20181288 윤성호

1\. (i) $\alpha(t)$ is not a parametric differentiable curve since
$z(t) = t^{\frac{13}{3}}$ is not differentiable, for $z^{(5)}(0)$ does not exist.

(ii) $\alpha(t)$ is a parametric differentiable curve, for the trigonometric functions, polynomials, and exponential functions are differentiable. Moreover, sum and composition of differentiable functions are differentiable.

(iii) $\alpha(t)$ is not a parametric differentiable curve since
$z(t) = (t-1)^{\frac{4}{3}}$ is not differentiable, for $z''(1)$ does not exist.
$x(t) = \int_0^t |t| dt$ and $y(t) = 3t^2 - 2$ are differentiable in $(0, \infty)$.

2\. (i) tangent vector : $\alpha'(t) = (-3\sin t, 3\sqrt{2}\cos t, -3\sin t)$

(ii) speed $= |\alpha'(t)| = \sqrt{(-3\sin t)^2 + (3\sqrt{2}\cos t)^2 + (-3\sin t)^2} = 3\sqrt{2}$

(iii) $s(t) = \int_0^t |\alpha'(t)| dt = \int_0^t 3\sqrt{2}\, dt = 3\sqrt{2}t$.

(iv) By (iii), $t = \frac{s}{3\sqrt{2}}$ then $\tilde{\alpha}(s) = \left(s + 3\cos\frac{s}{3\sqrt{2}}, 3\sqrt{2}\sin\frac{s}{3\sqrt{2}}, -s + 3\cos\frac{s}{3\sqrt{2}}\right)$

(v) $\vec{t}(s) = \tilde{\alpha}'(s) = \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{3\sqrt{2}}, \cos\frac{s}{3\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin\frac{s}{3\sqrt{2}}\right)$

(vi) $\tilde{\alpha}''(s) = \left(-\frac{1}{6}\cos\frac{s}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\sin\frac{s}{3\sqrt{2}}, -\frac{1}{6}\cos\frac{s}{3\sqrt{2}}\right)$

$k(s) = |\tilde{\alpha}''(s)| = \sqrt{\left(-\frac{1}{6}\cos\frac{s}{3\sqrt{2}}\right)^2 + \left(-\frac{1}{3\sqrt{2}}\sin\frac{s}{3\sqrt{2}}\right)^2 + \left(-\frac{1}{6}\cos\frac{s}{3\sqrt{2}}\right)^2}$

$= \frac{\sqrt{2}}{6}$ $\longrightarrow$ $k(s) > 0$: $\tilde{\alpha}$ really curves and is not a straight line.
$k(s)$ is constant : $\tilde{\alpha}$ curves in the same way everywhere.

(vii) $\vec{n}(s) = \dfrac{\vec{\alpha}''(s)}{k(s)} = \dfrac{6}{\sqrt{2}}\left(-\frac{1}{6}\cos\frac{s}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\sin\frac{s}{3\sqrt{2}}, -\frac{1}{6}\cos\frac{s}{3\sqrt{2}}\right)$

$$= \left(-\frac{1}{\sqrt{2}}\cos\frac{s}{3\sqrt{2}}, -\sin\frac{s}{3\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos\frac{s}{3\sqrt{2}}\right)$$

