7) Bonus

Let $\vec{v}_1 = \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ be the position of dart 1

$\vec{v}_2 = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$ ________ 2

where X_1, Y_1, X_2, Y_2 are iids of $U([-1, 1])$

·) <u>Find the pdf of $X_1 - X_2$:</u>

We have $f_{X_1-X_2}(t) \overset{\text{Inde}}{=} \int_{\mathbb{R}} f_{X_1}(s+t)\, f_{X_2}(s)\, ds$

$= \int_{\mathbb{R}} \frac{1}{2} \mathbb{1}_{[-1,1]}(s+t) \cdot \frac{1}{2} \mathbb{1}_{[-1,1]}(s)\, ds$

We have $\begin{cases} -1 \leq s \leq 1 \\ -1 \leq s+t \leq 1 \end{cases} \Rightarrow \begin{cases} -2 \leq t \leq 2 \\ -1 \leq s \leq 1 \\ -1-t \leq s \leq 1-t \end{cases}$

- If $-2 \leq t \leq 0$, then $\begin{cases} -1 \leq -1-t \leq 1 \\ 1 \leq 1-t \leq 3 \end{cases}$

$\Rightarrow -1-t \leq s \leq 1$

$\Rightarrow f_{X_1-X_2}(t) = \frac{1}{4} \int_{-1-t}^{1} ds = \frac{1}{4} \left(1 - (-1-t)\right) = \frac{1}{4}(2+t)$

- If $0 \leq t \leq 2$, then $\begin{cases} -3 \leq -1-t \leq -1 \\ -1 \leq 1-t \leq 1 \end{cases}$

$\Rightarrow -1 \leq s \leq 1-t$

$$\Rightarrow f_{X_1 - X_2}(t) = \frac{1}{4} \int_{-1}^{1-t} ds = \frac{1}{4}(1-t+1) = \frac{1}{4}(2-t)$$

Therefore, $f_{X_1-X_2}(t) = \frac{1}{4}(2 - |t|)$

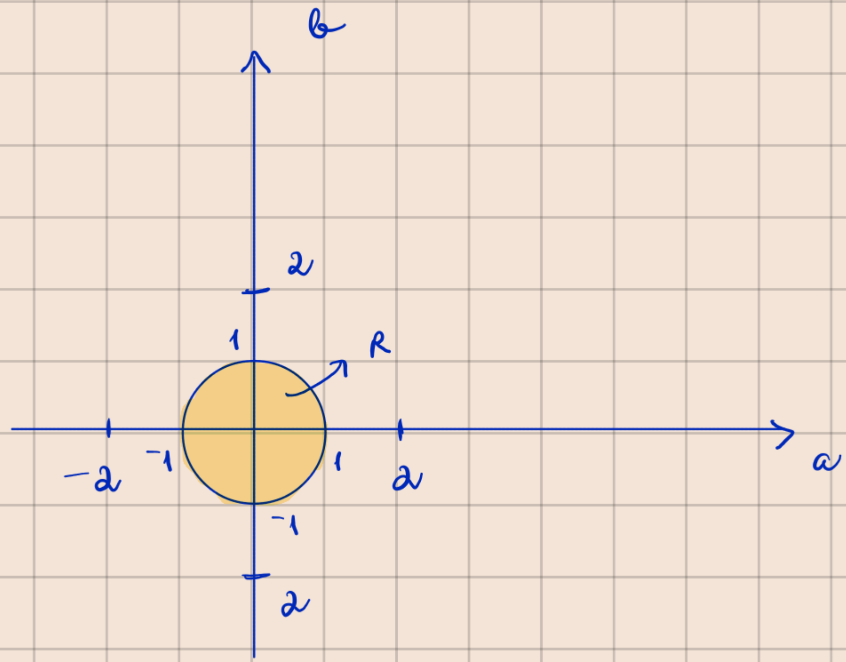
Similarly, we have $f_{Y_1-Y_2}(t) = \frac{1}{4}(2-|t|)$

•) Now, we want to find $P(d<1)$ where

$$d = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$$

Let $X_1 - X_2 = a$, $Y_1 - Y_2 = b$

We know that a and b are iids



$$P(d<1) \overset{\text{Inde}}{=} \iint_R f_{X_1-X_2}(a) \cdot f_{Y_1-Y_2}(b)\, db\, da$$

change from $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$:

$$P(d<1) = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} f_{X_1-X_2}(r\cos\theta)\, f_{Y_1-Y_2}(r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{1}{4^2}\left(2-|r\cos\theta|\right)\left(2-|r\sin\theta|\right) r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{1}{4^2}\left(2-r\cos\theta\right)\left(2-r\sin\theta\right) r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \left(\frac{1}{4} - \frac{1}{8} r(\sin\theta+\cos\theta) + \frac{r^2}{16}\sin\theta\cos\theta\right) r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \left(\frac{r^2}{8} - (\sin\theta+\cos\theta)\cdot\frac{r^3}{24} + \sin\theta\cos\theta\,\frac{r^4}{64}\right)\Big|_0^1 d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \left(\frac{1}{8} - \frac{1}{24}(\sin\theta+\cos\theta) + \frac{1}{64}\sin\theta\cos\theta\right) d\theta$$

$$= 4\left(\frac{1}{8}\theta\Big|_0^{\pi/2} - \frac{1}{24}\left(\sin\theta-\cos\theta\right)\Big|_0^{\pi/2} - \frac{1}{256}\cos 2\theta\Big|_0^{\pi/2}\right)$$

$$= 4\left(\frac{\pi}{16} - \frac{1}{24}\cdot 2 + \frac{1}{256}\cdot 2\right)$$

$$= 4\left(\frac{\pi}{16} - \frac{1}{12} + \frac{1}{128}\right) \approx 0{,}483$$