Optimizing Coupon Strike for an Autocallable Note with Local Volatility Model

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Introduction

Autocallable structured notes are popular financial instruments that offer enhanced coupons conditional on the performance of underlying assets, with early redemption features triggered by barrier events. This paper addresses the optimization of the coupon strike for an autocallable note tied to a basket of equity indices (Nikkei 225, S&P 500, HSI) using a local volatility model. We develop a Monte Carlo simulation engine that incorporates:

- A Local Volatility surface fitted via Dupire's equation and linear interpolation
- Correlated asset paths derived from historical returns
- Joint modeling of knock-in and knock-out events tied to the laggard index

Our primary objective is to identify the coupon strike that prices the note at 98% of its issue price. We implement a bisection algorithm and conduct sensitivity analyses on interest rates.

Contents

1 Note Structure

- Trade Date: 11 December, 2023
- Maturity Date: 2 years after the trade date
- Issue Price: 100% of note denomination
- Note Denomination: USD 10,000
- Underlying Indicies: Nikkei 225, S&P 500, HSI

1.1 Variable Interest

- At each semi-annual observation date, the note pays a maximum interest of 2% p.a. if the reference price is equal to or more than the Coupon Strike;
- Otherwise, a minimum interest of 0.01% p.a. will be paid
- Reference Price: Close price relevant to the initial spot of the laggard index
- Laggard Index: The index with the lowest value of S_n/S_0 on the observation date

1.2 Knock-Out Event

- A knock-out event is triggered if the closing price of the laggard index on the observation date is equal to or more than the knock-out price
- Knock-out Price: 110% of initial spot
- Knock-out Redemption: 100%$\times$ Note denomination, then the note expires (variable interest will still be paid)

1.3 Final Redemption

- A knock-in event is triggered if the closing price of the laggard index is equal to or less than the knock-in price any time
- Knock-out Price: 50% of initial spot
- If a knock-in has not occurred, each note is redeemed at the denomination
- If a knock-in has occurred, each note is redeemed at denomination $\times$ min(100%, Close price relevant to the initial spot of the laggard index)

The main objective is to identify the coupon rate that ensures the note's price is approximately 98% of the issue price. We will also perform a sensitivity analysis on various interest rates.

2 Data

Options data and risk-free rate information will be collected until November 11, 2023.

2.0.1 Option Data

Option data of SPX, HSI and NKY are fetched by OMON function available on the Bloomberg terminal. We removed option data entries that did not contain a record of the Last px (last price), ensuring that our analysis focused only on relevant and complete information.

2.0.2 Risk-free Rate

We collected Overnight Index Swap (OIS) rate data available on the Bloomberg terminal. Note that the OIS rate provided was assumed to be compounded on a daily basis; we converted it into a continuously compounded rate for the modelling of the volatility surface. Specifically, the OIS rates for Hong Kong and Japan were compounded on a 365-day basis, while the US OIS rates followed a 360-day convention. The transformation follows:

$$r_{continuous} = n * \ln\left(1 + \frac{r_{daily}}{n}\right)$$

2.1 Data Processing

2.1.1 At-The-Money Forward

To find K_{ATMF} we first locate the tightest interval $[K_1, K_2]$ such that

$$(K_1) > P(K_1) \quad \text{and} \quad C(K_2) < P(K_2).$$

Using put–call parity with continuous dividend yield q, $C(K) - P(K) = S\,e^{-qT} - K\,e^{-rT}$, and setting $C = P$ gives

$$K = S\,e^{(r-q)T},$$

which defines K_{ATMF}. We then approximate the first derivatives at K_1 by finite differences:

$$\frac{C(K_{\text{ATMF}}) - C(K_1)}{K_{\text{ATMF}} - K_1} \approx \left.\frac{\partial C}{\partial K}\right|_{K_1} \approx -N\big(d_2\,|_{K_1}\big), \tag{1}$$

$$\frac{P(K_{\text{ATMF}}) - P(K_1)}{K_{\text{ATMF}} - K_1} \approx \left.\frac{\partial P}{\partial K}\right|_{K_1} \approx N\big(d_2\,|_{K_1}\big), \tag{2}$$

$$\frac{C(K_{\text{ATMF}}) - C(K_1)}{K_{\text{ATMF}} - K_1} \approx \left.\frac{\partial C}{\partial K}\right|_{K_1} \approx \frac{C(K_2) - C(K_1)}{K_2 - K_1}. \tag{3}$$