(viii) $\vec{b}(s) = \vec{t}(s) \wedge \vec{n}(s)$

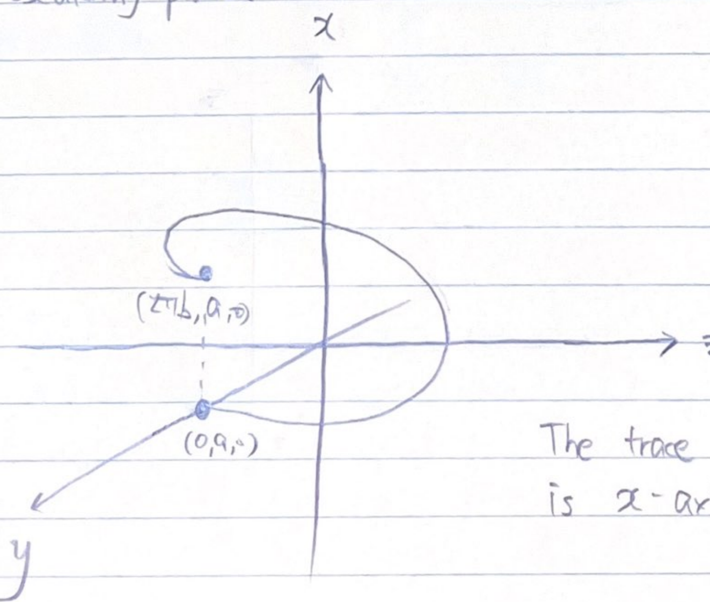
$$= \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ -\frac{1}{\sqrt{2}}\sin\frac{s}{3\sqrt{2}} & \cos\frac{s}{3\sqrt{2}} & -\frac{1}{\sqrt{2}}\sin\frac{s}{3\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\cos\frac{s}{3\sqrt{2}} & -\sin\frac{s}{3\sqrt{2}} & -\frac{1}{\sqrt{2}}\cos\frac{s}{3\sqrt{2}} \end{vmatrix}$$

$$= \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

(ix) $\vec{b}'(s) = \tau(s)\vec{n}(s) \rightarrow (0,0,0) = \tau(s)\left(-\frac{1}{\sqrt{2}}\cos\frac{s}{3\sqrt{2}}, -\sin\frac{s}{3\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos\frac{s}{3\sqrt{2}}\right)$

Thus, $\tau(s) = 0$. This mean that $\vec{\alpha}$ does not twist away from the osculating plane.

3. (i)



The trace is a helix, whose axis is x-axis.

(viii) $\vec{n}(s) = \frac{\vec{\alpha}''(s)}{k(s)} = \frac{a^2+b^2}{a}\left(0, -\frac{a}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}, -\frac{a}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}\right)$

$= \left(0, -\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}\right)$

(ix) $\vec{b}(s) = \vec{t}(s) \wedge \vec{n}(s)$

$$= \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \frac{b}{\sqrt{a^2+b^2}} & -\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}} \\ 0 & -\cos\frac{s}{\sqrt{a^2+b^2}} & -\sin\frac{s}{\sqrt{a^2+b^2}} \end{vmatrix}$$

$$= \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, -\frac{b}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}\right).$$

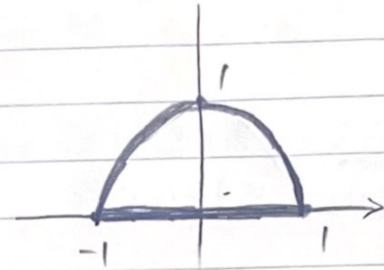
(x) $\vec{b}'(s) = \tau(s)\vec{n}(s)$

$\Rightarrow \left(0, \frac{b}{a^2+b^2}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{a^2+b^2}\sin\frac{s}{\sqrt{a^2+b^2}}\right) = \tau(s)\cdot\left(0, -\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}\right)$

Thus, $\tau(s) = -\frac{b}{a^2+b^2}$

Since $\tau(s) \neq 0$ ($\because b \neq 0$), $\vec{\alpha}$ is not a plane curve and twists away from the osculating plane.

4. (i)



The trace of α is a upper semicircle centered at $(0,0)$ and radius 1 and bounded by a straight line.

(ii) For $t \in [0,2]$, the curve is a straight line, whose length is 2 and for $t \in [2, 2+\pi]$, the curve is a half circle whose radius is 1, so the length of semisphere is π.

Thus the length of α is $2+\pi$.

(iii) $\int_0^2 (-1+t) \cdot (0)' \, dt = 0$. .... ①

$$\int_2^{2+\pi} \cos(t-2) \cdot (\sin(t-2))' \, dt = \int_2^{2+\pi} \cos^2(t-2) \, dt = \int_0^{\pi} \cos^2 t \, dt$$

$$= \int_0^{\pi} \frac{1}{2}(1+\cos 2t) \, dt = \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{\pi} = \frac{\pi}{2} \quad \cdots ②$$

Thus, the area is ① + ② $= 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

(iv) Isoperimetric inequality is stated below.

