Q1. (i) $\vec{\alpha} : \mathbb{R} \to \mathbb{R}^3$

$$\vec{\alpha}(t) = \left(\cos^2 t - \sin^3 t ,\ t^6 - t^2 + 8 ,\ e^{2t-3} \right)$$

A. This is a parametrized curve.

Let $x(t) = \cos^2 t - \sin^3 t$, $y(t) = t^6 - t^2 + 8$, $z(t) = e^{2t-3}$. $x(t)$, $y(t)$, and $z(t)$ are sum and composition of $\sin t$, $\cos t$, e^t and polynomials. Since $\sin t$, $\cos t$, e^t and polynomials are differentiable, sum and composition of differentiable functions are still differentiable by Chain rule. Therefore, $x(t)$, $y(t)$, and $z(t)$ are differentiable ∎

(ii) $\vec{\alpha} : \mathbb{R} \to \mathbb{R}^3$, $\vec{\alpha}(t) = (2t^3, t^{\frac{12}{5}}, 5)$

A. This is NOT a parametrized curve.

Let $x(t) = 2t^3$, $y(t) = t^{\frac{12}{5}}$, $z(t) = 5$. $x(t)$ and $z(t)$ are differentiable since they are polynomials. But, $y'''(t) = \dfrac{168}{125} t^{-\frac{3}{5}}$ does not exist at $t = 0 \in \mathbb{R}$, which makes $y(t)$ not differentiable ∎

(iii) $\vec{\alpha}:(0,\infty)\to\mathbb{R}^3$

$$\vec{\alpha}(t)=\left(3t^{\frac{11}{2}},\ (t-1)^{\frac{4}{3}},\ |t|\right)$$

A. This is NOT a parametrized curve.

Let $x(t)=3t^{\frac{11}{2}}$, $y(t)=(t-1)^{\frac{4}{3}}$, $z(t)=|t|$.

$x^{(n)}(t)$ $(n\geq 6, n\in\mathbb{N})$ and $z'(t)$ don't exist

at $t=0$ but $0\notin(0,\infty)$, so $x(t)$ and $z(t)$

are differentiable. But $y''(t)=\frac{4}{9}(t-1)^{-\frac{2}{3}}$

does not exist at $t=1\in(0,\infty)$, which makes

$y(t)$ not differentiable. ■

(iv) $\vec{\alpha}:\mathbb{R}\to\mathbb{R}^3$, $\vec{\alpha}(t)=\left(\cos(2t),\ \int_0^t|t|dt,\ 6\right)$

A. This is NOT a parametrized curve.

Let $x(t)=\cos(2t)$, $y(t)=\int_0^t|t|dt$, $z(t)=6$.

$x(t)$ and $z(t)$ are differentiable because compositions

of $\cos t$ and polynomials are differentiable. But,

$y''(t)$ does not exist at $t=0\in\mathbb{R}$, which makes $y(t)$

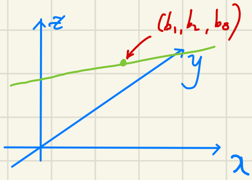
not differentiable. ■

Q2. $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^3$,

$$\vec{\alpha}(t) = (a_1 t^3 + b_1,\ a_2 t^3 + b_2,\ a_3 t^3 + b_3)$$ where

$(a_1, a_2, a_3) \neq \vec{0}$.

(i) Draw the trace of $\vec{\alpha}$



The trace of α is a line parallel to (a_1, a_2, a_3) and (b_1, b_2, b_3) is on the line. Even though trace of $\vec{\alpha}$ is same as trace of $(a_1 t + b_1,\ a_2 t + b_2,\ a_3 t + b_3)$, they are different because their speeds are not the same ■

(ii) $\vec{\alpha}'(t) = (3a_1 t^2,\ 3a_2 t^2 + 3a_3 t^2)$ ■

(iii) $\vec{\alpha}''(t) = (6a_1 t,\ 6a_2 t,\ 6a_3 t)$

$$|\vec{\alpha}''(t)| = \sqrt{(6a_1 t)^2 + (6a_2 t)^2 + (6a_3 t)^2}$$

$$= 6\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot |t|$$ ■

(iv) Speed of $\vec{\alpha}$ $= |\vec{\alpha}'(t)| = \sqrt{(3a_1t^2)^2 + (3a_2t^2)^2 + (3a_3t^2)^2}$

$$= \underline{3\sqrt{a_1^2+a_2^2+a_3^2}\cdot t^2} \ \blacksquare$$

(v) Arc length of $\vec{\alpha}$ from 0

$$s(t) = \int_0^t |\vec{\alpha}'(t)|\,dt = \int_0^t \left|3\sqrt{a_1^2+a_2^2+a_3^2}\cdot t^2\right| dt$$