From (1) and (2) we get

$$C\big(K_{\text{ATMF}}\big) \approx \frac{C(K_1) + P(K_1)}{2}.$$

Substituting this into (3) yields

$$K_{\text{ATMF}} \;\approx\; \frac{C(K_{\text{ATMF}}) - C(K_1)}{C(K_2) - C(K_1)}\,(K_2 - K_1) \;+\; K_1.$$

* Note that we assume this approximation is valid for the tightest bound on K_{ATMF}.

2.1.2 Implied Volatility

We implemented the bisection algorithm to estimate the implied volatility. See Appendix A for more details.

2.1.3 Risk-Free Rate Curve

The discrete OIS interest rate is interpolated using a cubic spline interpolation to generate a continuous OIS curve. See Appendix B for more details.

2.1.4 Implied Dividend Yield

Using the at-the-money forward relation $f_t = S_0 \, e^{(r-q)\,T}$, the implied dividend yield is estimated by

$$q_{\text{implied}} \approx \max\Bigl(0,\ r \;-\; \frac{\ln\bigl(K/S_0\bigr)}{T}\Bigr).$$

2.1.5 Local Volatility

Following the Dupire equation, the local volatility at strike K and maturity T is given by

$$\sigma^2_{\text{local}}(K,T) \;=\; \frac{\dfrac{\partial C}{\partial T}}{\frac{1}{2}\,K^2\,\dfrac{\partial^2 C}{\partial K^2}},$$

where $C = C(K,T)$ is the option price.

3 Volatility Surface

3.1 Fitting Volatility Curves

We fitted the implied and local volatility curves across forward moneyness using the following curve function:

$$\sigma_x^2 = \sigma_{ATMF}^2 + \delta \left(\frac{tanh(\kappa x)}{\kappa} \right) + \frac{\gamma}{2} \left(\frac{tanh(\kappa x)}{\kappa} \right)^2$$

, where x denotes the log-moneyness, δ denotes the skew, γ denotes the convexity, κ denotes the celerity.
After fitting the curves, we carefully examined each one and removed those that displayed an abnormal concave shape. This precaution was taken to prevent potential errors in subsequent curve fittings over time.

3.2 Fitting Volatility Surface

Next, we compared methodologies for interpolating discrete volatility curves into a continuous volatility surface.

3.2.1 Cubic Spline Interpolation

Firstly, we employed cubic spline interpolation, which provides a smoother surface due to its second-order differentiability. However, the polynomial nature of this method occasionally results in negative estimated variance ($\sigma^2_{estimated}$), particularly under extreme forward moneyness conditions. To ensure the robustness of our analysis, we aimed to minimize the occurrence of negative $\sigma^2_{estimated}$ values.

3.2.2 Linear Interpolation

We found that using linear interpolation to fit the local volatility surface was a preferable choice, especially when our intention was to project movements in indices. This approach allowed us to achieve a smoother and more reliable local volatility surface, enhancing the accuracy of our index movement projections.

3.2.3 Gatheral's Mapping

In addition to determining local volatility using the Dupire equation, we also explored an alternative approach by employing Gatheral's mapping:

$$\sigma^2_{local} = \frac{\frac{\partial \omega}{\partial T}}{1 - \frac{y}{\omega}\frac{\partial \omega}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{\omega} + \frac{y^2}{\omega^2}\right)\left(\frac{\partial \omega}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 \omega}{\partial y^2}}$$

,where $\omega = \sigma^2_{BS}T$ denotes the total volatility, and $y = \ln\left(\frac{K}{K_T}\right)$ denotes the log-forward moneyness. The fitted surfaces resulting from this mapping process can be found in the appendix. We observed that the fitted surfaces produced through this method were notably unusual. This discrepancy could be attributed to the fact that we utilized fitted implied volatility surfaces to derive the local volatility rather than fitting the local volatility surfaces directly. Consequently, this additional layer of work may amplify the error stemming from the data sources and estimation procedures.

