Rotations and angular momentum

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A rotation of a physical system is specified by the angle of rotation and the axis of rotation. The rotation can be either positive or negative. If a right-handed screw turned in the direction of rotation proceeds along the positive direction of the axis, the rotation is said to be positive. Thus, for example, $\phi\hat{z}$ denotes a positive rotation by an angle ϕ about the z-axis (figure 1).

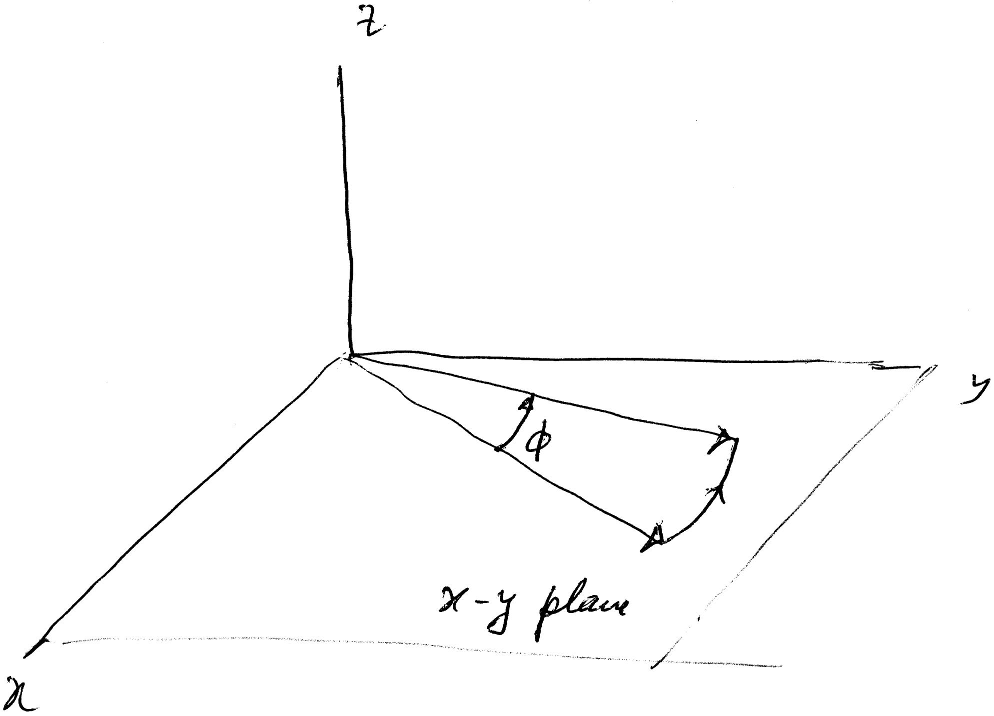


Fig 1: A positive rotation of the physical system by an angle ϕ about the z-axis

In our subsequent discussions we will consider <u>active</u> rotations, i.e., rotation of the physical system rather than the rotation of the coordinates.

Now, finite rotations about different axes do not commute, i.e., the change in the coordinates of a point in the physical system depends on the order the rotations are performed. To work out quantitatively the extent in which rotations about different axes fail to commute, we have to construct the matrices corresponding to rotations in the three-dimensional real space (x, y, z).

In each rotation a vector $\vec{r}$ with coordinates $(x\,y\,z)$ changes to a new vector $\vec{r}\,'$ with coordinates $(x'y'z')$. The matrix connecting $(x'y'z')$ with $(x\,y\,z)$ is the matrix corresponding to the rotation.

Thus

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} & & \\ & R & \\ & & \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (1)$$

where R is the 3×3 square matrix corresponding to the rotation. In a rotation, the length of the vector $\vec{s}$ remains unchanged, i.e.,

$$\sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + z^2} .$$

Therefore, the matrix R must be orthogonal, i.e.,

$$R^T R = R R^T = 1 .$$

Next, we will construct explicitly the rotation matrices R in the three-dimensional space corresponding to positive rotations about the x-axis, y-axis and z-axis.

First, consider a finite rotation by an angle ϕ in a positive sense about the z-axis (Fig. 2).