$$= \underline{\sqrt{a_1^2+a_2^2+a_3^2}\,t^3} \ \blacksquare$$

(vi) $s = \sqrt{a_1^2+a_2^2+a_3^2}\,t^3$, then $t = \left(\dfrac{s}{\sqrt{a_1^2+a_2^2+a_3^2}}\right)^{\frac{1}{3}}$

$$\underline{\vec{\alpha}(s) = \left(\frac{a_1 s}{\sqrt{a_1^2+a_2^2+a_3^2}}+b_1,\ \frac{a_2 s}{\sqrt{a_1^2+a_2^2+a_3^2}}+b_2,\ \frac{a_3 s}{\sqrt{a_1^2+a_2^2+a_3^2}}+b_3\right)} \ \blacksquare$$

(vii) We can find unit tangent vector as below.

$$\vec{t}(s) = \vec{\alpha}'(s) = \left(\frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}, \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}, \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}} \right)$$

(viii) $\vec{\alpha}''(s) = (0, 0, 0)$

$k(s) = |\vec{\alpha}''(s)| = 0.$

This means $\vec{\alpha}$ is a straight line and it does not curve at all.

Curvature tells us how much a trace curved. With this kind of perspective, curvature of a line should be 0 since it does not curve at all. But if we define curvature as $|\alpha''(t)|$, question 2 is the counterexample of the statement above. $|\alpha''(t)|$ is not 0 for all t. Therefore, it is not a great idea that we define curvature as $|\alpha''(t)|$. $\blacksquare$

Q3. $\vec{\alpha}:(0,\infty)\to\mathbb{R}^3,\ \vec{\alpha}(t)=(t,t^2,3)$

(i) $\alpha'(t)=(1,2t,0)$

(ii) Speed of $\alpha'(t)=|\alpha'(t)|=\sqrt{1+4t^2}$

(iii) Since $\alpha'(t)\neq 0$ for all $t\in(0,\infty)$,

$\alpha(t)$ is regular. Thus, we can get the arc length $s(t)$ as below.

$$s(t)=\int_0^t|\alpha'(t)|\,dt=\int_0^t\left|\sqrt{1+4t^2}\right|dt$$

$$=\int_0^t\sqrt{1+4t^2}\,dt\quad(\because\sqrt{1+4t^2}>0)$$

Substitute $t=\frac{1}{2}\tan\theta$ then we have $dt=\frac{1}{2}\sec^2\theta\,d\theta$

$$=\int_0^{\alpha}\frac{1}{2}\sec^3\theta\,d\theta\quad(\alpha=\tan^{-1}(2t))$$

$$=\frac{1}{4}\Big[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big]_0^{\alpha}$$

$$= \frac{1}{4}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\alpha}$$

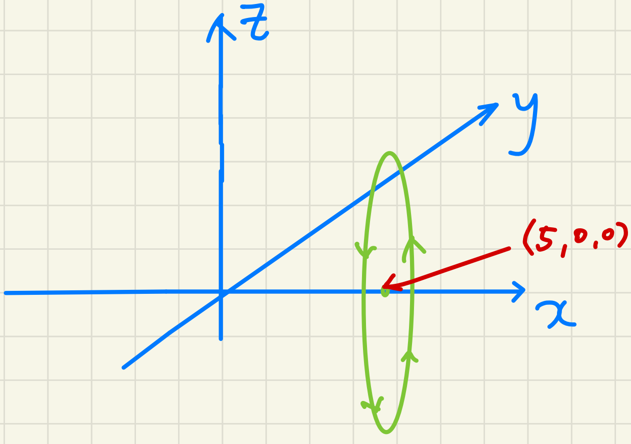
$$= \frac{1}{4}\Big(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|$$

$$- \sec 0\tan 0 - \ln|\sec 0 + \tan 0|\Big)$$

$$= \frac{1}{4}\left(\sqrt{1+4t^2}\cdot 2t + \ln\left|\sqrt{1+4t^2} + 2t\right|\right) \blacksquare$$

Q4. $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^3$, $\vec{\alpha}(t) = (5, 4\cos t, 4\sin t)$

(i)



The trace of $\vec{\alpha}$ is a circle parallel to yz-plane with center $(5,0,0)$ and radius 4 $\blacksquare$.

