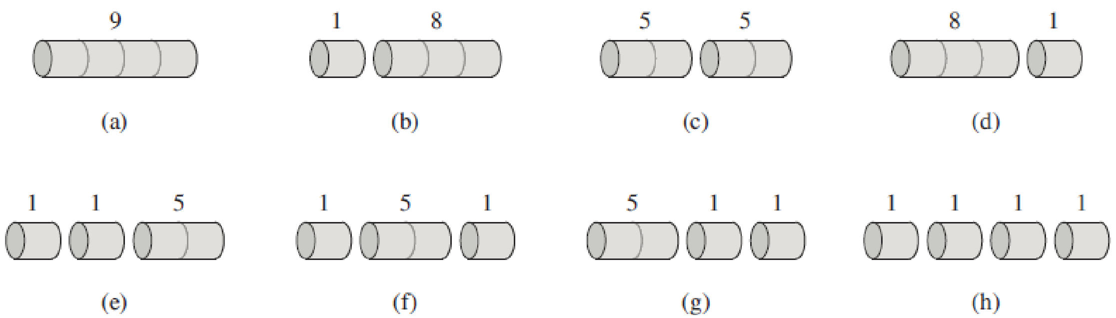
15 Dynamic Programming

- **Assembly lines scheduling (ALS)**
- **Steel rod cutting（钢条、钢管切割）**
- **Matrix-chain multiplication（矩阵链相乘，矩阵连乘）**
- **Characteristics of dynamic programming**

  （动态规划法的特征）
- **Longest common subsequence （最长相同子序列）**
- **Optimal binary search trees （最优二叉搜索树）**

15.1 Rod cutting



length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

家里有地?

好好学算法!

15.1 Rod cutting

家里有矿？　好好学算法！

Step 1: The structure of the optimal decomposition

length i	1	2	3	4	5	6	7	8	9	10
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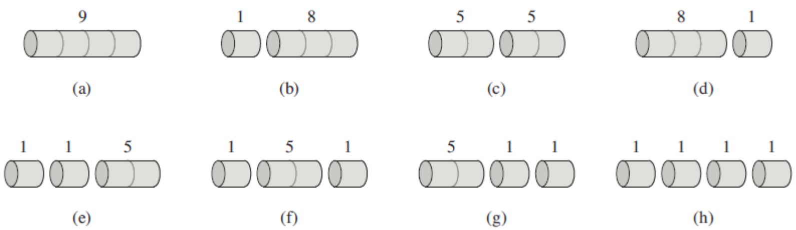

$$
\begin{aligned}
r_1 &= 1 && \text{from solution } 1 = 1 \quad \text{(no cuts)}\,, \\
r_2 &= 5 && \text{from solution } 2 = 2 \quad \text{(no cuts)}\,, \\
r_3 &= 8 && \text{from solution } 3 = 3 \quad \text{(no cuts)}\,, \\
r_4 &= 10 && \text{from solution } 4 = 2 + 2\,, \\
r_5 &= 13 && \text{from solution } 5 = 2 + 3\,, \\
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r_7 &= 18 && \text{from solution } 7 = 1 + 6 \text{ or } 7 = 2 + 2 + 3\,, \\
r_8 &= 22 && \text{from solution } 8 = 2 + 6\,, \\
r_9 &= 25 && \text{from solution } 9 = 3 + 6\,, \\
r_{10} &= 30 && \text{from solution } 10 = 10 \quad \text{(no cuts)}\,.
\end{aligned}
$$

Optimal substructure?

If an optimal solution cuts the rod into k pieces, for some $1 \le k \le n$, then an optimal decomposition

$$n = i_1 + i_2 + \cdots + i_k$$

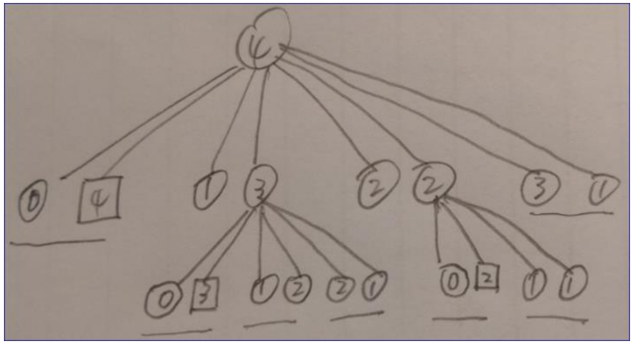
of the rod into pieces of lengths $i_1, i_2, \ldots, i_k$ provides maximum corresponding revenue

$$r_n = p_{i_1} + p_{i_2} + \cdots + p_{i_k}$$

r_n : n 米长rod的最优切割方式

（最优：价格最大）

$$r_n = \max\left(\underbrace{p_n}_{r_0 + p_n}, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\right) \qquad (15.1)$$



Step 1: The structure of the optimal decomposition

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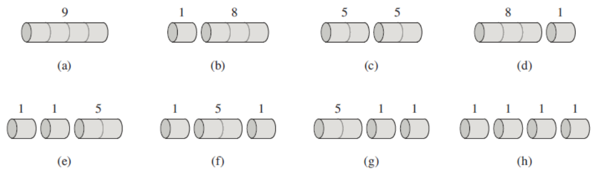

