20181288 윤성호 Homework 2.

Q1. (i) $\vec{x}_u = \left(1, \frac{-u}{\sqrt{1-u^2-v^2}}, 0\right)$

$\vec{x}_v = \left(0, \frac{-v}{\sqrt{1-u^2-v^2}}, 1\right)$

Since $p=(0,1,0)$, $u=v=0$.

$T_pS = \text{span}\{\vec{x}_u(0,0), \vec{x}_v(0,0)\}$

$= \text{span}\{(1,0,0), (0,0,1)\}$

$=$ zx-plane ■

(ii) $\vec{x}_u = (-\sin v \sin u, \sin v \cos u, 0)$

$\vec{x}_v = (\cos v \cos u, \cos v \sin u, -\sin v)$

Since $p=(0,1,0)$, $u=v=\frac{\pi}{2}$.

$T_pS = \text{span}\{\vec{x}_u(\frac{\pi}{2}, \frac{\pi}{2}), \vec{x}_v(\frac{\pi}{2}, \frac{\pi}{2})\}$

$= \text{span}\{(-1,0,0), (0,0,-1)\}$

$=$ zx-plane ■

Q2. Since S is a regular surface, there exists a neighborhood $V = \mathbb{R}^3$ of p, and if we set

$U = \{(u,v) \mid 0 < u < 2\pi,\ -\infty < v < \infty\}$, there exists a map $\vec{x} : U \to V \cap S$, since U is an open set.

Thus, we can parametrize the surface as below.

$$\vec{x}(u,v) = (\sqrt{1+v^2}\cos u, \sqrt{1+v^2}\sin u, v)$$

Since $p = (0,1,0)$, $u = \frac{\pi}{2}$, $v = 0$.

$$\vec{x_u} = (-\sqrt{1+v^2}\sin u, \sqrt{1+v^2}\cos u, 0)$$

$$\vec{x_v} = \left(\frac{v}{\sqrt{1+v^2}}\cos u, \frac{v}{\sqrt{1+v^2}}\sin u, 1\right)$$

$$T_pS = \operatorname{span}\left\{\vec{x_u}\left(\tfrac{\pi}{2},0\right), \vec{x_v}\left(\tfrac{\pi}{2},0\right)\right\}$$

$$= \operatorname{span}\{(-1,0,0), (0,0,1)\}$$

$$= zx\text{-plane}$$

Q3. (i) $\vec{x}_u = (-a\sin u, a\cos u, 0)$

$\vec{x}_v = (0, 0, 1)$

$E = \langle \vec{x}_u, \vec{x}_u \rangle = (-a\sin u)^2 + (a\cos u)^2 + 0^2 = a^2$

$F = \langle \vec{x}_u, \vec{x}_v \rangle = (-a\sin u)(0) + (a\cos u)(0) + (0)(1) = 0$

$G = \langle \vec{x}_v, \vec{x}_v \rangle = (0)(0) + (0)(0) + (1)(1) = 1$

(ii) Area of $\vec{x}(U) = \iint_U \sqrt{EG - F^2}\, du\,dv$

$= \int_0^b \int_0^{2\pi} \sqrt{a^2 \cdot 1 - 0}\, du\,dv = \int_0^b \int_0^{2\pi} a\, du\,dv \quad (\because a > 0)$

$= \int_0^b 2\pi a\, dv = 2\pi ab$

Q4. (i) $\vec{x}_u = \left(-(a+b\cos v)\sin u,\ (a+b\cos v)\cos u,\ 0\right)$

$\vec{x}_v = \left(-b\sin v\cos u,\ -b\sin v\sin u,\ b\cos v\right)$

Since $p = (0, a-b, 0)$, $u = \frac{\pi}{2}$, $v = \pi$

$T_pS = \text{span}\left\{\vec{x}_u\left(\frac{\pi}{2},\pi\right), \vec{x}_v\left(\frac{\pi}{2},\pi\right)\right\}$

