20181288 윤성호 미분기하학 기말고사

Q1. (i) $\vec{x}(u,v) = (\sqrt{1+v^2}\cos u, \sqrt{1+v^2}\sin u, v)$ where $(u,v) \in U = \{0<u<2\pi, -\infty<v<\infty\}$

Then $\vec{x}(\frac{\pi}{2}, 0) = (0,1,0) = p$.

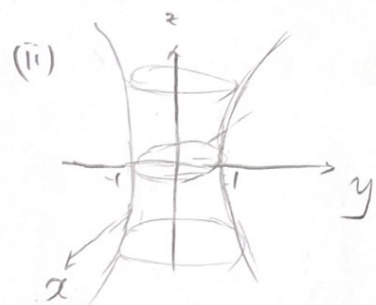
Therefore, $T_pS = \text{span}\{\vec{x}_u(\frac{\pi}{2},0), \vec{x}_v(\frac{\pi}{2},0)\}$

$\vec{x}_u = (-\sqrt{1+v^2}\sin u, \sqrt{1+v^2}\cos u, 0) \Rightarrow \vec{x}_u(\frac{\pi}{2},0) = (-1,0,0)$.

$\vec{x}_v = (\frac{v}{\sqrt{1+v^2}}\cos u, \frac{v}{\sqrt{1+v^2}}\sin u, 1) \Rightarrow \vec{x}_v(\frac{\pi}{2},0) = (0,0,1)$

$\rightarrow T_pS = \text{span}\{(-1,0,0), (0,0,1)\} =$ xz-plane

(ii)



Q2. (i) $\vec{x}(u,v) = (bv, a\cos u, a\sin u)$.

$\vec{x}_u = (0, -a\sin u, a\cos u)$

$\vec{x}_v = (b, 0, 0)$.

$E = \langle \vec{x}_u, \vec{x}_u \rangle = 0^2 + (-a\sin u)^2 + (a\cos u)^2 = a^2$

$F = \langle \vec{x}_u, \vec{x}_v \rangle = (0)(b) + (-a\sin u)(0) + (a\cos u)(0) = 0$

$G = \langle \vec{x}_v, \vec{x}_v \rangle = b^2 + 0^2 + 0^2 = b^2$

(ii) Area of $\vec{x}(U) = \iint_U \sqrt{EG - F^2}\, du dv$

$$= \int_0^1 \int_0^{2\pi} \sqrt{EG - F^2}\, du dv$$

$$= \int_0^1 \int_0^{2\pi} \sqrt{(a^2)(b^2) - 0^2}\, du dv$$

$$= ab \left(\int_0^1 dv\right)\left(\int_0^{2\pi} du\right) \quad (\because a, b > 0)$$

$$= ab \left[v\right]_0^1 \left[u\right]_0^{2\pi}$$

$$= 2\pi ab.$$

(iii) $\vec{x}_u \wedge \vec{x}_v = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & -a\sin u & a\cos u \\ b & 0 & 0 \end{vmatrix} = (0, ab\cos u, ab\sin u).$

$\|\vec{x}_u \wedge \vec{x}_v\| = \sqrt{0^2 + (ab\cos u)^2 + (ab\sin u)^2} = \sqrt{a^2b^2} = ab \quad (\because a, b > 0)$

$N(p) = \dfrac{\vec{x}_u \wedge \vec{x}_v}{\|\vec{x}_u \wedge \vec{x}_v\|} = (0, \cos u, \sin u)$ where $p = \vec{x}(u, v)$.

$\rightarrow N(p) = \frac{1}{a} \times$ (projection of p onto yz-plane).

(iv) Gauss map: $N : S \rightarrow S^2$

$p \mapsto \frac{1}{a} \times$ (projection of p onto yz-plane)

$p = (p_1, p_2, p_3)$

Thus, $\underline{N(p_1, p_2, p_3) = (0, \frac{1}{a}p_2, \frac{1}{a}p_3)}$.

Let $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$ with $\alpha(0) = p$ and $\alpha(t) = (x(t), y(t), z(t))$.

