Chapter 30

Polynomials, Convolution and the FFT

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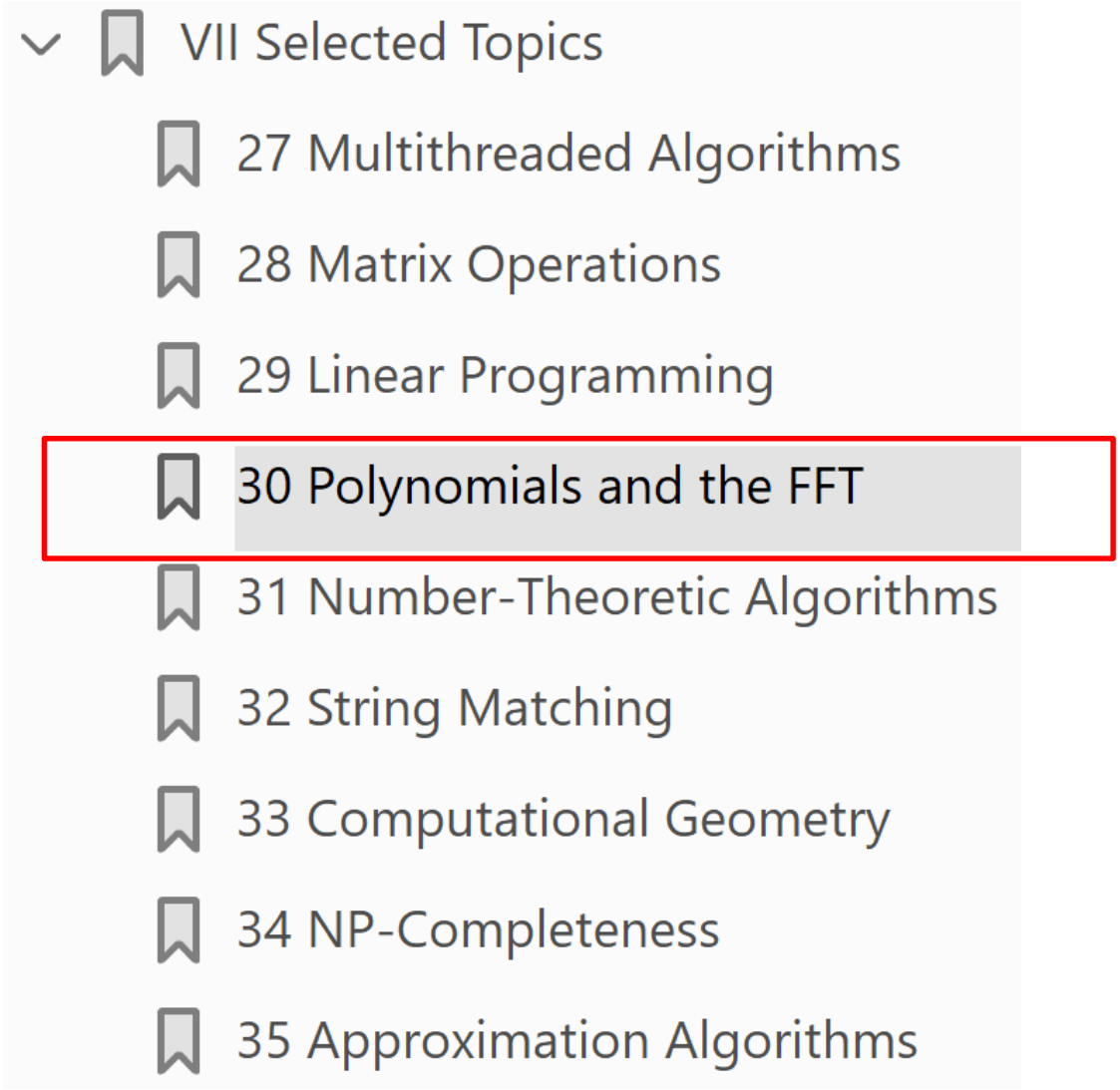
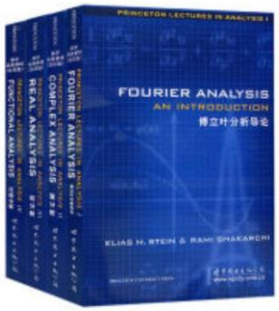
VII Selected Topics



30 Polynomials, Convolution and the FFT



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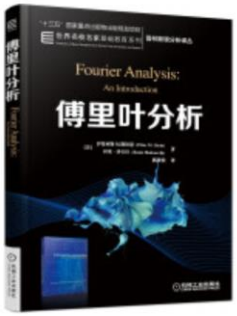

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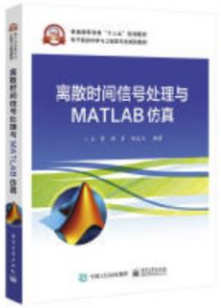

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30 Polynomials, Convolution and the FFT

$$A(x) = a_0 x^0 + a_1 x^1 + \ldots + a_{n-1} x^{n-1} = \sum_{j=0}^{n-1} a_j x^j \quad \text{and} \quad B(x) = \sum_{j=0}^{n-1} b_j x^j$$

$C(x) = A(x) + B(x)$?

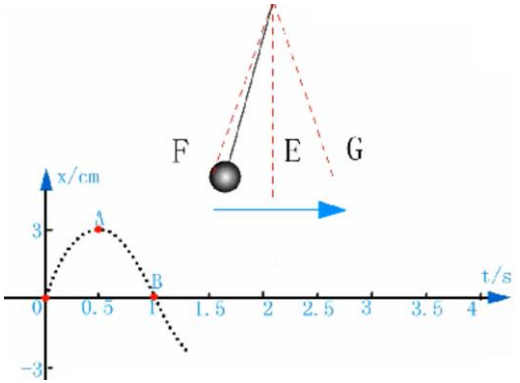
$C(x) = A(x)*B(x) = A(x) \cdot B(x) = A(x)B(x)$?

- **Straightforward method of adding two polynomials of degree *n* takes $\Theta(n)$ time**
- **Straightforward method of multiplying them takes $\Theta(n^2)$ time**
- **Fast Fourier Transform (FFT,〔快速傅里叶变换算法〕, can reduce the time to multiply polynomials to $\Theta(n \lg n)$**
- **Why? How?**

Fourier Transform (FT) and FFT

FT: the most common use is in signal processing

- ***s*(*t*): *time domain*** 〔信号的时域表示〕
- ***S*(*ω*): *frequency domain*** 〔频域〕



$$s(t)=\sin(\omega t)=\sin(60\pi t)=\sin(2\pi\cdot f\cdot t)$$

$$f=\frac{1}{T}=\frac{\omega}{2\pi}=\frac{60\pi}{2\pi}=30$$

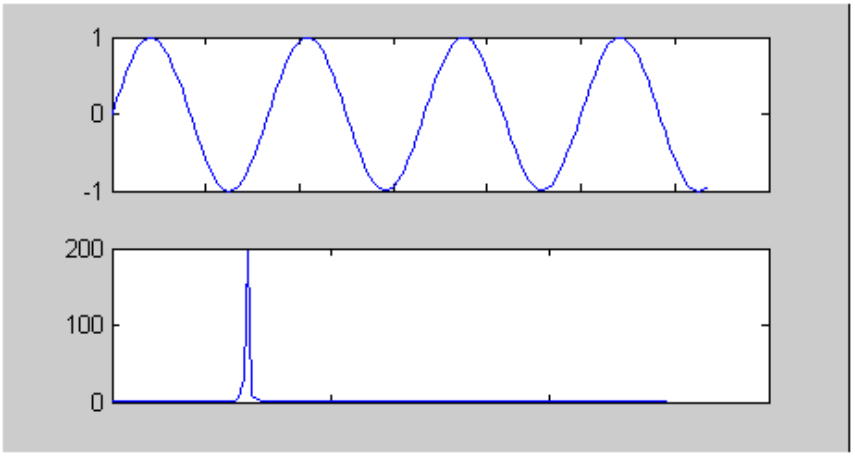
T: 周期，转一圈所需的时间

f: 频率，每秒转多少圈

ω: 角频率，每秒转多少（弧）度

$$S(\omega)=\int e^{-i\omega t}s(t)dt$$

$$S(f)=\int e^{-i2\pi f\cdot t}s(t)dt$$

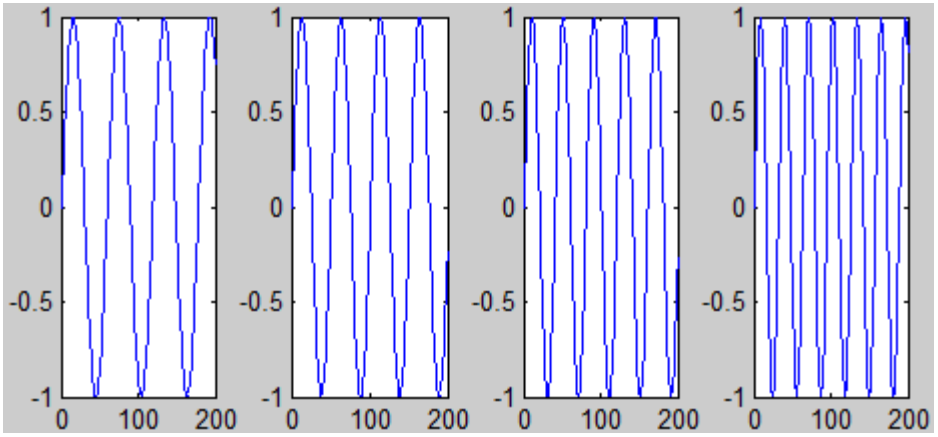


$$s(t)=\sin(60\pi t)+\sin(30\pi t)?$$

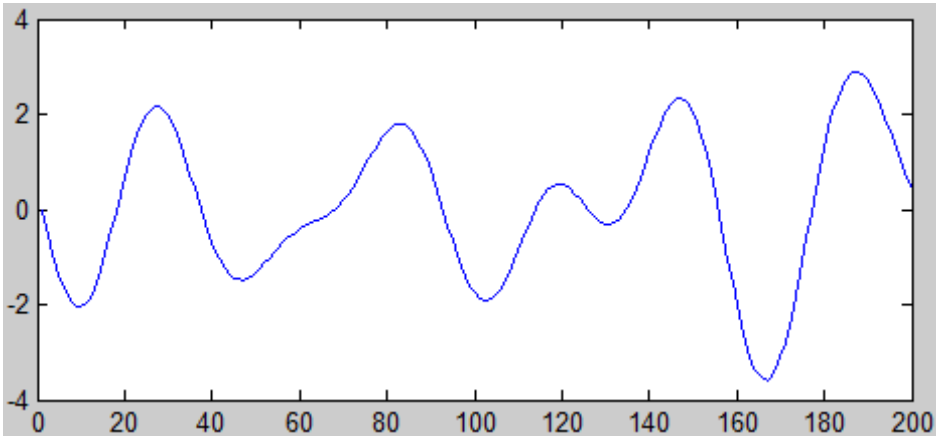
$$s(t)=\sin(60\pi t)+\sin(30\pi t)+\sin(90\pi t)?$$

Fourier Transform (FT) and FFT

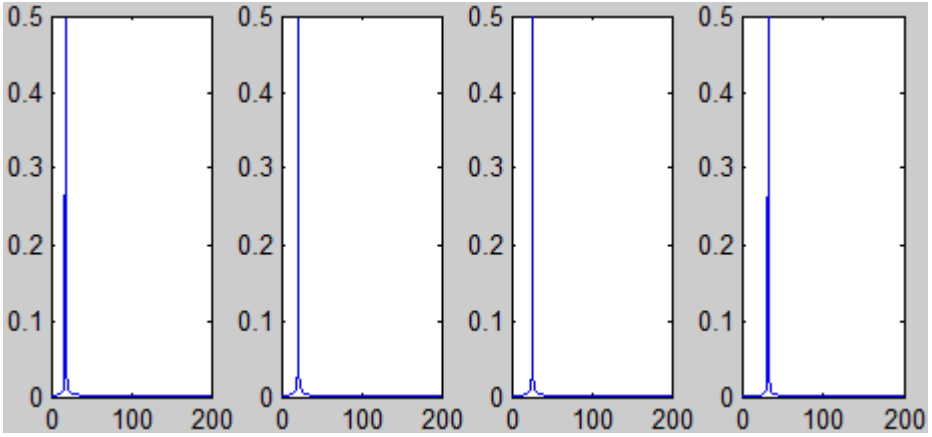
$s_i(t) = \sin(w_i t)$



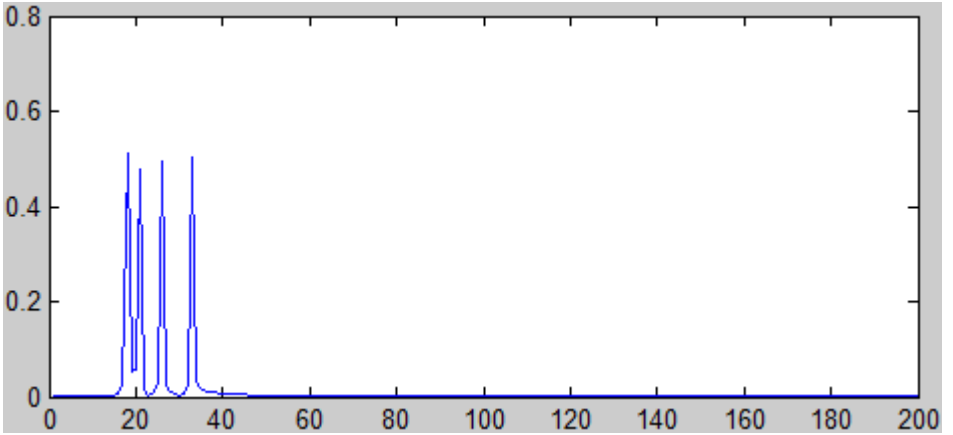
$s(t) = \sum a_i \cdot s_i(t)$



$S_i(\omega) = \mathrm{F}(s_i(t))$



$S(\omega) = \mathrm{F}(s(t))$



Fourier Transform (FT) and FFT

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T : 周期，转一圈所需的时间

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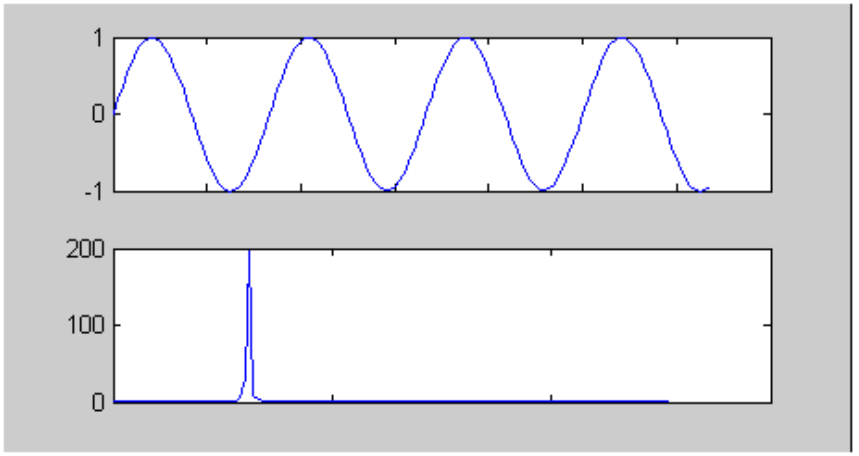
$$S(\omega)=\int e^{-i\omega t}s(t)dt$$

$$S(f)=\int e^{-i2\pi f\cdot t}s(t)dt$$



[计算科学中最重要的32个算法]

$$s(t)=\sin(60\pi t)+\sin(30\pi t)?$$

$$s(t)=\sin(60\pi t)+\sin(30\pi t)+\sin(90\pi t)?$$

Fourier Transform

$$S(\omega)=\int e^{-i\omega t}s(t)dt$$

让·巴普蒂斯·约瑟夫·傅里叶（Baron Jean Baptiste Joseph Fourier，1768-1830），男爵，法国数学家、物理学家，1768年3月21日生于欧塞尔，1830年5月16日卒于巴黎。

1817年当选为科学院院士。主要贡献是在研究《热的传播》和《热的分析理论》时创立了一套数学理论，对19世纪的数学和物理学的发展都产生了深远影响。

$$s(t)=\sin(60\pi t)+\sin(30\pi t)?$$

$$s(t)=\sin(60\pi t)+\sin(30\pi t)+\sin(90\pi t)?$$

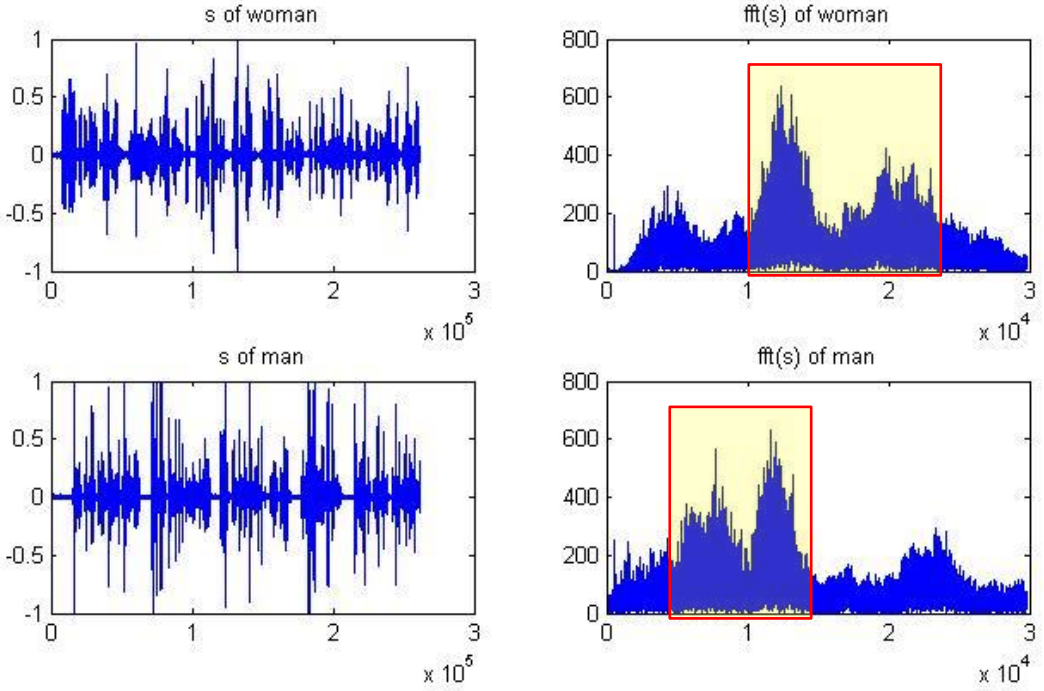
- **FT的基本思想首先由傅里叶提出，所以以其名字来命名以示纪念。傅里叶变换是一种特殊的积分变换。它能将满足一定条件的某个函数表示成正弦基函数的线性组合或者积分。在不同的研究领域，傅里叶变换具有多种不同的变体形式，如连续傅里叶变换和离散傅里叶变换。**
- **数学上看，用简单表示来对复杂函数的深入研究。正弦函数在物理上是被充分研究而相对简单的函数类，这一想法跟化学上的原子论想法何其相似！奇妙的是，现代数学发现傅里叶变换具有非常好的性质，使得它如此的好用和有用，让人不得不感叹造物的神奇。**
- **哲学上看，"分析主义"和"还原主义"，就是要通过对事物内部适当的分析达到增进对其本质理解的目的。比如近代原子论试图把世界上所有物质的本源分析为原子，而原子不过数百种而已，相对物质世界的无限丰富，这种分析和分类无疑为认识事物的各种性质提供了很好的手段。**

