# Abstract algebra Quiz ♯1
# (Mar. 28, 2022)

Provide a suitable explanation for all your answers. It must be **accurate** at every step. (10 points per each problem)

1. Suppose that a ring $R$ has a unique element $e$ such that $xe = x$ for all $x \in R$. Show that $ey = y$ for all $y \in R$.

   *Proof.* (idea)

   $ey = y \Longleftrightarrow ey - y = 0 \Longleftarrow ey - y + e = e \Longleftrightarrow x(ey - y + e) = x$ for all $x \in R$.

   (proof) Let $y$ be any element in $R$. For all $x \in R$, we have

   $$x(ey - y + e) = x(ey) - xy + xe = (xe)y - xy + x = x.$$

   By the uniqueness of $e$, $ey - y + e$ should be $e$, thus $ey = e$ for all $y \in R$.

2. Let $R$ be a ring with $1 \neq 0$ and $a, b \in R$. Assume that neither $a$ nor $b$ is a zero divisor. If $ab$ is a unit, show that $a$ and $b$ are units. then $a = b$.

   *Proof.* (i) Let $x$ be the unit of $ab$. Then $abx = 1$ (so $a(bx) = 1$). Multiplying $a$ on the right hand side of either side yields that $abxa = a$, thus $a(bxa - 1) = 0$. Since $a$ is nonzero and is not a zero divisor, $bxa - 1 = 0$, that is, $(bx)a = 1$. It says that $a$ is a unit and $a^{-1} = bx$.

   (ii) Since $x$ is the unit of $ab$, we also have that $xab = 1$. In (i), we showed that $bxa = 1$. It says that $b$ is a unit and $b^{-1} = xa$.

3. Describe **all** ring homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$.

   *Proof.* First, find necessary conditions for homomorphisms to satisfy. Let $f$ be a nonzero homomorphism. Since $(1,0)^2 = (1,0)$, we have that $f((1,0)) = f((1,0))^2$. Hence $f((1,0)) = 0$ or 1. Similarly, $f((0,1)) = 0$ or 1. Furthermore, since $(1,1)^2 = (1,1)$ and $(1,1) = (1,0) + (0,1)$, we also have that $f((1,1)) = f((1,1))^2$ and $f((1,1)) = f((1,0)) + f((0,1))$. So, the possible candidates are

   (i) $f((1,0)) = 0, f((0,1)) = 0$

   (ii) $f((1,0)) = 1, f((0,1)) = 0$

   (iii) $f((1,0)) = 0, f((0,1)) = 1$

   Next, show that all these cases indeed yield homomorphisms.

   In case of (i), $f(m,n) = 0$, hence $f$ is the zero map.

   In case of (ii), $f((m,n) = m$, hence $f$ is the projection to the first component.

   In case of (iii), $f((m,n) = n$, hence $f$ is the projection to the second component.

4. Prove or disprove that $2\mathbb{Z}$ is isomorphic to $3\mathbb{Z}$ as rings.

   *Proof.* Let $f : 2\mathbb{Z} \to 3\mathbb{Z}$ be any bijection. Letting $f(2) = 3m$, $f(4) = f(2+2) = 6m$ and $f(4) = f(2 \times 2) = f(2)^2 = 9m^2$. Since $6m = 9m^2$, it follows that $m = 0$. But this is absurd since $f(2n) = nf(2) = 0$ for all $n \in \mathbb{Z}$. Hence there are no isomorphisms.