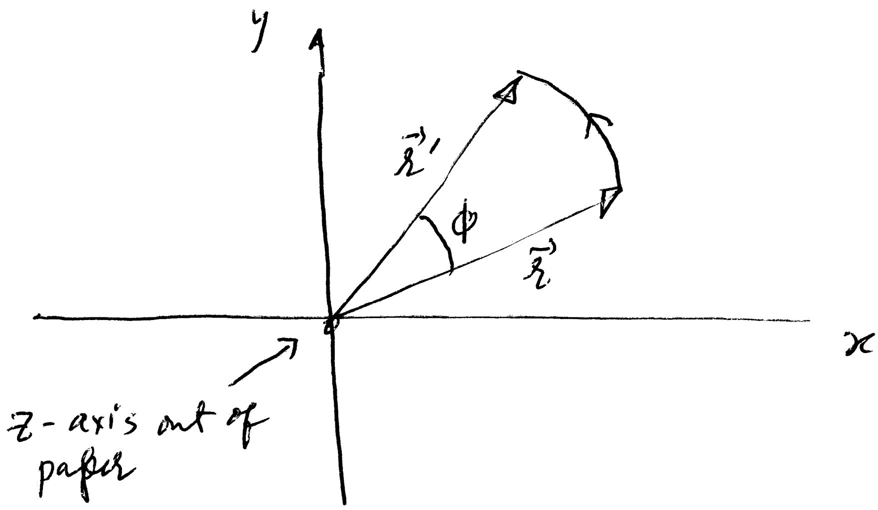


Fig 2: A positive rotation by an angle ϕ about the z-axis.

We have

$$x' = \cos\phi \, x - \sin\phi \, y$$
$$y' = \sin\phi \, x + \cos\phi \, y$$
$$z' = z.$$

In matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus, the rotation matrix corresponding to a positive rotation by an angle ϕ about the z-axis is

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{---- (4)}$$

Next, consider a rotation about the x-axis. The corresponding matrix is

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}. \quad \text{---- (5)}$$

Similarly

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}. \quad \text{---- (6)}$$

For infinitesimal rotation, i.e., $\phi = \epsilon$ (infinitesimal), the rotation matrices, up to second order in ϵ are:

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\epsilon^2/2 & -\epsilon \\ 0 & \epsilon & 1-\epsilon^2/2 \end{pmatrix}$$

$$R_y(\epsilon) = \begin{pmatrix} 1-\epsilon^2/2 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\epsilon^2/2 \end{pmatrix} \quad \cdots (7)$$

$$R_z(\epsilon) = \begin{pmatrix} 1-\epsilon^2/2 & -\epsilon & 0 \\ \epsilon & 1-\epsilon^2/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, matrix multiplication leads to (up to order ϵ^2)

$$R_x(\epsilon) R_y(\epsilon) = \begin{pmatrix} 1-\epsilon^2/2 & 0 & \epsilon \\ \epsilon^2 & 1-\epsilon^2/2 & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix} + O(\epsilon^3) \quad \cdots (8a)$$

and

$$R_y(\epsilon)R_x(\epsilon) = \begin{pmatrix} 1-\epsilon^2/2 & \epsilon^2 & \epsilon \\ 0 & 1-\epsilon^2/2 & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix} \quad \text{- - - - - (8b)}$$

From (8a) and (8b) we have

$$R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon)R_x(\epsilon) = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= R_z(\epsilon^2) - 1 \quad \text{- - - - - (9)}$$

In these calculations all terms higher than ϵ^2 have been ignored. Eq. (9) leads to the important result that infinitesimal rotations about different axes ~~do not~~ do commute up to first order.

Now, we have

$$1 = R_{any}(0) \quad \text{- - - - - - - - (10)}$$

where 'any' stands for any rotation axis. Thus Eq. (9) can be written as

$$\left[R_x(\epsilon),\, R_y(\epsilon)\right] = R_z(\epsilon^2) - R_{any}(0). \qquad (11)$$

Similarly we have

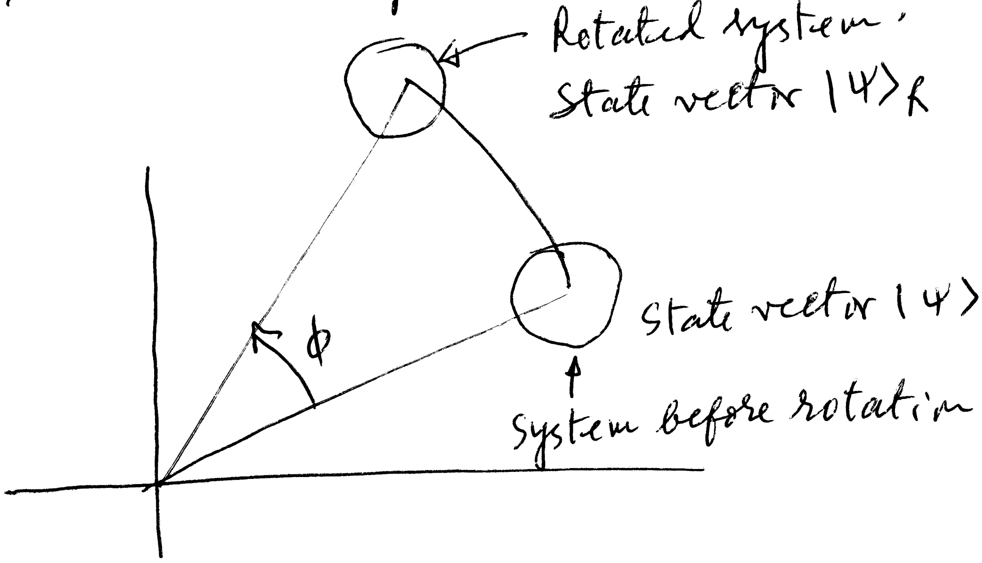
$$\left[R_y(\epsilon),\, R_z(\epsilon)\right] = R_x(\epsilon^2) - R_{any}(0) \qquad (12)$$

$$\left[R_z(\epsilon),\, R_x(\epsilon)\right] = R_y(\epsilon^2) - R_{any}(0) \qquad (13),$$