30 Polynomials, Convolution and the FFT

- **FT: the most common use is in signal processing**
  - ***s*(*t*): *time domain*** 〔信号的时域表示〕
  - ***S*(*w*): *frequency domain*** 〔频域〕
- **FFT: Fast Fourier Transform**

$$S(\omega) = \int e^{-i\omega t} s(t) dt$$

$$S(f) = \int e^{-i2\pi f \cdot t} s(t) dt$$



Polynomials

$$A(x) = \sum_{j=0}^{n-1} a_j x^j \qquad x \in F, \quad (R,\ C)$$

$$(a_0, a_1, \cdots, a_{n-1})$$

- **polynomial *coefficients* : $a_0, a_1, \cdots, a_{n-1}, \in F$.**
- **polynomial *degree* 〔维数〕 : $A(x)$ is said to have degree k if its highest nonzero coefficient is a_k .**
- **polynomial *degree-bound* 〔维界〕 : any integer strictly greater than the degree.**

- **Therefore, the degree of a polynomial of degree-bound n may be any integer between 0 and n-1, inclusive.**

Polynomials addition〔多项式相加〕

- $A(x)$, $B(x)$, degree-bound n
- $C(x) = A(x) + B(x)$, also degree-bound n

$$A(x) = \sum_{j=0}^{n-1} a_j x^j \quad \text{and} \quad B(x) = \sum_{j=0}^{n-1} b_j x^j$$

$$\text{then} \quad C(x) = \sum_{j=0}^{n-1} c_j x^j$$

where $c_j = a_j + b_j$ for $j = 0, 1, \cdots, n - 1$.

For example, if the polynomials $A(x) = 6x^3+7x^2-10x+9$

and $B(x) = -2x^3+4x-5$,

then $C(x) = 4x^3+7x^2-6x+4$.

- Running time: $T(n) = \Theta(n)$

Polynomials multiplication〔多项式相乘〕

- $A(x)$, $B(x)$, degree-bound n
- $C(x) = A(x)*B(x)$, degree-bound $2n-1$

$$= \sum_{j=0}^{2n-2} c_j x^j$$

For example, multiply $A(x) = 6x^3 + 7x^2 - 10x + 9$

and $B(x) = -2x^3 + 4x - 5$

$$\begin{array}{rrrrrrr}
 & & & 6x^3 & + 7x^2 & - 10x & + 9 \\
 & & & - 2x^3 & & + 4x & - 5 \\
\hline
 & & & - 30x^3 & - 35x^2 & + 50x & - 45 \\
 & & 24x^4 & + 28x^3 & - 40x^2 & + 36x & \\
 - 12x^6 & - 14x^5 & + 20x^4 & - 18x^3 & & & \\
\hline
 - 12x^6 & - 14x^5 & + 44x^4 & - 20x^3 & - 75x^2 & + 86x & - 45
\end{array}$$

- Running time: $T(n) = \Theta(n^2)$

Polynomials multiplication〔多项式相乘〕

- **$A(x)$, $B(x)$, degree-bound n** $A(x)=\sum_{j=0}^{n-1}a_j x^j$ and $B(x)=\sum_{j=0}^{n-1}b_j x^j$
- **$C(x) = A(x)*B(x)$, degree-bound $2n$-1**

Another way to express the product $C(x)$ is

Convolution 卷积

$$C(x)=\sum_{j=0}^{2n-2}c_j x^j \quad (30.1) \text{ , where } c_j=\sum_{k=0}^{j}a_k b_{j-k} \quad (30.2)$$

$$a \otimes b = (a_0, a_1, \cdots, a_{n-1}) \otimes (b_0, b_1, \cdots, b_{n-1}) = (c_0, c_1, \cdots, a_{2n-2})$$

where, $c_j = a_0 b_j + a_1 b_{j-1} + \ldots + a_k b_{j-k} + \ldots + a_j b_0$

Note that degree(C) = degree(A) + degree(B), implying

degree-bound(C) = degree-bound(A) + degree-bound(B) -1

≤ degree-bound(A) + degree-bound(B).

We also say: degree-bound(C)=degree-bound(A)+degree-bound(B)

Polynomials multiplication〔多项式相乘〕

- $A(x)$, $B(x)$, degree-bound n $\quad A(x)=\sum_{j=0}^{n-1}a_j x^j$ and $B(x)=\sum_{j=0}^{n-1}b_j x^j$
- $C(x) = A(x)*B(x)$, degree-bound $2n-1$

  Another way to express the product $C(x)$ is

Convolution 卷积

$$C(x)=\sum_{j=0}^{2n-2}c_j x^j \qquad (30.1)\ \text{, where} \quad c_j=\sum_{k=0}^{j}a_k b_{j-k} \qquad (30.2)$$

$$a \otimes b = (a_0, a_1, \cdots, a_{n-1}) \otimes (b_0, b_1, \cdots, b_{n-1}) = (c_0, c_1, \cdots, a_{2n-2})$$

$$\text{where, } c_j = a_0 b_j + a_1 b_{j-1} + \ldots + a_k b_{j-k} + \ldots + a_j b_0$$

- **Running time: $T(n) = \Theta(n^2)$, since each coefficient in A must be multiplied by each coefficient in B.**
- **Can we accelerate the computation? Yes!**

Chapter outline

- **30.1, presents two ways to represent polynomials:**
  - **coefficient representation**
  - **point-value representation**
- **Adding polynomials**
- **The straightforward methods for multiplying polynomials**
  - **$\Theta(n^2)$, polynomials are represented in coefficient form**
  - **$\Theta(n)$, when they are represented in point-value form**
- **Multiply polynomials using the coefficient representation in only $\Theta(n \lg n)$ time? FFT and its inverse**
- **30.2, DFT and FFT**
- **30.3, how to implement the FFT quickly**

30.1 Representation of polynomials

- **equivalence : The coefficient and point-value representations of polynomials are in a sense equivalent.**

- **That is, a polynomial in point-value form has a unique counterpart in coefficient form.**

30.1.1 Coefficient representation

- **A *coefficient representation* of a polynomial** $A(x) = \sum_{j=0}^{n-1} a_j x^j$

  $$a = (a_0, a_1, \cdots, a_{n-1})^T, \text{ column vectors}$$

- **Convenient for certain operations. For example,**
  - ***evaluating*** $A(x)$ **at a given point** x_0 **, takes time** $\Theta(n)$ **using *Horner's rule*:**

    $$A(x_0) = a_0 + x_0(a_1 + x_0(a_2 + \cdots + x_0(a_{n-2} + x_0(a_{n-1}))\ )).$$

  - ***adding*** **two polynomials represented** $a = (a_0, a_1, \cdots, a_{n-1})^T$ **and** $b = (b_0, b_1, \cdots, b_{n-1})^T$ **, takes** $\Theta(n)$ **time: we just produce the coefficient vector** $c = (c_0, c_1, \cdots, c_{n-1})$**, where** $c_j = a_j + b_j$ **for** $j = 0, 1, \cdots, n - 1$.

30.1.1 Coefficient representation

- **multiplication of $A(x)$ and $B(x)$**

$$A(x)=\sum_{j=0}^{n-1}a_j x^j,\quad a=(a_1,a_2,\cdots,a_{n-1})^T;\qquad B(x)=\sum_{j=0}^{n-1}b_j x^j,\quad b=(b_1,b_2,\cdots,b_{n-1})^T$$

$$C(x)=\sum_{j=0}^{2n-2}c_j x^j \qquad (30.1)$$

$$\text{where}\quad c_j=\sum_{k=0}^{j}a_k b_{j-k} \qquad (30.2)$$

- **takes time $\Theta(n^2)$, why?**
- **more difficult than adding two polynomials.**
- **vector c, given by equation (30.2), is also called the *convolution*(卷积) of a and b, denoted by $c = a \odot b$ ($a \otimes b$) .**
- **Multiplying polynomials and computing convolutions are fundamental computational problems, and important.**

30.1.2 Point-value representation

- **a *point-value representation* of a polynomial**

$$y = A(x) = \sum_{j=0}^{n-1} a_j x^j$$

$\{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$, a set of n point-value pairs

such that all of the x_k are distinct and

$$y_k = A(x_k),\ k = 0, 1, \cdots, n - 1 \qquad (30.3)$$

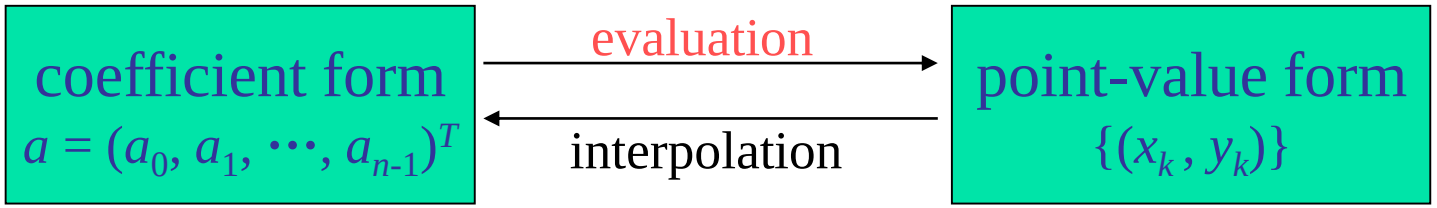
- **any set of n distinct points $x_0, x_1, \cdots, x_{n-1}$ can be used as a basis for a polynomial representation, which is not unique.〔多项式的点值表示不唯一〕**

30.1.2 Point-value representation

- **polynomial representation** $y = A(x) = \sum_{j=0}^{n-1} a_j x^j$

  $a = (a_0, a_1, \cdots, a_{n-1})^T$, column vectors

  $\{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$, a set of n point-value pairs



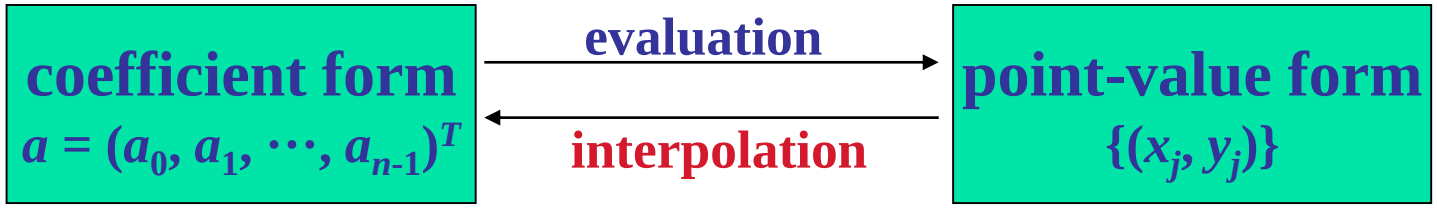
- **Evaluation**
  - **select n distinct points $x_0, x_1, \cdots, x_{n-1}$,**
  - **evaluate $A(x_k)$ for $k = 0, 1, \cdots, n - 1$.**
  - **the n-point evaluation takes time $\Theta(n^2)$. (Horner method)**
  - **if choose the x_k cleverly, the computation only needs $\Theta(n \lg n)$.**

30.1.2 Point-value representation

- **polynomial representation** $A(x) = \sum_{j=0}^{n-1} a_j x^j$

  $a = (a_0, a_1, \cdots, a_{n-1})^T$, column vectors

  $\{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$, a set of n point-value pairs



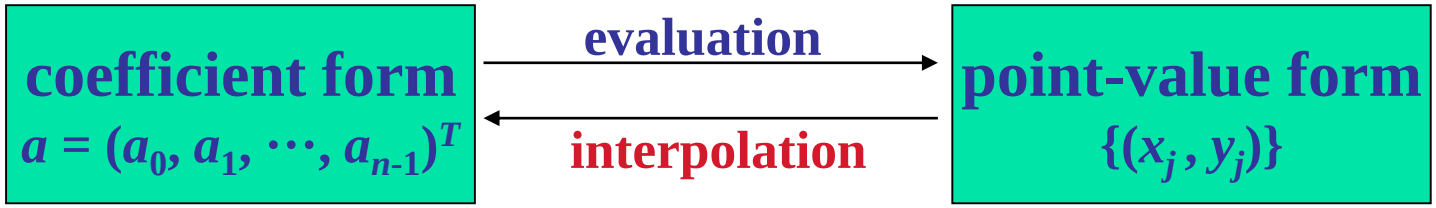
- **Interpolation**〔插值〕, The inverse of evaluation

- *Theorem* 30.1 (**Uniqueness** of an interpolating polynomial)

  For any set $\{(x_k, y_k)\}$ of n point-value pairs, all the x_k values are distinct, there is a unique polynomial $A(x)$ of degree-bound n such that $y_k = A(x_k)$ for $k = 0,1, \cdots, n - 1$.

30.1.2 Point-value representation

- **polynomial representation** $y = A(x) = \sum_{j=0}^{n-1} a_j x^j$



- ***Theorem* 30.1 For any set $\{(x_k, y_k)\}$ of n point-value pairs, all the x_k values are distinct, there is a unique polynomial $A(x)$ of degree-bound n such that $y_k = A(x_k)$ for $k = 0, 1, \cdots, n-1$.**

  Proof **Equation (30.3) "$y_k = A(x_k)$, $k = 0, 1, \cdots, n-1$" is equivalent to**

$$\begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ & & \cdots\cdots & & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{n-1} \end{pmatrix} = V(x_0, x_1, \cdots, x_{n-1}) \cdot a \qquad (30.4)$$

  V is Vander-monde matrix, has determinant $\prod_{0 \le j < k \le n-1} (x_k - x_j)$, therefore, V^{-1} exists. Thus, $a = V(x_0, x_1, \cdots, x_{n-1})^{-1} \cdot y$. (chap28, LU decomposition, $O(n^3)$)

30.1.2 Point-value representation

- **polynomial representation** $y = A(x) = \sum_{j=0}^{n-1} a_j x^j$

- ***Theorem* 30.1 For any set $\{(x_k, y_k)\}$ of n point-value pairs, all the x_k values are distinct, there is a unique polynomial $A(x)$ of degree-bound n such that $y_k = A(x_k)$ for $k = 0,1, \cdots, n - 1$.**

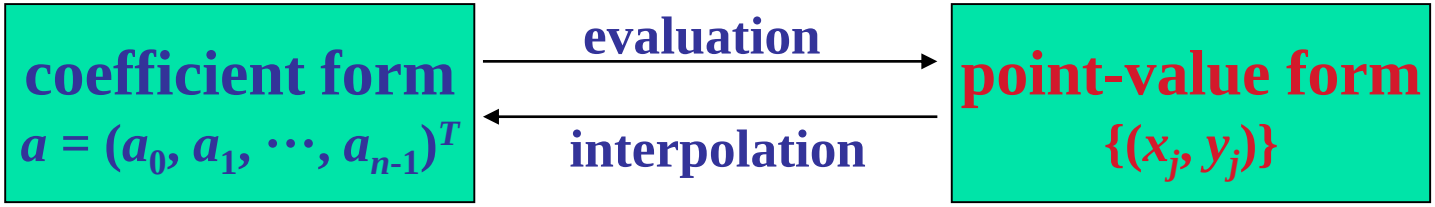
- ***Lagrange's formula*: A faster algorithm for n-point interpolation**

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Computing a of A using Lagrange's formula takes time $\Theta(n^2)$.