3.2.4 Conclusion

Therefore, we have determined that the most suitable approach is to utilize Dupire's Equation in conjunction with linear interpolation since it provides the lowest likelihood of returning a negative $\sigma^2_{estimated}$.

3.2.5 Limitaions

There are specific limitations and potential sources of error to consider:

1. First, the last price of the options obtained from Bloomberg may not accurately reflect the most recent trades. As a result, discrepancies between this price, the current time to maturity, and the spot price can introduce noise and uncertainty when fitting the volatility surface.

2. Additionally, the option data collected for our analysis is limited to a time horizon of approximately T $\approx$ 2. Including option data with longer expirations, extending over several years, could provide a more comprehensive dataset and potentially yield more valuable information during the surface fitting process.

3. Another factor to consider is the use of OIS rates as a proxy for risk-free rates. While these rates are commonly used, they may not perfectly align with the true risk-free rate in all situations.

4. Furthermore, when solving both Dupire's Equation and Gatheral's mapping, numerical differentiation techniques were employed. The choice of the differential step size, denoted as dx, could introduce minor errors in the calculations.

4 Monte Carlo Simulation

4.1 Correlated Simulation Path

In the process of simulating index prices for SPX, HSI and NKY, we use the latest five years of historical returns to compute the correlation matrix. A Cholesky factorization then yields three correlated Gaussian processes Z_1, Z_2, Z_3 from three independent processes W_1, W_2, W_3. See Appendix C for more details.

$$\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.2422 & 0.9702 & 0 \\ 0.2319 & 0.4710 & 0.8517 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

4.2 Simulation Formula

For each simulation path i, the index price evolves on the grid $0 = t_0 < t_1 < \cdots < t_N = T$ as

$$S_i(t_{j+1}) = S_i(t_j)\,\exp\Big(\big(r_{t_j \to t_{j+1}} - \tfrac{1}{2}\,\sigma_x^2(t_j)\big)\,(t_{j+1} - t_j) \;+\; \sigma_x(t_j)\,Z_{k,i,j}\,\sqrt{t_{j+1} - t_j}\,\Big),$$

,where:

- $r_{t_j \to t_{j+1}} = \frac{t_{j+1}\,r_{t_{j+1}} - t_j\,r_{t_j}}{t_{j+1} - t_j}$ is the forward rate from t_j to t_{j+1}
- $\sigma_x^2(t_j)$ is the local variance at forward moneyness x at t_j
- Z_k is the correlated increment ($k = 1$ for SPX, 2 for HSI, 3 for NKY).

If at any t_j the estimate $\sigma_x^2(t_j)$ is negative, we replace it by the smallest non-negative value observed up to t_j. This adjustment is most frequent for HSI due to fitting errors in the local-volatility surface.

5 Autocallable Pricing

The present value of the autocallable note is the sum of the three components:

$$\text{Autocallable Value} = \text{Interest} \;+\; \text{KO Redemption} \;+\; \text{Final Redemption}\,.$$

5.1 Variable Interest

At each semi-annual observation date t_i, interest is paid on the note of denomination N. Define

$$1_{\max,i} = \begin{cases} 1, & \text{if reference price at } t_i \geq \text{coupon strike}, \\ 0, & \text{otherwise}, \end{cases}$$

$$1_{\text{pay},i} = \begin{cases} 0, & \text{if a knock-out has occurred on or before } t_i, \\ 1, & \text{otherwise.} \end{cases}$$

Let r_+ and r_- be the maximum (2% p.a.) and minimum (0.01% p.a.) interest rates, and $D(t_i)$ the discount factor to t_i. Then the total variable interest is

$$\text{Interest} \;=\; \sum_{i=1}^{M} N\, D(t_i)\, \big[\, 1_{\max,i}\, r_+ \;+\; (1 - 1_{\max,i})\, r_- \big]\, 1_{\text{pay},i}\,.$$

5.2 Knock-out Redemption

Define the knock-out indicator at t_i:

$$1_{\text{KO},i} = \begin{cases} 1, & \text{if laggard closing price at } t_i \geq 110\% \times S_0, \\ 0, & \text{otherwise.} \end{cases}$$

