]$. x and $y \in C$, and having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

***Proof idea*: take the tree T representing an arbitrary optimal prefix code, and modify it to make a tree representing another optimal prefix code such that x and y appear as sibling leaves**（姐妹叶） **of maximum depth in the new tree.**



16.3.3 Correctness of Huffman's algorithm*





Lemma 16.2

$c \in C$ has frequency $f[c]$. x, $y \in C$, having the lowest frequencies. Then, exist an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

$$B(T) = \sum_{c \in C} f(c) d_T(c) \qquad (16.5)$$

Proof : Let a and b are sibling leaves of max depth in optimal T. Assume that $f[a] \le f[b]$, $f[x] \le f[y]$. $f[x]$ and $f[y]$ are the two lowest leaf frequencies, $f[a]$, $f[b]$ are two arbitrary frequencies, in order,⇒ $f[x] \le f[a]$, $f[y] \le f[b]$. Exchange the pos in T of a and x to produce a tree T'. By (16.5), we have

$$\begin{aligned}
B(T) - B(T') &= \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) \\
&= f[x] d_T(x) + f[a] d_T(a) - f[x] d_{T'}(x) - f[a] d_{T'}(a) \\
&= f[x] d_T(x) + f[a] d_T(a) - f[x] d_T(a) - f[a] d_T(x) \\
&= (f[x] - f[a]) d_T(x) + (f[a] - f[x]) d_T(a) \\
&= (f[a] - f[x])(d_T(a) - d_T(x)) \ \ge\ 0
\end{aligned}$$

字符的频率：$f \le F$,(x为f , a为F)

字符在编码树中的高度：$d \le D$

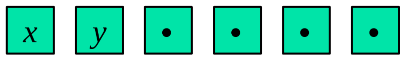
	x	a
in T	fd	FD
in T’	fD	Fd

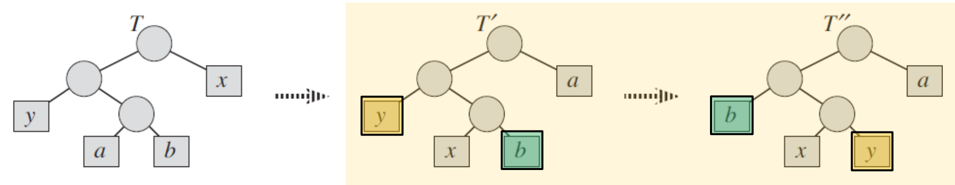
$B(T) - B(T') = fd + FD - fD - Fd$

$= f(D - d) + F(D - d)$

$= (F - f)(D - d) \ge 0$

16.3.3 Correctness of Huffman's algorithm*





Lemma 16.2

$c \in C$ has frequency $f[c]$. x , $y \in C$, having the lowest frequencies. Then, exist an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

$$B(T) = \sum_{c \in C} f(c) d_T(c) \qquad (16.5)$$

Proof : Let a and b are sibling leaves of maximum depth in optimal T. Assume that $f[a] \le f[b]$, $f[x] \le f[y]$. $f[x]$ and $f[y]$ are the two lowest leaf frequencies, $f[a]$, $f[b]$ are two arbitrary frequencies, in order,$\Rightarrow f[x] \le f[a]$, $f[y] \le f[b]$. Exchange the positions in T of a and x to produce a tree T' , and then exchange the positions in T' of b and y to produce a tree T'' . By (16.5), we have

Similarly, $B(T') - B(T'') \ge 0$,

therefore, $B(T'') \le B(T)$.

Since T is optimal, $B(T) \le B(T'')$.

Then $B(T'') = B(T)$.

Thus, T'' is an optimal tree.

引理16.2的重要意义：

贪心选择 x 和 y 是安全的（ x 和 y具有最长编码，且这种编码在最优解中）。

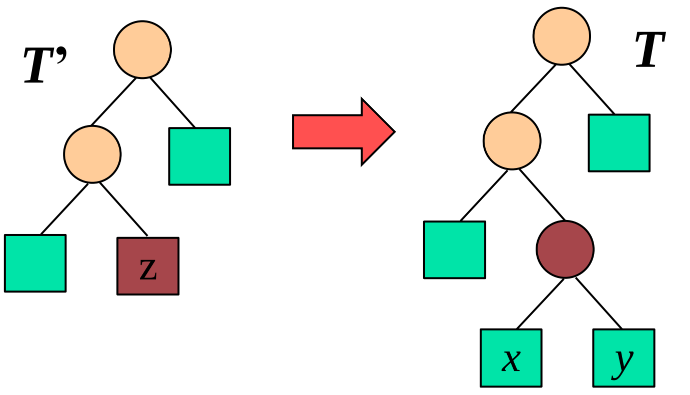
16.3.3 Correctness of Huffman's algorithm*

Lemma 16.3 (optimal-substructure property): Alphabet C , each character $c \in C$ has frequency $f[c]$. x and $y \in C$, and having the lowest frequencies. $C' = C - \{x, y\} \cup \{z\}$. Define f for C' as for C, except that $f[z]=f[x]+f[y]$. Let T' be any tree representing an optimal prefix code for the alphabet C'. Then the tree T, obtained from T' by replacing the leaf node for z with an internal node having x and y as children, represents an optimal prefix code for C.