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r_n : n 米长rod的最优切割方式

（最优：价格最大）

$$r_n = \max\left(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\right) \tag{15.1}$$

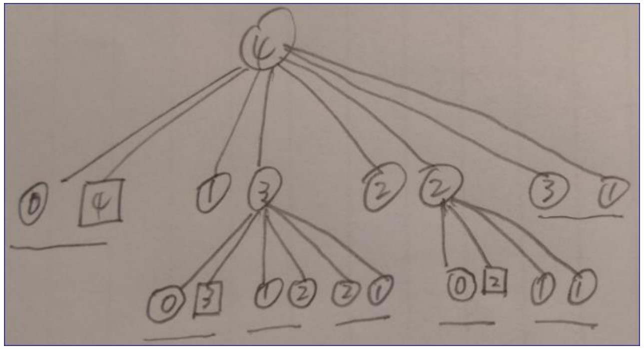
$\boldsymbol{r_0 + p_n}$

Optimal substructure? 例：

原问题 r_7： 一个最优切割（最优解）是2+2+3（对应的最优值为18）；

子问题r_4：(因为 r_4+r_3 ∈ r_7)，那么，可行解2+2一定是 r_4 的一个最优切割（最优解）。

最优子结构：原问题的解（2+2+3）包括子问题的解（2+2）。



Step 2: A recursive solution

length i	1	2	3	4	5	6	7	8	9	10
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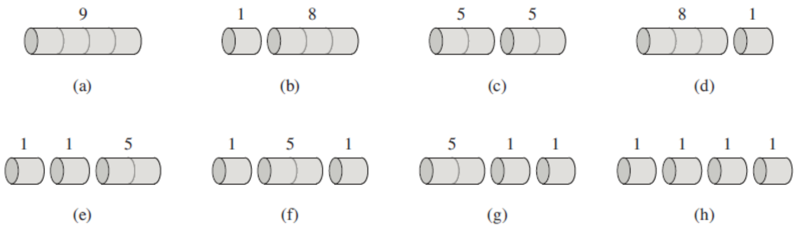

(a) (b) (c) (d)
(e) (f) (g) (h)

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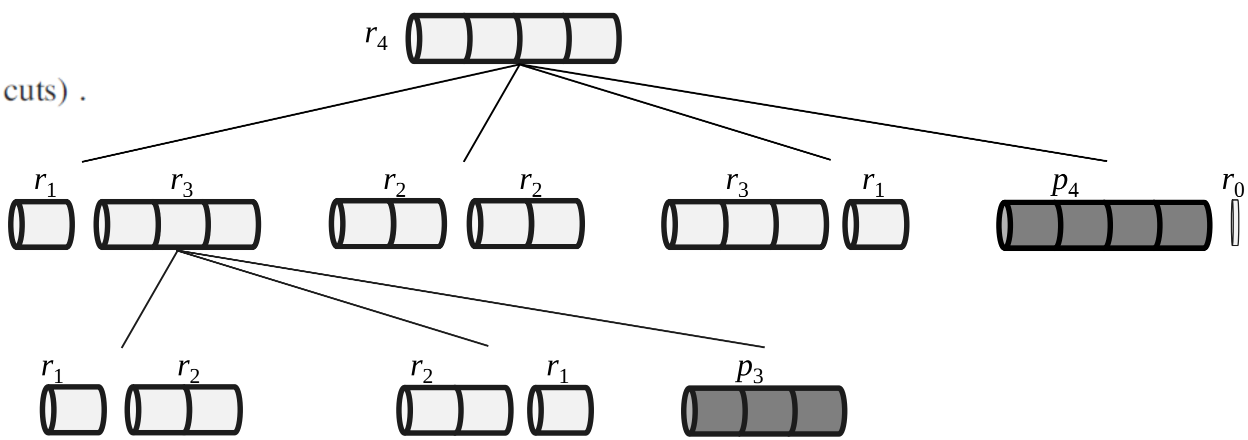
$$r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1) \qquad (15.1)$$

$$\mathbf{r_n = \max(r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1, p_n + r_0)}$$

（写成这样更好理解？）

可以根据15.1直接写递归方程进行求解，但效率很低!



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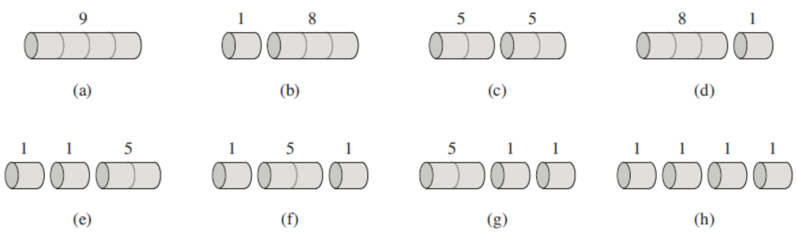

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（最优：价格最大）

$$r_n = \max\left(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\right) \qquad (15.1)$$

?

$$\boxed{r_n = \max_{1 \le i \le n}\left(p_i + r_{n-i}\right)} \qquad (15.2)$$

不用递归计算式(15.1)中 $r_i + r_{n-i}$ 部分的 r_i （因为 $r_i + r_{n-i}$ 与 $r_{n-i} + r_i$ 在式(15.1)中是对称的，r_i 从一个方向递归，就可以遍历所有切割情况。）