Let t_{i^*} be the first observation date at which $1_{\text{KO},i^*} = 1$. The knock-out redemption is paid at t_{i^*}:

$$\text{KO Redemption} \;=\; N\, D\big(t_{i^*}\big) \;=\; \sum_{i=1}^{M} N\, D(t_i)\, 1_{\text{KO},i}\,.$$

5.3 Final Redemption

If no knock-out occurs by maturity T, a final redemption at T is paid according to whether a knock-in barrier was breached. Define

$$1_{\text{KI}} = \begin{cases} 1, & \min_{0\leq t\leq T}\{\text{laggard price at } t\} \leq 50\% \times S_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then at T the redemption on the note N is

$$\text{Final Redemption} = N\, D(T)\, \big(1 - \max_i 1_{\text{KO},i}\big)\, \Big[1_{\text{KI}} \min\Big(100\%,\, \tfrac{S_{\text{laggard}}(T)}{S_{\text{laggard}}(0)}\Big) \;+\; (1 - 1_{\text{KI}})\, 100\%\Big].$$

6 Coupon Strike Optimzation

To find the coupon strike K_{coupon} that makes the note's value equal to 98% of its issue price, we employ the bisection method:

1. Let $V(K)$ be the computed autocallable value as a function of the coupon strike K.
2. Initial bracket: $K_{\text{low}} = 1$, $K_{\text{high}} =$ some upper bound.
3. Tolerance on $|V(K) - 0.98|$ is set to 0.1%.
4. Maximum iterations: 100. If the tolerance is not met after 100 trials, return "no result."

Using the original interest rates ($r_+ = 2\%$, $r_- = 0.01\%$), no root was found within tolerance by 100 iterations. We then varied r_+ and r_-, observing that for fixed price, the optimal coupon strike increases with both the maximum and the minimum interest rates.

		Min interest														
		0.0001	0.0008	0.0015	0.0022	0.0029	0.0036	0.0043	0.0051	0.0058	0.0065	0.0072	0.0079	0.0086	0.0093	0.0100
Max interest	0.020	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result
	0.033	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result	No result
	0.046	0.563	0.564	0.565	0.566	0.567	0.569	0.569	0.571	0.571	0.573	0.573	0.575	0.576	0.577	0.578
	0.059	0.781	0.782	0.783	0.785	0.786	0.788	0.789	0.791	0.793	0.794	0.796	0.798	0.800	0.801	0.803
	0.071	0.840	0.842	0.843	0.845	0.846	0.848	0.849	0.850	0.852	0.854	0.855	0.857	0.858	0.860	0.862
	0.084	0.875	0.876	0.878	0.879	0.880	0.882	0.883	0.885	0.887	0.888	0.889	0.891	0.893	0.894	0.896
	0.097	0.898	0.900	0.901	0.903	0.904	0.906	0.907	0.909	0.910	0.911	0.913	0.914	0.916	0.917	0.919
	0.110	0.916	0.918	0.919	0.920	0.922	0.923	0.924	0.926	0.927	0.928	0.930	0.931	0.933	0.934	0.936
	0.123	0.931	0.932	0.933	0.934	0.936	0.937	0.938	0.940	0.941	0.943	0.945	0.946	0.947	0.949	0.950
	0.136	0.943	0.944	0.946	0.947	0.948	0.949	0.951	0.952	0.954	0.955	0.956	0.958	0.959	0.961	0.962
	0.149	0.953	0.955	0.956	0.957	0.958	0.959	0.961	0.962	0.963	0.965	0.966	0.968	0.969	0.971	0.972
	0.161	0.962	0.963	0.964	0.966	0.967	0.968	0.969	0.971	0.972	0.974	0.975	0.976	0.978	0.979	0.981
	0.174	0.970	0.971	0.972	0.974	0.975	0.976	0.977	0.979	0.980	0.981	0.983	0.984	0.986	0.987	0.989
	0.187	0.977	0.978	0.979	0.981	0.982	0.983	0.985	0.986	0.987	0.989	0.990	0.991	0.993	0.995	0.996
	0.200	0.984	0.985	0.986	0.987	0.989	0.990	0.991	0.992	0.994	0.996	0.997	0.998	1.000	1.001	1.003

