Coordinate and momentum representations in Quantum Mechanics.

Coordinate representation.

Let us consider a single point particle and for simplicity we assume that the particle has no internal degrees of freedom like spin. What is a suitable basis we can use to describe the state of the particle? We could take any dynamical variable of the particle, like position, momentum or energy. The hermitian operators representing each of these dynamical variables have a complete set of eigenvectors and hence can be chosen as a basis set.

Let us choose position as the dynamical variable and suppose that the particle moves in one dimension. The generalization to three-dimensional motion is straightforward.

Let the position operator for motion on the x-axis be denoted by $\hat{X}$. The eigenvalue equation for $\hat{X}$ can be written as

$$\hat{X}\,|x\rangle = x\,|x\rangle. \quad \cdots \quad (1)$$

The state $|x\rangle$ corresponds to the particle being localized at x. The eigenvalues x form a continuum and so the eigenvectors $\{|x\rangle\}$ form a nondenumerable infinite set of basis vectors. The completeness condition for the basis set $\{|x\rangle\}$ can be written as

$$\int dx\; |x\rangle\langle x| = \hat{\mathbb{1}}. \quad \cdots \quad (2)$$

Now consider any arbitrary state $|a\rangle$ of the particle. The ket $|a\rangle$ can be expanded as

$$|a\rangle = \int dx\; |x\rangle\langle x|a\rangle \quad \cdots \quad (3)$$

The quantity $\langle x|a\rangle$ can be thought of as the component of $|a\rangle$ along $|x\rangle$. This quantity is a complex-valued function of the ~~state~~ real variable x and is called the wave function of the particle in the state $|a\rangle$. Thus we introduce the wave function $\psi_a(x)$ as

$$\psi_a(x) \overset{\text{def}}{\equiv} \langle x|a\rangle \quad \cdots\cdots \quad (4).$$

The wave function of a state $|a\rangle$ is nothing but the representation of $|a\rangle$ in the coordinate basis. Since the coordinate basis is nondenumerable, the representation of $|a\rangle$ is a function rather than a column matrix.

According to one of the postulates of Quantum Mechanics, the wave function $\psi_a(x) \equiv \langle x|a\rangle$ is the probability amplitude for finding the particle at x, i.e.,

$|\langle x|a\rangle|^2 dx$ = probability for finding the particle within x and $x+dx$ when the particle is in the state $|a\rangle$.

The quantity $|\langle x|a|^2 \equiv |\psi_a(x)|^2$ is called the probability density.

Next, let us ask what is the value of the scalar product $\langle x|x'\rangle$? To find the answer we proceed by expanding an arbitrary ket $|a\rangle$ as in Eq. (3):

$$|a\rangle = \int dx' \; |x'\rangle\langle x'|a\rangle .$$

Taking the scalar product of both sides with $\langle x|$ we obtain

$$\langle x|a\rangle = \int dx' \; \langle x|x'\rangle \langle x'|a\rangle$$

or $$\psi_a(x) = \int dx' \; \langle x|x'\rangle \, \psi_a(x') \quad \cdots\cdots (5)$$

Since $\psi_a(x)$ is an arbitrary function, we must have

$$\langle x|x'\rangle = \delta(x-x') . \quad \cdots\cdots (6)$$

Thus the eigenkets $\{|x\rangle\}$ of the position operator are not normalizable since

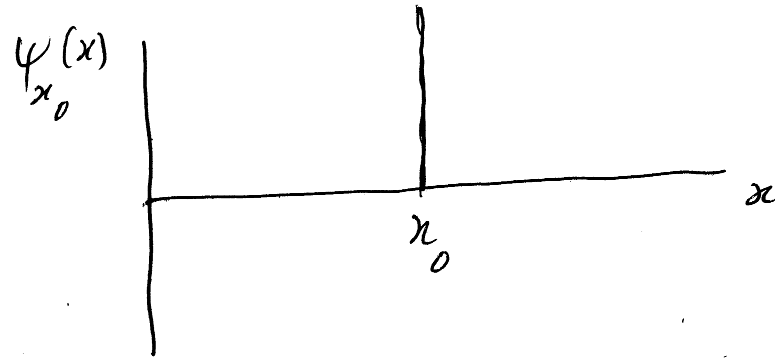
$$\langle x|x\rangle = \delta(0) \to \infty$$

i.e., the norm $\| \, |x\rangle \, \| \equiv \sqrt{\langle x|x\rangle} = \infty$.

Since $\langle x|x\rangle \neq$ finite, these basis states do not really belong to the Hilbert space. The problem is that these states represent the states of a perfectly ~~localized~~ localized particle. If the state of the particle is one of the eigenstates of $\hat{x}$, say $|x_0\rangle$, then the wavefunction corresponding to this state is

$$\psi_{x_0}(x) = \langle x|x_0\rangle = \delta(x - x_0)$$

Thus the probability amplitude of finding the particle is zero everywhere except at x_0. In other words, the particle is localized at x_0. A plot of $\psi_{x_0}(x)$ shows a delta-function spike at x_0.



The fact that the basis kets $\{|x\rangle\}$ are not normalizable is not really a difficulty since the only use we will have for these perfectly localized states is as basis states. The actual state of the particle $|a\rangle$ is never perfectly localized since such a particle would have infinite kinetic energy according to the uncertainty principle.

<u>Generalization to three dimensions.</u>

Generalization to three-dimensions is obvious. We introduce the vector position operator $\hat{\vec{R}}$ as

$$\hat{\vec{R}} = \hat{i}\,\hat{X} + \hat{j}\,\hat{Y} + \hat{k}\,\hat{Z}$$

Then the eigenstates of $\hat{\vec{R}}$ are written as

$$|\vec{r}\rangle \equiv |x\,y\,z\rangle = |x\rangle|y\rangle|z\rangle$$

and the eigenvalue equation is

$$\hat{\vec{R}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle.$$

The completeness and orthogonality conditions of the kets $\{|\vec{r}\rangle\}$ are expressed as

$$\hat{\mathbb{1}} = \int d^3r\; |\vec{r}\rangle\langle\vec{r}|$$

and

$$\langle\vec{r}|\vec{r}\,'\rangle = \delta(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

Any ket $|a\rangle$ representing the state of the particle can be expanded as

$$|a\rangle = \int d^3r \; |\vec{r}\rangle\langle\vec{r}|a\rangle$$

where the wave function is

$$\psi_a(\vec{r}) \equiv \langle\vec{r}|a\rangle .$$

Scalar product in coordinate representation.