$\rightarrow N(\alpha(t)) = \frac{1}{a}(0, y(t), z(t)) = (0, \frac{y(t)}{a}, \frac{z(t)}{a})$

Differentiating it with respect to t and evaluating it at $t=0$.

$$\frac{d}{dt} N(\alpha(t))\Big|_{t=0} = \left(0, \frac{1}{a}y'(0), \frac{1}{a}z'(0)\right).$$

// chain rule.

$dN_{\alpha(0)}(\alpha'(0))$ $\quad = \quad \frac{1}{a}\times$(projection of $\vec{\alpha}'(0)$ onto yz-plane).

$= dN_p(\vec{\alpha}'(0))$

Therefore, $dN_p : T_pS \to T_pS$ is given by

$dN_p(\vec{v}) = \frac{1}{a}\times$(projection of $\vec{v}$ onto yz-plane) for any $\vec{v}\in T_pS$. ∎

(v) By (iv), $-dN_p : T_pS \to T_pS$ such that $-dN_p(\vec{v}) = (-\frac{1}{a})\times$(projection of $\vec{v}$ onto yz-plane)

$$-dN_p((v_1, v_2, v_3)) = \left(0, -\frac{1}{a}v_2, -\frac{1}{a}v_3\right).$$

Let $\vec{v}_1 = \frac{1}{a}\vec{x}_u = (0, -\sin u, \cos u)$, $\vec{v}_2' = \frac{1}{b}\vec{x}_v = (1, 0, 0)$

Claim $\vec{e}_1' = \vec{v}_1$, $\vec{e}_2' = \vec{v}_2'$

(pf) $\langle \vec{e}_1', \vec{e}_2' \rangle = (0)(1) + (-\sin u)(0) + (\cos u)(0) = 0$.

$\langle \vec{e}_1', \vec{e}_1' \rangle = 0^2 + (-\sin u)^2 + (\cos u)^2 = 1$.

$\langle \vec{e}_2', \vec{e}_2' \rangle = 1^2 + 0^2 + 0^2 = 1$.

Set $\{\vec{e}_1', \vec{e}_2'\}$ be orthonormal basis

$$-dN_p(\vec{e}_1') = -dN_p(\vec{v}_1') = -dN_p(0, -\sin u, \cos u) = \left(0, \frac{1}{a}\sin u, -\frac{1}{a}\cos u\right) = -\frac{1}{a}\vec{e}_1' = k_1\vec{e}_1'$$

$$-dN_p(\vec{e}_2') = -dN_p(\vec{v}_2') = -dN_p(1,0,0) = (0,0,0) = 0\cdot\vec{e}_2' = k_2\vec{e}_2'$$

Thus $k_1 = -\frac{1}{a}$ and $k_2 = 0$.

Gaussian curvature $K = k_1k_2 = (-\frac{1}{a})(0) = 0$.

Mean curvature $H = \frac{k_1+k_2}{2} = -\frac{1}{2a}$ ∎

Q3. (i) $\vec{x}(u,v) = (u, v, F(u,v))$ where $(u,v) \in U$.

(ii) $\vec{x}_u' = (1, 0, F_u)$ $\quad\rightarrow\quad$ $E = \langle \vec{x}_u', \vec{x}_u' \rangle = 1^2 + 0^2 + F_u^2 = 1 + F_u^2$

$\vec{x}_v' = (0, 1, F_v)$ $\qquad$ $F = \langle \vec{x}_u', \vec{x}_v' \rangle = (1)(0) + (0)(1) + (F_u)(F_v) = F_u F_v$

$G = \langle \vec{x}_v', \vec{x}_v' \rangle = 0^2 + 1^2 + F_v^2 = 1 + F_v^2$

(iii)

$$\vec{x}_u' \wedge \vec{x}_v' = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & F_u \\ 0 & 1 & F_v \end{vmatrix} = (-F_u, -F_v, 1).$$

$$\|\vec{x}_u' \wedge \vec{x}_v'\| = \sqrt{(-F_u)^2 + (-F_v)^2 + 1^2} = \sqrt{F_u^2 + F_v^2 + 1},$$

$$N(p) = \frac{\vec{x}_u' \wedge \vec{x}_v'}{\|\vec{x}_u' \wedge \vec{x}_v'\|} = \frac{(-F_u, -F_v, 1)}{\sqrt{F_u^2 + F_v^2 + 1}} \quad \text{where } p = \vec{x}(u,v)$$