Appendix A - Bisection Algorithm

Assumption: $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$
By Intermediate value theorem, if $f(a)f(b) < 0, \exists x* \in (a, b)$ s.t. $f(x^*) = 0$

Algorithm

1. $x_k = \frac{a_k+b_k}{2}$ for $k - 0, 1, 2, \ldots$

2. If $|f(x_k)| < \varepsilon$, stop and output x_k
   If $f(x_k) > 0$, set $a_{k+1} = a_k, b_{k+1} = x_k$
   If $f(x_k) < 0$, set $a_{k+1} = x_k, b_{k+1} = b_k$

3. If $|b_{k+1} - a_{k+1}| < \delta$, stop and output x_k

Convergence Analysis

1. $b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1}) = \cdots = \frac{1}{2^k}(b_0 - a_0)$, hence, $x^* = \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$

2. $f(a_k)f(b_k) < 0$ implies $f(x^*) \leq 0$, hence, $f(x^*) = 0$

3. $|x_k - x^*| \leq \frac{1}{2}(b_k - a_k) = \cdots = \frac{1}{2^{(k+1)}}(b - a)$

Convergence Rate: $\frac{|x_{k+1}-x^*|}{|x_k-x^*|} = \frac{1}{2}$

Appendix B - Cubic Spline

Define $s_k(x) = s_{k0} + s_{k1}(x - x_k) + s_{k2}(x - x_k)^2 + s_{k3}(x - x_k)^3$
Natural cubic spline: $s''(x_0) = s''(x_n) = 0$

Uniqueness and Existence

Denote $s''(x_k) = m_k, \quad h_k = x_{k+1} - x_k$ $(known)$

$$s''_k(x) = s''(x_k)\frac{x - x_{k+1}}{x_k - x_{k+1}} + s''(x_{k+1})\frac{x - x_k}{x_{k+1} - x_k} = \frac{m_k}{h_k}(x_{k+1} - x) + \frac{m_{k+1}}{h_k}(x - x_k)$$

$$\begin{aligned} s_k(x) &= \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + c_k x + d_k \\ &= \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + p_k(x_{k+1} - x) + q_k(x - x_k) \end{aligned}$$

$$f_k = s_k(x_k) = \frac{m_k h_k^2}{6} + p_k h_k \quad \Rightarrow \quad p_k = \frac{f_k}{h_k} - \frac{m_k h_k}{6}$$

$$f_{k+1} = s_k(x_{k+1}) = \frac{m_{k+1} h_k^2}{6} + q_k h_k \quad \Rightarrow \quad q_k = \frac{f_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}$$

$$s_k(x) = \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + \left(\frac{f_k}{h_k} - \frac{m_k h_k}{6}\right)(x_{k+1} - x) + \left(\frac{f_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}\right)(x - x_k)$$

$$s'_k(x) = \frac{m_k}{2h_k}(x_{k+1} - x)^2 + \frac{m_{k+1}}{2h_k}(x - x_k)^2 - \left(\frac{f_k}{h_k} - \frac{m_k h_k}{6}\right) + \left(\frac{f_{k+1}}{h_k} - \frac{m_{k+1} h_k}{6}\right)$$

Denote $d_k = \frac{f_{k+1}-f_k}{h_k}$ $(known)$ and $u_k = 6(d_k - d_{k-1})$ $(known)$

$$s'_k(x_k) = -\frac{m_k}{3}h_k - \frac{m_{k+1}}{6}h_k + d_k$$
$$s'_{k-1}(x_k) = \frac{m_k}{3}h_{k-1} + \frac{m_{k-1}}{6}h_{k-1} + d_{k-1}$$
$$s'_k(x_k) = s'_{k-1}(x_k), \Rightarrow h_{k-1}m_{k-1} + 2(h_{k-1}+h_k)m_k + h_k m_{k+1} = u_k \quad -(*)$$

$(*)$ in matrix form is diagonally dominant, hence non-singular, i.e. solution $\{m_i\}_{i=0}^n$ exists and is unique