Let $|a\rangle$ and $|b\rangle$ two arbitrary vectors in the Hilbert space. The scalar product of $|a\rangle$ and $|b\rangle$ in the coordinate representation is

$$\langle a|b\rangle = \int d^3r \; \langle a|\vec{r}\rangle\langle\vec{r}|b\rangle$$

$$= \int d^3r \; \psi_a^*(\vec{r})\,\psi_b(\vec{r}) \qquad \text{- - - (7)}$$

The norm of a vector is then given by

$$\| \, |a\rangle \|^2 = \langle a|a\rangle = \int d^3r \; \psi_a^*(\vec{r})\,\psi_a(\vec{r})$$

$$= \int d^3r \; |\psi_a(\vec{r})|^2 = \text{finite}.$$

Operators in coordinate space.

We will now discuss how various operators in the Hilbert space can be expressed in coordinate representation. Consider the equation

$$|b\rangle = \hat{O}\,|a\rangle \quad \text{- - - - - - -} \quad (8)$$

where the operator $\hat{O}$ transforms the ket $|a\rangle$ to the ket $|b\rangle$. In coordinate representation

$$\langle \vec{r}\,|b\rangle = \langle r|\hat{O}\,|a\rangle$$

$$= \int d^3r' \,\langle \vec{r}\,|\hat{O}\,|\vec{r}\,'\rangle \langle \vec{r}\,'|a\rangle$$

or, in terms of the wave function

$$\psi_b(\vec{r}) = \int d^3r' \,\langle \vec{r}\,|\hat{O}\,|\vec{r}\,'\rangle \,\psi_a(\vec{r}\,') \quad \text{- -} (9)$$

The quantity $\langle \vec{r}\,|\hat{O}\,|\vec{r}\,'\rangle$ is called the matrix element of $\hat{O}$ in the coordinate basis, the indices taking on a continuum of values. We can think of $\langle \vec{r}|\hat{O}|\vec{r}'\rangle$ as a function, $O(\vec{r},\vec{r}')$, of $\vec{r}$ and $\vec{r}'$ and The matrix element is also called the kernel of the operator $\hat{O}$.

Matrix element of the coordinate operator in the coordinate representation.

If the operator $\hat{O}$ is taken to be coordinate operator $\hat{\vec{R}}$, then the matrix element of $\vec{R}$ in the coordinate basis is

$$\langle \vec{r}|\hat{\vec{R}}|\vec{r}'\rangle = \vec{r}'\langle \vec{r}|\vec{r}'\rangle = \vec{r}'\,\delta(\vec{r}-\vec{r}') \quad \cdots (10)$$

In one-dimension we would write

$$\langle x|\hat{x}|x'\rangle = x'\langle x|x'\rangle = x'\,\delta(x-x') \quad \cdots (11)$$

The matrix element of the coordinate operator in the coordinate representation is diagonal.

The effect of the coordinate operator on a wave function is simply to multiply the wave function by r. To see this consider

$$|\psi'\rangle = \hat{\vec{R}}\,|\psi\rangle$$

In coordinate representation

$$\langle\vec{r}|\psi'\rangle = \int \langle\vec{r}|\hat{\vec{R}}|\vec{r}\,'\rangle\langle\vec{r}\,'|\psi\rangle\, d^3r'$$

$$= \int \vec{r}\,'\,\delta(\vec{r}-\vec{r}\,')\langle\vec{r}\,'|\psi\rangle\, d^3r'$$

$$= \vec{r}\,\langle\vec{r}|\psi\rangle$$

i.e.,

$$\psi'(\vec{r}) = r\,\psi(\vec{r})$$

Thus

$$\hat{\vec{R}} \longrightarrow \vec{r}$$

in the coordinate representation.

Since $\hat{\vec{R}}$ is a hermitian operator and $\hat{\vec{R}}|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$, we also have

$$\langle\vec{r}|\hat{\vec{R}} = \langle\vec{r}|\,\vec{r} \quad \text{- - - - - - - (12)}$$

that is the bra $\langle\vec{r}|$ is an eigenbra of $\hat{\vec{R}}$ with eigenvalue $\vec{r}$. Equation (12) gives the effect of $\hat{\vec{R}}$ on a wave function immediately. Taking the scalar product of (12) with an arbitrary ket $|\psi\rangle$ we have

$$\langle\vec{r}|\hat{\vec{R}}|\psi\rangle = r\langle\vec{r}|\psi\rangle$$

or $\quad \psi'(\vec{r}) = r\,\psi(\vec{r})$

where $\quad |\psi'\rangle = \hat{\vec{R}}\,|\psi\rangle$.

The wave function of the ket $\hat{\vec{R}}|\psi\rangle$ is simply the wave function of the ket $|\psi\rangle$ multiplied by r.

If we consider an operator $V(\hat{\vec{R}})$ which is a function of the position operator, then in coordinate representation $V(\hat{\vec{R}})$ is also diagonal, i.e.,

$$\langle \vec{r} | V(\hat{\vec{R}}) | \vec{r}\,' \rangle = V(\vec{r}\,') \, \delta(\vec{r} - \vec{r}\,')$$

If $V(\hat{\vec{R}})$ operates on a ket to give a new ket, then the wave function of the new ket is simply $V(\vec{r})$ times the wave function of the old ket. Thus,

if

$$|\psi'\rangle = V(\hat{\vec{R}}) \, |\psi\rangle$$

then

$$\langle \vec{r} | \psi' \rangle = \langle \vec{r} | V(\hat{\vec{R}}) | \psi \rangle$$

$$= V(\vec{r}) \langle \vec{r} | \psi \rangle$$

or

$$\psi'(\vec{r}) = V(\vec{r}) \, \psi(\vec{r}).$$

Therefore, in coordinate representation

$$V(\hat{\vec{R}}) \longrightarrow V(\vec{r}) .$$

As an example, if

$$V(\hat{\vec{R}}) = \hat{\vec{R}}^2 = \hat{\vec{R}} \cdot \hat{\vec{R}}$$

Then

$$\langle \vec{r} \,|\, V(\hat{\vec{R}}) \,|\, \psi \rangle = r^2 \langle \vec{r} \,|\, \psi \rangle .$$

Momentum operator in coordinate representation.