(iv) $\vec{x}_{uu} = (0, 0, F_{uu})$ $\quad$ $\vec{x}_{uv} = (0, 0, F_{uv})$ $\quad$ $\vec{x}_{vv} = (0, 0, F_{vv})$

$$e = \langle \vec{x}_{uu}, N \rangle = (0)\left(\frac{-F_u}{\sqrt{F_u^2 + F_v^2 + 1}}\right) + (0)\left(\frac{-F_v}{\sqrt{F_u^2 + F_v^2 + 1}}\right) + (F_{uu})\left(\frac{1}{\sqrt{F_u^2 + F_v^2 + 1}}\right)$$

$$= \frac{F_{uu}}{\sqrt{F_u^2 + F_v^2 + 1}}.$$

$$f = \langle \vec{x}_{uv}', N \rangle = (0)\left(\frac{-F_u}{\sqrt{F_u^2 + F_v^2 + 1}}\right) + (0)\left(\frac{-F_v}{\sqrt{F_u^2 + F_v^2 + 1}}\right) + (F_{uv})\left(\frac{1}{\sqrt{F_u^2 + F_v^2 + 1}}\right)$$

$$= \frac{F_{uv}}{\sqrt{F_u^2 + F_v^2 + 1}}.$$

$$g = \langle \vec{x}_{vv}, N \rangle = (0)\left(\frac{-F_u}{\sqrt{F_u^2 + F_v^2 + 1}}\right) + (0)\left(\frac{-F_v}{\sqrt{F_u^2 + F_v^2 + 1}}\right) + (F_{vv})\left(\frac{1}{\sqrt{F_u^2 + F_v^2 + 1}}\right).$$

$$= \frac{F_{vv}}{\sqrt{F_u^2 + F_v^2 + 1}}.$$

(v) Gaussian Curvature

$$k = \frac{eg - f^2}{EG - F^2} = \frac{\left(\frac{F_{uu}}{\sqrt{F_u^2+F_v^2+1}}\right)\left(\frac{F_{vv}}{\sqrt{F_u^2+F_v^2+1}}\right) - \left(\frac{F_{uv}}{\sqrt{F_u^2+F_v^2+1}}\right)^2}{(1+F_u^2)(1+F_v^2) - (F_uF_v)^2} = \frac{F_{uu}F_{vv} - (F_{uv})^2}{(F_u^2+F_v^2+1)^2}$$

Mean Curvature

$$H = \frac{eG + gE - 2fF}{2(EG - F^2)} = \frac{\left(\frac{F_{uu}}{\sqrt{F_u^2+F_v^2+1}}\right)(1+F_v^2) + \left(\frac{F_{vv}}{\sqrt{F_u^2+F_v^2+1}}\right)(1+F_u^2) - 2\cdot\left(\frac{F_{uv}}{\sqrt{F_u^2+F_v^2+1}}\right)(F_uF_v)}{2\left((1+F_u^2)(1+F_v^2) - (F_uF_v)^2\right)}$$

$$= \frac{F_{uu} + F_{uu}F_v^2 + F_{vv} + F_{vv}F_u^2 - 2F_{uv}F_uF_v}{2(1+F_u^2+F_v^2)^{\frac{3}{2}}}$$

(vi) By (i),

$F_u = a \quad , F_v = b$

$F_{uu} = F_{uv} = F_{vv} = 0$

$$k = \frac{F_{uu}F_{vv} - (F_{uv})^2}{(F_u^2+F_v^2+1)^2} = \frac{(0)(0) - (0)^2}{(a^2+b^2+1)^2} = 0$$

$$H = \frac{F_{uu} + F_{uu}F_v^2 + F_{vv} + F_{vv}F_u^2 - 2F_{uv}F_uF_v}{2(1+F_u^2+F_v^2)^{\frac{3}{2}}} = \frac{0 + (0)(b)^2 + 0 + (0)(a)^2 - 2(0)(a)(b)}{2(1+a^2+b^2)^{\frac{3}{2}}} = 0$$

Consider $k^2 - 2Hk + K = 0$. Solving this quadratic equation gives us the principal curvature k_1, k_2.