30.1.2 Point-value representation

- **polynomial representation** $A(x) = \sum_{j=0}^{n-1} a_j x^j$



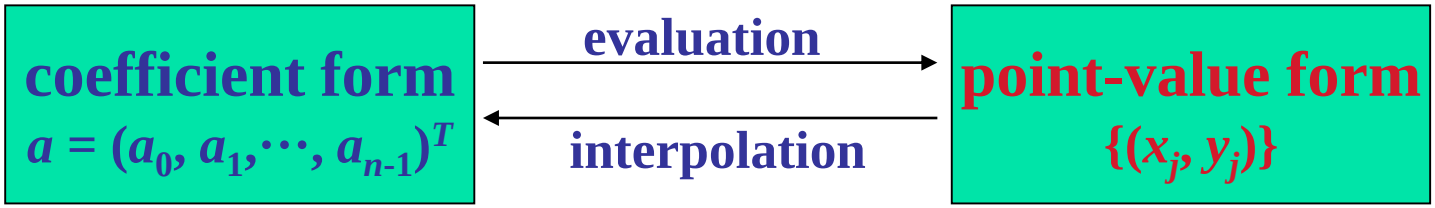
- **quite convenient for many operations on polynomials,**
  - ***addition*, if $C(x)=A(x)+B(x)$, then $C(x_k)=A(x_k)+B(x_k)$ for any point x_k .**
  - **More precisely, if we have a point-value representation for $A : \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$, and**

    for $B : \{(x_0, z_0), (x_1, z_1), \cdots, (x_{n-1}, z_{n-1})\}$, then

    $$C : \{(x_0, y_0+z_0), (x_1, y_1+z_1), \cdots, (x_{n-1}, y_{n-1}+z_{n-1})\}$$
  - **takes time $\Theta(n)$.**

30.1.2 Point-value representation

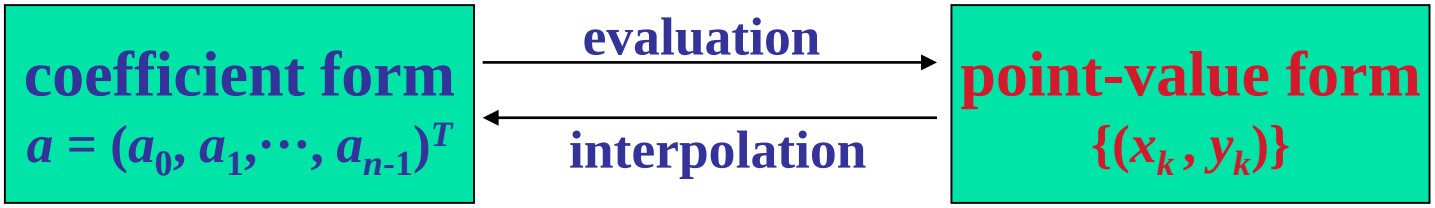
- **polynomial representation** $A(x) = \sum_{j=0}^{n-1} a_j x^j$



- **quite convenient for many operations on polynomials,**
  - ***multiplying*, if $C(x)=A(x)B(x)$, then $C(x_k)=A(x_k)B(x_k)$ for any point x_k, however**
  - **degree-bound(C)=degree-bound(A)+degree-bound(B)**
  - **multiplying A and B gives n point-value pairs for C,**
  - **but, degree-bound(C) = $2n$**
  - **need $2n$ point-value pairs for C**
  - **must "extended" point-value for A and B consisting of $2n$ point-value pairs each.**

30.1.2 Point-value representation

- **polynomial representation** $y = A(x) = \sum_{j=0}^{n-1} a_j x^j$



- **quite convenient for many operations on polynomials,**
  - ***multiplying***, $C(x)=A(x)B(x)$ ⇨ $C(x_k)=A(x_k)B(x_k)$ **for any** x_k
  - **"extended" point-value for *A* and *B***
  - **Given an extended point-value representation for *A* and *B*,**

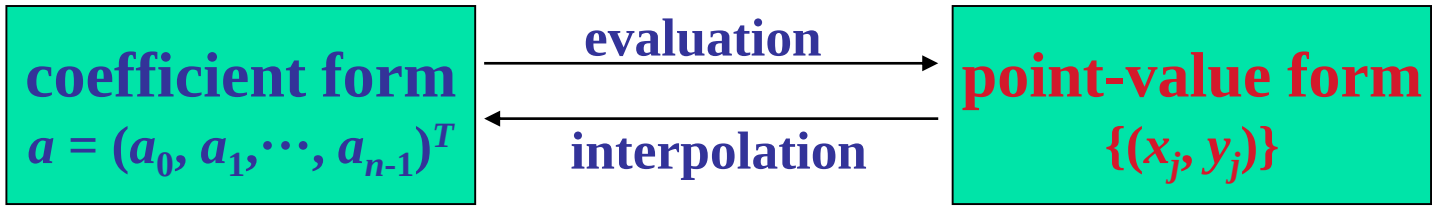
    $A : \{(x_0, y_0), (x_1, y_1), \cdots, (x_{2n-1}, y_{2n-1})\}$, **and**

    $B : \{(x_0, z_0), (x_1, z_1), \cdots, (x_{2n-1}, z_{2n-1})\}$, **then**

    $C : \{(x_0, y_0 z_0), (x_1, y_1 z_1), \cdots, (x_{2n-1}, y_{2n-1} z_{2n-1})\}$
  - **takes time $\Theta(n)$, much less than the time required to multiply polynomials in coefficient form.**

30.1.2 Point-value representation

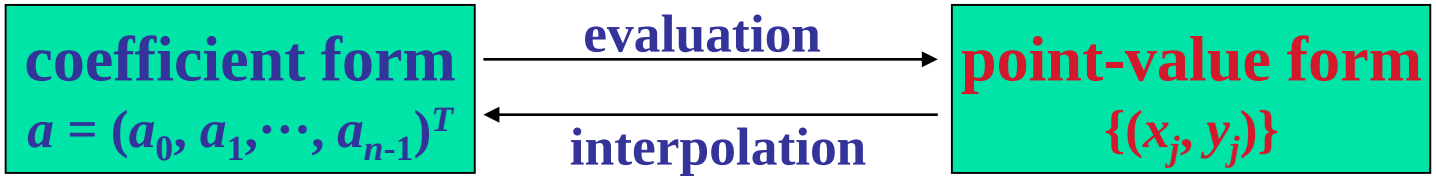
- **polynomial representation** $y = A(x) = \sum_{j=0}^{n-1} a_j x^j$



- **How to evaluate a polynomial given in point-value form at a new point?**
  给定多项式的点值表示形式，如何求多项式在新点处的值？
- **There is apparently no approach that is simpler than converting the polynomial to coefficient form first, and then evaluating it at the new point.**
  先将多项式转换为系数表示形式，再求多项式在新点处的值。

30.1.3 Fast multiplication of polynomials

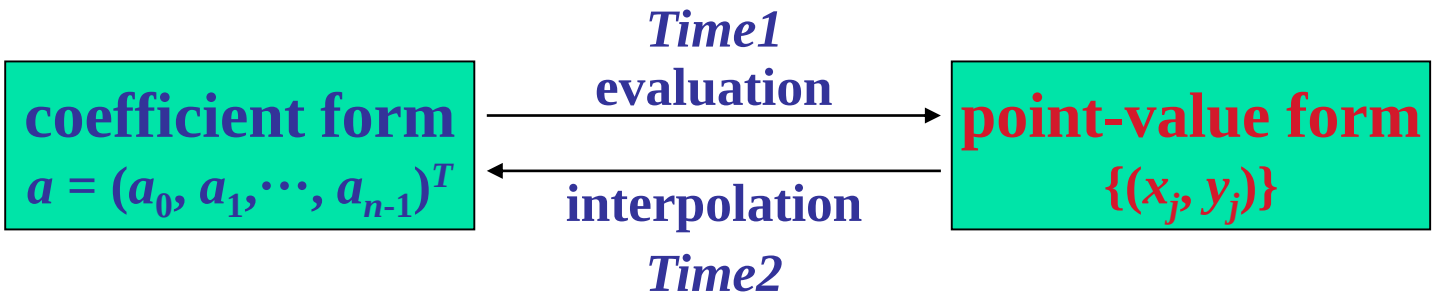
- **polynomial representation** $A(x) = \sum_{j=0}^{n-1} a_j x^j$



- *Multiplying polynomial* $C(x)=A(x)B(x)$
  - **point-value form, $\Theta(n)$**
  - **coefficient form, $\Theta(n^2)$**
- **Can we use the linear-time multiplication method for polynomials in point-value form to expedite(加速) polynomial multiplication in coefficient form?**
  对于多项式乘法，能否依据点值形式的线性乘法来改进系数形式的乘法？
- **The answer hinges on our ability to convert a polynomial quickly from coefficient form to point-value and vice-versa.**
  该问题依赖于将多项式从系数表示快速转换为点值表示的能力，或相反。

30.1.3 Fast multiplication of polynomials

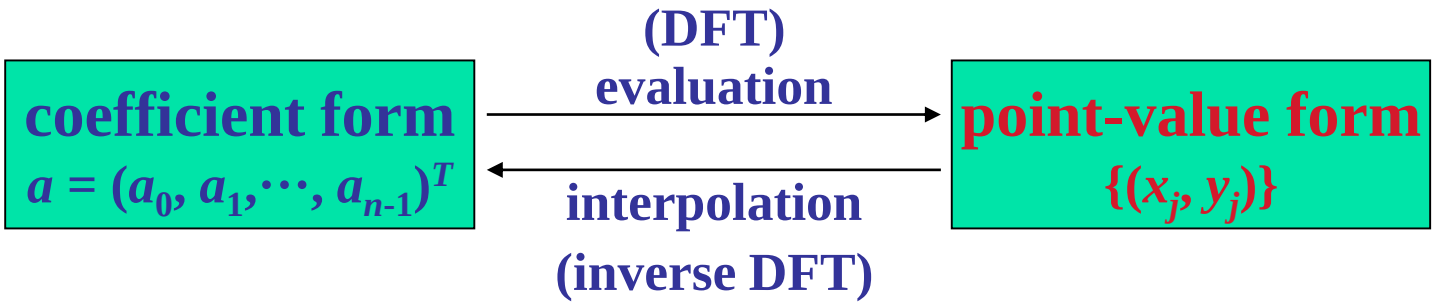
polynomial multiplication



If $Time1 + Time2 + n < n^2$

30.1.3 Fast multiplication of polynomials

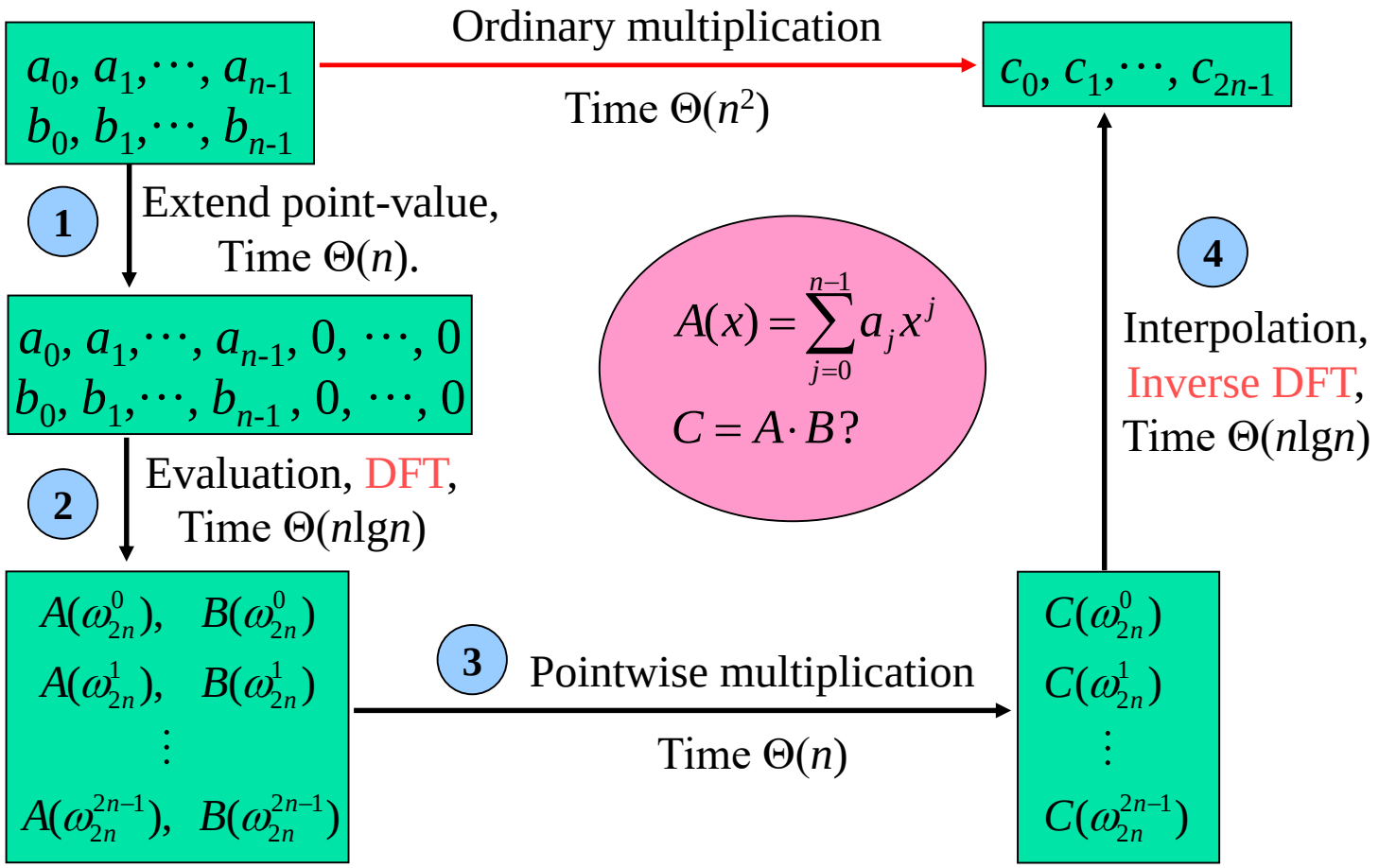
- **polynomial representation** $A(x) = \sum_{j=0}^{n-1} a_j x^j$



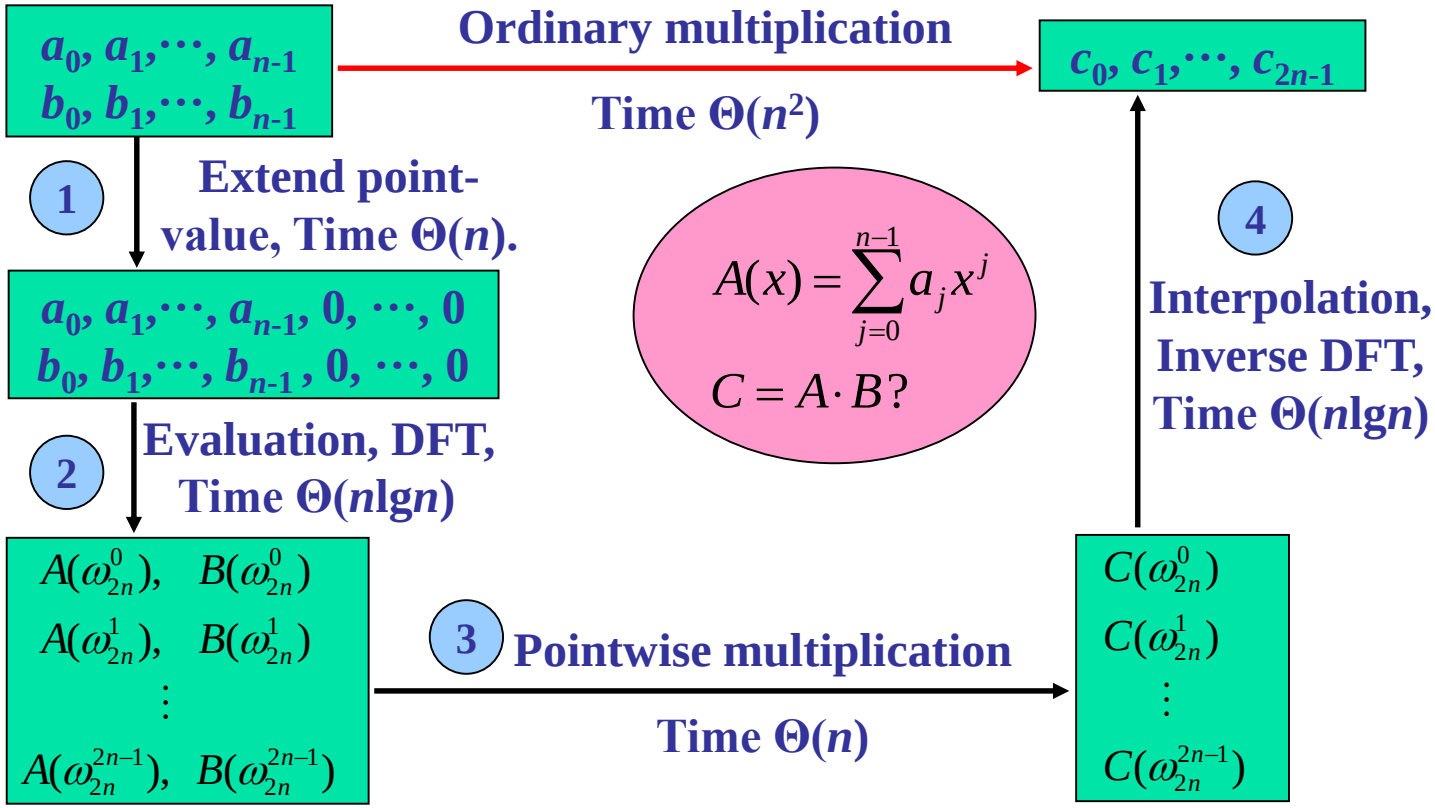
- **We can use any points we want as evaluation points,**
- **but by choosing "complex roots of unity" as the evaluation points, we can convert between representations in only $\Theta(n \lg n)$ time.** 〔选单位复数根作为求值点〕
- **30.2, FFT performs the DFT and inverse DFT operations in $\Theta(n \lg n)$ time.**

30.1.3 Fast multiplication of polynomials

A graphical outline of an efficient polynomial-multiplication process



30.1.3 Fast multiplication of polynomials



***Theorem* 30.2** **The product of two polynomials of degree-bound n can be computed in time $\Theta(n \lg n)$, with both the input and output representations in coefficient form.**

Exercises

30.1-2

Another way to evaluate a polynomial $A(x)$ of degree-bound n at a given point x_0

$$A(x) = q(x)(x - x_0) + r$$

30.2 The DFT and FFT

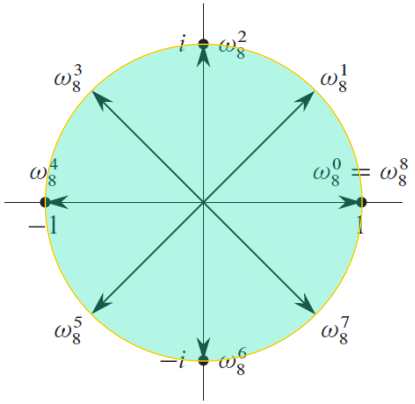
$$S(\omega) = \int e^{-i\omega t} s(t) dt \qquad A(x) = \sum_{j=0}^{n-1} a_j x^j$$