Step 2: A recursive solution

1 $$\boxed{r_n = \max\left(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\right)} \quad (15.1)$$

2 $$\boxed{r_n = \max_{1 \le i \le n}\left(p_i + r_{n-i}\right)} \quad (15.2)$$

length i	1	2	3	4	5	6	7	8	9	10
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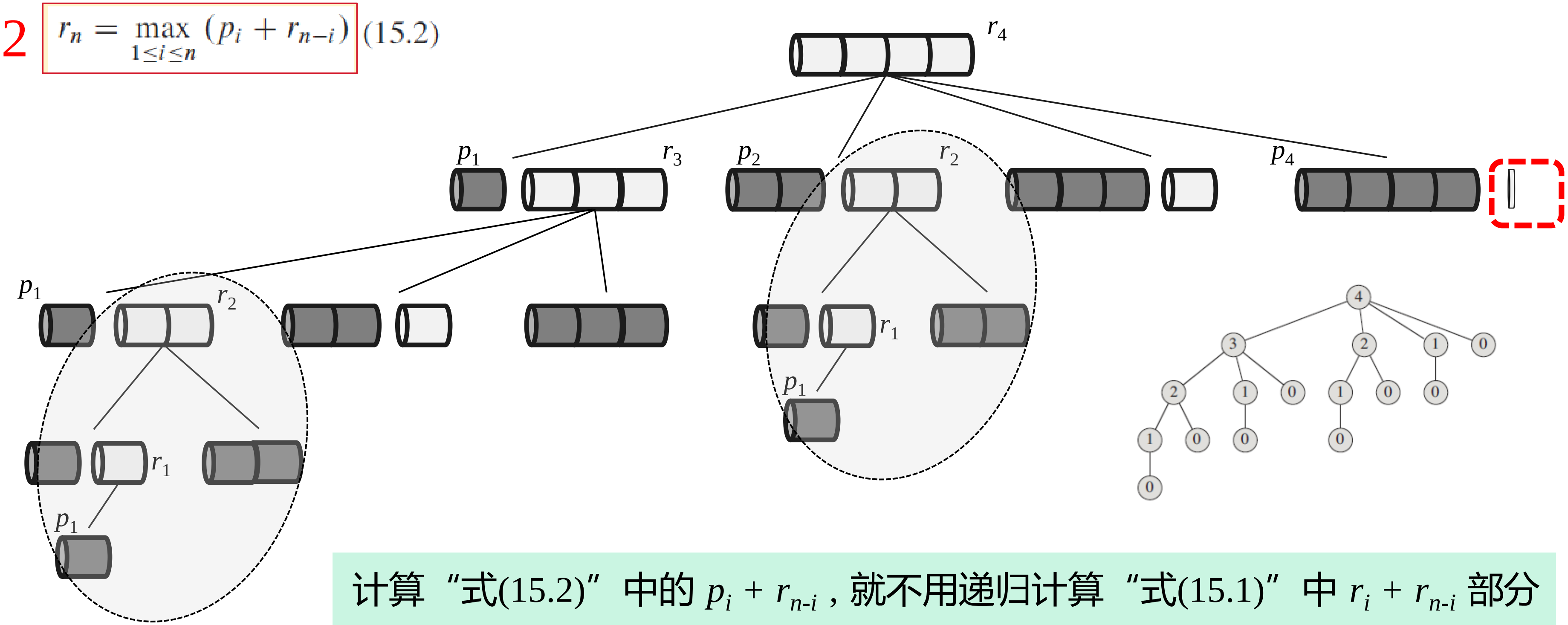


计算“式(15.2)”中的 $p_i + r_{n-i}$，就不用递归计算“式(15.1)”中 $r_i + r_{n-i}$ 部分的 r_i （对称情况，取其一）

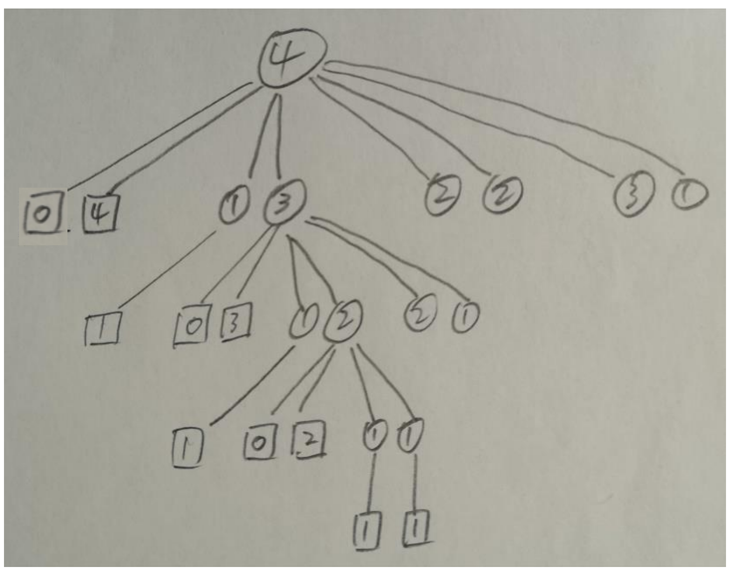
浅色钢管需要进行递归计算，深色钢管直接返回（不用切割）。即便是式15.2，采用递归，也有冗余计算，从图中还能看出，r_2 仍然被递归调用了两次。