$\Leftrightarrow k^2 - 2(0)k + 0 = 0 \Leftrightarrow k^2 = 0 \Leftrightarrow k_1 = k_2 = 0$. ∎

(vii) $F(u,v) = 4+u^2+v^2$

$F_u = 2u$, $\quad F_{uu} = 2 \quad F_{vv} = 2$.

$F_v = 2v \quad F_{uv} = 0$

$$K = \frac{F_{uu}F_{vv} - (F_{uv})^2}{(F_u^2+F_v^2+1)^2} = \frac{(2)(2) - 0}{((2u)^2+(2v)^2+1)^2} = \frac{4}{(4u^2+4v^2+1)^2}$$

$$H = \frac{F_{uu}+F_{uu}F_v^2+F_{vv}+F_{vv}F_u^2-2F_{uv}F_uF_v}{2(1+F_u^2+F_v^2)^{\frac{3}{2}}} = \frac{2+2(2v)^2+2+(2)(2u)^2-2(0)(2u)(2v)}{2(1+(2u)^2+(2v)^2)^{\frac{3}{2}}}$$

$$= \frac{2+4u^2+4v^2}{(4u^2+4v^2+1)^{\frac{3}{2}}}$$

(viii) $p = x(u,v)$ is an umbilical point $\Leftrightarrow H^2 = K$.

By (vii), $\left(\frac{2+4u^2+4v^2}{(4u^2+4v^2+1)^{\frac{3}{2}}}\right)^2 = \frac{4}{(4u^2+4v^2+1)^2}$

$$\Leftrightarrow \frac{(1+2u^2+2v^2)^2}{(4u^2+4v^2+1)^3} = \frac{1}{(4u^2+4v^2+1)^2}$$

$$\Leftrightarrow 1+4u^4+4v^4+4u^2+4v^2+8u^2v^2 = 4u^2+4v^2+1$$

$$\Leftrightarrow u^4+2u^2v^2+v^4=0 \Leftrightarrow (u^2+v^2)^2=0 \Leftrightarrow u^2+v^2=0$$

$$\Leftrightarrow u=v=0.$$

Thus, $x(0,0) = (0,0,4)$ is the only umbilical point of S

Q4. (i) $\vec{x}(u,v) = (a\sin v\cos u,\ b\sin v\sin u,\ c\cos v)$

$\vec{x}_u' = (-a\sin v\sin u,\ b\sin v\cos u,\ 0)$

$\vec{x}_v' = (a\cos v\cos u,\ b\cos v\sin u,\ -c\sin v)$

$E = \langle \vec{x}_u, \vec{x}_u \rangle = (-a\sin v\sin u)^2 + (b\sin v\cos u)^2 + 0^2$

$= a^2\sin^2 v\sin^2 u + b^2\sin^2 v\cos^2 u$

$F = \langle \vec{x}_u, \vec{x}_v \rangle = (-a\sin v\sin u)(a\cos v\cos u) + (b\sin v\cos u)(b\cos v\sin u) + (0)(-c\sin v)$

$= (-a^2+b^2)\sin v\cos v\sin u\cos u.$

$G = \langle \vec{x}_v, \vec{x}_v \rangle = (a\cos v\cos u)^2 + (b\cos v\sin u)^2 + (-c\sin v)^2$

$= a^2\cos^2 v\cos^2 u + b^2\cos^2 v\sin^2 u + c^2\sin^2 v.$

(ii) $\vec{x}_u \wedge \vec{x}_v = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -a\sin v\sin u & b\sin v\cos u & 0 \\ a\cos v\cos u & b\cos v\sin u & -c\sin v \end{vmatrix}$