In Quantum Mechanics we have the fundamental commutation relation

$$[\hat{X}, \hat{P}] = i\hbar \hat{\mathbb{1}} \quad \text{- - - - - - (12)}$$

where $\hat{P}$ is the operator corresponding to the momentum conjugate to x. Taking the matrix element of (12) between $\langle x|$ and $|x'\rangle$ we have

$$\langle x|[\hat{X}, \hat{P}]|x'\rangle = i\hbar \langle x|\hat{\mathbb{1}}|x'\rangle$$

or $$\langle x|\hat{X}\hat{P} - \hat{P}\hat{X}|x'\rangle = i\hbar \langle x|x'\rangle$$

or $$(x-x')\langle x|\hat{P}|x'\rangle = i\hbar\, \delta(x-x') \quad \text{- - - (13)}$$

We now use the identity

$$q\,\delta'(q) = -\delta(q).$$

Therefore, in Eq. (13) we must choose

$$\langle x|\hat{P}|x'\rangle = -i\hbar\, \delta'(x-x') \quad \cdots\cdots (14)$$

where the prime on the delta function denotes differentiation of the delta function with respect to its full argument $(x-x')$, or, what ~~amounts~~ amounts to the same thing, differentiation with respect to the first argument x. Therefore, we can write

$$\begin{aligned} \langle x|\hat{P}|x'\rangle &= -i\hbar \frac{\partial}{\partial x}\,\delta(x-x') \\ &= i\hbar \frac{\partial}{\partial x'}\,\delta(x-x') \end{aligned} \quad \cdots (15)$$

Using Eq. (15) we can easily derive the effect of the momentum on a wave function.

Let

$$|\psi'\rangle = \hat{P}\,|\psi\rangle$$

Then

$$\langle x|\psi'\rangle = \langle x|\hat{p}|\psi\rangle$$

$$= \int \langle x|\hat{p}|x'\rangle\langle x'|\psi\rangle\, dx'$$

$$= \int (-i\hbar)\frac{\partial}{\partial x}\delta(x-x')\langle x'|\psi\rangle\, dx'$$

$$= -i\hbar\frac{\partial}{\partial x}\int \delta(x-x')\langle x'|\psi\rangle\, dx'$$

i.e.,

$$\psi'(x) = -i\hbar\frac{\partial}{\partial x}\psi(x) \quad \cdots\cdots (16)$$

Thus

$$\hat{p} \longrightarrow -i\hbar\frac{\partial}{\partial x}$$

in coordinate representation. We can formally write

$$\langle x|\hat{p} = -i\hbar\frac{\partial}{\partial x}\langle x| \quad \cdots\cdots (17)$$

where it is understood that this equation acts on an arbitrary ket $|\psi\rangle$.

In three dimensions we have

$$\hat{\vec{p}} \longrightarrow -i\hbar\vec{\nabla}\,.$$

As an example consider the operator $\hat{p}^n$ where n is an integer greater than zero. In coordinate representation

$$\langle x|\hat{p}^n|\psi\rangle = \left(-i\hbar\frac{\partial}{\partial x}\right)^n\langle x|\psi\rangle$$

i.e $\hat{p}^n \rightarrow \left(-i\hbar\frac{\partial}{\partial x}\right)^n$.

Hermiticity of momentum operator

We recall that an operator $\hat{A}$ is hermitian if it satisfies the following condition

$$\langle f | \hat{A} | g \rangle = \langle g | \hat{A} | f \rangle^*$$

for any pair of states $|f\rangle$ and $|g\rangle$ in the Hilbert space.

Consider now the momentum operator $\hat{P}$. Using the pair $|x'\rangle$ and $|x\rangle$ of the eigenstates of the coordinate operator we have

$$P^*_{x'x} = \langle x' | \hat{P} | x \rangle^* = \left[-i\hbar\, \delta'(x'-x) \right]^*$$

$$= i\hbar\, \delta'(x'-x) \quad \text{(delta function is real)}$$

$$= -i\hbar\, \delta'(x-x')$$

$$= P_{xx'}$$

It turns out that inspite of the above, the operator $\hat{P}$ is not guaranteed to be hermitian unless the

Hilbert space is defined appropriately. To see this consider two arbitrary kets $|f\rangle$ and $|g\rangle$ with their respective wave functions $f(x)$ and $g(x)$ defined over the interval $a-b$. If $\hat{p}$ is hermitian, it must satisfy

$$\langle f|\hat{p}|g\rangle = \langle g|\hat{p}|f\rangle^*$$

Let us check whether the above relation is satisfied. Starting with the left hand side of the above equation we can proceed as follows:

$$LHS = \langle f|\hat{p}|g\rangle$$

$$= \int_a^b \langle f|x\rangle\langle x|\hat{p}|x'\rangle\langle x'|g\rangle\, dx\, dx'$$

$$= \int_a^b f^*(x)\,(-i\hbar)\frac{\partial}{\partial x}\,\delta(x-x')\,g(x')\,dx'\,dx$$

$$= -i\hbar\int_a^b f^*(x)\,\frac{\partial g}{\partial x}\,dx$$

Integrating by parts

LHS $= \langle f | \hat{P} | g \rangle$

$$= -i\hbar \int_a^b f^*(x) \frac{\partial g}{\partial x}\, dx$$

$$= -i\hbar \left[f^*(x) g(x) \Big|_a^b - \int_a^b \frac{d f^*(x)}{dx} g(x)\, dx \right]$$

$$= -i\hbar \left(f^*(x) g(x) \right)_a^b + i\hbar \int_a^b g(x) \frac{d f^*(x)}{dx}\, dx$$

$$= -i\hbar \left(f^* g \right)_a^b + \left[-i\hbar \int_a^b g^*(x) \frac{d f(x)}{dx} \right]^*$$

$$= -i\hbar \left(f^* g \right)_a^b + \langle g | \hat{P} | f \rangle^*$$

So $\hat{P}$ is hermitian only if the surface terms vanish. Surface terms will vanish if for any function $f(x)$

$$f(a) = f(b).$$

Usually, we take the entire space $-\infty < x < \infty$ for the domain of x and restrict the Hilbert space to square integrable functions for which $f(x) \to 0$ as $|x| \to \infty$.

The surface terms are then zero at either end and $\hat{p}$ becomes hermitian.

Momentum representation.