Step 2: A recursive solution

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

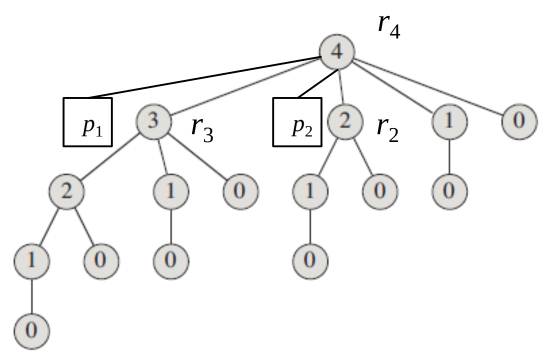
1 $r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1)$

?



子问题多，递归过程中的冗余计算很多。

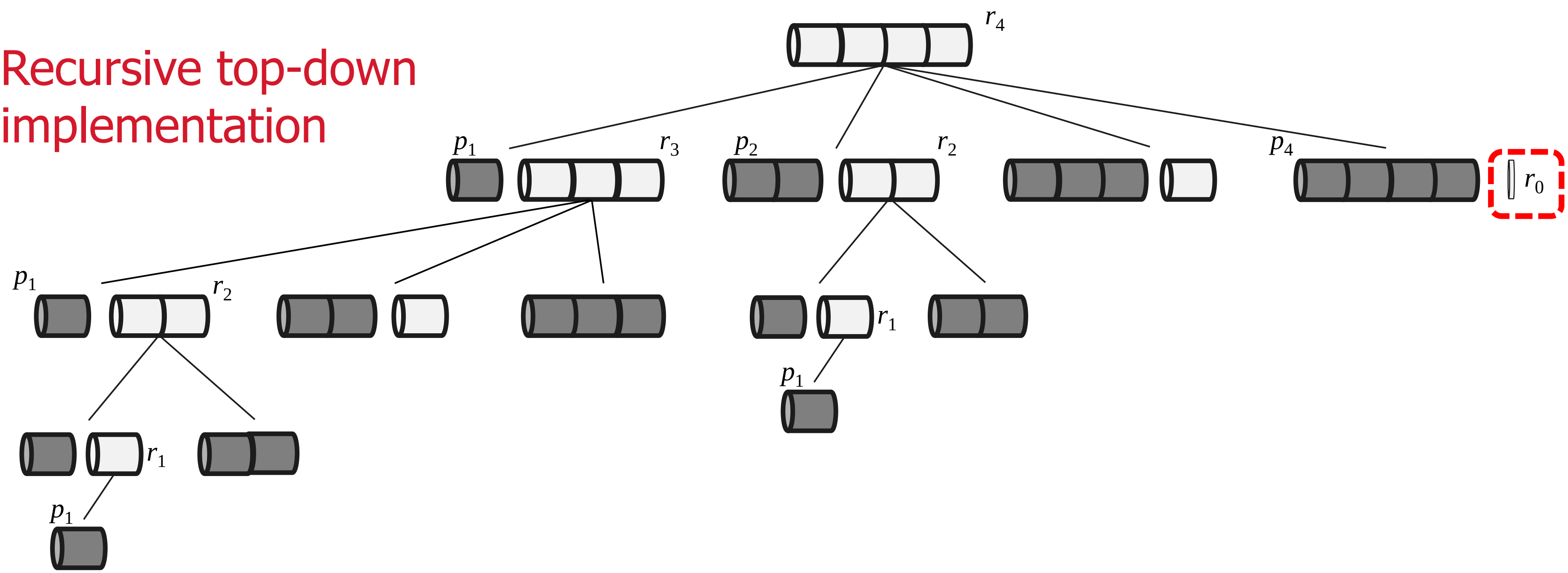
2 $r_n = \max\limits_{1 \le i \le n} (p_i + r_{n-i})$



子问题减少，效率提升。
递归过程中的冗余计算仍然不少。

Step 2: A recursive solution

Recursive top-down implementation



$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

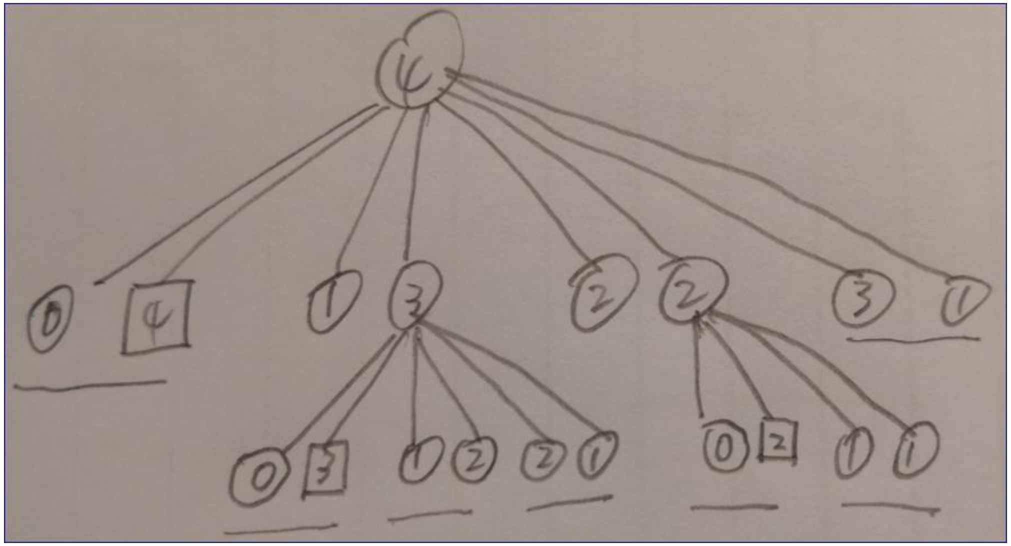
length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

```
CUT-ROD(p, n)
    if n == 0
        return 0
    q = -∞
    for i = 1 to n
        q = max(q, p[i] + CUT-ROD(p, n - i))
    return q
```