$= (-bc\sin^2 v\cos u,\ -ac\sin^2 v\sin u,\ -ab\sin v(\cos v+\cos u))$

$\|\vec{x}_u \wedge \vec{x}_v\| = \sqrt{(-bc\sin^2 v\cos u)^2 + (-ac\sin^2 v\sin u)^2 + (-ab\sin v(\cos v+\cos u))^2}$

$= abc\sin v\sqrt{\dfrac{\cos^2 u}{a^2} + \dfrac{\sin^2 u}{b^2} + \dfrac{(\cos v+\cos u)^2}{c^2}}$ $(\because \sin v > 0,\ a,b,c > 0)$

$$H(p) = \frac{\left(-\dfrac{\sin v\cos u}{a},\ -\dfrac{\sin v\sin u}{b},\ -\dfrac{(\cos v+\cos u)}{c}\right)}{\sqrt{\dfrac{\cos^2 u}{a^2} + \dfrac{\sin^2 u}{b^2} + \dfrac{(\cos v+\cos u)^2}{c^2}}}$$

(7)

Q5. (i) $\vec{x}(u,v) = (b\sin v,\ (a+b\cos v)\cos u,\ (a+b\cos v)\sin u)$

$$\vec{x}_u' = (0,\ -(a+b\cos v)\sin u,\ (a+b\cos v)\cos u)$$

$$\vec{x}_v' = (b\cos v,\ -b\sin v\cos u,\ -b\sin v\sin u)$$

$$E = \langle \vec{x}_u', \vec{x}_u' \rangle = 0^2 + (-(a+b\cos v)\sin u)^2 + ((a+b\cos v)\cos u)^2 = (a+b\cos v)^2$$

$$F = \langle \vec{x}_u', \vec{x}_v' \rangle = (0)(b\cos v) + (-(a+b\cos v)\sin u)(-b\sin v\cos u) + ((a+b\cos v)\cos u)(-b\sin v\sin u)$$

$$= 0.$$

$$G = (b\cos v)^2 + (-b\sin v\cos u)^2 + (-b\sin v\sin u)^2 = b^2.$$

(ii) Area of $\vec{x}(U) = \iint_U \sqrt{EG-F^2}\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} \sqrt{(a+b\cos v)^2 \cdot b^2 - 0}\,du\,dv.$

$$= \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos v)\,du\,dv \qquad (\because a+b\cos v \geq a-b>0,\ b>0).$$

$$= b\left(\int_0^{2\pi} a+b\cos v\,dv\right)\left(\int_0^{2\pi} du\right)$$

$$= b\left[av+b\sin v\right]_0^{2\pi}\left[u\right]_0^{2\pi} = 4\pi^2 ab$$

(iii) $$\vec{x}_u \wedge \vec{x}_v = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & -(a+b\cos v)\sin u & (a+b\cos v)\cos u \\ b\cos v & -b\sin v\cos u & -b\sin v\sin u \end{vmatrix}$$

$$= ((a+b\cos v)b\sin v,\ (a+b\cos v)b\cos v\cos u,\ (a+b\cos v)b\cos v\sin u).$$

$$\|\vec{x}_u \wedge \vec{x}_v\| = \sqrt{((a+b\cos v)b\sin v)^2 + ((a+b\cos v)b\cos v\cos u)^2 + ((a+b\cos v)b\cos v\sin u)^2}$$

$$= b(a+b\cos v) \quad (\because b>0 \text{ and } a+b\cos v \geq a-b>0).$$

$$N(p) = \frac{\vec{x}_u \wedge \vec{x}_v}{\|\vec{x}_u \wedge \vec{x}_v\|} = (\sin v,\ \cos u\sin v,\ \sin u\cos v) \quad \text{where } p = \vec{x}(u,v)$$

(iv) $\vec{x}_{uu} = (0, -(a+b\cos v)\cos u, -(a+b\cos v)\sin u)$.