- **DFT (Discrete Fourier Transform, 离散傅里叶变换)**
- **how the FFT (Fast Fourier Transform, 快速傅里叶变换) computes the DFT and its inverse in just $\Theta(n \lg n)$ time**
- **complex roots of unity and their properties**
  〔单位复数根及其性质〕

30.2.1 Complex roots of unity

- $\omega^n = 1$, complex number ω is a complex nth root of unity
  〔ω称为 1 的 n 次复根，或单位 n 次复根〕
- $\omega_n = e^{2\pi i/n}$: principal nth root of unity 〔单位 n 次复根的基元〕
- $\omega_n^k = (\omega_n)^k = e^{2\pi ik/n},\ k = 0,1,\cdots,n-1$

  exactly n complex nth roots of unity
  〔单位 1 有 n 个 n 次复根〕



- $e^{iu} = \cos(u) + i\sin(u)$:

  the exponential of a complex number

$$\omega_n^k = e^{2\pi ik/n} = \cos(2\pi k/n) + i\sin(2\pi k/n),$$

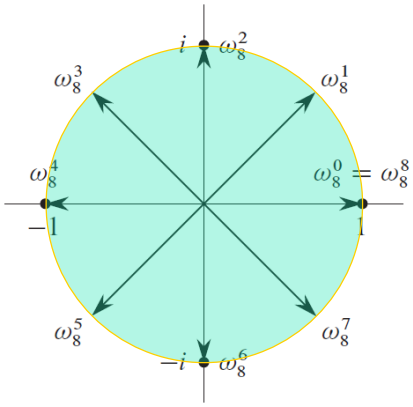
$$k = 0,1,\cdots,n-1$$

- All of the other complex nth roots of unity are powers of ω_n . The n complex nth roots of unity $\{\omega_n^k,\ k = 0,\cdots,n-1\}$ form a group. 〔 n 个单位 n 次复根组成一个群（伽罗瓦，法，1811~1832）〕

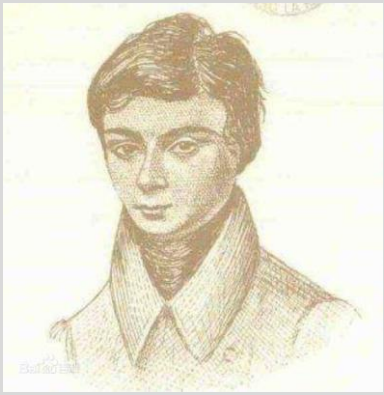
30.2.1 Complex roots of unity

- $\omega_n = e^{2\pi i/n}$: principal *n*th root of unity 〔单位 *n* 次复根的基元〕
- $e^{iu} = \cos(u) + i\sin(u)$ 〔欧拉公式〕
  the exponential of a complex number

$$\omega_n^k = e^{2\pi ik/n} = \cos(2\pi k/n) + i\sin(2\pi k/n),$$
$$k = 0, 1, \cdots, n-1$$



- All of the other complex *n*th roots of unity are powers of ω_n . The *n* complex *n*th roots of unity $\{\omega_n^k,\ k = 0, \cdots, n-1\}$ form a group.



埃瓦里斯特·伽罗瓦（1811年10月25日-1832年5月31日），1811年10月25日生，法国数学家。现代数学中的分支学科群论的创立者。用群论彻底解决了根式求解代数方程的问题，而且由此发展了一整套关于群和域的理论，人们称之为伽罗瓦理论，并把其创造的“群”叫作伽罗瓦群（Galois Group）。在世时在数学上研究成果的重要意义没被人们所认识，曾呈送科学院3篇学术论文，均被退回或遗失。后转向政治，支持共和党，曾两次被捕。21岁时死于一次决斗。

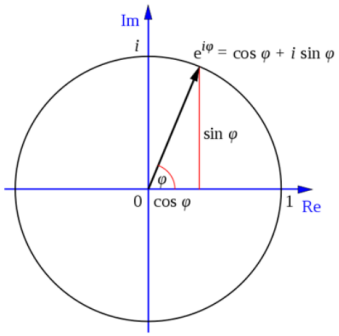
个人成就：使用群论的想法去讨论方程式的可解性，整套想法现称为伽罗瓦理论，是当代代数与数论的基本支柱之一。他系统化地阐释了为何五次以上之方程式没有公式解，而四次以下有公式解。他解决了古代三大尺规作图问题中的两个：“三等分任意角不可能”，“倍立方不可能”。

30.2.1 Complex roots of unity

最美丽的公式：欧拉公式

$$e^{iu} = \cos(u) + i\sin(u)$$

$$e^{i\pi} + 1 = 0$$

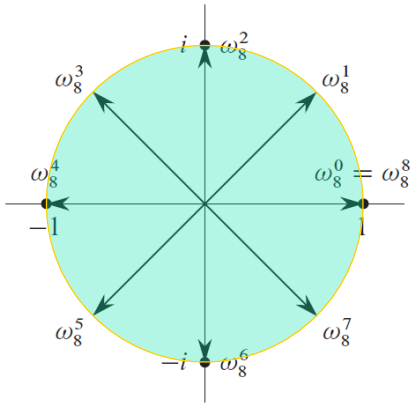


理由如下：

1. 最重要的自然数的“e”含于其中。自然对数的底，大到飞船的速度，小至蜗牛的螺线，谁能够离开它?
2. 最重要的常数 π 含于其中。世界上最完美的平面对称图形是圆。“最伟大的公式”能够离开圆周率吗？（还有π和e是两个最重要的无理数！）
3. 最重要的运算符号 + 含于其中。加号最重要：因为其余符号都是由加号派生而来，减号是加法的逆逆运算，乘法是累计的加法……
4. 最重要的两个元在里面。零元0，单位1，是构造群，环，域的基本元素。如果你看了有关《近世代数》的书，你就会体会到它的重要性。
5. 最重要的虚单位 i 也在其中。虚单位 i 使数轴上的问题扩展到了平面，而在哈密尔的 4 元数与 凯莱的 8 元数中也离开不了它。
6. 这个公式的美在于其精简。她没有多余的字符，却联系着几乎所有的数学知识。有了加号，可以得到其余运算符号；有了0，1，就可以得到其他的数字；有了 π 就有了圆函数，也就是三角函数；有了 i 就有了虚数，平面向量与其对应，也就有了哈密尔的 4 元数，现实的空间与其对应；有了 e 就有了微积分，就有了和工业革命时期相适宜的数学。

30.2.1 Complex roots of unity

some properties



1) $\omega_n^n = \omega_n^0 = 1$

($\omega_n^n = (e^{2\pi i/n})^n = \cos 2\pi + i \sin 2\pi$)

2) $\omega_n^j \omega_n^k = \omega_n^{j+k} = \omega_n^{(j+k) \bmod n}$

(if $j + k = mn + r$, then

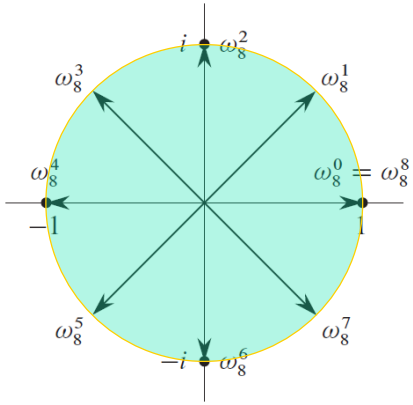
$\omega_n^{j+k} = \omega_n^{mn+r} = \omega_n^{mn} \omega_n^r = (\omega_n^n)^m \omega_n^r = \omega_n^r = \omega_n^{(j+k) \bmod n}$)

3) $\omega_n^{-1} = \omega_n^{n-1}$

(proof by using $\omega_n^k = e^{2\pi ik/n}$, $k = 0, 1, \cdots, n-1$)

30.2.1 Complex roots of unity

- ***Lemma* 30.3 (Cancellation lemma, 可约（相消）引理)**
  For any integers $n \geq 0$, $k \geq 0$, and $d > 0$,



$$\omega_{dn}^{dk} = \omega_n^k. \qquad (30.7)$$

$$\text{Proof} \quad \omega_{dn}^{dk} = (e^{2\pi i/dn})^{dk} = (e^{2\pi i/n})^k = \omega_n^k.$$

$$(\ \omega_8^2 = \omega_4^1\ ,\ \ \omega_8^4 = \omega_4^2\ ,\ \ \omega_8^6 = \omega_4^3\ ,\ \ \omega_8^8 = \omega_4^4\ )$$

- ***Corollary* 30.4 For any even integer $n > 0$,**

$$\omega_n^{n/2} = \omega_2 = -1.$$

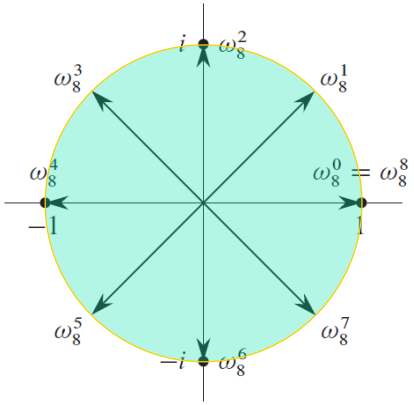
$$\omega_n^{n/2} = \omega_{(n/2)\cdot 2}^{n/2} = \omega_2 = e^{2\pi i/2} = \cos\pi + i\sin\pi = -1.$$

30.2.1 Complex roots of unity

***Lemma* 30.5 (Halving lemma, 等分引理)**

If $n > 0$ is even, then the squares of the n complex nth roots of unity are the $n/2$ complex $(n/2)$th roots of unity.

〔n 个单位 n 次复方根的平方是 $n/2$ 个 $n/2$ 次复方根〕



Proof idea

$$(\omega_n^k)^2 = \omega_{2(n/2)}^{2k} = \omega_{n/2}^k \ , \ k = 0,1,\cdots,\frac{n}{2}-1$$

$$(\omega_n^{k+n/2})^2 = \omega_n^{2k+n} = \omega_n^{2k}\omega_n^n = \omega_n^{2k} = (\omega_n^k)^2 = \omega_{n/2}^k \ , \ k = 0,1,\cdots,\frac{n}{2}-1$$

$$(\omega_n^0)^2 \ , (\omega_n^1)^2 \ , \cdots \ , (\omega_n^{n/2-1})^2$$
$$(\omega_n^{0+n/2})^2 , (\omega_n^{1+n/2})^2 , \cdots , (\omega_n^{n/2-1+n/2})^2$$

$$\omega_{n/2}^0 \ , \omega_{n/2}^1 \ , \cdots , \omega_{n/2}^{n/2-1}$$

2

-3, -2, -1, 1, 2, 3

9, 4, 1

30.2.1 Complex roots of unity

Lemma 30.6 (Summation lemma, 求和引理)

For any integer $n \geq 1$ and nonnegative integer k not divisible by n , ($k \neq m \cdot n$), we have

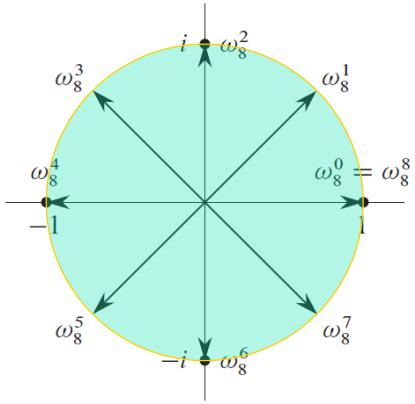
$$\sum_{j=0}^{n-1} (\omega_n^k)^j = 0.$$

Proof

$$\sum_{j=0}^{n-1} (\omega_n^k)^j = \frac{(\omega_n^k)^n - 1}{\omega_n^k - 1} = \frac{(\omega_n^n)^k - 1}{\omega_n^k - 1} = \frac{(1)^k - 1}{\omega_n^k - 1} = 0, \ k \neq mn.$$

(证明思想：利用实数的等幂级数求和性质)

30.2.2 The DFT



$$A(x)=\sum_{j=0}^{n-1}a_j x^j \qquad \begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ & & \cdots\cdots & & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{n-1} \end{pmatrix} = V(x_0, x_1, \cdots, x_{n-1}) \cdot a \qquad (30.4)$$

- **wish to evaluate a polynomial $A(x)=\sum_{j=0}^{n-1}a_j x^j$ at $x = \omega_n^0, \omega_n^1, \omega_n^2, \cdots, \omega_n^{n-1}$**
- **without loss of generality, assume that $n = 2^m$, if not, let $a_{n+k} = 0$**
- **Discrete Fourier Transform (DFT)**

  let A is given in coefficient form: $a = (a_0, a_1, \cdots, a_{n-1})^T$,

  let $x_k = \omega_n^k$, define y_k , for $k = 0, 1, \cdots, n-1$, by

$$y_k = A(\omega_n^k) = \sum_{j=0}^{n-1} a_j \omega_n^{kj} \ , \qquad \begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \cdots & \omega_n^{n-1} \\ & & \cdots\cdots & & \\ 1 & \omega_n^{(n-1)\cdot 1} & \omega_n^{(n-1)\cdot 2} & \cdots & \omega_n^{(n-1)\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{n-1} \end{pmatrix}$$

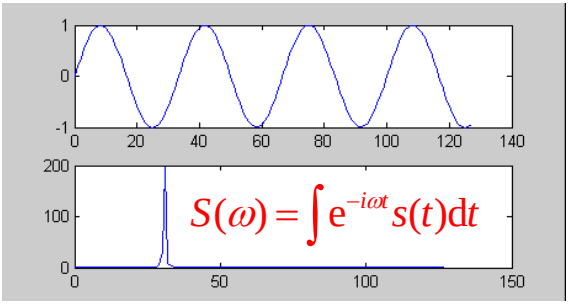
also write $y = \mathrm{DFT}_n(a)$. Take time $\Theta(n^2)$ to compute straightforward? （慢）

$y = \mathrm{DFT}_n(a)$：数学意义上，多项式在特殊点的取值（单位复根）

Application of DFT

- **wish to evaluate a polynomial** $A(x)=\sum_{j=0}^{n-1}a_j x^j$ **of degree-bound *n* at** $x=\omega_n^0,\omega_n^1,\omega_n^2,\cdots,\omega_n^{n-1}$ **.**
- **Discrete Fourier Transform (DFT, 离散傅里叶变换)**

$$y_k=A(\omega_n^k)=\sum_{j=0}^{n-1}a_j\omega_n^{kj}\ ,\qquad \begin{pmatrix}y_0\\y_1\\\cdots\\y_{n-1}\end{pmatrix}=\begin{pmatrix}1&1&1&\cdots&1\\1&\omega_n^1&\omega_n^2&\cdots&\omega_n^{n-1}\\&&\cdots\cdots&&\\1&\omega_n^{(n-1)\cdot1}&\omega_n^{(n-1)\cdot2}&\cdots&\omega_n^{(n-1)\cdot(n-1)}\end{pmatrix}\begin{pmatrix}a_0\\a_1\\\cdots\\a_{n-1}\end{pmatrix}$$



Signal *s*(*t*): discrete, $a_i = s(t_i)$

Spcetrum *S*(*w*): $S(w) = \mathrm{DFT}_n(s)$

$y = \mathrm{DFT}_n(a)$：物理意义上，从一个矢量（时域）变换到另一个矢量（频域）

30.2.3 The FFT

An algorithm for the machine calculation of complex Fourier series

JW Cooley, JW Tukey - Mathematics of computation, 1965 - JSTOR

An efficient method for the calculation of the interactions of a 2'factorial ex-periment was introduced by Yates and is widely known by his name. The generalization to 3'was given by Box et al.[1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an N-vector by an NXN matrix which can be factored into m sparse matrices, where m is proportional to log N …

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30.2.3 The FFT

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$$

Fast Fourier Transform (FFT , 快速傅里叶变换)

- **takes advantage of the special properties of the complex roots of unity**
- **we can compute $DFT_n(a)$ in time $\Theta(n \lg n)$**
- **employs a divide-and-conquer strategy,**

  even-index, $A^{[0]}(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{n/2-1}$

  odd-index, $A^{[1]}(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{n/2-1}$
- **$A[0]$ contains all the even-index coefficients of A (the binary representation of the index ends in 0)**
- **$A[1]$ contains all the odd-index coefficients (the binary representation of the index ends in 1). It follows that**

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2) \qquad (30.9)$$

30.2.3 The FFT

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

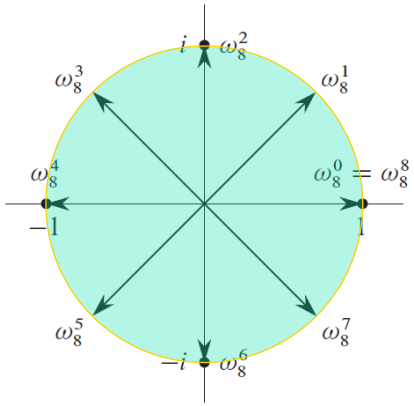
- **Fast Fourier Transform (FFT)**

  employs a divide-and-conquer strategy,

  even-index, $A^{[0]}(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{n/2-1}$

  odd-index, $A^{[1]}(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{n/2-1}$

  $$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2) \qquad (30.9)$$



$$A(\omega_n^k) = A^{[0]}((\omega_n^k)^2) + \omega_n^k A^{[1]}((\omega_n^k)^2)$$

- **the problem of evaluating** $A(x)$ **at** $\omega_n^0, \omega_n^1, \omega_n^2, \cdots, \omega_n^{n-1}$ **reduces to**

  1) evaluating the $A^{[0]}(x)$ **and** $A^{[1]}(x)$ **at the points**

$$(\omega_n^0)^2 \quad , (\omega_n^1)^2 \quad , \cdots \quad , (\omega_n^{n/2-1})^2$$
$$(\omega_n^{0+n/2})^2 , (\omega_n^{1+n/2})^2 , \cdots \quad , (\omega_n^{n/2-1+n/2})^2 \qquad (32.10)$$

  and then

$$\omega_{n/2}^0 \quad , \omega_{n/2}^1 \quad , \ \ldots \ , \omega_{n/2}^{n/2-1}$$

  2) combining the results according to equation (30.9).

