2) We have the condition for least squares :

$$\| \vec{Y} - \hat{Y} \|^2 \quad \min$$

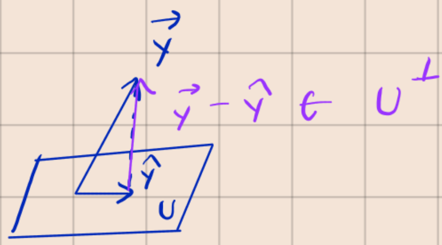
$$\Rightarrow \quad SSR = \| \vec{Y} - \hat{Y} \|^2 \quad \min \quad (1)$$

where $\hat{Y} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ & \vdots & \\ 1 & x_n & x_n^2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = A \cdot \hat{\beta}$

Let $U = \text{Span}\left(\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} \right)$, we have $\hat{Y} \in U$

From (1) $\Rightarrow \hat{Y} = \text{proj}_U \vec{Y}$

$$\Rightarrow \vec{Y} - \hat{Y} = \text{proj}_{U^\perp} \vec{Y}$$



From lecture, we have $\hat{\beta} = (A^T A)^{-1} A^T \vec{Y}$ ($A^T A$ is invertible when $x_i = i \ \forall i$)

$$\Rightarrow A\hat{\beta} = \hat{Y} = \underbrace{A}_{3 \times n}\underbrace{(A^T A)^{-1}}_{3 \times 3}\underbrace{A^T}_{n \times 3} \vec{Y}$$

$$\Rightarrow \vec{Y} - \hat{Y} = \left(I_n - A (A^T A)^{-1} A^T \right) \vec{Y}$$

Therefore, we have $\text{proj}_{U^\perp} = I_n - A (A^T A)^{-1} A^T = B_1$

and $\text{proj}_U = A (A^T A)^{-1} A^T = B_2$

•) Consider $\left(A(A^TA)^{-1}A^T\right)^T = A\,(A^TA)^{-T}A^T \quad (2)$

We have $(A^TA)^T = A^TA \Rightarrow A^TA$ is symmetric

$\Rightarrow \exists$ an orthogonal matrix P and a diagonal matrix D s.t

$$PDP^T = A^TA$$

From (2) $\Rightarrow \left(A(A^TA)^{-1}A^T\right)^{-1} = A\,.\,(PDP^T)^{-T}A^T$

$$= A\,(PD^{-1}P^T)^T A^T = A\,(PD^{-1}P^T)\,A^T$$

$$= A\,(A^TA)^{-1}A^T$$

Therefore B_1 and B_2 are symmetric

$$\|B_1\vec{Y}\|^2 = (B_1\vec{Y})^T(B_1\vec{Y}) = \vec{Y}^T B_1^T B_1 Y \quad (3)$$

•) Let $\vec{Y} \sim \mathcal{N}(\vec{\mu}_Y, \Sigma_Y)$

where $\vec{\mu}_Y = \begin{pmatrix} \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 \\ \vdots \\ \beta_0 + \beta_1 x_n + \beta_2 x_n^2 \end{pmatrix}$

$$\Sigma_Y = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\Rightarrow \vec{Z} = \left(\sqrt{\Sigma_Y}\right)^{-1}(Y - \vec{\mu}_Y) \sim \mathcal{N}(0, 1)$

$$\Rightarrow \vec{Y} = I_n \cdot \vec{Z} + \vec{\mu}_Y = \vec{Z} + \vec{\mu}_Y$$

From (3) $\Rightarrow$

$$SSR = (B_1 \vec{Y})^T B_1 \vec{Y}$$

$$= \vec{Y}^T B_1^T B_1 \vec{Y} = (\vec{Z} + \vec{\mu}_Y)^T B_1^T B_1 (\vec{Z} + \vec{\mu}_Y)$$

Because B_1 is symmetric and is a projection matrix,

$$B_1^T B_1 = B_1^2 = B_1$$

$$\Rightarrow SSR = (\vec{Z} + \vec{\mu}_Y)^T B_1 (\vec{Z} + \vec{\mu}_Y) = \vec{Z}^T B_1 \vec{Z} + \vec{Z}^T B_1 \vec{\mu}_Y$$
$$+ \vec{\mu}_Y^T B_1 \vec{Z} + \vec{\mu}_Y^T B_1 \vec{\mu}_Y$$

We have $B_1 = \text{proj}_{U^\perp}$ and $\vec{\mu}_Y \in U$

$$\Rightarrow B_1 \vec{\mu}_Y = 0 \quad \text{and} \quad \vec{\mu}_Y^T B_1 \vec{v} = \langle \vec{\mu}_Y, B_1 \vec{v} \rangle = 0$$
$\forall v \in \mathbb{R}^n$

Then $SSR = \vec{Z}^T B_1 \vec{Z}$

We have $\text{rank}(B_1) + \text{rank}(B_2) = \dim U + \dim U^\perp = n$

B_1, B_2 are symmetric and $B_1 + B_2 = I_n$

$\vec{Z} \sim \mathcal{N}(0, I_n)$

By Cochran's Theorem, we have $SSR \sim \chi^2(n-3)$