$= \text{span}\left\{(-a+b, 0, 0), (0, 0, -b)\right\}$

$= zx\text{-plane}$ ■

(ii) $E = \left(-(a+b\cos v)\sin u\right)^2 + \left((a+b\cos v)\cos u\right)^2 + 0^2$

$= (a+b\cos v)^2$

$F = \left(-(a+b\cos v)\sin u\right)\left(-b\sin v\cos u\right)$

$+ \left((a+b\cos v)\cos u\right)\left(-b\sin v\sin u\right) + (0)(b\cos v)$

$= 0$

$G = (-b\sin v\cos u)^2 + (-b\sin v\sin u)^2 + (b\cos v)^2$

$= b^2$

(iii) Area of $\vec{x}(U) = \iint_U \sqrt{EG-F^2}\,du\,dv$

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{(a+b\cos v)^2 b^2 - 0^2}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} (a+b\cos v)b\,du\,dv \quad (\because a>b>0)$$

$$= \int_0^{2\pi} 2\pi b(a+b\cos v)\,dv$$

$$= \left[2\pi b a v + 2\pi b^2 \sin v\right]_0^{2\pi} = 4\pi^2 ab$$

(iv)

$$\vec{x}_u' \wedge \vec{x}_v' = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -(a+b\cos v)\sin u & (a+b\cos v)\cos u & 0 \\ -b\sin v\cos u & -b\sin v\sin u & b\cos v \end{vmatrix}$$

$$= \left(b(a+b\cos v)\cos u\cos v,\ b(a+b\cos v)\sin u\cos v,\ b(a+b\cos v)\sin v\right)$$

$$\|\vec{x}_u' \wedge \vec{x}_v'\| = b(a+b\cos v)$$

Unit normal vector at p: $N(p) = \dfrac{\vec{x}_u' \wedge \vec{x}_v'}{\|\vec{x}_u' \wedge \vec{x}_v'\|}$

$$= (\cos u\cos v,\ \sin u\cos v,\ \sin v)$$

Q5. (i) $\vec{x}_u' = (-a\sin u, a\cos u, 0)$

$\vec{x}_v' = (0, 0, e^v)$

$$\vec{x}_u' \wedge \vec{x}_v' = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & e^v \end{vmatrix}$$

$$= (a\cos u e^v, a\sin u e^v, 0)$$

$\|\vec{x}_u' \wedge \vec{x}_v'\| = ae^v$

Unit normal $N(p)$ at $p = \vec{x}(u,v)$

$$\Rightarrow N(p) = \frac{\vec{x}_u' \wedge \vec{x}_v'}{\|\vec{x}_u' \wedge \vec{x}_v'\|} = (\cos u, \sin u, 0)$$

$=$ projection of $\frac{1}{a}p$ onto xy-plane $\blacksquare$

(ii) Let the surface $\vec{x}(u,v)$ be S.

$$N(p) = (\cos u, \sin u, 0)$$

$$= \text{projection of } \frac{1}{a}p \text{ onto } xy\text{-plane.}$$

Gauss map $N : S \to S^2$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ p \end{array} \to \text{projection of } \frac{1}{a}p \text{ onto } xy\text{-plane.}$$

Consider $\vec{\alpha} : (-\varepsilon, \varepsilon) \to S$ with $\vec{\alpha}(0) = p$

$$\vec{\alpha}(t) = (x(t), y(t), z(t))$$

$$N(\vec{\alpha}(t)) = \left(\frac{1}{a}x(t), \frac{1}{a}y(t), 0\right)$$

$$\frac{d}{dt}N(\vec{\alpha}(t))\Big|_{t=0} = \left(\frac{1}{a}x'(0), \frac{1}{a}y'(0), 0\right)$$

$$\overset{\parallel}{dN_{\vec{\alpha}(0)}(\vec{\alpha}'(0))}$$

$$\overset{\parallel}{dN_p}(\vec{\alpha}'(0))$$

$\therefore dN_p : T_pS \to T_pS$

$$dN_p(\vec{v}) = \text{projection of } \frac{1}{a}\vec{v} \text{ onto } xy\text{-plane} \blacksquare$$