(ii) $\vec{\alpha}'(t) = (0, -4\sin t, 4\cos t)$

(iii) Speed of $\vec{\alpha}(t) = |\vec{\alpha}'(t)| = \sqrt{0^2 + (-4\sin t)^2 + (4\cos t)^2}$

$= 4$ ■

(iv) $S(t) = \int_0^t |\alpha'(t)|\, dt = \int_0^t 4\, dt = 4t$ ■

(v) We've found that arclength function of $\vec{\alpha}$ is $S = 4t$.

then $\vec{\alpha}(s) = (s, 4\cos(\frac{s}{4}), 4\sin(\frac{s}{4}))$ ■

(vi) Unit tangent vector $\vec{T}(s) = \vec{\alpha}'(s) = (0, -\sin\frac{s}{4}, \cos\frac{s}{4})$

(vii) $\vec{\alpha}''(s) = (0, -\frac{1}{4}\cos\frac{s}{4}, -\frac{1}{4}\sin\frac{s}{4})$, then

$$k(s) = |\vec{\alpha}''(s)| = \sqrt{0^2 + (-\frac{1}{4}\cos\frac{s}{4})^2 + (-\frac{1}{4}\sin\frac{s}{4})^2}$$

$$= \frac{1}{4}$$

We need to use torsion to know this is a plane curve. If so, we could say that $\vec{\alpha}$ is really a circle!

The curvature of $\vec{\alpha}$ is $\frac{1}{4}$ which is constant.

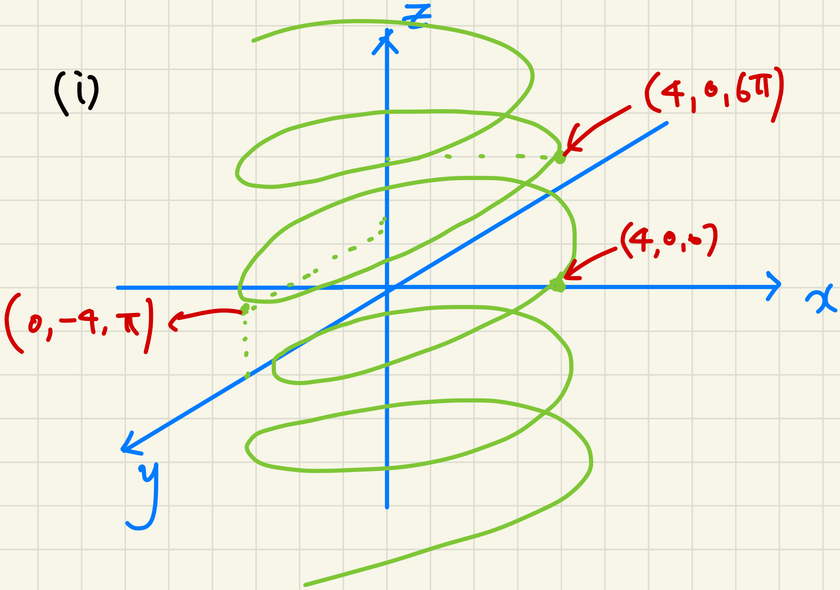
This means that $\vec{\alpha}$ curves exactly the same way everywhere. $\vec{\alpha}$ is a circle

center: $(s, 0, 0)$

radius: 4 ■

Q5. $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^3$, $\vec{\alpha}(t) = (4\cos t, 4\sin t, 3t)$

(i)



The trace of α is helix whose axis is z-axis, and whose diameter is 8 ∎

(ii) $\vec{\alpha}'(t) = (-4\sin t, 4\cos t, 3)$

(iii) Speed of $\vec{\alpha}(t) = |\vec{\alpha}'(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}$

$= 5$

(iv) arclength function $s(t) = \int_0^t |\alpha'(t)|\,dt = \int_0^t 5\,dt$

$= 5t$ $\quad \therefore\ s = 5t$ ∎

(v) We've found that arclength function is $s = 5t$, then $t = \frac{s}{5}$.

$$\vec{\alpha}(s) = \left(4\cos\frac{s}{5},\ 4\sin\frac{s}{5},\ \frac{3}{5}s\right)$$

(vi) Unit tangent vector $\vec{T}(s) = \vec{\alpha}'(s)$

$$= \left(-\frac{4}{5}\sin\frac{s}{5},\ \frac{4}{5}\cos\frac{s}{5},\ \frac{3}{5}\right)$$

(vii) $\vec{\alpha}''(s) = \left(-\frac{4}{25}\cos\frac{s}{5},\ -\frac{4}{25}\sin\frac{s}{5},\ 0\right)$

$$k(s) = |\vec{\alpha}''(s)| = \sqrt{\left(-\frac{4}{25}\cos\frac{s}{5}\right)^2 + \left(-\frac{4}{25}\sin\frac{s}{5}\right)^2 + 0^2}$$

$$= \frac{4}{25}$$

The curvature is $\frac{4}{25}$ which is constant. This means that $\vec{\alpha}$ curves exactly the same way everywhere. We can say that $\vec{\alpha}$ is a helix.