Eqs. (11) - (13) are examples of the commutation relations between rotational matrices in the three-dimensional real space. These commutation relations will be used later to derive the angular momentum commutation relations.

'Rotations' in Hilbert space (ket space)

Consider a physical system with state vector $|\psi\rangle$ in Hilbert space.



If the system is now rotated by a certain angle about a certain axis, the state vector changes to $|\psi\rangle_R$. Thus there exists an operator $U(R)$ in Hilbert space which carries the state $|\psi\rangle$ to $|\psi\rangle_R$, i.e.,

$$|\psi\rangle_R = U(R)\,|\psi\rangle . \quad \text{------} \quad (14)$$

The operator $U(R)$ is unitary so that normalization of the states remain unaltered.

What we have done is to establish a correspondence between a rotation R in the real three-dimensional space and a unitary operator $U(R)$ in the Hilbert space,

$$R \longleftrightarrow U(R)$$

Rotation in 3-space — Transformation in Hilbert space.

Note that R is a 3×3 orthogonal matrix acting on the components of a classical vector in 3-space, while $U(R)$ is a unitary operator acting on the 'vectors' of a Hilbert space (ket space). We could also find a matrix representation of the operator $U(R)$ in the Hilbert space by choosing an appropriate set of basis kets. If the number of kets in the basis set is N, then the matrix representation of $U(R)$ would be $N \times N$ dimensional.

For example, if we consider a spin-1/2 particle with no other degrees of freedom, then $N=2$ and $U(R)$ would be a 2×2 unitary matrix. For a spin-3/2 particle with no other degrees of freedom, $N=4$, and $U(R)$ would be a 4×4 unitary matrix.

Now, we will construct the unitary operator $U(R)$. To do so, it is advantageous to consider infinitesimal rotations of the physical system. To be specific, suppose the physical system is rotated by an infinitesimal angle $d\varphi$ about the z-axis. Therefore

$$R = \hat{z}\, d\phi .$$

We can then write $U(R)$ as

$$U(d\phi\, \hat{z}) = 1 - \frac{J_z}{\hbar} d\phi \quad \text{------} \quad (15)$$

where J_z is a hermitian operator with dimensions of action (i.e., dimensions of $\hbar$: Energy $\times$ Time). At this stage J_z is not yet identified with the z-component of the total angular momentum operator. This identification will be made later after we derive the commutations of J_z with other generators.

In Eq. (15), J_z is the generator of the unitary operator $U(d\phi\hat{z})$. The operator $U(\phi\hat{z})$ corresponding to a finite positive rotation ϕ about the z-axis can be obtained by successively compounding infinitesimal rotations about the same axis. Thus

$$U_z(\phi) \equiv \lim_{N\to\infty}\left[1 - \frac{iJ_z}{\hbar}\frac{\phi}{N}\right]^N$$

$$= e^{-iJ_z\phi/\hbar}$$

$$= 1 - \frac{iJ_z\phi}{\hbar} - \frac{J_z^2\phi^2}{2\hbar^2} \cdots\cdots \quad (16)$$

Similarly we can write

$$U_x(\phi) = e^{-i J_x \phi/\hbar} \quad \text{- - - - - - - (17)}$$

$$U_y(\phi) = e^{-i J_y \phi/\hbar} . \quad \text{- - - - - - - (18)}$$

In general, for a positive rotation by an angle ϕ about an axis $\hat{n}$, we have

$$U_{\hat{n}}(\phi) = e^{-i \vec{J} \cdot \hat{n} \phi/\hbar} . \quad \text{- - - - - (19)}$$

From Eqs. (16), (17) and (18) we note that the hermitian operators J_x, J_y and J_z are the generators of the unitary transformation operators in the Hilbert space if the system is rotated about the x-, y- and z-axis, respectively. We will now show that the three generators J_x, J_y and J_z obey the commutation relations of angular momentum operators.

Commutation relations of J_x, J_y and J_z.