In momentum representation, the eigenvectors $|p\rangle$ of the momentum operator $\hat{P}$ are used as the basis set. The eigenvalue equation is

$$\hat{P}\,|p\rangle = p\,|p\rangle \,. \quad \text{- - - - - - (1)}$$

The momentum eigenvalues p vary over a continuum. The completeness condition can be written as

$$\int dp\; |p\rangle\langle p| = \hat{\mathbb{1}} \,. \quad \text{- - - - - (2)}$$

We adopt the normalization

$$\langle p|p'\rangle = \delta(p-p') \,. \quad \text{- - - - - (3)}$$

Note :

If the completeness condition is written as in Eq. (2) then the normalization must be written as in eq. (4). Then the right hand side of (2) would be the identity operator. To show this, ~~we must~~ first note that ~~we all~~ ~~Also~~

$$\hat{\mathbb{1}}^2 = \hat{\mathbb{1}}$$

$$\therefore \int dp\, dp' \; |p\rangle\langle p|p'\rangle\langle p'| = \int dp \; |p\rangle\langle p|$$

If the left hand side is to be equal to the right hand side, we must have

$$\langle p|p'\rangle = \delta(p-p').$$

Next, if we choose a different normalization, say

$$\langle p|p'\rangle = C\,\delta(p-p')$$

where C is a constant, then the completeness condition must be modified accordingly. The completeness condition would now be

$$\frac{1}{c}\int dp\ |p\rangle\langle p| = \hat{\mathbb{1}} .$$

To check this, see whether $\hat{\mathbb{1}}^2$ equals $\hat{\mathbb{1}}$.

Check:

$$\hat{\mathbb{1}}^2 = \hat{\mathbb{1}} . \hat{\mathbb{1}}$$

$$\ldots = \frac{1}{c^2}\int dp\,dp'\ |p\rangle\langle p|p'\rangle\langle p'|$$

$$= \frac{1}{c^2}\int dp\,dp'\ |p\rangle\, c\,\delta(p-p')\langle p'|$$

$$= \frac{1}{c}\int dp\ |p\rangle\langle p|$$

$$= \hat{\mathbb{1}} \quad (\mathrm{OK}).$$

Momentum eigenstates in coordinate representation.

The eigenvalue equation of the momentum operator, (Eq. (1)), expressed in coordinate representation is

$$\langle x|\hat{P}|p\rangle = p\,\langle x|p\rangle$$

or $$-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\,\langle x|p\rangle$$

or $$-i\hbar\frac{\partial\phi_p(x)}{\partial x} = p\,\phi_p(x) \quad \text{------} \quad (4)$$

The solution of this equation is straightforward:

$$\phi_p(x) = C\,e^{ipx/\hbar} \quad \text{.....} \quad (5)$$

The constant C has to be found using the normalization condition

$$\langle p|p'\rangle = \delta(p-p')$$

i.e., $$\int dx\,\langle p|x\rangle\langle x|p'\rangle = \delta(p-p')$$

or $$\int dx\,\phi_p^*(x)\phi_{p'}(x) = \delta(p-p')$$

or,

$$|C|^2 \int_{-\infty}^{\infty} dx \; e^{-\frac{i}{\hbar}(p-p')x} = \delta(p-p')$$

or $\quad |C|^2 \, 2\pi \, \delta\left(\frac{p-p'}{\hbar}\right) = \delta(p-p') \quad \Big| \; \int_{-\infty}^{\infty} e^{\pm i(k-k')x} dx = 2\pi\,\delta(k-k')$

or, $\quad |C|^2 \, 2\pi\hbar \, \delta(p-p') = \delta(p-p') \quad \Big| \; \delta(ax) = \frac{1}{|a|}\delta(x)$

or $\quad |C|^2 = \dfrac{1}{2\pi\hbar}$,

i.e., $\quad |C| = \dfrac{1}{\sqrt{2\pi\hbar}}$.

We take

$$C = \frac{1}{\sqrt{2\pi\hbar}}.$$

Thus

$$\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \quad \ldots\ldots\ldots (6).$$

In three dimensions we can write

$$\phi_{\vec{p}}(\vec{r}) = \langle \vec{r} | \vec{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}\cdot\vec{r}/\hbar} \quad \ldots\ldots (7)$$

Note:

The momentum operator may not be hermitian, i.e., the operator can have complex eigenvalues in an arbitrary space of functions. Looking back to the eigenvalue equation (Eq. (4)) we see that the equation does have solutions for complex p and the solutions as in Eq. (5) both for real p and complex p. However, if p is complex, i.e.,

$$p = \mathrm{Re}\,p + i\,\mathrm{Im}\,p$$

then the corresponding eigenfunction is

$$\phi_p(x) = C\, e^{i\,\mathrm{Re}\,p\, x} \cdot e^{-\mathrm{Im}\,p\, x}$$

and so $\phi_p(x) \longrightarrow \infty$ as $x \to \infty$ or $x \to -\infty$ depending on whether $\mathrm{Im}\,p < 0$ or $\mathrm{Im}\,p > 0$. Thus the eigenfunctions $\phi_p(x)$ would diverge at infinity ($|x| \to \infty$) if p were complex. Such divergent functions are excluded from the Hilbert space which consists of normalizable functions. These normalizable or square-integrable functions tend to zero sufficiently rapidly as $|x| \to \infty$. It is only in the Hilbert space of square integrable functions, the momentum operator is hermitian and its eigenvalues are real.

However, the eigenfunctions of the momentum operator do not belong to the ~~Hilb~~ ~~Hilbert~~ Hilbert space of square integrable functions. The eigenfunctions are of the form $\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$, with p real. These functions are oscillatory and do not tend to zero as $|x| \to \infty$. Such functions, even though are not normalizable in the strict mathematical sense, they are 'normalizable' to the delta function. They can act as a basis in terms of which an arbitrary normalizable vector of the Hilbert space can be expanded.

We can include these oscillatory functions in our Hilbert space and the augmented Hilbert space is called the physical Hilbert space.

