20181288 윤성훈 추상대수학 4차과제

8.1.30. Let $\pi = \{Kb_i a_j \mid 1 \le i \le m, 1 \le j \le n\}$

Claim 1: π covers G.

pf) Let $x \in G$. Then there exists j ($j \in \mathbb{N}, 1 \le j \le n$) such that $x \in Ha_j$. This means, $x = ha_j$ for some $h \in H$. Also, there exists i ($i \in \mathbb{N}, 1 \le i \le m$) such that $\in Kb_i$. This means, $h = kb_i$ for some $k \in K$. Then, $x = ha_j = kb_i a_j \in Kb_i a_j \in \pi$. Thus, the claim holds. **So, π contains all** right cosets of K on G.

Claim 2: $|\pi| = mn$.

pf) Suppose $Kb_p a_q = Kb_r a_s$. Then $kb_p, kb_r \in H$ thus we know that $kb_p a_q \in Ha_q$ and $kb_r a_s \in Ha_s$. But by "Hint" on the problem, Ha_q and Ha_s are disjoint sets if $q \ne s$. $Ha_q \cap Ha_s \ne \emptyset$, thus $q = s$. ⋯ ①

By above,
$$\begin{aligned} kb_p = kb_p(a_q \cdot a_q^{-1}) &= kb_p a_q (a_q^{-1}) \\ &= kb_r a_s (a_q^{-1}) \ (\because \text{assumption}) \\ &= kb_r a_s (a_s^{-1}) \ (\because ①) \\ &= kb_r \end{aligned}$$

$kb_p = kb_r$ means $kb_p \cap kb_r \ne \emptyset$. Thus $p = r$. Thus, the claim holds.

Thus, by claim 1, 2, $[G:K] = |\pi| = mn = [H:K][G:H] = [G:H][H:K]$ ∎

8.1.37 Since $(k,n)=1$, there exists $x,y\in\mathbb{Z}$ such that $kx+ny=1$. $\cdots(*)$.

(i) Homomorphism

Let $a,b\in G$, then $f(ab)=(ab)^k=a^kb^k=f(a)f(b)$

(ii) Injectivity.

Let $a,b\in G$ and suppose $f(a)=f(b)$.

$$
\begin{aligned}
& f(a)=f(b) \\
\Leftrightarrow\ & a^k=b^k \\
\Leftrightarrow\ & a^{kx}=b^{kx} \quad (\because x\in\mathbb{Z}) \\
\Leftrightarrow\ & a^{1-ny}=b^{1-ny} \quad (\because y\in\mathbb{Z} \text{ and by } (*)). \\
\Leftrightarrow\ & a\cdot(a^n)^{-y}=b\cdot(b^n)^{-y} \\
\Leftrightarrow\ & a\cdot e^{-y}=b\cdot e^{-y} \quad (\because \text{by consequence of Lagrange thm}). \\
\Leftrightarrow\ & a=b.
\end{aligned}
$$

(iii) Surjectivity.

For any $a\in G$,
$$
\begin{aligned}
a &= a^{kx-ny} \quad (\because x,y\in\mathbb{Z} \text{ and by } (*)). \\
&= a^{kx} \quad (\because \text{by consequence of Lagrange thm}). \\
&= f(a^x). \longrightarrow a^x\in G.
\end{aligned}
$$

By (i), (ii) and (iii), f is an isomorphism.

8.2.14 Let $M = \{v, r_0\}$, $N = \{h, v, r_2, r_0\}$, $G = D_4 = \{d, t, h, v, r_0, r_1, r_2, r_3\}$

S.t r_0, r_1, r_2, r_3 are $0°, 90°, 180°, 270°$ rotations respectively and d, t, h, v are x-axis, y-axis, $y=x$, $y=-x$ reflections respectively.

(i) $M \trianglelefteq N$

$hM = \{h, r_2\} = Mh$ $\qquad$ $r_2 M = \{r_2, h\} = Mr_2$

$vM = \{v, r_0\} = Mv$ $\qquad$ $r_0 M = \{v, r_0\} = Mr_0$

All the left cosets are equivalent to right cosets,

thus, $M \trianglelefteq N$.

