6) Bonus :

- We have $S_N = X_1 + \dots + X_n$ is the time of the n-th crash

$\forall i, j \in \mathbb{N}$ s.t $i < j$, we have $S_i < S_j$

Therefore $\mathbb{P}(N_{10} = k) = \mathbb{P}(k = \max\{n : S_n < 10\})$

$= \mathbb{P}(S_k < 10 \text{ and } S_k + X_{k+1} \geq 10) = \mathbb{P}(S_k = u < 10, X_{k+1} = v \geq 10 - u)$

- Find the characteristic function of T :

$$M_T(z) = \mathbb{E}(\exp(zT)) = \int_{\mathbb{R}} \exp(zt) f_T(t)\, dt$$

$$= \int_0^\infty \exp(zt) \cdot \lambda e^{-\lambda t}\, dt = \lambda \int_0^\infty \exp((z-\lambda)t)\, dt$$

$$= \frac{\lambda}{z-\lambda} \cdot \lim_{a \to \infty} \left(e^{(z-\lambda)t} \right)\Big|_0^a$$

If $z < \lambda$, then $M_T(z) = \dfrac{\lambda}{z-\lambda} \cdot (0 - 1) = \dfrac{\lambda}{\lambda - z}$

We can conclude that $\varphi_T(\tau) = M_T(i\tau) = \dfrac{\lambda}{\lambda - i\tau}$

$(\tau \in \mathbb{R})$

•) We have $\varphi_{S_n}(\tau) = \mathbb{E}\left(\exp(i\tau S_n)\right) = \mathbb{E}\left(\exp(i\tau(X_1 + \dots + X_n))\right) = \mathbb{E}\left(\prod_{i=1}^{n} \exp(i\tau X_i)\right)$

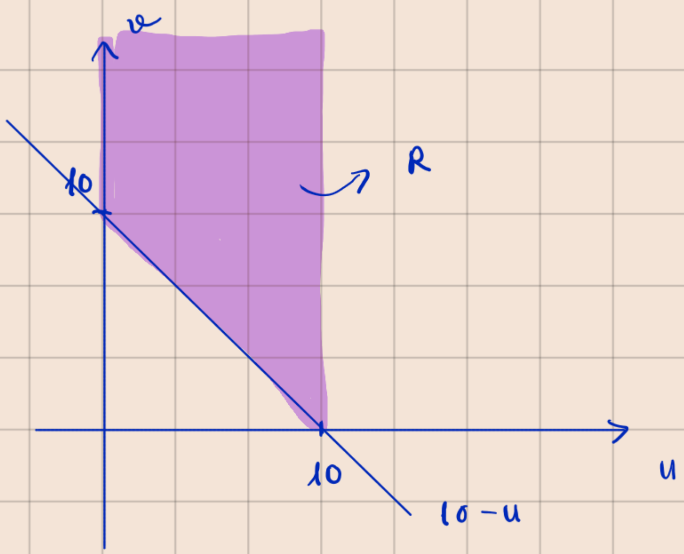
Because X_i's are iid of $Exp(\lambda)$, $\varphi_{S_n}(\tau) = \prod_{i=1}^{n} \mathbb{E}(\exp i\tau X_i)$

$$= \prod_{i=1}^{n} \varphi_{X_i}(\tau) = \prod_{i=1}^{n}\left(\frac{\lambda}{\lambda - i\tau}\right) = \frac{\lambda^n}{(\lambda - i\tau)^n}$$

We see that φ_{S_n} is the same of $\varphi_{Gamma(n,\lambda)}$

$\Rightarrow$ $S_n \sim Gamma(n, \lambda)$ (Fourier Black box)

•) We now evaluate $\mathbb{P}(N_{10} = k) = \mathbb{P}(S_k = u < 10 \text{ and } X_{k+1} = v \geq 10 - u)$



Because X_{k+1} and S_k are inde, $\mathbb{P}(N_{10} = k) = \iint_R f_{S_k}(u) \cdot f_{X_{k+1}}(v) \, dv \, du$

$$= \int_0^{10} \int_{10-u}^{\infty} \frac{\lambda^k}{\Gamma(k)} u^{k-1} e^{-\lambda u} \cdot \lambda e^{-\lambda v} \, dv \, du$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^{10} u^{k-1} \cdot e^{-\lambda u} \left(\int_{10-u}^{\infty} e^{-\lambda v} \, d(\lambda v) \right) du$$

$$= \frac{-\lambda^k}{\Gamma(k)} \int_0^{10} u^{k-1} e^{-\lambda u} \cdot \left(\lim_{a \to \infty} e^{-\lambda v} \Big|_{10-u}^{a} \right) du$$

$$= \frac{-\lambda^k}{\Gamma(k)} \int_0^{10} u^{k-1} \cdot e^{-\lambda u} \cdot \left(0 - e^{\lambda(u-10)} \right) du$$

$$= \frac{\lambda^k}{\underset{\in \mathbb{N}}{\Gamma(k)}} \int_0^{10} u^{k-1} \cdot e^{-10\lambda} \, du = \frac{\lambda^k \cdot e^{-10\lambda}}{\Gamma(k)} \cdot \frac{u^k}{k} \Big|_0^{10}$$

$$= \frac{\lambda^k \cdot e^{-10\lambda}}{(k-1)!} \cdot \frac{10^k}{k} = \frac{(10\lambda)^k \cdot e^{-10\lambda}}{k!}$$

•) $\mathbb{E}(N_{10}) = \sum_{k=0}^{\infty} P(N_{10} = k) \cdot k = \sum_{k=0}^{\infty} \frac{(10\lambda)^k \cdot e^{-10\lambda}}{(k-1)!}$

$$= 10\lambda \cdot e^{-10\lambda} \cdot \sum_{k=0}^{\infty} \frac{(10\lambda)^{k-1}}{(k-1)!} = 10\lambda \cdot e^{-10\lambda} \cdot e^{10\lambda} = 10\lambda$$

$$= \frac{10}{3}$$