30.2.3 The FFT

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

Fast Fourier Transform (FFT): divide-and-conquer strategy

even-index, $A^{[0]}(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{n/2-1}$

odd-index, $A^{[1]}(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{n/2-1}$

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2) \qquad (30.9)$$

$$A(x_r) = A^{[0]}(x_r^{\,2}) + x_r A^{[1]}(x_r^{\,2})$$

$$= A(\omega_n^r) = A^{[0]}((\omega_n^r)^2) + (\omega_n^r) \cdot A^{[1]}((\omega_n^r)^2)$$

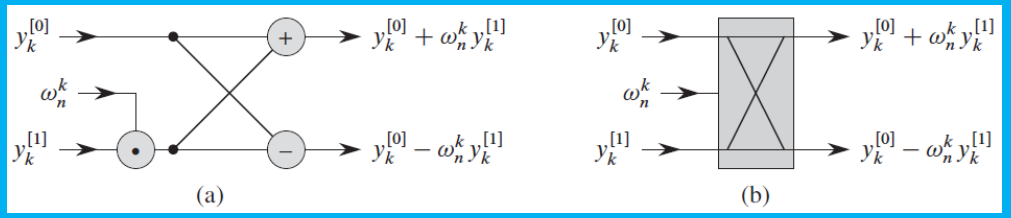
$$= A^{[0]}(\omega_{n/2}^r) + (\omega_n^r) \cdot A^{[1]}(\omega_{n/2}^r),\ r = 0,\ \ldots,\ n/2-1,\ n/2,\ \ldots,\ n-1$$

let $k = 0,\ \ldots,\ n/2-1,$ then

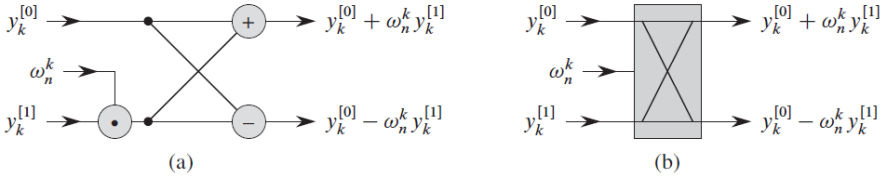
$$A(\omega_n^k) = A^{[0]}(\omega_{n/2}^k) + (\omega_n^k) \cdot A^{[1]}(\omega_{n/2}^k),$$

$$A(\omega_n^{n/2+k}) = A^{[0]}(\omega_{n/2}^{n/2+k}) + (\omega_n^{n/2+k}) \cdot A^{[1]}(\omega_{n/2}^{n/2+k})$$

$$= A^{[0]}(\omega_{n/2}^k) - (\omega_n^k) \cdot A^{[1]}(\omega_{n/2}^k)$$



30.2.3 The FFT



$$A(x)=\sum_{j=0}^{n-1}a_j x^j$$

$$\begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \cdots & \omega_n^{n-1} \\ & & \cdots\cdots & & \\ 1 & \omega_n^{(n-1)\cdot 1} & \omega_n^{(n-1)\cdot 2} & \cdots & \omega_n^{(n-1)\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{n-1} \end{pmatrix}$$

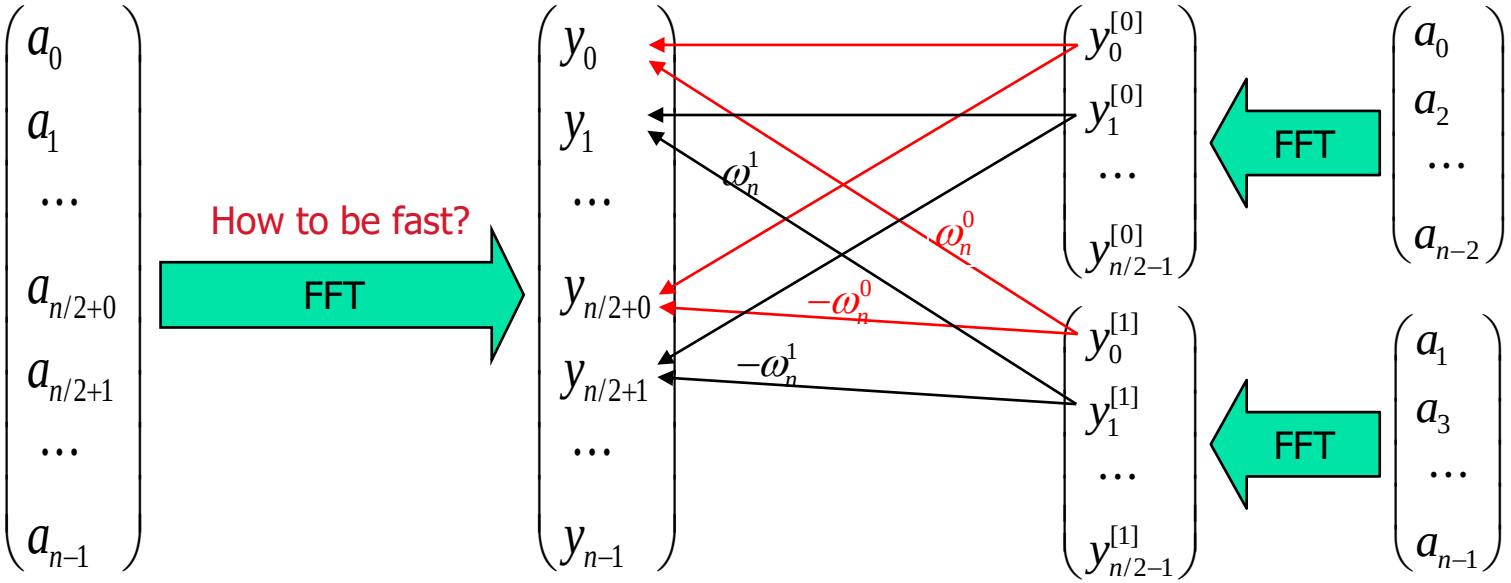
even: $A^{[0]}(x) = a_0+a_2x+a_4x^2+\cdots+a_{n-2}x^{n/2-1}$

odd: $A^{[1]}(x) = a_1+a_3x+a_5x^2+\cdots+a_{n-1}x^{n/2-1}$

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$$

let $k=0,\ \ldots,\ n/2-1,$ then

$$y_k = A(\omega_n^k) = A^{[0]}(\omega_{n/2}^k)+(\omega_n^k)\cdot A^{[1]}(\omega_{n/2}^k)$$
$$= y_k^{[0]}+(\omega_n^k)\cdot y_k^{[1]}$$
$$y_{k+n/2} = A(\omega_n^{n/2+k}) = A^{[0]}(\omega_{n/2}^{n/2+k})+(\omega_n^{n/2+k})\cdot A^{[1]}(\omega_{n/2}^{n/2+k})$$
$$= y_k^{[0]}-(\omega_n^k)\cdot y_k^{[1]}$$



30.2.3 The FFT

even, $A^{[0]}(x)=a_0+a_2x+a_4x^2+\cdots+a_{n-2}x^{n/2-1}$

odd, $A^{[1]}(x)=a_1+a_3x+a_5x^2+\cdots+a_{n-1}x^{n/2-1}$

$$A(x)=\sum_{j=0}^{n-1}a_j x^j$$

$$A(x)=A^{[0]}(x^2)+xA^{[1]}(x^2)$$

ω_n^k $\qquad (\omega_n^k)^2=\omega_{n/2}^k$

$$A(\omega_n^k)=A^{[0]}(\omega_{n/2}^k)+(\omega_n^k)\cdot A^{[1]}(\omega_{n/2}^k),$$
$$A(\omega_n^{n/2+k})=A^{[0]}(\omega_{n/2}^k)-(\omega_n^k)\cdot A^{[1]}(\omega_{n/2}^k),$$
$$k=0,\ldots,n/2-1$$

```
RECURSIVE-FFT(a)
 1  n = a.length                    // n is a power of 2
 2  if n == 1
 3       return a
 4  ω_n = e^{2πi/n}
 5  ω = 1
 6  a^[0] = (a_0, a_2, ..., a_{n-2})
 7  a^[1] = (a_1, a_3, ..., a_{n-1})
 8  y^[0] = RECURSIVE-FFT(a^[0])
 9  y^[1] = RECURSIVE-FFT(a^[1])
10  for k = 0 to n/2 - 1
11       y_k = y_k^[0] + ω y_k^[1]
12       y_{k+(n/2)} = y_k^[0] - ω y_k^[1]
13       ω = ω ω_n
14  return y                        // y is assumed to be a column vector
```

$(\omega_n^0)^2 \quad ,(\omega_n^1)^2 \quad ,\cdots \quad ,(\omega_n^{n/2-1})^2$

$(\omega_n^{0+n/2})^2,(\omega_n^{1+n/2})^2,\cdots \quad ,(\omega_n^{n/2-1+n/2})^2$

$\omega_{n/2}^0 \quad ,\omega_{n/2}^1 \quad , \ \ldots \ , \omega_{n/2}^{n/2-1}$

30.2.3 The FFT

even, $A^{[0]}(x)=a_0+a_2x+a_4x^2+\cdots+a_{n-2}x^{n/2-1}$

odd, $A^{[1]}(x)=a_1+a_3x+a_5x^2+\cdots+a_{n-1}x^{n/2-1}$

$$A(x)=\sum_{j=0}^{n-1}a_j x^j$$

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$$

ω_n^k $\qquad (\omega_n^k)^2=\omega_{n/2}^k$

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 8  y^[0] = RECURSIVE-FFT(a^[0])
 9  y^[1] = RECURSIVE-FFT(a^[1])
10  for k = 0 to n/2 − 1
11       y_k = y_k^[0] + ω y_k^[1]
12       y_{k+(n/2)} = y_k^[0] − ω y_k^[1]
13       ω = ω ω_n
14  return y
```

$(\omega_n^0)^2, (\omega_n^1)^2, \cdots, (\omega_n^{n/2-1})^2$

$(\omega_n^{0+n/2})^2, (\omega_n^{1+n/2})^2, \cdots, (\omega_n^{n/2-1+n/2})^2$

$\omega_{n/2}^0, \omega_{n/2}^1, \ldots, \omega_{n/2}^{n/2-1}$

Line 2-3

$y_0 = a_0\omega_1^0 = a_0 \cdot 1 = a_0$

30.2.3 The FFT

even, $A^{[0]}(x)=a_0+a_2x+a_4x^2+\cdots+a_{n-2}x^{n/2-1}$

odd, $A^{[1]}(x)=a_1+a_3x+a_5x^2+\cdots+a_{n-1}x^{n/2-1}$

$$A(x)=\sum_{j=0}^{n-1}a_j x^j$$

$$A(x)=A^{[0]}(x^2)+xA^{[1]}(x^2)$$

ω_n^k $\qquad (\omega_n^k)^2=\omega_{n/2}^k$

$(\omega_n^0)^2, (\omega_n^1)^2, \cdots, (\omega_n^{n/2-1})^2$

$(\omega_n^{0+n/2})^2, (\omega_n^{1+n/2})^2, \cdots, (\omega_n^{n/2-1+n/2})^2$

$\omega_{n/2}^0, \omega_{n/2}^1, \ldots, \omega_{n/2}^{n/2-1}$

```
RECURSIVE-FFT(a)
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 5  ω = 1
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 7  a^[1] = (a_1, a_3, ..., a_{n-1})
 8  y^[0] = RECURSIVE-FFT(a^[0])
 9  y^[1] = RECURSIVE-FFT(a^[1])
10  for k = 0 to n/2 - 1
11       y_k = y_k^[0] + ω y_k^[1]
12       y_{k+(n/2)} = y_k^[0] - ω y_k^[1]
13       ω = ω ω_n
14  return y
```

Line 8-9

$$y_k^{[0]}=A^{[0]}(\omega_{n/2}^k)=A^{[0]}(\omega_n^{2k}),$$

$$y_k^{[1]}=A^{[1]}(\omega_{n/2}^k)=A^{[1]}(\omega_n^{2k}).$$

Line 11

$$y_k=y_k^{[0]}+\omega_n^k y_k^{[1]}$$

$$=A^{[0]}(\omega_n^{2k})+\omega_n^k A^{[1]}(\omega_n^{2k})=A(\omega_n^k)$$

30.2.3 The FFT

even, $A^{[0]}(x)=a_0+a_2x+a_4x^2+\cdots+a_{n-2}x^{n/2-1}$

odd, $A^{[1]}(x)=a_1+a_3x+a_5x^2+\cdots+a_{n-1}x^{n/2-1}$

$$A(x)=\sum_{j=0}^{n-1}a_j x^j$$

$$A(x)=A^{[0]}(x^2)+xA^{[1]}(x^2)$$

ω_n^k $\qquad (\omega_n^k)^2=\omega_{n/2}^k$

$(\omega_n^0)^2, (\omega_n^1)^2, \cdots, (\omega_n^{n/2-1})^2$

$(\omega_n^{0+n/2})^2, (\omega_n^{1+n/2})^2, \cdots, (\omega_n^{n/2-1+n/2})^2$

$\omega_{n/2}^0, \omega_{n/2}^1, \ldots, \omega_{n/2}^{n/2-1}$

```
RECURSIVE-FFT(a)
 1  n = a.length
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 8  y^[0] = RECURSIVE-FFT(a^[0])
 9  y^[1] = RECURSIVE-FFT(a^[1])
10  for k = 0 to n/2 - 1
11       y_k = y_k^[0] + ω y_k^[1]
12       y_{k+(n/2)} = y_k^[0] - ω y_k^[1]
13       ω = ω ω_n
14  return y
```

Line 12

$$y_{k+n/2}=y_k^{[0]}-\omega_n^k y_k^{[1]}$$
$$=A^{[0]}(\omega_n^{2k})+\omega_n^{k+n/2}A^{[1]}(\omega_n^{2k})$$
$$=A^{[0]}(\omega_n^{2k+n})+\omega_n^{k+n/2}A^{[1]}(\omega_n^{2k+n})$$
$$=A(\omega_n^{k+n/2})$$

30.2.3 The FFT

$$A(x)=\sum_{j=0}^{n-1}a_j x^j$$

even, $A^{[0]}(x)=a_0+a_2x+a_4x^2+\cdots+a_{n-2}x^{n/2-1}$

odd, $A^{[1]}(x)=a_1+a_3x+a_5x^2+\cdots+a_{n-1}x^{n/2-1}$

$$A(x)=A^{[0]}(x^2)+xA^{[1]}(x^2)$$

ω_n^k $\qquad (\omega_n^k)^2=\omega_{n/2}^k$

$$A(\omega_n^k)=A^{[0]}(\omega_{n/2}^k)+(\omega_n^k)\cdot A^{[1]}(\omega_{n/2}^k),$$
$$A(\omega_n^{n/2+k})=A^{[0]}(\omega_{n/2}^k)-(\omega_n^k)\cdot A^{[1]}(\omega_{n/2}^k),$$
$$k=0,\ldots,n/2-1$$