Running time?

Exercise

$$r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1)$$

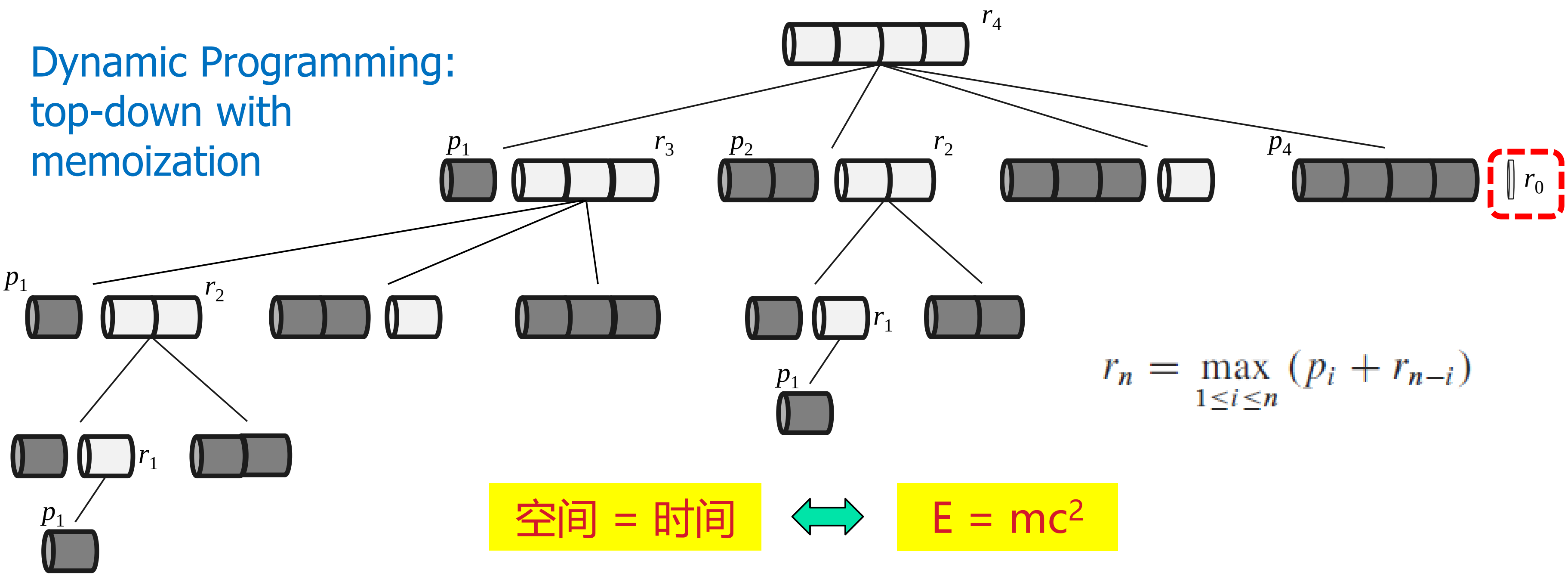


Recursive Algorithm?

Running time?