(ii) $N \trianglelefteq G$

$dN = \{d, t, r_1, r_3\} = Nd$ $\qquad$ $r_0 N = \{h, v, r_2, r_0\} = Nr_0$

$tN = \{t, d, r_1, r_3\} = Nt$ $\qquad$ $r_1 N = \{t, d, r_3, r_1\} = Nr_1$

$hN = \{h, v, r_0, r_2\} = Nh$ $\qquad$ $r_2 N = \{v, h, r_0, r_2\} = Nr_2$

$vN = \{v, h, r_0, r_2\} = Nv$ $\qquad$ $r_3 N = \{d, t, r_1, r_3\} = Nr_3$

All the left cosets are equivalent to right cosets,

thus, $N \trianglelefteq G$.

However, we can see that M is not a normal subgroup of G because $\{t, r_1\} = Mt \neq tM = \{t, r_3\}$.

Thus, normality isn't transitive ∎

8.2.20 (a) Let $n, n' \in N$, $k, k' \in K$ then $nk \in NK$ thus $NK \neq \phi$.

(i) $(nk)(n'k') = n(kn')k' = n(n_1 k)k'$ ($\because$ since N is normal, $\exists\, n_1 \in N$ s.t $kn' = n_1 k$)

$= (nn_1)(kk') \in NK$.

(ii) $(nk)^{-1} = k^{-1}n^{-1} = n_2 k^{-1}$ ($\because$ since N is normal, $\exists\, n_2 \in N$ s.t $k^{-1}n^{-1} = n_2 k^{-1}$)

$\in NK$

$\therefore NK$ is a subgroup of G $\blacksquare$

(b) Claim: For all $a \in G$, $n \in N$, $k \in K$, $a(nk)a^{-1} \in NK$.

pf) Since N and K are normal, there exists $n' \in N$, $k' \in K$ such that $an = n'a$ and $ka^{-1} = a^{-1}k'$.

By above, $a(nk)a^{-1} = (an)(ka^{-1}) = (n'a)(a^{-1}k')$

$= n'k' \in NK$.

Thus, the claim holds and this says that NK is a normal subgroup of G $\blacksquare$

8.3.28 Define $\pi : G \to (G/M) \times (G/N)$ by $\pi(g) = (gM, gN)$

(i) Homomorphism

Let $a, b \in G$ then $\pi(ab) = (abM, abN) = (aMbM, aNbN)$

$$= (aM, aN) \cdot (bM, bN)$$

$$= \pi(a)\pi(b)$$

(ii) Injectivity

Let $a, b \in G$ and suppose $\pi(a) = \pi(b)$.

$$\pi(a) = \pi(b)$$

$\Leftrightarrow (aM, aN) = (bM, bN)$

$\Leftrightarrow aM = bM$ and $aN = bN$

$\Leftrightarrow ab^{-1} \in M$ and $ab^{-1} \in N$

$\Leftrightarrow ab^{-1} \in M \cap N = \{e\}$

$\Leftrightarrow a = b$.

Let $S = \operatorname{Im} \pi$, which is a subgroup of $(G/M) \times (G/N)$

If we edit the map as $\pi : G \to S$, then this makes the the map is also surjective.

Then π induces an isomorphism $G \cong S$ ∎

8.3.29 Let $g \in G$. Then there exists $r > 0$ such that $(gN)^r = N$ since the order of $gN \in G/N$ is finite. $(gN)^r = g^r N = N$, thus $g^r \in N$.

This means that there exists $s > 0$ such that $(g^r)^s = g^{rs} = e$.

Thus, every element of G has finite order. ∎

8.4.30. We know that $\mathrm{Im} f$ is a subgroup of H and by Lagrange's theorem, $|\mathrm{Im} f| \mid |H|$.

Let K be $\ker f$. Then apply first isomorphism theorem.

Since $f: G \to \mathrm{Im} f$ is a surjective homorphism, $G/K \cong \mathrm{Im} f$.

$$
\begin{aligned}
\text{Then, } |\mathrm{Im} f| &= |G/K| \quad (\because \text{isomorphism}) \\
&= [G:K] \quad (\because \text{number of cosets}) \\
&= |G|/|K|. \quad (\because \text{Lagrange's theorem})
\end{aligned}
$$

Thus, $|\mathrm{Im} f| \mid |G|$ ∎