```
RECURSIVE-FFT(a)
 1  n = a.length
 2  if n == 1
 3       return a
 4  ω_n = e^{2πi/n}
 5  ω = 1
 6  a^[0] = (a_0, a_2, ..., a_{n-2})
 7  a^[1] = (a_1, a_3, ..., a_{n-1})
 8  y^[0] = RECURSIVE-FFT(a^[0])
 9  y^[1] = RECURSIVE-FFT(a^[1])
10  for k = 0 to n/2 - 1
11       y_k = y_k^[0] + ω y_k^[1]
12       y_{k+(n/2)} = y_k^[0] - ω y_k^[1]
13       ω = ω ω_n
14  return y
```

in line 11-12, each $y_k^{[1]}$ is multiplied by ω_n^k, the product is both *added to* and *substracted* from $y_k^{[0]}$. each ω_n^k is used in both its positive and negative forms, ω_n^k is called *twiddle factors*.

旋转因子

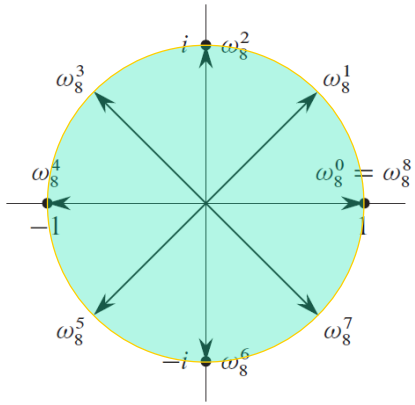
Running Time?

$$T(n)=2T(n/2)+\Theta(n)=\Theta(n\lg n)$$

30.2.4 Interpolation at the complex roots of unity

DFT

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \cdots & \omega_n^{n-1} \\ & & \cdots\cdots & & \\ 1 & \omega_n^{(n-1)\cdot 1} & \omega_n^{(n-1)\cdot 2} & \cdots & \omega_n^{(n-1)\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{n-1} \end{pmatrix} = V_n a$$



$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

$$x_k = \omega_n^k$$

inverse DFT

$$a = \mathrm{DFT}_n^{-1}(y) = V_n^{-1} y$$

***Theorem* 30.7**

For *j*, *k* = 0, 1, . . . , *n*-1, the (*j*, *k*) entry of V_n^{-1} is ω_n^{-kj} / n .

30.2.4 Interpolation at the complex roots of unity

- ***Lemma* 30.6 (Summation lemma, 求和引理)**

  For any integer $n \geq 1$ and nonnegative integer k not divisible by n , ($k \neq m \cdot n$), we have

$$\sum_{j=0}^{n-1} (\omega_n^k)^j = 0.$$

- ***Theorem* 30.7**

  For $j, k = 0, 1, \ldots, n-1$, the (j, k) entry of V_n^{-1} is ω_n^{-kj} / n .

  Proof We show that $V_n^{-1} V_n = I_n$

$$[V_n^{-1} V_n]_{j\,j'} = \sum_{k=0}^{n-1} (\omega_n^{-kj} / n)(\omega_n^{kj'}) = \sum_{k=0}^{n-1} \omega_n^{k(j'-j)} / n = \begin{cases} 1 & \text{if } j' = j \\ 0 & \text{if } j' \neq j \end{cases}$$

since $-(n-1) < j' - j < n-1$, apparently $j' - j \neq mn$, if $m \neq 0$

30.2.4 Interpolation at the complex roots of unity

$$y = A(x) = \sum_{j=0}^{n-1} a_j x^j$$

DFT

$$y_k = A(\omega_n^k) = \sum_{j=0}^{n-1} a_j \omega_n^{kj} \ , (k = 0,1,\cdots,n-1) \qquad (30.8)$$

inverse DFT

$$a_j = \frac{1}{n}\sum_{k=0}^{n-1} y_k \omega_n^{-kj} \ , \qquad (j = 0,1,\cdots,n-1) \qquad (30.11)$$

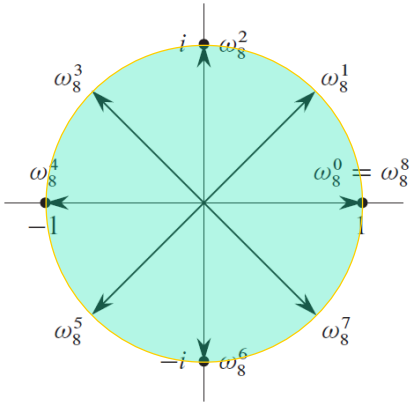
DFT

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \cdots & \omega_n^{n-1} \\ & & \cdots\cdots & & \\ 1 & \omega_n^{(n-1)\cdot 1} & \omega_n^{(n-1)\cdot 2} & \cdots & \omega_n^{(n-1)\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{n-1} \end{pmatrix} = V_n a$$

inverse DFT

$$a = \mathrm{DFT}_n^{-1}(y) = V_n^{-1} y$$

The inverse DFT （逆DFT） can be computed in $\Theta(n\lg n)$ time, for (30.8), by replacing ω_n by ω_n^{-1} , and divide each element of the result by n.



30.2.4 Interpolation at the complex roots of unity

DFT

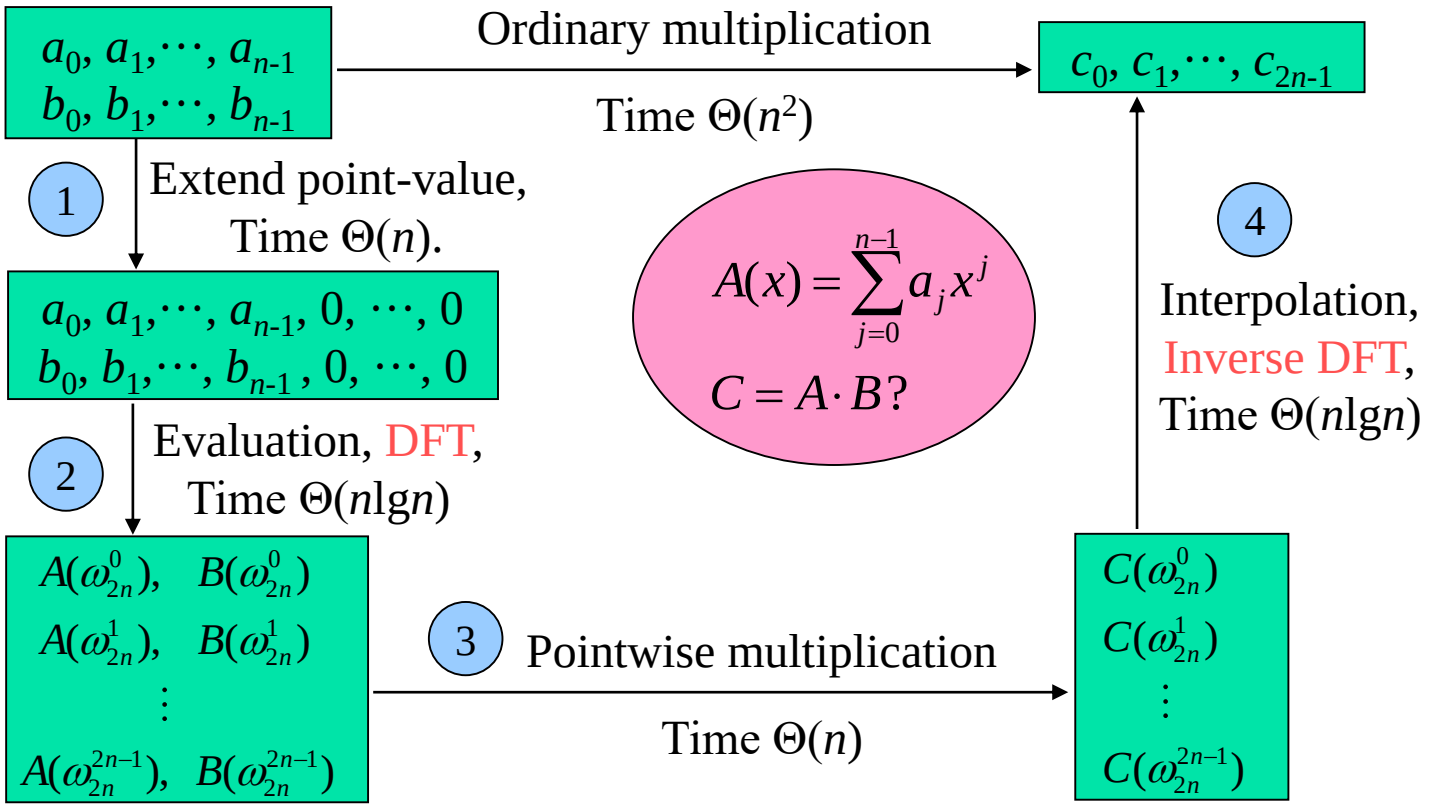
$$y_k = A(\omega_n^k) = \sum_{j=0}^{n-1} a_j \omega_n^{kj} \ , (k = 0,1,\cdots,n-1) \qquad (30.8)$$

inverse DFT

$$a_j = \frac{1}{n}\sum_{k=0}^{n-1} y_k \omega_n^{-kj} \ , \qquad (j = 0,1,\cdots,n-1) \qquad (30.11)$$

$$y = A(x) = \sum_{j=0}^{n-1} a_j x^j$$

30.2.4 Interpolation at the complex roots of unity



***Theorem* 30.8 (Convolution theorem, 卷积定理)**

For any two vectors *a* and *b* of length *n*, where *n* is a power of 2,

$$a \otimes b = \mathrm{DFT}_{2n}^{-1}(\mathrm{DFT}_{2n}(a) \cdot \mathrm{DFT}_{2n}(b))$$

where the vectors *a* and *b* are padded with 0's to length 2*n* and · denotes the component-wise product of two 2*n*-element vectors.

30.3 Efficient FFT implementations

- **The practical applications of the DFT, such as signal processing, demand the utmost speed.**

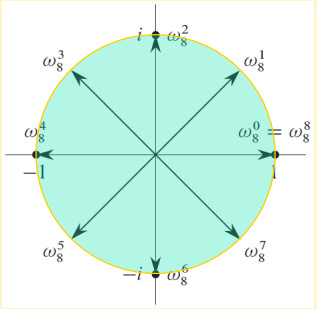
- **Two efficient FFT implementations**
  - **iterative FFT algorithm**
  - **butterfly operation algorithm (parallel FFT circuit)**

30.3.1 An iterative FFT implementation

$$A(x)=\sum_{j=0}^{n-1}a_j x^j = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}$$

最美丽的公式:

$$e^{i\pi} + 1 = 0$$



$$x = \omega_n^0, \omega_n^1, \omega_n^2, \cdots, \omega_n^{n-1};$$

$$\omega_n = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$$

$$y_k = A(\omega_n^k) = \sum_{j=0}^{n-1} a_j \omega_n^{kj} \ ,$$

$$\begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \cdots & \omega_n^{n-1} \\ & & \cdots\cdots & & \\ 1 & \omega_n^{(n-1)\cdot 1} & \omega_n^{(n-1)\cdot 2} & \cdots & \omega_n^{(n-1)\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{n-1} \end{pmatrix}$$

even, $A^{[0]}(x)=a_0+a_2x+a_4x^2+\cdots+a_{n-2}x^{n/2-1}$;

odd, $A^{[1]}(x)=a_1+a_3x+a_5x^2+\cdots+a_{n-1}x^{n/2-1}$.

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$$

ω_n^k $\qquad (\omega_n^k)^2 = \omega_{n/2}^k$

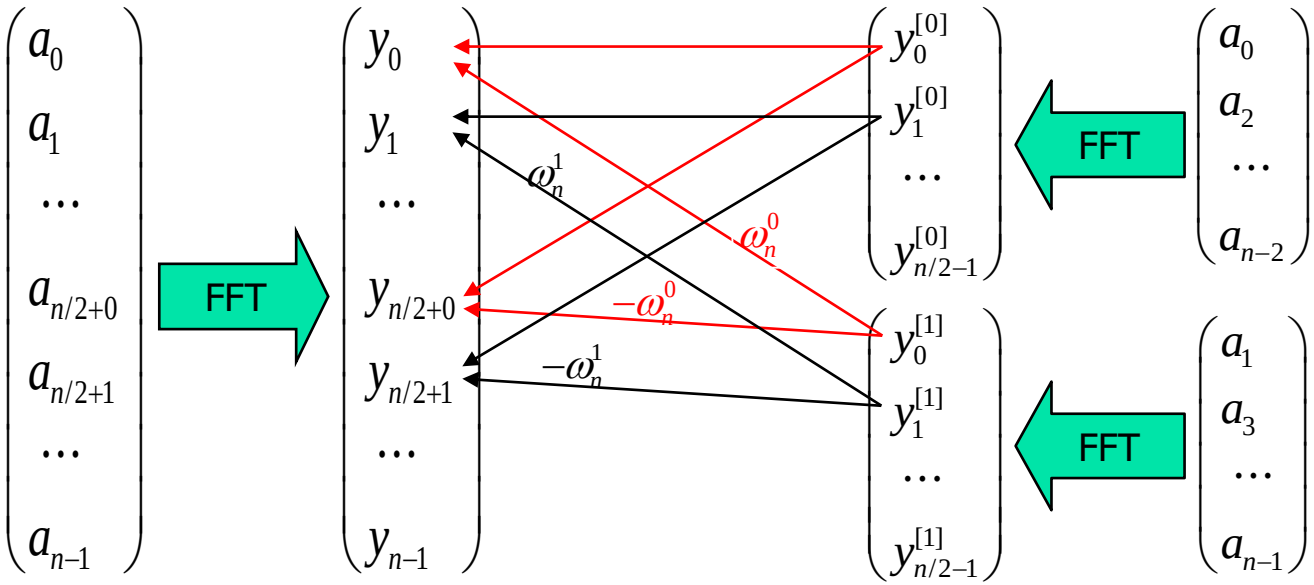
let $k = 0, \ldots, n/2-1,$ then

$$A(\omega_n^k) = A^{[0]}(\omega_{n/2}^k) + (\omega_n^k)\cdot A^{[1]}(\omega_{n/2}^k), \qquad y_k = y_k^0 + \omega_n^k y_k^1,$$

$$A(\omega_n^{n/2+k}) = A^{[0]}(\omega_{n/2}^k) - (\omega_n^k)\cdot A^{[1]}(\omega_{n/2}^k) \qquad y_{k+n/2} = y_k^0 - \omega_n^k y_k^1$$

30.3.1 An iterative FFT implementation

lines 10-13, RECURSIVE-FFT involves computing the value $\omega_n^k y_k^{[1]}$ twice, change the loop to compute it only once.



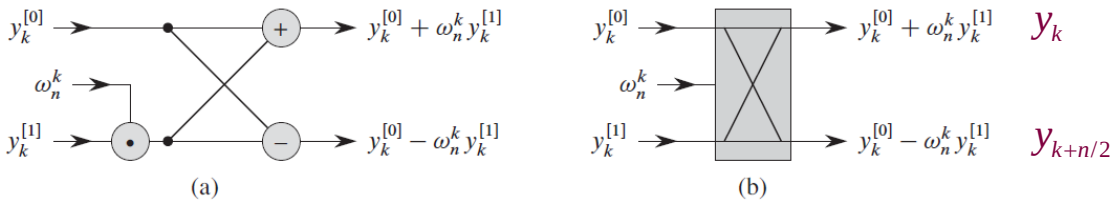

```
RECURSIVE-FFT(a)
 1  n = a.length
 2  if n == 1
 3       return a
 4  ω_n = e^{2πi/n}
 5  ω = 1
 6  a^[0] = (a_0, a_2, ..., a_{n-2})
 7  a^[1] = (a_1, a_3, ..., a_{n-1})
 8  y^[0] = RECURSIVE-FFT(a^[0])
 9  y^[1] = RECURSIVE-FFT(a^[1])
10  for k = 0 to n/2 - 1
11       y_k = y_k^[0] + ω y_k^[1]
12       y_{k+(n/2)} = y_k^[0] - ω y_k^[1]
13       ω = ω ω_n
14  return y
```

butterfly operation （蝶型操作）
adding and subtracting t from $y_k^{[0]}$

```
for k = 0 to n/2 - 1
    t = ω y_k^[1]
    y_k = y_k^[0] + t
    y_{k+(n/2)} = y_k^[0] - t
    ω = ω ω_n
```

30.3.1 An iterative FFT implementation

$(k = 0,1,\cdots,n/2-1)$



RECURSIVE-FFT(a)

```
1  n = a.length
2  if n == 1
3      return a
4  ω_n = e^{2πi/n}
5  ω = 1
6  a^[0] = (a_0, a_2, ..., a_{n-2})
7  a^[1] = (a_1, a_3, ..., a_{n-1})
8  y^[0] = RECURSIVE-FFT(a^[0])
9  y^[1] = RECURSIVE-FFT(a^[1])
10 for k = 0 to n/2 − 1
11     y_k = y_k^[0] + ω y_k^[1]
12     y_{k+(n/2)} = y_k^[0] − ω y_k^[1]
13     ω = ω ω_n
14 return y
```

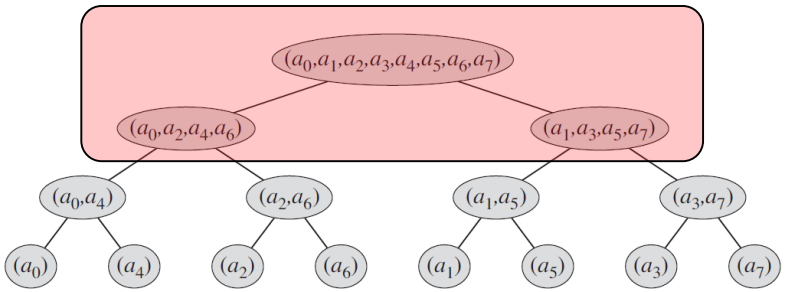
butterfly operation
adding and subtracting t from $y_k^{[0]}$

```
for k = 0 to n/2 − 1
    t = ω y_k^[1]
    y_k = y_k^[0] + t
    y_{k+(n/2)} = y_k^[0] − t
    ω = ω ω_n
```

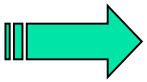
30.3.1 An iterative FFT implementation

Make the FFT algorithm iterative rather than recursive in structure.

Recursive calls of RECURSIVE-FFT in a tree structure.




```
RECURSIVE-FFT(a)
 1  n = a.length
 2  if n == 1
 3      return a
 4  ω_n = e^{2πi/n}
 5  ω = 1
 6  a^[0] = (a_0, a_2, ..., a_{n-2})
 7  a^[1] = (a_1, a_3, ..., a_{n-1})
 8  y^[0] = RECURSIVE-FFT(a^[0])
 9  y^[1] = RECURSIVE-FFT(a^[1])
10  for k = 0 to n/2 - 1
11      y_k = y_k^[0] + ω y_k^[1]
12      y_{k+(n/2)} = y_k^[0] - ω y_k^[1]
13      ω = ω ω_n
14  return y
```

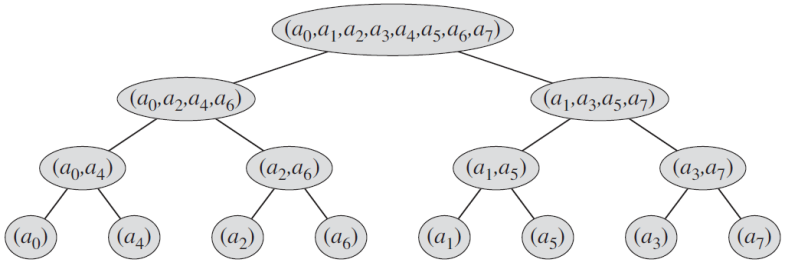


```
for k = 0 to n/2 - 1
    t = ω y_k^[1]
    y_k = y_k^[0] + t
    y_{k+(n/2)} = y_k^[0] - t
    ω = ω ω_n
```

30.3.1 An iterative FFT implementation

iterative FFT algorithm

Recursive calls of RECURSIVE-FFT in a tree structure.



arrange the elements of *a* into the order in which they appear in the leaves,

1. **compute the DFT of each pair using one butterfly operation, replace the pair with its DFT, the vector then holds (*n*/2)th 2-element DFT's;**
2. **take the (*n*/2)th DFT's in pairs, compute the DFT of the four vector elements, 2 butterfly operations, the new vector holds (*n*/4)th 4-element DFT's;**
3. **continue in this manner.**

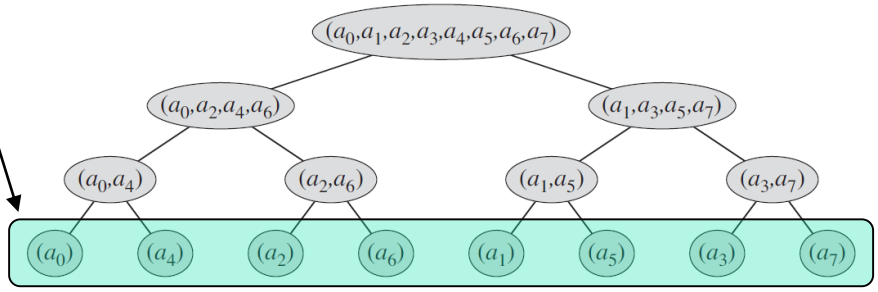
for $k = 0$ **to** $n/2 - 1$
$t = \omega\, y_k^{[1]}$
$y_k = y_k^{[0]} + t$
$y_{k+(n/2)} = y_k^{[0]} - t$
$\omega = \omega\, \omega_n$

- 第一层循环，分层：高度（从下往上）（如果是递归的话，相当于递归的深度）
- 第二层循环，分组：求m个数的FFT(即每组有m个数)，共有n/m组数（即每组有m个数，共有n个数，跟源数据个数相等）
- 第三层循环，蝶算：蝶形操作（每组有m个数，有m/2个蝶形操作，一共n/m组数，所有蝶形操作(m/2)*(n/m)=n/2 个 ）

initial $A[0..n-1]$