Step 3: Compute the optimal value

Dynamic Programming:
top-down with
memoization



$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

空间 = 时间 ⟺ E = mc^2

time-memory trade-off ?
开辟空间，存储中间值，
减少递归，节省时间。

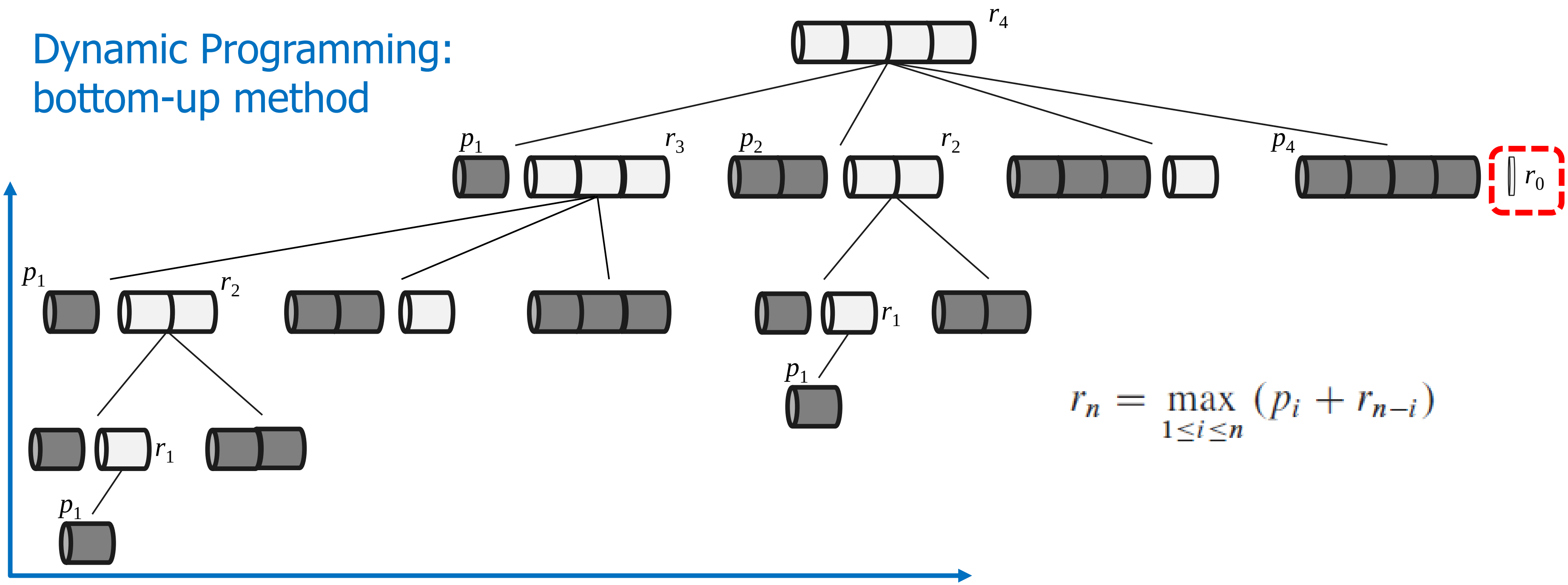
```
MEMOIZED-CUT-ROD(p, n)
  let r[0..n] be a new array
  for i = 0 to n
      r[i] = −∞
  return MEMOIZED-CUT-ROD-AUX(p, n, r)
```

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
  if r[n] ≥ 0              // 已经被计算出结果
      return r[n]          // 直接返回，不重复计算
  if n == 0
      q = 0
  else q = −∞
      for i = 1 to n
          q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
  r[n] = q
  return q
```

Running time?

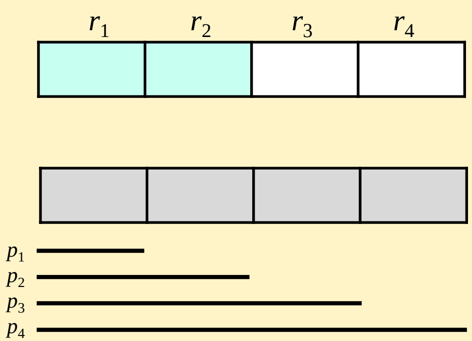
Step 3: Compute the optimal value

Dynamic Programming: bottom-up method



$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

$r_1 = p_1$
$r_2 = \max(p_1+r_1, p_2)$
$r_3 = \max(p_1+r_2, p_2+r_1, p_3)$



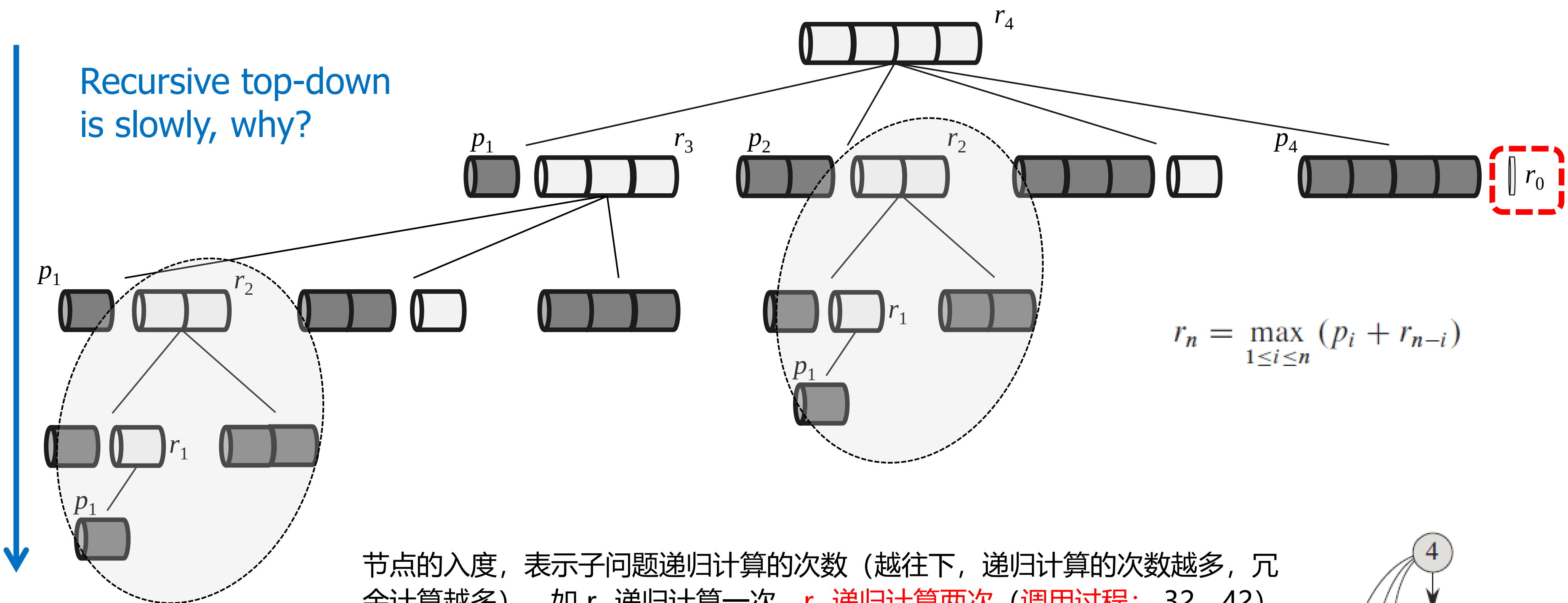
BOTTOM-UP-CUT-ROD(p, n)