Momentum representation of a state vector

Let $|a\rangle$ denote a normalizable ket in the Hilbert space corresponding to a particular physical state of a particle. We can expand $|a\rangle$ in the $\{|\vec{p}\rangle\}$ basis. The orthogonality and completeness relations for this basis set are written as

$$\langle \vec{p}|\vec{p}'\rangle = \delta(\vec{p}-\vec{p}')$$

and
$$\int d^3p \, |\vec{p}\rangle\langle\vec{p}| = \hat{\mathbb{1}} \, .$$

We can now write

$$|a\rangle = \hat{\mathbb{1}}|a\rangle = \int d^3p \, |\vec{p}\rangle\langle\vec{p}|a\rangle$$
$$= \int d^3p \, |\vec{p}\rangle \, \tilde{\psi}_a(\vec{p}) \quad \ldots\ldots (8)$$

where
$$\tilde{\psi}_a(\vec{p}) \equiv \langle\vec{p}|a\rangle$$

is a complex-valued functions of the real variables $\vec{p}$ (i.e., p_x, p_y and p_z). The quantity $\tilde{\psi}_a(\vec{p})$ is called the wavefunction of the state $|a\rangle$ in momentum representation.

According to one of the postulates of quantum mechanics, we interpret $\tilde{\psi}_a(\vec{p})$ as the probability amplitude for finding the particle with momentum $\vec{p}$ when the state of the particle is $|a\rangle$. More precisely,

$$|\tilde{\psi}_a(\vec{p})|^2 d^3p = |\langle \vec{p}|a\rangle|^2 d^3p$$

= probability that the momentum of the particle lies in the range $\vec{p}$ to $\vec{p}+d\vec{p}$ when the particle is in the state $|a\rangle$.

Now, if the physical state vector $|a\rangle$ is normalized to ~~una~~ unity, we have

$$\langle a|a\rangle = 1.$$

In momentum representation, we can write this equation as

$$\int d^3p \; \langle a|\vec{p}\rangle\langle\vec{p}|a\rangle = 1$$

or $$\int d^3p \; \tilde{\psi}_a^*(\vec{p})\,\tilde{\psi}_a(\vec{p}) = 1$$

i.e., $$\int d^3p \; |\tilde{\psi}_a(\vec{p})|^2 = 1.$$

Operators in momentum representation.

Let
$$|\psi'\rangle = \hat{O}|\psi\rangle. \quad \cdots\cdots\cdots (9)$$

In momentum representation we can write (assuming one-dimensional motion)
$$\langle p|\psi'\rangle = \langle p|\hat{O}|\psi\rangle = \int dp' \langle p|\hat{O}|p'\rangle\langle p'|\psi\rangle$$

or,
$$\tilde{\psi}'(p) = \int dp' \langle p|\hat{O}|p'\rangle \tilde{\psi}(p'). \quad \cdots\cdots (10)$$

The quantities $\langle p|\hat{O}|p'\rangle$ are the matrix elements of $\hat{O}$ in the momentum representation.

• Momentum operator in momentum representation.

In momentum representation, the matrix of the momentum operator $\hat{p}$ is diagonal, i.e.,
$$\langle p|\hat{P}|p'\rangle = p\,\delta(p-p') \quad \cdots\cdots (11).$$

Therefore, we can write
$$\hat{p} = \int dp\, dp' \; |p\rangle\langle p|\hat{P}|p'\rangle\langle p'| = \int dp \; p\,|p\rangle\langle p|.$$

Now, let us consider the effect of the momentum operator on the wavefunction in momentum representation.

Let

$$|\psi'\rangle = \hat{P}\,|\psi\rangle$$

Then

$$\langle p|\psi'\rangle = \langle p|\hat{P}|\psi\rangle$$

$$= \int dp' \,\langle p|\hat{P}|p'\rangle \langle p'|\psi\rangle$$

$$= \int dp' \, p' \,\delta(p-p') \langle p'|\psi\rangle$$

$$= p \,\langle p|\psi\rangle$$

or $\quad \tilde{\psi}'(p) = p\,\psi(p)$.

The effect of $\hat{P}$ in momentum representation is simply multiplication by p. Thus

$$\hat{P} \longrightarrow p \quad (\text{in momentum representation})$$

Similarly

$$f(\hat{P}) \rightarrow f(p). \quad (\text{in momentum representation}).$$

Coordinate operator in momentum representation.

Let us now find the matrix elements of the coordinate operator $\hat{x}$ in momentum representation, i.e.,

$$\langle p|\hat{x}|p'\rangle.$$

Method 1

We use the fundamental commutation relation

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\hat{1}$$

$$\Rightarrow \quad \langle p|\hat{x}\hat{p} - \hat{p}\hat{x}|p'\rangle = i\hbar\langle p|p'\rangle$$

$$\Rightarrow \quad (p'-p)\langle p|\hat{x}|p'\rangle = i\hbar\,\delta(p-p')$$

$$\Rightarrow \quad -(p-p')\langle p|\hat{x}|p'\rangle = i\hbar\,\delta(p-p').$$

Using the identity

$$x\,\delta'(x) = -\delta(x)$$

we have

$$\langle p|\hat{x}|p'\rangle = i\hbar\,\delta'(p-p') \quad \cdots\cdots (12)$$

where the prime denotes the derivative with respect to the argument ~~derivative~~ of the delta function. We can also write

$$\boxed{\langle p|\hat{x}|p'\rangle = i\hbar \frac{\partial}{\partial p}\,\delta(p-p')} \quad \cdots\cdots (13)$$

Method 2

$$\langle p|\hat{x}|p'\rangle = \int \langle p|x\rangle\langle x|\hat{x}|x'\rangle\langle x'|p'\rangle\, dx\, dx'$$

Now

$$\langle x|\hat{x}|x'\rangle = x\,\delta(x-x')$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}$$

Therefore we have:

$$\langle p|\hat{x}|p'\rangle = \int \langle p|x\rangle\, x\, \delta(x-x')\, \langle x'|p\rangle\, dx\, dx'$$

$$= \int \langle p|x\rangle\, x\, \langle x|p'\rangle\, dx$$

$$= \frac{1}{2\pi\hbar}\int e^{-\frac{i}{\hbar}px}\, x\, e^{\frac{i}{\hbar}p'x}\, dx$$

$$= \frac{1}{2\pi\hbar}\int \left(-\frac{\hbar}{i}\right)\left(\frac{\partial}{\partial p} e^{-ipx/\hbar}\right) e^{ip'x/\hbar}\, dx$$

$$= \frac{1}{2\pi\hbar}\left(-\frac{\hbar}{i}\right)\frac{\partial}{\partial p}\int e^{-\frac{i}{\hbar}(p-p')x}\, dx$$

$$= \frac{1}{2\pi\hbar}(i\hbar)\frac{\partial}{\partial p}\, 2\pi\, \delta\left(\frac{p-p'}{\hbar}\right)$$

$$= \frac{1}{2\pi\hbar}(i\hbar)\frac{\partial}{\partial p}\, 2\pi\hbar\, \delta(p-p')$$

$$= i\hbar\frac{\partial}{\partial p}\delta(p-p')$$

which is the result obtained earlier using the fundamental commutation relation.

