DIFFERENTIAL GEOMETRY FINAL EXAM

Instructions: Do all the questions. You need to provide all the steps in order to get the full credits.

Full mark is 90.

Question 1. Define

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$$

which is a regular surface.
(i) (5 points) For $p = (0, 1, 0) \in S$, find the tangent plane of S at p.
(ii) (1 point) Draw S.

Question 2. For $a, b > 0$, let

$$\overrightarrow{x}(u, v) = (bv, a\cos u, a\sin u)$$

where $(u, v) \in U = \{0 < u < 2\pi, 0 < v < 1\}$.
(i) (4 points) Find the coefficients of the first fundamental form.
(ii) (3 points) Hence, find the area of $\overrightarrow{x}(U)$.
(iii) (3 points) Find the unit normal $N(p)$ at $p = \overrightarrow{x}(u, v)$.
(iv) (5 points) Find the Gauss map $N : S \to \mathbb{S}^2$, and find dN_p the differential of the Gauss map at $p = \overrightarrow{x}(u, v)$.
(v) (5 points) Using (iv), find the principal curvatures. Hence, find the Gaussian curvature and the mean curvature.

Question 3. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$. Let

$$S = \{(u, v, F(u, v)) | (u, v) \in U\}$$

be the graph of F.
(i) (1 point) Write down the parametrization of S.
(ii) (4 points) Find the coefficients of the first fundamental form in terms of F.
(iii) (3 points) Find the unit normal $N(p)$ at $p = \overrightarrow{x}(u, v)$ in terms of F.
(iv) (4 points) Find the coefficients of the second fundamental form in terms of F.
(v) (2 points) Find the Gaussian curvature and the mean curvature in terms of F.
(vi) (2 points) Using (v), find the Gaussian curvature and the mean curvature, and hence find the principal curvatures, when $F(u, v) = au + bv + c$ where a, b, c are constants.
(vii) (3 points) Using (v), find the Gaussian curvature and the mean curvature when $F(u, v) = 4 + u^2 + v^2$.
(viii) (4 points) Find all the umbilical point(s) of S when $F(u, v) = 4 + u^2 + v^2$.

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Question 4. For $a, b, c > 0$, let

$$\overrightarrow{x}(u,v) = (a\sin v\cos u, b\sin v\sin u, c\cos v)$$

where $(u,v) \in U = \{0 < u < 2\pi, 0 < v < \pi\}$.
(i) (4 points) Find the coefficients of the first fundamental form.
(ii) (4 points) Find the unit normal $N(p)$ at $p = \overrightarrow{x}(u,v)$.
(iii) (4 points) Find the coefficients of the second fundamental form.
(iv) (2 point) Find the Gaussian curvature at $p = \overrightarrow{x}(u,v)$. Show that p is an elliptic point for any $p = \overrightarrow{x}(u,v)$.

Question 5. For $a > b > 0$, let

$$\overrightarrow{x}(u,v) = (b\sin v, (a + b\cos v)\cos u, (a + b\cos v)\sin u)$$

where $(u,v) \in U = \{0 < u < 2\pi, 0 < v < 2\pi\}$.
(i) (4 points) Find the coefficients of the first fundamental form.
(ii) (4 points) Hence, find the area of $\overrightarrow{x}(U)$.
(iii) (4 points) Find the unit normal $N(p)$ at $p = \overrightarrow{x}(u,v)$.
(iv) (4 points) Find the coefficients of the second fundamental form.
(v) (3 points) Find the Gaussian curvature and the mean curvature.
(vi) (4 points) Determine when $p = \overrightarrow{x}(u,v)$ is an elliptic, a hyperbolic, or a parabolic point.
(vii) (4 points) Use the Gauss-Bonnet formula to find the Euler-characteristics of $S = \overrightarrow{x}(U)$.