Equality Constraints

1. $s'_j(x_{j+1}) = s'_{j+1}(x_{j+1})$ for $j = 0, 1, 2, \dots, n-2$
2. $s''_j(x_{j+1}) = s''_{j+1}(x_{j+1})$ for $j = 0, 1, 2, \dots, n-2$

1. $s_0(x_0) = f_0$
2. $s_j(x_{j+1}) = s_{j+1}(x_{j+1}) = f(x_{j+1})$ for $j = 0, 1, 2, \dots, n-2$
3. $s_{n-1}(x_n) = f_n$

Construction $\quad s_k(x) = s_{k0} + s_{k1}(x - x_k) + s_{k2}(x - x_k)^2 + s_{k3}(x - x_k)^3$

1. $s_{k0} = f_k$
2. $s_{k1} = s'_k(x_k) = -\frac{m_k}{3}h_k - \frac{m_{k+1}}{6}h_k + \frac{f_{k+1}-f_k}{h_k}$
3. $s_{k2} = s''_k(x_k) = \frac{m_k}{2}$
4. By $s''_k(x_{k+1}) = 2s_{k2} + 6s_{k3}h_k$, $s_{k3} = \frac{m_{k+1}-m_k}{6h_k}$

Appendix C - Cholesky Factorization

If A is a SPD matrix and $A = U^T U$ where U is an upper triangular matrix. If all diagonal entries of U are positive, the factorization is unique and called Cholesky factorization

Algorithm

1. $u_{11} = \sqrt{\alpha}$
2. $\boldsymbol{r}^T = \boldsymbol{a}^T / u_{11}$
3. $\hat{A}_{11} \triangleq U_{11}^T U_{11} = A_{11} - \boldsymbol{r}\boldsymbol{r}^T$
4. Repeat for the sub-matrix $\hat{A}_{11}$, which is also SPD

Appendix D - Simulation Methodology

We simulate the autocallable payoff over M Monte Carlo paths and N equally-spaced observation dates $t_1, \ldots, t_N$, with notional $N_0 = \$10\,000$, coupon strike K_{CS}, knock-out barrier 110%, knock-in barrier 50%, maximum rate $r_+ = 2\%$, minimum rate $r_- = 0.01\%$. At the end we discount each path's terminal payoff by the USD risk-free curve and average.

1. **For each observation date t_i, $i = 1, \ldots, N$:**

   (a) For each path j, compute each underlying's total return

   $$R_{j,k}(t_i) \;=\; \frac{S_{j,k}(t_i)}{S_{j,k}(0)}, \quad k \in \{\text{SPX,HSI,NKY}\}.$$

   (b) Let $R_j^{\min}(t_i) = \min_k R_{j,k}(t_i)$.

   (c) *Accrue interest:*

   $$\Delta V_j(t_i) = N_0 \;\times \begin{cases} \frac{r_+}{2}, & R_j^{\min}(t_i) \geq K_{\mathrm{CS}}, \\ \frac{r_-}{2}, & R_j^{\min}(t_i) < K_{\mathrm{CS}}. \end{cases}$$

   (d) *Check knock-out:* If $R_j^{\min}(t_i) \geq 1.10$, then

   $$V_j(t_i) \;=\; V_j(t_i^-) + N_0, \quad \text{mark path } j \text{ as expired,}$$

   and skip all future dates for this path.

2. **Monitor knock-in:** On each t_i, for each *active* path j, if

   $$R_j^{\min}(t_i) \leq 0.50,$$

   then mark path j as "knock-in."

3. **At maturity $T = t_N$, for each *active* (non–knock-out) path j:**

   - If path j was *knock-in*, pay
   $$N_0 \,\min\bigl(1,\; R_j^{\min}(T)\bigr).$$
   - Otherwise (no knock-in), pay N_0.

4. **Present value and averaging:**

   $$\text{Price} \;=\; \frac{1}{M} \sum_{j=1}^{M} D_j \,\bigl[\text{terminal payoff of path } j\bigr],$$

   where D_j is the USD discount factor to T.

5. **Optimization goal:** Adjust K_{CS} (via bisection or similar) so that Price $\approx \$9\,800$.