To summarize, we have obtained

$$\langle p|\hat{x}|p'\rangle = i\hbar \frac{\partial}{\partial p}\delta(p-p').$$

Formally, we can write

$$\langle p|\hat{x} = i\hbar \frac{\partial}{\partial p}\langle p|$$

or, in three-dimensions

$$\langle \vec{p}\,|\hat{\vec{R}} = i\hbar \vec{\nabla}_p \langle \vec{p}\,| .$$

Next, consider the operator $\hat{x}$ acting on the ket $|\psi\rangle$ giving a new ket $|\psi'\rangle$, i.e.,

$$|\psi'\rangle = \hat{x}\,|\psi\rangle .$$

Hence

$$\langle p|\psi'\rangle = \langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

i.e., $$\tilde{\psi}'(p) = i\hbar\frac{\partial}{\partial p}\tilde{\psi}(p),$$

or, in other words, the momentum-space wave function of the new ket $|\psi'\rangle$ is obtained by having the differential operator $i\hbar\frac{\partial}{\partial p}$ act on the momentum-space wave function of the old ket $|\psi\rangle$. Thus, in momentum representation

$$\hat{x} \rightarrow i\hbar\frac{\partial}{\partial p}$$

Therefore, $$\hat{x}^n \rightarrow \left(i\hbar\frac{\partial}{\partial p}\right)^n . \quad (n = +\text{ve integer}).$$

In three-dimensions

$$\hat{\vec{R}} \rightarrow i\hbar\vec{\nabla}_{\vec{p}} .$$

Next, consider the operator $V(\hat{x})$ which is an arbitrary function of the position operator $\hat{x}$. The matrix elements of $V(\hat{x})$ can be written as

$$\langle p|V(\hat{x})|p'\rangle$$

$$= \int \langle p|x\rangle\langle x|V(\hat{x})|x'\rangle\langle x'|p'\rangle \, dx\, dx'$$

$$= \int \langle p|x\rangle \; V(x)\,\delta(x-x')\langle x'|p'\rangle \, dx\, dx'$$

$$= \int \langle p|x\rangle V(x) \langle x|p'\rangle \, dx$$

$$= \frac{1}{2\pi\hbar} \int e^{-ipx/\hbar} \, V(x) \, e^{+ip'x/\hbar} \, dx$$

$$= \frac{1}{2\pi\hbar} \int e^{-i(p-p')x/\hbar} \, V(x)\, dx$$

$$\equiv \tilde{V}(p-p') \qquad \text{. . . . . (14)}$$

where we have defined

$$\tilde{V}(p-p') \equiv \frac{1}{2\pi\hbar} \int e^{-i(p-p')x/\hbar} \, V(x)\, dx \, .$$

If $V(\hat{X}) = \hat{X}^n$ (n = integer), then

$$\langle p|\hat{X}^n|p'\rangle = \frac{1}{2\pi\hbar}\int e^{-\frac{i}{\hbar}(p-p')x} x^n dx$$

$$= \frac{1}{2\pi\hbar}\int\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^n e^{-\frac{i}{\hbar}(p-p')x} dx$$

$$= \frac{1}{(2\pi\hbar)}\left(i\hbar\frac{\partial}{\partial p}\right)^n \int e^{-\frac{i}{\hbar}(p-p')x} dx$$

$$= \frac{1}{(2\pi\hbar)}\left(i\hbar\frac{\partial}{\partial p}\right)^n 2\pi\hbar\,\delta(p-p')$$

$$= \left(i\hbar\frac{\partial}{\partial p}\right)^n \delta(p-p')$$

For any arbitrary ket $|\psi\rangle$

$$\langle p|\hat{X}^n|\psi\rangle = \int dp'\,\langle p|\hat{X}^n|p'\rangle\langle p'|\psi\rangle$$

$$= \left(i\hbar\frac{\partial}{\partial p}\right)^n \int dp'\,\delta(p-p')\langle p'|\psi\rangle$$

$$= \left(i\hbar\frac{\partial}{\partial p}\right)^n \langle p|\psi\rangle$$

$$= \left(i\hbar\frac{\partial}{\partial p}\right)^n \tilde{\psi}(p)$$

where $\tilde{\psi}(p)$ is the wave function corresponding to $|\psi\rangle$ in momentum representation.

Thus in momentum representation

$$\hat{X} \rightarrow i\hbar \frac{\partial}{\partial p}$$

$$\hat{X}^n \rightarrow \left(i\hbar \frac{\partial}{\partial p}\right)^n$$

a result which we have derived earlier.

Schrödinger equation in coordinate representation.

The Schrödinger equation in Hilbert space is written as

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle \quad \cdots\cdots\cdots (1)$$

where the ~~the~~ Hamiltonian operator $\hat{H}$ for a single particle is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad \cdots\cdots\cdots (2)$$

In coordinate representation, the Schrödinger equation can be written as

$$i\hbar \frac{\partial}{\partial t}\langle x|\psi(t)\rangle = \langle x|\hat{H}|\psi(t)\rangle$$

$$= \int \langle x|\hat{H}|x'\rangle\langle x'|\psi(t)\rangle\, dx'$$

i.e.,

$$i\hbar\frac{\partial}{\partial t}\langle x|\psi(t)\rangle$$

$$= \int\langle x|\frac{\hat{p}^2}{2m}|x'\rangle\langle x'|\psi(t)\rangle dx' + \int\langle x|V(\hat{x})|x'\rangle\langle x'|\psi(t)\rangle dx'$$

$$= \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\int\delta(x-x')\langle x'|\psi(t)\rangle dx' + \int V(x)\,\delta(x-x')\langle x'|\psi(t)\rangle dx'$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\langle x|\psi(t)\rangle + V(x)\langle x|\psi(t)\rangle$$

We define the time-dependent wavefunction of the particle is

$$\psi(x,t) \equiv \langle x|\psi(t)\rangle$$

Hence

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t)\,. \quad \cdots (4)$$

This is the Schrödinger equation in coordinate representation.

Schrödinger equation in momentum representation.

The Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

where the Hamiltonian operator $\hat{H}$ for a single particle is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

In momentum representation, the operator $\hat{p}$ can be replaced by its eigenvalues p, i.e.,

$$\hat{p} \longrightarrow p \quad (\text{in momentum representation}).$$

More formally,

$$\langle p|\hat{p}|p'\rangle = p\,\delta(p-p'). \qquad \ldots\ldots (5)$$

However, $V(\hat{x})$ is not a multiplicative operator in momentum representation. Previously we have shown (Eq. 14, page 34) that

$$\langle p|V(\hat{x})|p'\rangle = \frac{1}{2\pi\hbar}\int e^{-i(p-p')x/\hbar}\, V(x)\,dx$$

$$\equiv V(p-p'). \qquad \ldots\ldots (6)$$

Now, in momentum representation the Schrödinger equation becomes

$$i\hbar \frac{\partial}{\partial t} \langle p|\psi(t)\rangle = \langle p|\frac{\hat{p}^2}{2m}|\psi(t)\rangle + \langle p|V(\hat{x})|\psi(t)\rangle$$

$$= \int dp' \langle p|\frac{\hat{p}^2}{2m}|p'\rangle\langle p'|\psi(t)\rangle$$

$$+ \int dp' \langle p|V(\hat{x})|p'\rangle\langle p'|\psi(t)\rangle$$

$$= \int dp' \frac{p^2}{2m}\delta(p-p') \langle p'|\psi(t)\rangle$$

$$+ \int dp' \, V(p-p') \langle p'|\psi(t)\rangle$$

or,

$$\boxed{i\hbar \frac{\partial}{\partial t}\tilde{\psi}(p,t) = \frac{p^2}{2m}\tilde{\psi}(p,t) + \int dp' \, V(p-p')\,\tilde{\psi}(p',t)} \quad \cdots (7).$$

In the special case when $V(\hat{x}) = c\,\hat{x}^n$, we can make the replacement

$$\hat{x} \to i\hbar\frac{\partial}{\partial p}, \quad \text{i.e.,} \quad \langle p|\hat{x}|p'\rangle = i\hbar\frac{\partial}{\partial p}\delta(p-p')$$

therefore,

$$\hat{x}^n \to \left(i\hbar\frac{\partial}{\partial p}\right)^n$$

i.e., $\langle p|c\hat{x}^n|p'\rangle \equiv V(p-p') = c\left(i\hbar\frac{\partial}{\partial p}\right)^n \delta(p-p')$.

and the Schrödinger eq. in momentum representation becomes

$$i\hbar \frac{\partial}{\partial t} \tilde{\psi}(p,t) = \frac{p^2}{2m} \tilde{\psi}(p,t) + c\left(i\hbar \frac{\partial}{\partial p}\right)^n \tilde{\psi}(p,t) \,,$$

Example.

One-dimensional harmonic oscillator.

The Hamiltonian of a one-dimensional harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 ,$$

In coordinate representation

$$\hat{p} \to -i\hbar \frac{\partial}{\partial x}$$

$$\hat{x} \to x$$

Therefore the Schrödinger equation in coordinate representation is

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi(x,t).$$

In momentum representation

$$\hat{p} \rightarrow p$$

$$\hat{x} \rightarrow i\hbar \frac{\partial}{\partial p}$$

Therefore, the Schrödinger equation for the harmonic oscillator in momentum representation is

$$i\hbar \frac{\partial \tilde{\Psi}(p,t)}{\partial t} = \frac{p^2}{2m}\tilde{\Psi}(p,t) + \frac{1}{2}m\omega^2\left(i\hbar\frac{\partial}{\partial p}\right)^2\tilde{\Psi}(p,t)$$

or,

$$\boxed{i\hbar \frac{\partial \tilde{\Psi}(p,t)}{\partial t} = \frac{p^2}{2m}\tilde{\Psi}(p,t) - \frac{1}{2}m\omega^2\hbar^2\frac{\partial^2\tilde{\Psi}(p,t)}{\partial p^2}}.$$

Translation operator (An example of an operator that depends continuously on a single parameter).

Let us define an operator $\hat{T}(a)$ as follows:

$$\hat{T}\,|x\rangle = |x+a\rangle \quad \cdots\cdots (1)$$

Therefore,

$$\langle x|\,\hat{T}^{+}(a) = \langle x+a| \quad \cdots\cdots (2).$$

Here a is a ~~real~~ real number. Now from (1) it follows that

$$\hat{T}(a)\,\hat{T}(b) = T(a+b) \quad \cdots\cdots (3)$$

$$\hat{T}^{-1}(a) = \hat{T}(-a) \quad \cdots\cdots (4).$$

Next, we ask what is the effect of the operator $\hat{T}(a)$ on an arbitrary vector $|\psi\rangle$?. Let

$$|\psi'\rangle = \hat{T}(a)\,|\psi\rangle \quad \cdots\cdots (5)$$

In coordinate representation, Eq. (5) is

$$\langle x|\psi'\rangle = \langle x|\hat{T}(a)|\psi\rangle$$

$$= \int \langle x|\hat{T}(a)|x'\rangle\langle x'|\psi\rangle\, dx'$$

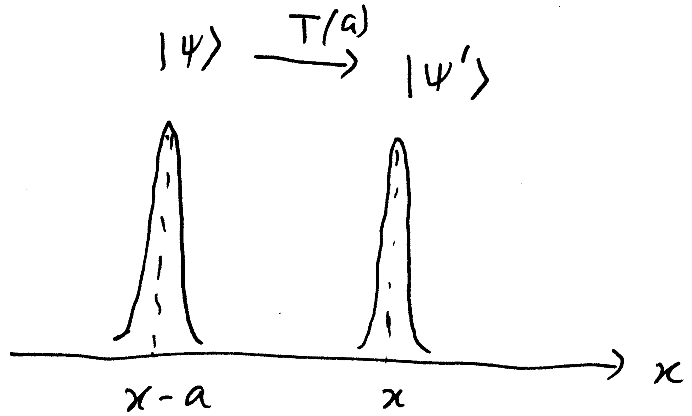
$$= \int \langle x|x'+a\rangle\langle x'|\psi\rangle\, dx'$$

$$= \int \delta(x-x'-a)\,\langle x'|\psi\rangle\, dx'$$

or

$$\psi'(x) = \psi(x-a), \quad \cdots\cdots\cdots (6)$$

i.e., the wavefunction of the new ket $|\psi'\rangle$ at an arbitrary point x ~~is the same~~ has the same value as the wavefunction of the old ket $|\psi\rangle$ at the point $x-a$. In other words, if $|\psi\rangle$ is a ket representing a particular state of the system, then the ket $\hat{T}(a)|\psi\rangle$ represents the state of the system displaced by a.