$\vec{x}_{uv} = (0, b\sin v\sin u, -b\sin v\cos u)$

$\vec{x}_{vv} = (-b\sin v, -b\cos v\cos u, -b\cos v\sin u)$

$e = \langle \vec{x}_{uu}, N\rangle = (0)(\sin v) + (-(a+b\cos v)\cos u)(\cos v\sin u) + (-(a+b\cos v)\sin u)(\sin u\cos v) =$

$= -(a+b\cos v)\cos v)$

$f = \langle \vec{x}_{uv}, N\rangle = (0)(\sin v) + (b\sin v\sin u)(\cos u\cos v) + (-b\sin v\cos u)(\sin u\cos v)$

$= 0.$

$g = \langle \vec{x}_{vv}, N\rangle = (-b\sin v)(\sin v) + (-b\cos v\cos u)(\cos v\cos u) + (-b\cos v\sin u)(\sin u\cos v)$

$= -b.$

(v) $K = \dfrac{eg - f^2}{EG - F^2} = \dfrac{(-(a+b\cos v)\cos v)(-b) - 0^2}{(a+b\cos v)^2 b^2 - 0^2} = \dfrac{\cos v}{b(a+b\cos v)}$ $(\because b>0,\ a+b\cos v \geq a-b > 0)$

$H = \dfrac{eG+gE-2fF}{2(EG-F^2)} = \dfrac{(-(a+b\cos v)\cos v)(b^2) + (-b)(a+b\cos v)^2 - 2(0)(0)}{2((a+b\cos v)^2 b^2 - 0^2)} = -\dfrac{a+2b\cos v}{2b(a+b\cos v)}$

$(\because b>0,\ a+b\cos v \geq a-b>0)$.

(vi) $p = \vec{x}(u,v)$ is elliptic $(\because b>0,\ a+b\cos v \geq a-b>0)$

$\Leftrightarrow K = \dfrac{\cos v}{b(a+b\cos v)} > 0 \Leftrightarrow \cos v > 0 \Leftrightarrow \boxed{0<v<\frac{\pi}{2},\ \frac{3}{2}\pi < v < 2\pi,\ 0<u<2\pi}$.

$p = \vec{x}(u,v)$ is hyperbolic $(\because b>0,\ a+b\cos v \geq a-b>0)$

$\Leftrightarrow K = \dfrac{\cos v}{b(a+b\cos v)} < 0 \Leftrightarrow \cos v < 0 \Leftrightarrow \boxed{\frac{\pi}{2} < v < \frac{3}{2}\pi,\ 0<u<2\pi}$

$p = \vec{x}(u,v)$ is parabolic $(\because b>0,\ a+b\cos v \geq a-b>0)$

$\Leftrightarrow K = \dfrac{\cos v}{b(a+b\cos v)} = 0 \Leftrightarrow \cos v = 0 \Leftrightarrow v = \frac{\pi}{2}$ or $v = \frac{3}{2}\pi$

If $\cos v = 0$, $H = \dfrac{-a+2b0}{2b(a+b(0))} = -\dfrac{a}{2ab}$

$(\because a>0) = -\dfrac{1}{2b} \neq 0$

$\boxed{v = \frac{\pi}{2} \text{ or } v = \frac{3}{2}\pi,\ 0<u<2\pi}$

(vii) Gauss-Bonnet formula.

$$\iint_S k\sqrt{EG-F^2}\,du\,dv = 2\pi\chi(S)$$

$$\to \chi(S) = \frac{1}{2\pi}\iint_S k\sqrt{EG-F^2}\,du\,dv$$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi}\int_0^{2\pi} \frac{\cos v}{b(a+b\cos v)} \times (a+b\cos v)b\,du\,dv\right) \quad (\because \sqrt{EG-F^2} = \|\vec{x}_u \wedge \vec{x}_v\|)$$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi}\cos v\,dv\right)\left(\int_0^{2\pi}du\right) = \frac{1}{2\pi}[\sin v]_0^{2\pi}[u]_0^{2\pi} = 0.$$

Thus, Euler-characteristics of S is 0 ∎