Let C be a simple closed curve with length ℓ, and let A be the area of the interior of C. Then $A \leq \frac{\ell^2}{4\pi}$, and equality holds if and only if C is a circle.

For length $2+\pi$, the largest area with this length would be $\frac{(2+\pi)^2}{4\pi}$.

The area of $\alpha = \frac{\pi}{2} = \frac{2\pi^2}{4\pi} = \frac{\pi^2+\pi^2}{4\pi} < \frac{\pi^2+4\pi}{4\pi} < \frac{\pi^2+4\pi+4}{4\pi} = \frac{(2+\pi)^2}{4\pi}$, so true.

5. Pick $p \in (x, -\sqrt{1-x^2-y^2}, y)$. Then $V = \mathbb{R}^3$ which is a neighborhood of p.

Let $\vec{x}: U \subseteq \mathbb{R}^2 \to S \cap V = S$

$\vec{x}(u,v) = (u, -\sqrt{1-u^2-v^2}, v)$

(i) $\vec{x}$ is differentiable, since $x(u,v) = u$, $y(u,v) = -\sqrt{1-u^2-v^2}$, $z(u,v) = v$ are differentiable on $u^2+v^2 < 1$.

(ii) $\vec{x}$ is a homeomorphism, since $\vec{x}: U \to S \cap V = S$

$\vec{x}(u,v) = (u, -\sqrt{1-u^2-v^2}, v)$ is continuous.

$\vec{x}^{-1}(u, -\sqrt{1-u^2-v^2}, v) = (u,v)$

for any $(u, -\sqrt{1-u^2-v^2}, v) \in S \cap V$ is continuous since it is a projection.

(iii) $\begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{u}{\sqrt{1-u^2-v^2}} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{v}{\sqrt{1-u^2-v^2}} \\ 1 \end{bmatrix}$

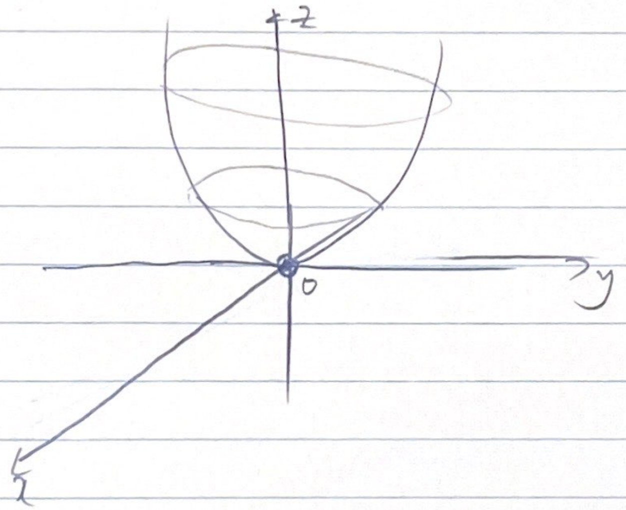
Two vectors are linearly independent since for $t, s \in \mathbb{R}$.

$$t \begin{bmatrix} 1 \\ \frac{u}{\sqrt{1-u^2-v^2}} \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ \frac{v}{\sqrt{1-u^2-v^2}} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow t = s = 0.$$

Thus, S is a regular surface.

6. Let $f(x,y)=x^4+y^4$ then $S=\{(x,y,z)\in\mathbb{R}^3 \mid z=x^4+y^4\}$ would be the graph of f since f is differentiable because it is polynomial of degree 4.

Thus, S is a regular surface.



7. $S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1\}=F^{-1}(1)$
where $F:\mathbb{R}^3\to\mathbb{R}$, $F(x,y,z)=x^2+y^2$ which is differentiable since it is a polynomial of degree 2.

1 is a regular value of F because for any $p=(x,y,z)\in F^{-1}(1)$
$\to F(x,y,z)=1$
$\to x^2+y^2=1 \to$ either x or y is not zero.

$$\left[\frac{\partial f}{\partial x}(p),\ \frac{\partial f}{\partial y}(p),\ \frac{\partial f}{\partial z}(p)\right]=[2x,2y,0]\neq(0,0,0)$$

$S=F^{-1}(1)$ is a regular surface.

The surface is a cylinder.