Now we will prove that $T(a)$ is a unitary operator, i.e., $T^{\dagger}(a) = T^{-1}(a) = T(-a)$. First consider $T(a)\,T^{\dagger}(a)$.

$$T(a)\,T^{\dagger}(a) = \int dx \; T(a)\,|x\rangle\langle x|\,T^{\dagger}(a)$$

$$= \int_{-\infty}^{\infty} dx \; |x+a\rangle\langle x+a|$$

$$= \int_{-\infty}^{\infty} dx' \; |x'\rangle\langle x'|$$

$$= \hat{\mathbb{1}} \, .$$

Next consider $T^\dagger(a)\,T(a)$.

$$T^\dagger(a)T(a) = \int dx\,dx' \; |x\rangle \langle x| T^\dagger(a)\,T(a)|x'\rangle\langle x'|$$

$$= \int dx\,dx' \; |x\rangle \; \langle x+a|x'+a\rangle \; \langle x'|$$

$$= \int dx\,dx' \; |x\rangle \; \delta(x-x') \; \langle x'|$$

$$= \int dx \; |x\rangle\langle x|$$

$$= \hat{\mathbb{1}}$$

Thus we have shown that

$$T(a)T^\dagger(a) = T^\dagger(a)\,T(a) = \hat{\mathbb{1}}.$$

Hence

$$T^\dagger(a) = T^{-1}(a),$$

i.e, $T(a)$ is a unitary operator.

Transformation of kets by a unitary operator preserves the value of the scalar product between any two kets. To see this, consider two kets $|\psi\rangle$ and $|\phi\rangle$ each transformed by a unitary operator $\hat{U}$ to new kets $|\psi'\rangle$ and $|\phi'\rangle$, respectively, i.e.,

$$|\psi'\rangle = \hat{U}|\psi\rangle$$

and

$$|\phi'\rangle = \hat{U}|\phi\rangle .$$

Then

$$\langle\psi'|\phi'\rangle = \langle\psi|\hat{U}^{+}\hat{U}|\phi\rangle$$

$$= \langle\psi|\hat{U}^{-1}\hat{U}|\phi\rangle \quad (\because \hat{U} \text{ is unitary, } \hat{U}^{+} = \hat{U}^{-1})$$

$$= \langle\psi|\phi\rangle .$$

In particular, the norm of a vector $\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$ remains unchanged under a unitary transformation

Construction of $\hat{T}(a)$

Let
$$|\psi'\rangle = T(a)|\psi\rangle.$$

In coordinate representation

$$\psi'(x) = \psi(x-a)$$
$$= \psi(x) - a\frac{\partial}{\partial x}\psi(x) + \frac{1}{2!}a^2\frac{\partial^2}{\partial x^2}\psi(x) + \cdots$$
$$= \psi(x) - \left(\frac{ia}{\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi(x) + \frac{1}{2!}\left(\frac{ia}{\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)^2\psi(x)$$
$$+ \cdots$$

$$= e^{-\frac{ia}{\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)}\psi(x)$$

or
$$\langle x|\psi'\rangle = e^{-\frac{ia}{\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)}\langle x|\psi\rangle$$
$$= \langle x|e^{-\frac{ia}{\hbar}\hat{p}}|\psi\rangle \quad \cdots\cdots\cdots (7)$$

In Eq. (7) we have used
$$\langle x|\hat{p} = -i\hbar\frac{\partial}{\partial x}\langle x|$$

and
$$\langle x|f(\hat{p}) = f\left(-i\hbar\frac{\partial}{\partial x}\right)\langle x|$$

We can write Eq. (7) in representation independent manner as

$$|\psi'\rangle = e^{-\frac{ia}{\hbar}\hat{p}}\,|\psi\rangle. \quad \cdots\cdots (8)$$

Thus, we have proved

$$\hat{T}(a) = e^{-\frac{ia}{\hbar}\hat{p}}. \quad \cdots\cdots (9)$$

Note that $\hat{T}(a)$ is unitary because $\hat{p}$ is hermitian.

Ex Show that

$$\hat{T}^{+}(a)\,\hat{x}\,\hat{T}(a) = \hat{x} + a\hat{\mathbb{1}}$$

where

$$\hat{T}(a) = e^{-ia\hat{p}/\hbar} .$$

Ans Recall the Baker - Hausdorff - Campbell identity (Lecture 4, page 8)

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots$$

If $[\hat{A},\hat{B}]$ is a complex number times the identity operator, i.e., if

$$[\hat{A},\hat{B}] = c\hat{\mathbb{1}}$$

where c is a complex (or real) number, then

$$[\hat{A},[\hat{A},\hat{B}]] = [\hat{A},[\hat{A},[\hat{A},\hat{B}]]] = \cdots = 0 .$$

Therefore

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] .$$

$$(\text{if } [\hat{A},\hat{B}] = c\hat{\mathbb{1}})$$

Now,

$$\hat{T}(a) = e^{-ia\hat{p}/\hbar}$$

Therefore

$$\hat{T}^{+}(a) = \hat{T}^{-1}(a) = e^{ia\hat{p}/\hbar}.$$

We have to show

$$\hat{T}^{+}(a)\,\hat{x}\,\hat{T}(a) = \hat{x} + a\hat{\mathbb{1}}$$

i.e.,

$$e^{ia\hat{p}/\hbar}\,\hat{x}\,e^{-ia\hat{p}/\hbar} = \hat{x} + a\hat{\mathbb{1}}.$$

Let

$$\hat{A} = \frac{ia}{\hbar}\hat{p}$$

$$\hat{B} = \hat{x}$$

$$\therefore [\hat{A}, \hat{B}] = \frac{ia}{\hbar}[\hat{p}, \hat{x}] = -\frac{ia}{\hbar}[\hat{x}, \hat{p}]$$

$$= -\frac{ia}{\hbar}(i\hbar\hat{\mathbb{1}}) = a\hat{\mathbb{1}}$$

Therefore

$$e^{ia\hat{p}/\hbar}\,\hat{x}\,e^{-ia\hat{p}/\hbar} = \hat{x} + a\hat{\mathbb{1}} \quad (\text{Proved}).$$