```
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        q = max(q, p[i] + r[j - i])
    r[j] = q      // programming: filling table
return r[n]
```

Running time?

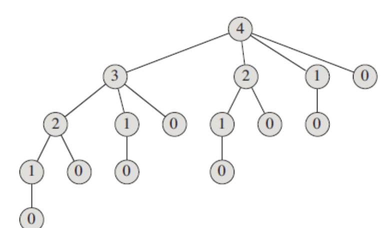
Step 3: Compute the optimal value

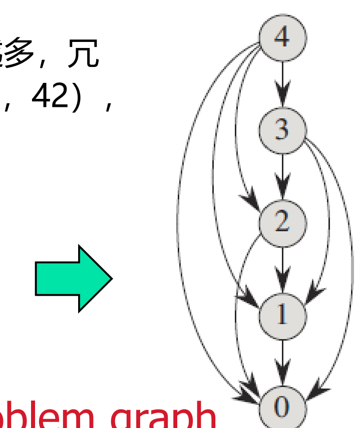
Recursive top-down is slowly, why?



$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

节点的入度，表示子问题递归计算的次数（越往下，递归计算的次数越多，冗余计算越多）。如 r_3 递归计算一次，r_2 递归计算两次（调用过程： 32，42），r_1 递归计算三次（调用过程：21，31，41）



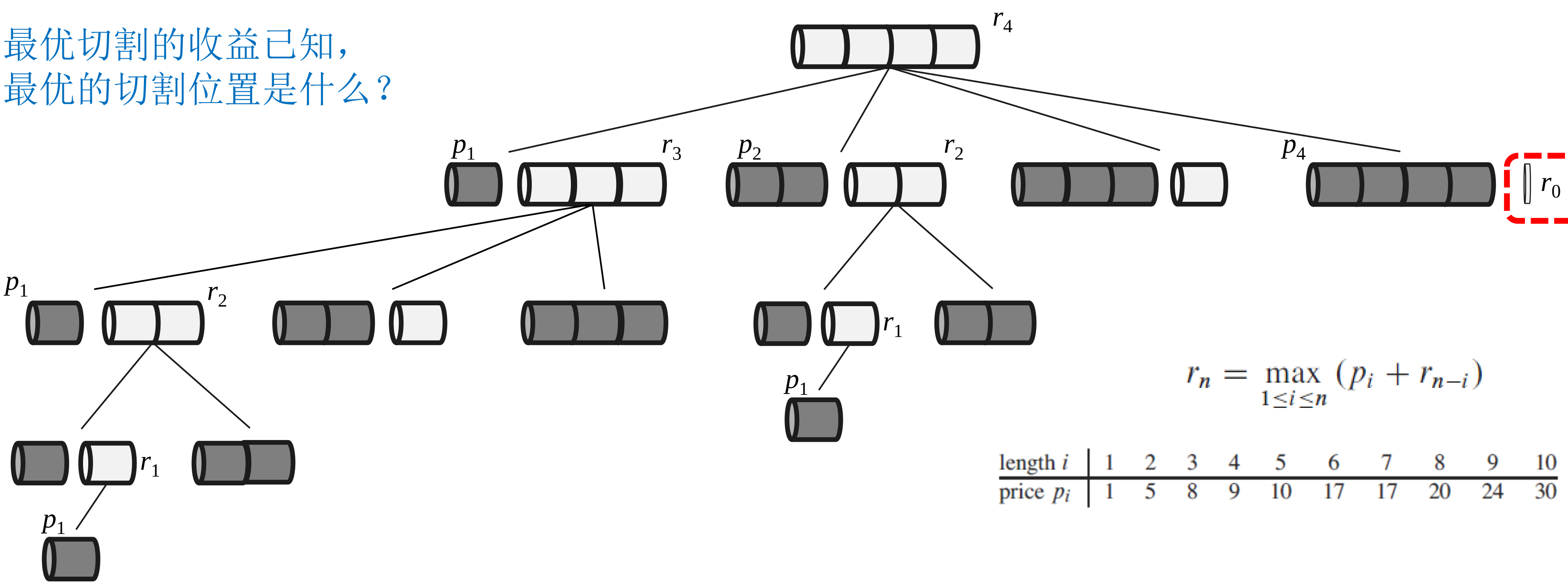


Subproblem graph

1. 递归调用（自顶向下）：拓扑序，顶点之间的调用关系；每个顶点多次被访问（每次访问都重新进行递归计算）。
2. 带备忘录的递归调用（自顶向下）：每个顶点在递归调用中被访问多次，但只计算一次，有值后，其余访问直接返回回值（不重复递归计算）。
3. 填表方法（自底向上）：拓扑逆序（把图中边的方向反向），每个顶点计算一次，每条边被访问一次（备忘录方法同理）。

Step 4: Reconstructing a solution

最优切割的收益已知，
最优的切割位置是什么？



$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

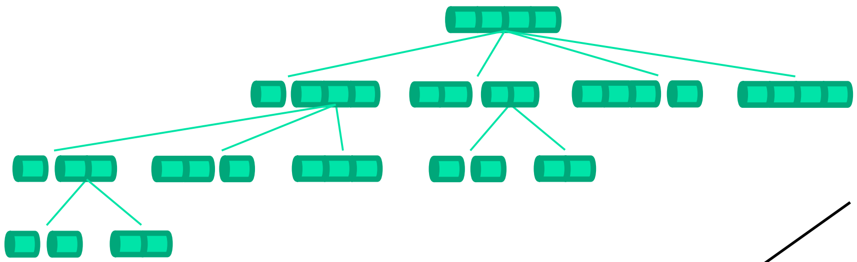
length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

