20181288 윤성호 추상대수학 중간고사

1. (a) Let 0_R be the additive identity of R.

$$a \oplus 0_R = a \Leftrightarrow a + 0_R - 1 = a \Leftrightarrow (-a) + a + 0_R - 1 = (-a) + a.$$

$$\Leftrightarrow 0_R - 1 = 0 \Leftrightarrow 0_R = 1$$

$$\boxed{\therefore\ 0_R = 1}$$

(b) Yes. $a \odot 1_R = a1_R - a - 1_R + 2 = a$.

$\rightarrow a1_R - 2a - 1_R + 2 = 0$

$\rightarrow 1_R(a-1) - 2(a-1) = 0$

$\rightarrow (1_R - 2)(a-1) = 0$

$\rightarrow 1_R = 2$ or $a = 1$ (since $\mathbb{Z}$ is an integral domain.)

$\rightarrow 1_R = 2$ (since a is a nonzero element)

(c) By (a), $0_R = 1$. Then for $a, b \in R$, suppose $a \odot b = 1$ then $ab - a - b + 2 = 1 \Rightarrow (a-1)(b-1) = 0$ In $\mathbb{Z}$ forcing $a = 1$ or $b = 1$. Since $\mathbb{Z}$ is an integral domain. Thus R is an integral domain.

2. $\boxed{\text{(a) None.}}$

$\boxed{\text{(b) } 2, 3, 4, 6.}$

$\boxed{\text{(d) } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ s.t } ad - bc = 0 \ (a, b, c, d \in \mathbb{R}, \text{ not all of them are zeros})}$

$\boxed{\text{(c) } (0,1), (0,2), (0,3), (1,0), (2,0), (1,2), (2,2)}$

$\boxed{\text{(e) None}}$

3. (a) Let $a, b, c \in F$ s.t $c = a - b$. and let $f(a) = f(b)$.

Since f is a nonzero homomorphism,

$f(c) = f(a-b) = f(a) - f(b) = 0$ → this means that $c = 0$.

Therefore, $a = b$ ∎

(b) There is none.

(c)

4. (a) Consider monic polynomials of degree 2 that are reducible in $\mathbb{Z}_p[x]$.

$(x+a)(x+b)$ would be the form.

(i) $a=b$.

There are p situations.

(ii) $a \neq b$.

There are $_pC_2 = \frac{p(p-1)}{2}$ situations.

By (i), (ii) there are $p + \frac{p(p-1)}{2} = \frac{p(p+1)}{2}$ monic reducible polynomials of degree 2.

The number of monic polynomials of degree 2 is p^2 since we can think that the form would be $x^2 + cx + d$.

Thus, there are $p^2 - \frac{p(p+1)}{2} = \frac{p(p-1)}{2}$ monic irreducible polynomials of degree 2 ∎

(b) Let $f(x) = x^7 - x$ then we can see that.

$f(0) = 0$ $\quad f(3) = 3^7 - 3 = 0$ $\quad f(6) = 6^7 - 6 = 0$.

$f(1) = 1^7 - 1 = 0$ $\quad f(4) = 4^7 - 4 = 0$

$f(2) = 2^7 - 2 = 0$ $\quad f(5) = 5^7 - 5 = 0$

By factor theorem $x^7 - x = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$ ∎

5. Lemma. Every irreducible polynomial $p(x) \in \mathbb{R}[x]$ is of degree 1 or 2.

pf) Factorize $p(x)$ over $\mathbb{C}$. Let $\alpha \in \mathbb{C}$ be a root of $p(x)$
Then $\bar{\alpha}$ is also a root of $p(x)$ since it is the only way that the coefficient of $p(x)$ could all be in $\mathbb{R}$.

If $\alpha = \bar{\alpha}$, then $\alpha \in \mathbb{R}$ and $x - \alpha \mid p(x)$. Since $x - \alpha \in \mathbb{R}[x]$, $p(x)$ is an associate of $x - \alpha$.

If $\alpha \neq \bar{\alpha}$, then $(x-\alpha)(x-\bar{\alpha}) \mid p(x)$. Since $x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} \in \mathbb{R}[x]$, $p(x)$ is an associate of $(x-\alpha)(x-\bar{\alpha})$.

By Lemma, we can say as below.

① $ax^2 + bx + c$ s.t $a, b, c \in \mathbb{R}$ and $b^2 - 4ac < 0$ when $a \neq 0$.
② $dx + e$ s.t $d, e \in \mathbb{R}$ when $d \neq 0$.

① and ② are irreducible polynomials. ■

6. (a) $U_{27} = \{a \mid (n,a)=1,\ 0 \le a \le n\}$

$= \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26\}$

Thus $|U_{27}| = 18$ ▪

(b) By (a), Yes. We can see that $8^6 = 1$.

Thus the order of 8 is 6 ▪

7. (a) $aba^{-1} = b^2 \rightarrow ab = b^2 a$

(b) By (a), $a^5 b = b^{32} a^5$ and $a^5 = e$ (e is the identity).

$\rightarrow b = b^{32} \rightarrow e = b^{31}$ thus $|b| = 31$ ▪