$C' = C - \{x, y\} \cup \{z\}$, 除 $f[z]=f[x]+f[y]$ 以外，f 在 C' 中的定义与在 C 中相同。若 T' 为 C' 的最优编码，则编码 T' 加上贪心选择 ($z = x \cup y$) 得到的 T 为关于 C 的最优编码。

$C : \{c_1, \ldots, c_m, x, y\}$, $C' : \{c_1, \ldots, c_m, z\}$,

T' is optimal ⇒ ***T*** is optimal



16.3.3 Correctness of Huffman's algorithm*

Lemma 16.3 (optimal-substructure property)

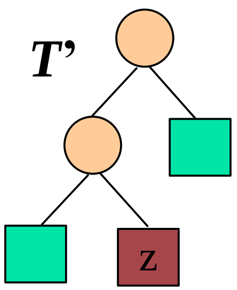
Proof : For each $c \in C$-$\{x, y\}$, we have $d_T(c) = d_{T'}(c)$, then

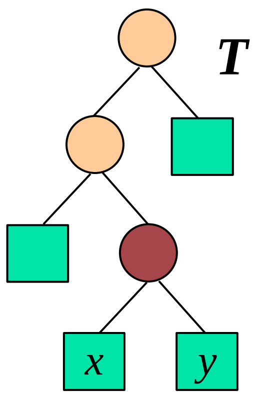
$f[c]d_T(c) = f[c]d_{T'}(c)$. Since $d_T(x)=d_T(y)=d_{T'}(z)+1$, we have

$$f[x]d_T(x)+f[y]d_T(y)=(f[x]+f[y])(d_{T'}(z)+1)= f[z]d_{T'}(z)+(f[x]+f[y]),$$

from which we conclude that $B(T) = B(T') + f[x] + f[y]$.

$(B(T) = f[x]d_T(x)+f[y]d_T(y) + f[c]d_T(c), B(T') = f[z]d_{T'}(z) + f[c]d_T(c))$

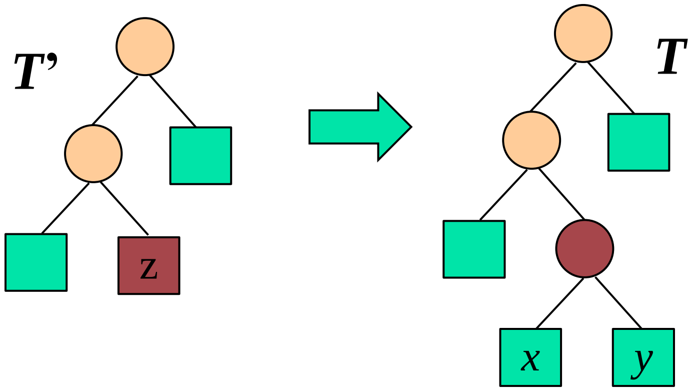




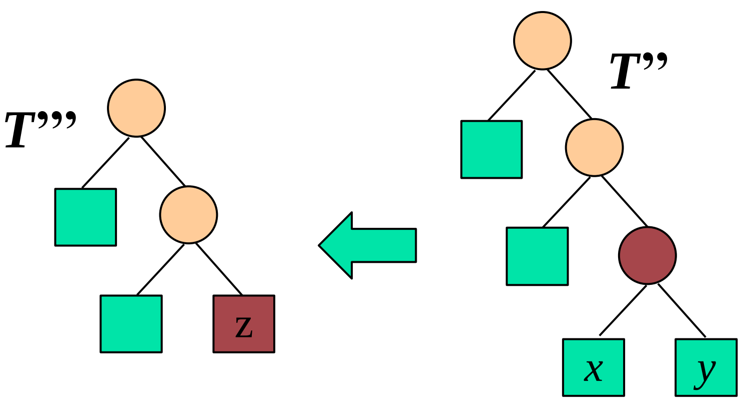
16.3.3 Correctness of Huffman's algorithm*

Lemma 16.3 (optimal-substructure property)

C : $\{c_1, \ldots, c_m, x, y\}$, C': $\{c_1, \ldots, c_m, z\}$, **T'** is optimal ⇒ **T** is optimal