```
BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = −∞
    for i = 1 to j
        q = max(q, p[i] + r[j − i])
    r[j] = q
return r[n]
```

不仅仅求最大值 q，把取得最大值的位置 i 也记录下来（记录在 $s[j]$ 中）。

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] and s[0..n] be new arrays
r[0] = 0
for j = 1 to n
    q = −∞
    for i = 1 to j
        if q < p[i] + r[j − i]
            q = p[i] + r[j − i]
            s[j] = i
    r[j] = q
return r and s
```

Step 4: Reconstructing a solution



$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

length i	1	2	3	4	5	6	7	8	9	10
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输入

输出（填表）

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] and s[0..n] be new arrays
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        if q < p[i] + r[j - i]
            q = p[i] + r[j - i]
            s[j] = i
    r[j] = q
return r and s
```

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

输入

$r_1 = 1$ from solution $1 = 1$ (no cuts) ,
$r_2 = 5$ from solution $2 = 2$ (no cuts) ,
$r_3 = 8$ from solution $3 = 3$ (no cuts) ,
$r_4 = 10$ from solution $4 = 2 + 2$,
$r_5 = 13$ from solution $5 = 2 + 3$,
$r_6 = 17$ from solution $6 = 6$ (no cuts) ,
$r_7 = 18$ from solution $7 = 1 + 6$ or $7 = 2 + 2 + 3$,
$r_8 = 22$ from solution $8 = 2 + 6$,
$r_9 = 25$ from solution $9 = 3 + 6$,
$r_{10} = 30$ from solution $10 = 10$ (no cuts) .

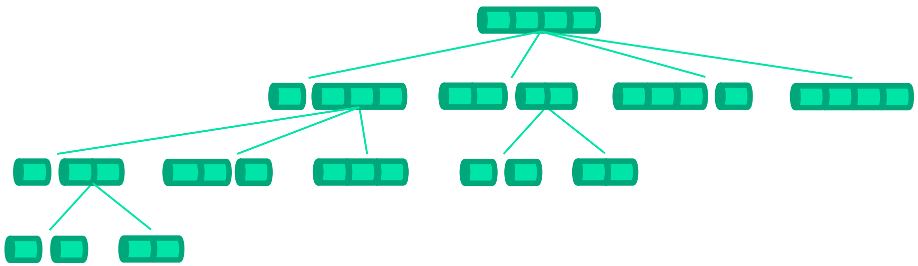
```
PRINT-CUT-ROD-SOLUTION(p, n)
(r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while n > 0
    print s[n]
    n = n - s[n]
```

输出

PRINT程序中实际上取参数 *s* 即可

输出示例：
1. 长度为10米时，整卖最好，不需要切割，能卖30元；
2. 长度为9米时，两截，分别为3米和6米，共卖25元；
3. 长度为8米时，两截，分别为2米和6米，共卖22元；
……

In-class Exercise



i	1	2	3	4	5	6
p_i	2	5	6	8	9	10

i	0	1	2	3	4	5	6
$r[i]$				?			
$s[i]$							

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] and s[0..n] be new arrays
r[0] = 0
for j = 1 to n
    q = −∞
    for i = 1 to j
        if q < p[i] + r[j − i]
            q = p[i] + r[j − i]
            s[j] = i
    r[j] = q
return r and s
```

```
PRINT-CUT-ROD-SOLUTION(p, n)
(r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while n > 0
    print s[n]
    n = n − s[n]
```

长度为6米时，如何切割收益最高？收益是多少？

Exercise

Rod cutting 中的最优子结构、重叠子问题，怎么体现?

For the Recursive Top-down Implementation as (15.1), what is the algorithm? what is the Running time?

$$r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1) \qquad (15.1)$$