```
ITERATIVE-FFT (a)
1  BIT-REVERSE-COPY (a, A)
2  n←length[a]  // n is 2^k
3  for s←1 to lg n
4      m←2^s
5      ω_m←e^(2πi/m)
6      for k←0 to n-1 by m    // n/m组
7          ω←1
8          for j←0 to m/2 -1  // 每组m个数
9              t←ωA[k+j+m/2]
10             u←A[k+j]
11             A[k+j]←u+t
12             A[k+j+m/2]←u-t
13             ω←ω · ω_m
```

Recursive calls of RECURSIVE-FFT in a tree structure.



$s=1$
$m=2^s=2$
$k=0, 2,.., n-1$
$j=0$
$A[0]$ $A[1]$
$A[2]$ $A[3]$
…

$s=2$
$m=4$
$k=0, 4,.., n-1$
$j=0, 1$
$A[0]$ $A[2]$
$A[1]$ $A[3]$
…

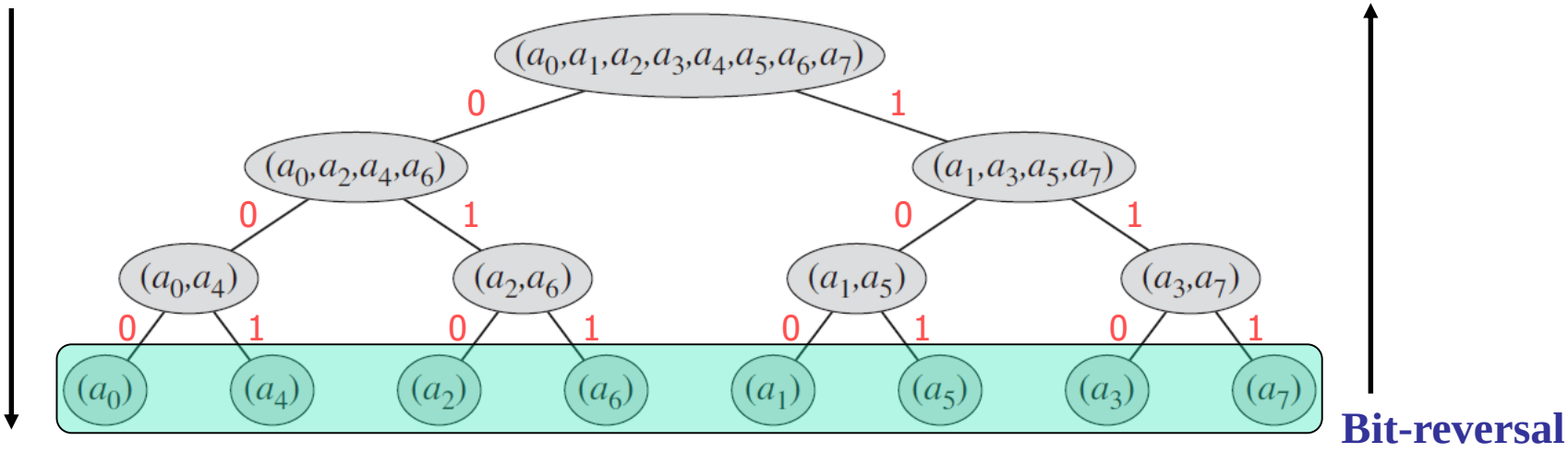
$s=3$
$m=8$
$k=0, 8,.., n-1$
$j=0, 1, 2, 3$
$A[0]$ $A[4]$
$A[1]$ $A[5]$
…

30.3.1 An iterative FFT implementation

- 递归到最底层时，按原始输入的什么顺序开始计算？
- 按位逆置换 (Bit-reversal permutation)

ITERATIVE-FFT (*a*)
1 BIT-REVERSE-COPY (*a*, *A*)
…

Original order



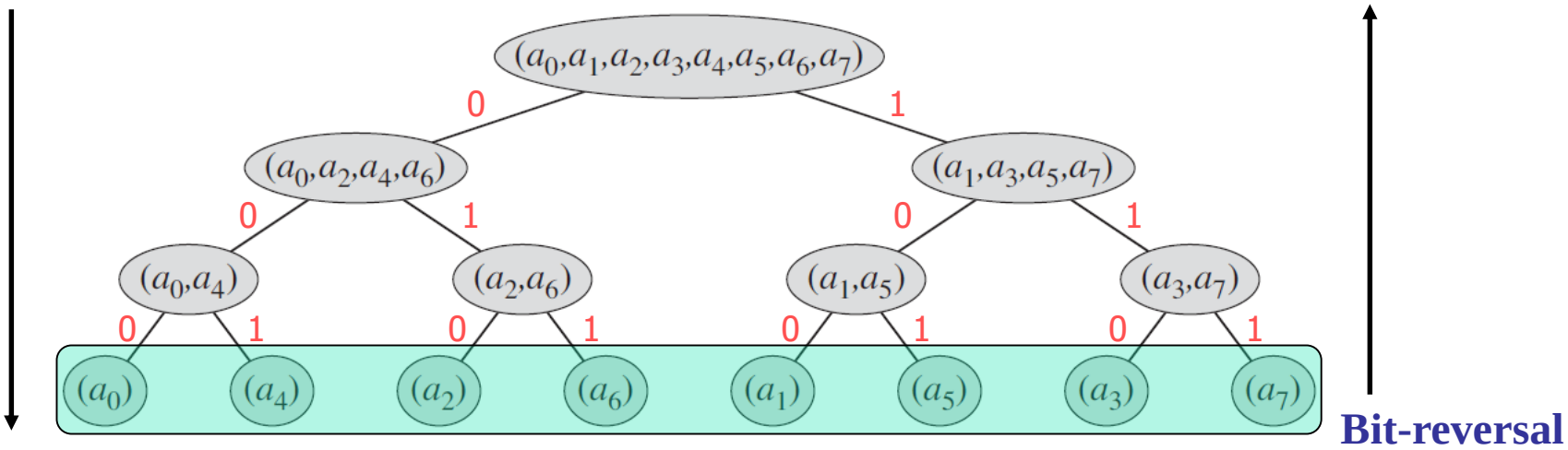
0	1	2	3	4	5	6	7
000	001	010	011	100	101	110	111
000	100	010	110	001	101	011	111
0	4	2	6	1	5	3	7

30.3.1 An iterative FFT implementation

Bit-reversal permutation

```
ITERATIVE-FFT (a)
1  BIT-REVERSE-COPY (a, A)
…
```

Original order



0	1	2	3	4	5	6	7
000	001	010	011	100	101	110	111
000	100	010	110	001	101	011	111
0	4	2	6	1	5	3	7

```
BIT-REVERSE-COPY(a, A)  //Θ(nlgn)
1  n←length[a]
2  for k←0 to n–1                 //Θ(n)
3       do A[rev(k)]←a_k          //Θ(lgn)
```

30.3.1 An iterative FFT implementation

iterative FFT algorithm

```
ITERATIVE-FFT (a)
1  BIT-REVERSE-COPY (a, A)
2  n←length[a]  // n is a power of 2.
3  for s←1 to lg n
4       m←2^s
5       ω_m←e^(2πi/m)
6       for k←0 to n-1 by m
7            ω←1
8            for j←0 to m/2 -1
9                 t←ωA[k+j+m/2]
10                u←A[k+j]
11                A[k+j]←u+t
12                A[k+j+m/2]←u-t
13                ω←ω · ω_m
```

running time?

30.3.1 An iterative FFT implementation

iterative FFT algorithm