Here, $B(T) = B(T') + f[x] + f[y]$



Suppose that T is not optimal, T'' is. Then $B(T'') < B(T)$. Without loss of generality (by Lemma 16.2), T'' has x and y as siblings. Let T''' be the tree T'' with the common parent of x and y replaced by a leaf z with frequency $f[z]=f[x]+f[y]$. Then

$$B(T''') = B(T'')-f[x]-f[y]$$
$$< B(T)-f[x]-f[y] = B(T'),$$

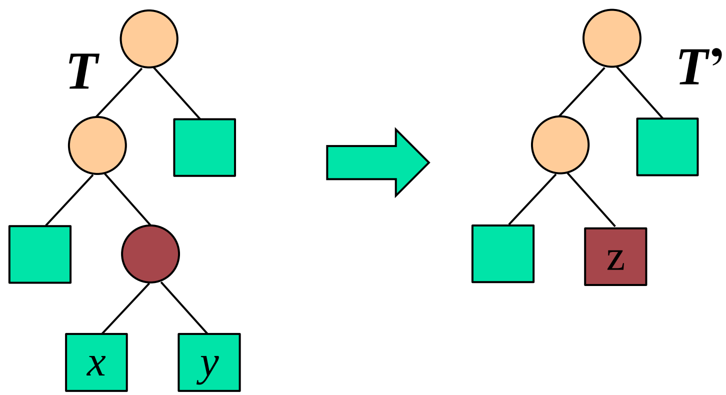
yielding a contradiction to the assumption that T' represents an optimal prefix code for C'. Thus, T must represent an optimal prefix code for the alphabet C.

16.3.3 Correctness of Huffman's algorithm*

Lemma 16.3 的另一种解释 (optimal-substructure property)

$C : \{c_1, \ldots, c_m, x, y\}$, $C' : \{c_1, \ldots, c_m, z\}$,

T is optimal ➡ ***T'*** is optimal



引理16.3的重要意义：

通过贪心选择 x 和 y 以后（根据引理16.2，贪心选择是安全的），把求解问题 C（最优解为T）变为求解子问题 C'（最优解为T'），而由 T'能构造出原问题的解 T。

16.3.3 Correctness of Huffman's algorithm*

Theorem 16.4

Procedure HUFFMAN produces an optimal prefix code.

Proof **Immediate from Lemmas 16.2（每一次选择是贪婪的、是正确的） and Lemmas 16.3（确保由子问题的最优解能构造原问题的最优解）.**

Exercises

$$c[i,j] = \begin{cases} 0 & , \quad \text{if } S_{ij} = \varnothing, \\ \max\limits_{i<k<j}\{c[i,k] + c[k,j] + 1\} & , \quad \text{if } S_{ij} \neq \varnothing. \end{cases} \qquad (16.3)$$

- **It may be easy to design an algorithm to the problem based on recurrence (16.3).**
  ① **Direct recursion algorithm (pseudo-code)? complexity?**
  ② **Dynamic programming algorithm (pseudo-code)? complexity?**
- **For (16.3):**
  ③ **How many choices?**
  ④ **How many subproblems for a choice?**
  ⑤ **How many subproblems totally?**

Exercise ?

Seven questions.
Email to me.

- **Can we simplify our solution?**

16.1.3 Converting a DP solution to a greedy solution

Exercises

16.1-1,

16.1-2（最晚开始优先原则）

16.2-2

Give a dynamic-programming solution to the 0-1 knapsack problem that runs in $O(nW)$ time, where n is number of items and W is the maximum weight of items that the thief can put in his knapsack.

16.3-2（课堂作业）

What is an optimal Huffman code for the following set of frequencies, based on the first 8 Fibonacci numbers?

a:1 b:1 c:2 d:3 e:5 f:8 g:13 h:21

又一些大作业或讨论题提示：

- 活动安排、Huffman code等等，是否都能描述为背包问题?
- 本书有多少算法是用的贪心策略（小论文：贪心算法十个经典问题？）
- 哈夫曼编码（用哈夫曼压缩方法，设计一个压缩软件：测试一下，算法导论这本书的压缩率能到多少？）
- OBST（最优二叉搜索树构建（以某本书里的词汇为基础？））

- 活动安排、分数背包
- 0-1背包、钢管切割、ALS、MCM、LCS、最短路径
- 雇佣（雇佣多少人）、取帽子、相同生日
- RSA加密解密、FFT、串匹配、计算几何、最大流

- 算法实验室（问题求解工具、算法效果展示平台、多种算法时间复杂度对比分析、多种算法空间复杂度对比分析、IO导入、……。支持穷举、递归、回溯、分治、DP、贪心；排序、查找；随机；图、树；等等若干方法（算法）的仿真演示。）