```
ITERATIVE-FFT (a)
1  BIT-REVERSE-COPY (a, A)                    //Θ(nlgn)
2  n←length[a]  // n is a power of 2.
3  for s←1 to lg n                             // lgn
4       m←2^s
5       ω_m←e^(2πi/m)
6       for k←0 to n-1 by m                    // n/m=n/2^s
7            ω←1
8            for j←0 to m/2 -1                 // m/2=2^s/2=2^(s-1)
9                 t←ωA[k+j+m/2]
10                u←A[k+j]
11                A[k+j]←u+t
12                A[k+j+m/2]←u-t
13                ω←ω · ω_m
```

$$L(n)=\sum_{s=1}^{\lg n}\frac{n}{2^s}2^{s-1}$$
$$=\sum_{s=1}^{\lg n}\frac{n}{2}$$
$$=\Theta(n\lg n)$$

思考题(email to me)：本算法是否能用于归并排序的非递归实现？如果可以，请写出C程序（输入n个数，迭代的归并排序，输出n个排序的数），并与递归版的归并排序比较（运行实际输入，对比运行时间）。

启示录：1951年，MIT的老师Robert M. Fano给学生们布置了一个题目“寻找最有效的二进制编码”。1952年，David A. Huffman 发表了《一种构建极小多余编码的方法》一文，于是，诞生了一个著名算法：Huffman编码。从Prof. Song布置的系列作业里，你悟出了什么？

30.3.2 A parallel FFT circuit

FFT的并联图

- **See book**

30.3.3 A butterfly operation

$$\text{DFT}, \quad \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \cdots & \omega_n^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{(n-1)\cdot 1} & \omega_n^{(n-1)\cdot 2} & \cdots & \omega_n^{(n-1)\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}, \quad \Theta(n^2)$$

$$\text{FFT}, \quad \Theta(n \lg n) \quad ?$$

$$\text{FFT}, \quad \begin{pmatrix} y_0 \\ y_4 \\ y_2 \\ y_6 \\ y_1 \\ y_5 \\ y_3 \\ y_7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{4\cdot 1} & \omega_n^{4\cdot 2} & \cdots & \omega_n^{4\cdot(n-1)} \\ 1 & \omega_n^{2\cdot 1} & \omega_n^{2\cdot 2} & \cdots & \omega_n^{2\cdot(n-1)} \\ 1 & \omega_n^{6\cdot 1} & \omega_n^{6\cdot 2} & \cdots & \omega_n^{6\cdot(n-1)} \\ 1 & \omega_n^{1\cdot 1} & \omega_n^{1\cdot 2} & \cdots & \omega_n^{1\cdot(n-1)} \\ 1 & \omega_n^{5\cdot 1} & \omega_n^{5\cdot 2} & \cdots & \omega_n^{5\cdot(n-1)} \\ 1 & \omega_n^{3\cdot 1} & \omega_n^{3\cdot 2} & \cdots & \omega_n^{3\cdot(n-1)} \\ 1 & \omega_n^{7\cdot 1} & \omega_n^{7\cdot 2} & \cdots & \omega_n^{7\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = W_1 W_2 W_3 a, \quad \Theta(n \lg n)$$

稀疏矩阵?

30.3.3 A butterfly operation

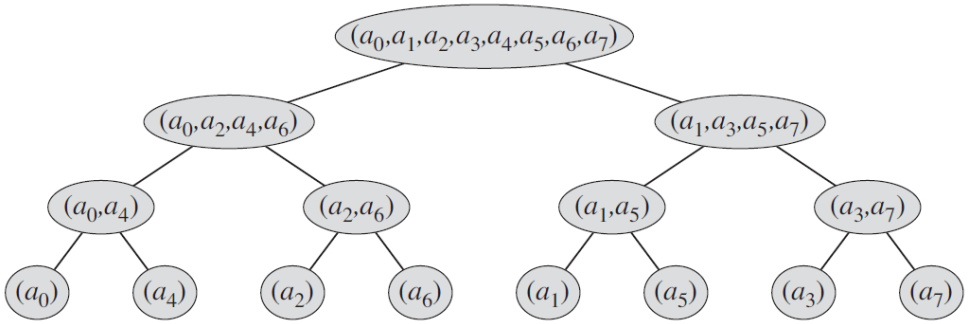
$$\omega = e^{2\pi i/8}$$

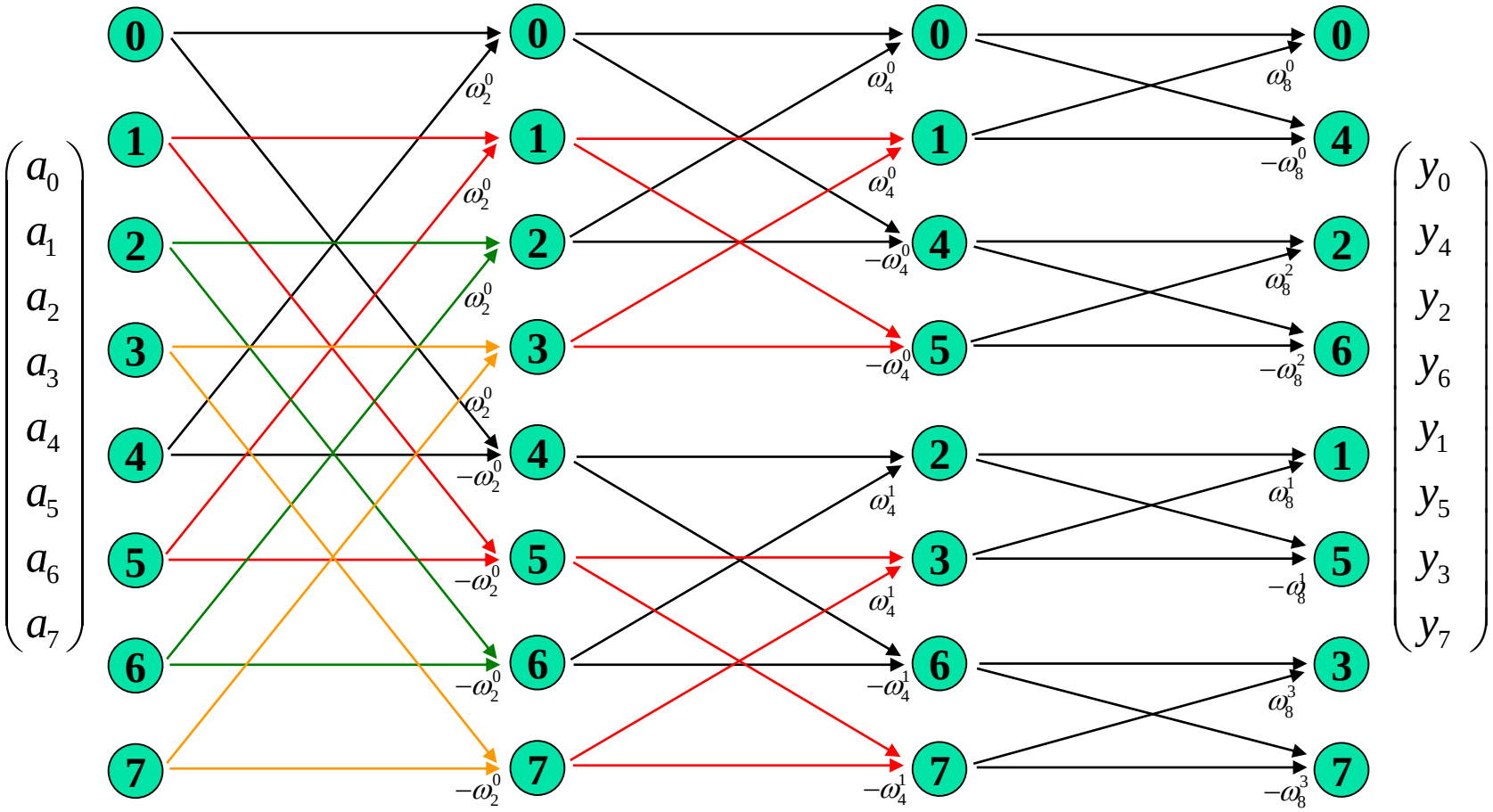
$$\begin{pmatrix} y_0 \\ y_4 \\ y_2 \\ y_6 \\ y_1 \\ y_5 \\ y_3 \\ y_7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{4\cdot 1} & \omega_n^{4\cdot 2} & \cdots & \omega_n^{4\cdot(n-1)} \\ 1 & \omega_n^{2\cdot 1} & \omega_n^{2\cdot 2} & \cdots & \omega_n^{2\cdot(n-1)} \\ 1 & \omega_n^{6\cdot 1} & \omega_n^{6\cdot 2} & \cdots & \omega_n^{6\cdot(n-1)} \\ 1 & \omega_n^{1\cdot 1} & \omega_n^{1\cdot 2} & \cdots & \omega_n^{1\cdot(n-1)} \\ 1 & \omega_n^{5\cdot 1} & \omega_n^{5\cdot 2} & \cdots & \omega_n^{5\cdot(n-1)} \\ 1 & \omega_n^{3\cdot 1} & \omega_n^{3\cdot 2} & \cdots & \omega_n^{3\cdot(n-1)} \\ 1 & \omega_n^{7\cdot 1} & \omega_n^{7\cdot 2} & \cdots & \omega_n^{7\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \omega^0 & & & & & & \\ 1 & \omega^4 & & & & & & \\ & & 1 & \omega^2 & & & & \\ & & 1 & \omega^6 & & & & \\ & & & & 1 & \omega^1 & & \\ & & & & 1 & \omega^5 & & \\ & & & & & & 1 & \omega^3 \\ & & & & & & 1 & \omega^7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \omega^0 & 0 & & & & \\ 0 & 1 & 0 & \omega^0 & & & & \\ 1 & 0 & \omega^4 & 0 & & & & \\ 0 & 1 & 0 & \omega^4 & & & & \\ & & & & 1 & 0 & \omega^2 & 0 \\ & & & & 0 & 1 & 0 & \omega^2 \\ & & & & 1 & 0 & \omega^6 & 0 \\ & & & & 0 & 1 & 0 & \omega^6 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & \omega^0 & & & \\ & 1 & & & & \omega^0 & & \\ & & 1 & & & & \omega^0 & \\ & & & 1 & & & & \omega^0 \\ 1 & & & & \omega^4 & & & \\ & 1 & & & & \omega^4 & & \\ & & 1 & & & & \omega^4 & \\ & & & 1 & & & & \omega^4 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}$$

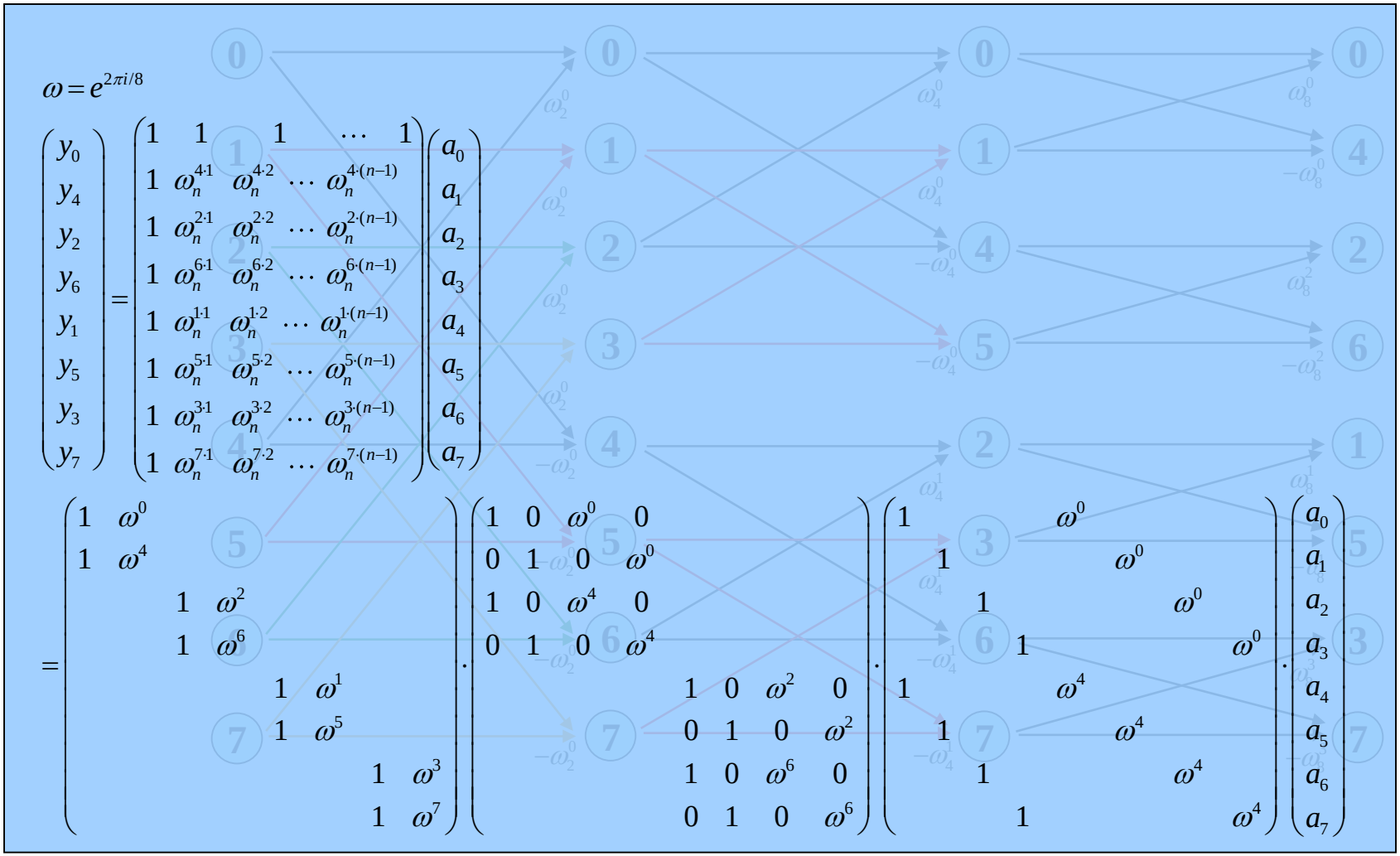
30.3.3 A butterfly operation

for $k = 0$ **to** $n/2 - 1$
$t = \omega\, y_k^{[1]}$
$y_k = y_k^{[0]} + t$
$y_{k+(n/2)} = y_k^{[0]} - t$
$\omega = \omega\, \omega_n$





30.3.3 A butterfly operation



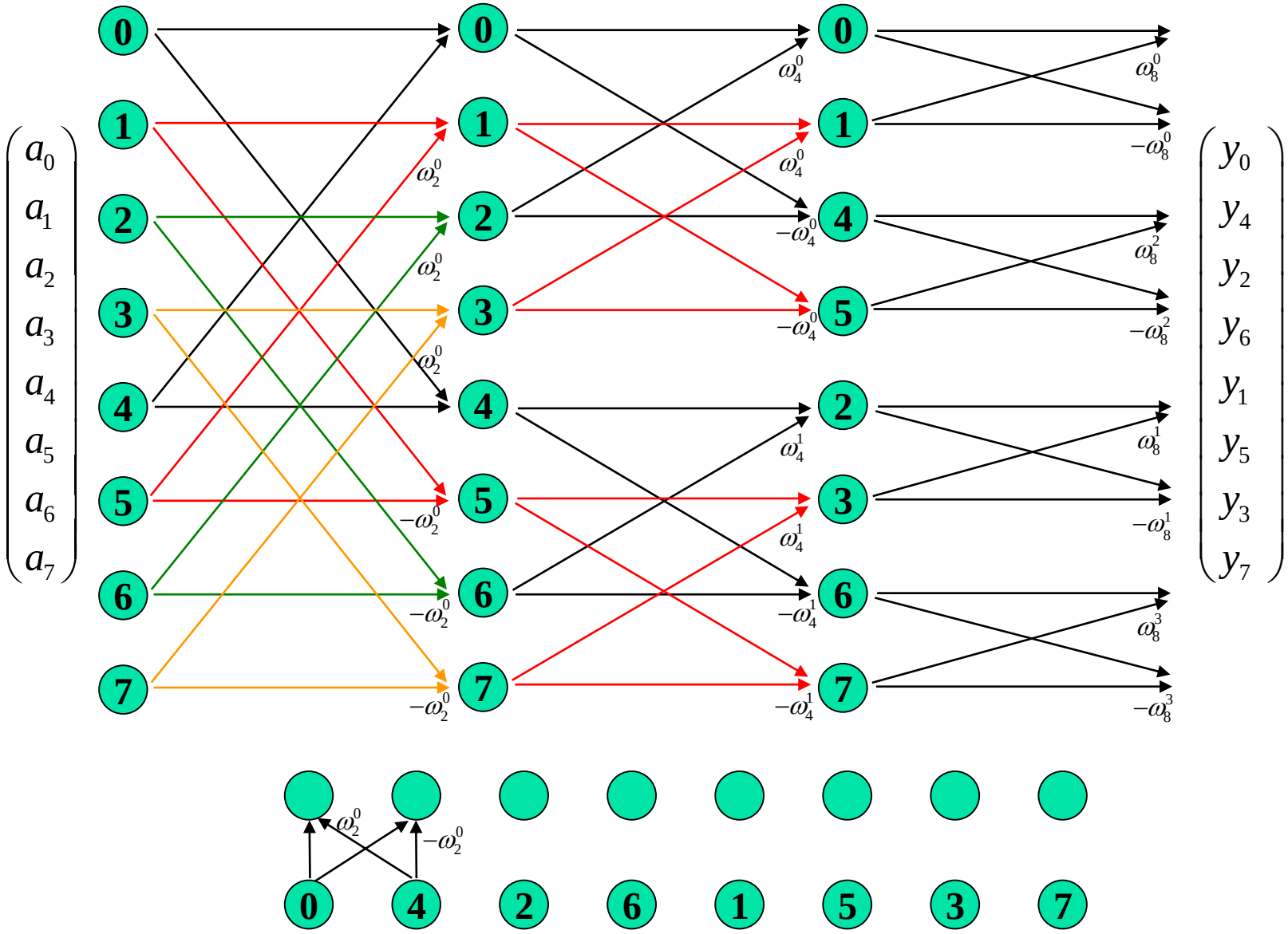
30.3.3 A butterfly operation

$$\omega = e^{2\pi i/8}$$

$$\begin{pmatrix} y_0 \\ y_4 \\ y_2 \\ y_6 \\ y_1 \\ y_5 \\ y_3 \\ y_7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{4\cdot 1} & \omega_n^{4\cdot 2} & \cdots & \omega_n^{4\cdot(n-1)} \\ 1 & \omega_n^{2\cdot 1} & \omega_n^{2\cdot 2} & \cdots & \omega_n^{2\cdot(n-1)} \\ 1 & \omega_n^{6\cdot 1} & \omega_n^{6\cdot 2} & \cdots & \omega_n^{6\cdot(n-1)} \\ 1 & \omega_n^{1\cdot 1} & \omega_n^{1\cdot 2} & \cdots & \omega_n^{1\cdot(n-1)} \\ 1 & \omega_n^{5\cdot 1} & \omega_n^{5\cdot 2} & \cdots & \omega_n^{5\cdot(n-1)} \\ 1 & \omega_n^{3\cdot 1} & \omega_n^{3\cdot 2} & \cdots & \omega_n^{3\cdot(n-1)} \\ 1 & \omega_n^{7\cdot 1} & \omega_n^{7\cdot 2} & \cdots & \omega_n^{7\cdot(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \omega^0 & & & & & & \\ 1 & \omega^4 & & & & & & \\ & & 1 & \omega^2 & & & & \\ & & 1 & \omega^6 & & & & \\ & & & & 1 & \omega^1 & & \\ & & & & 1 & \omega^5 & & \\ & & & & & & 1 & \omega^3 \\ & & & & & & 1 & \omega^7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \omega^0 & 0 & & & & \\ 0 & 1 & 0 & \omega^0 & & & & \\ 1 & 0 & \omega^4 & 0 & & & & \\ 0 & 1 & 0 & \omega^4 & & & & \\ & & & & 1 & 0 & \omega^2 & 0 \\ & & & & 0 & 1 & 0 & \omega^2 \\ & & & & 1 & 0 & \omega^6 & 0 \\ & & & & 0 & 1 & 0 & \omega^6 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & \omega^0 & & & \\ & 1 & & & & \omega^0 & & \\ & & 1 & & & & \omega^0 & \\ & & & 1 & & & & \omega^0 \\ 1 & & & & \omega^4 & & & \\ & 1 & & & & \omega^4 & & \\ & & 1 & & & & \omega^4 & \\ & & & 1 & & & & \omega^4 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}$$

30.3.4 A new butterfly operation?



30.3.4 A new butterfly operation?

(1)

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$$

even, $A^{[0]}(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{n/2-1}$

odd, $A^{[1]}(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{n/2-1}$

(2)

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$$

$$\omega_n^k \qquad (\omega_n^k)^2 = \omega_{n/2}^k$$

$$A(\omega_n^k) = A^{[0]}(\omega_{n/2}^k) + (\omega_n^k) \cdot A^{[1]}(\omega_{n/2}^k),$$

$$A(\omega_n^{n/2+k}) = A^{[0]}(\omega_{n/2}^k) - (\omega_n^k) \cdot A^{[1]}(\omega_{n/2}^k),$$

$$k = 0, 1, \ldots, n/2 - 1$$

(3)

$$\mathrm{DFT}(a_0, a_1, \cdots, a_{n-1}) = A(\omega_n^k),\ k = 0, 1, \cdots, n-1$$

$$\mathrm{DFT}(a_0, a_2, \cdots, a_{n-2}) = A^{[0]}(\omega_{n/2}^k),\ k = 0, 1, \cdots, n/2-1$$

$$\mathrm{DFT}(a_1, a_3, \cdots, a_{n-1}) = A^{[1]}(\omega_{n/2}^k),\ k = 0, 1, \cdots, n/2-1$$

(4)

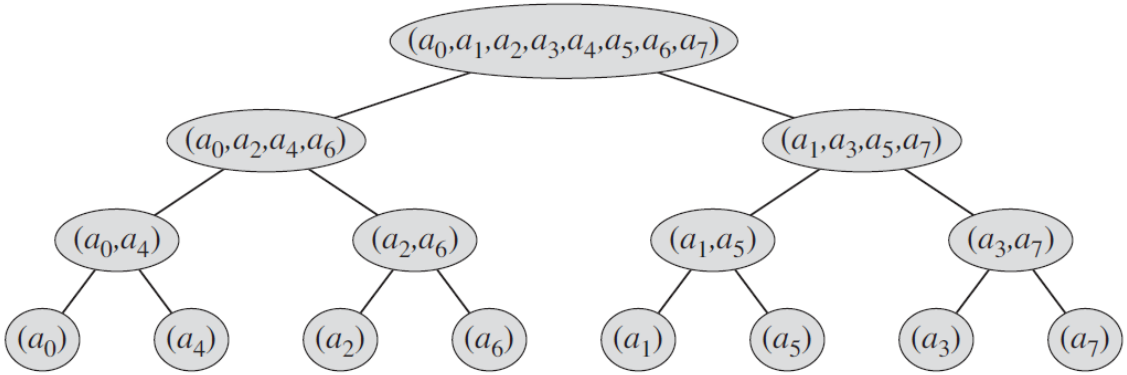
for $k = 0$ **to** $n/2 - 1$

$t = \omega\, y_k^{[1]}$

$y_k = y_k^{[0]} + t$

$y_{k+(n/2)} = y_k^{[0]} - t$

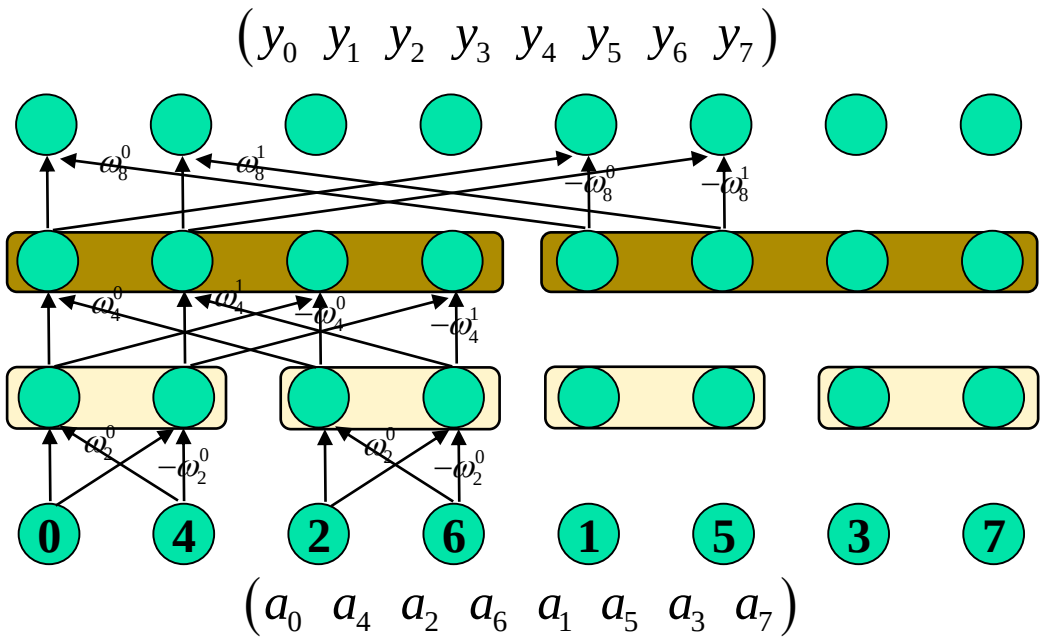
$\omega = \omega\, \omega_n$

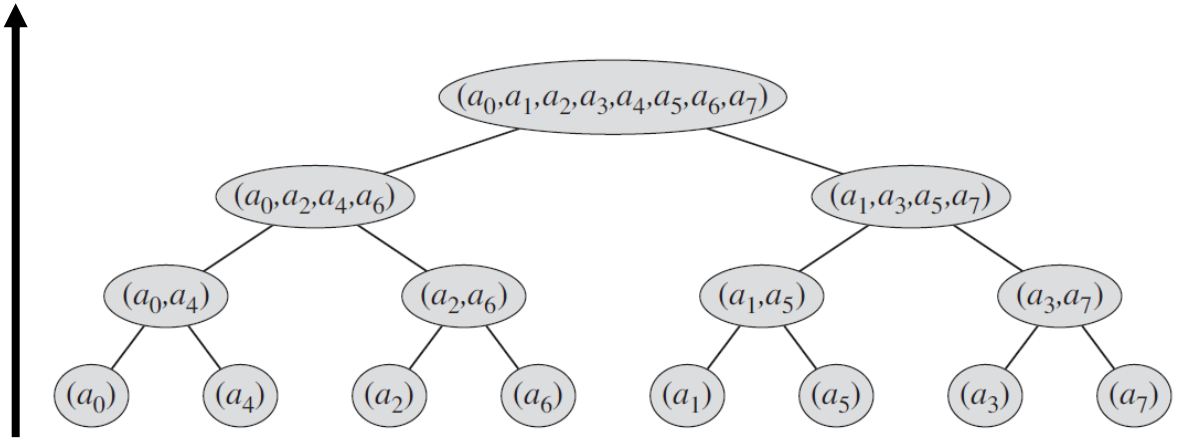


30.3.4 A new butterfly operation?

for $k = 0$ **to** $n/2 - 1$

$$t = \omega\, y_k^{[1]}$$
$$y_k = y_k^{[0]} + t$$
$$y_{k+(n/2)} = y_k^{[0]} - t$$
$$\omega = \omega\, \omega_n$$





Some Applications of FFT

- **Polynomials operation**
- **Multiplication of two big integers**
- **Signal processing (phonic, image, video, ⋯)**
- **⋯**

Exercises

对下列输入向量$\boldsymbol{a}$，给出用FFT求 输出向量 $\boldsymbol{y}$ 的蝶形操作过程和结果

$\boldsymbol{a}$ = (0,1)

$\boldsymbol{a}$ = (1, 0)

$\boldsymbol{a}$ = (1, 1, 0, 1)

$\boldsymbol{a}$ = (0, 1, 2, 3)