To obtain the commutation relations between J_x, J_y and J_z, we need the concept of a group. A group G is a collection of objects $g_1, g_2, \cdots$, satisfying the following properties:

1. If $g_1 \in G$ and $g_2 \in G$, then

$$g_1 \cdot g_2 \in G$$

where the symbol '$\cdot$' denotes group 'multiplication'. In other words, a binary operation between the group elements is defined (the binary operation called 'multiplication') such that the product of two group elements leads to another group element. This property of group multiplication is called the ~~clau~~ closure property.

2. There exists an element in the group, called the unit element, and denoted by 1 such that

$$1 \cdot g = g \cdot 1 = g$$

for any $g \in G$.

3. For every $g \in G$, there exists another element in the group, called the inverse of g and denoted by g^{-1} such that

$$g \cdot g^{-1} = g^{-1} \cdot g = 1 .$$

4. Group multiplication is associative:

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$$

Now, rotations form a group. The group 'multiplication' is the application of two rotations successively. To see that the set of <u>all</u> rotations of a physical system form a group, we note that two successive rotations is equivalent to one single rotation. The inverse of a rotation $\phi\hat{n}$ is $-\phi\hat{n}$. The unit element is no rotation at all.

To every rotation of the physical system there corresponds a 3×3 orthogonal matrix R acting on the coordinates of a classical vector, and a unitary operator $U(R)$ acting on the state kets in the Hilbert space. We say that the 3×3 orthogonal ~~matrix R~~ matrices R is a representation of the rotation group in the ordinary 3-space.

The set of unitary operators $U(R)$ is also a representation of the rotation group but in the Hilbert space of state vectors. Thus we may postulate $U(R)$ has the same group properties as R:

	3-space		Hilbert space
identity	$1 \cdot R = R \cdot 1 = R$ (• in 3-space means matrix multiplication)	$\Longrightarrow$	$1\, U(R) = U(R)\, 1 = U(R)$
closure	$R_1 R_2 = R_3$	$\Longrightarrow$	$U(R_1)\, U(R_2) = U(R_3)$
Inverses	$R \cdot R^{-1} = 1$ $R^{-1} \cdot R = 1$	$\Longrightarrow$	$U(R)\, U(R^{-1}) = \hat{1}$ $U(R^{-1})\, U(R) = \hat{1}$ $\therefore U(R^{-1}) = U^{-1}(R)$.
Associativity	$R_1 \cdot (R_2 \cdot R_3)$ $= (R_1 \cdot R_2) \cdot R_3$ $= R_1 \cdot R_2 \cdot R_3$		$U(R_1)\big(U(R_2)\, U(R_3)\big)$ $= \big(U(R_1)\, U(R_2)\big)\, U(R_3)$ $= U(R_1)\, U(R_2)\, U(R_3)$.

Therefore, to any equation involving the orthogonal matrices R, there will be a corresponding equation involving the unitary operators U(R). Now consider Eq (9):

$$R_x(\epsilon) R_y(\epsilon) - R_y(\epsilon) R_x(\epsilon) = R_z(\epsilon^2) - 1, \quad \cdots (9)$$

The analog of this equation in Hilbert space is

$$U_x(\epsilon) U_y(\epsilon) - U_y(\epsilon) U_x(\epsilon) = U_z(\epsilon^2) - 1, \quad \cdots (20)$$

Eqs. (9) and (20) are valid up to second order in ϵ. We therefore expand Eq. (20) up to second order obtaining

$$\left(1 - \frac{i J_x \epsilon}{\hbar} - \frac{J_x^2 \epsilon^2}{2\hbar^2}\right)\left(1 - \frac{i J_y \epsilon}{\hbar} - \frac{J_y^2 \epsilon^2}{2\hbar^2}\right)$$

$$- \left(1 - \frac{i J_y \epsilon}{\hbar} - \frac{J_y^2 \epsilon^2}{2\hbar^2}\right)\left(1 - \frac{i J_x \epsilon}{\hbar} - \frac{J_x^2 \epsilon^2}{2\hbar^2}\right)$$

$$= 1 - \frac{i J_z \epsilon^2}{\hbar} - 1 \qquad (21)$$

Terms of the order ϵ automatically drop out. Equating terms of order ϵ^2 on both sides of Eq. (21) we obtain

$$[J_x, J_y] = i\hbar J_z \; . \qquad (22)$$

Repeating this kind of arguments with rotations about other axes, we obtain

$$[J_y, J_z] = i\hbar J_x \qquad (23)$$

$$[J_z, J_x] = i\hbar J_y \qquad (24)$$

Equations (22) - (24) are the fundamental commutation relations of angular momentum operators. We thus conclude that the generators of the unitary transformations of state vectors in Hilbert space corresponding to rotations of the physical system are nothing but the angular momentum operators.

Rotation operator applied to a spinless particle.

Suppose that a system is rotated by an angle ϕ about an axis $\hat{n}$ in the positive sense. The state of the system changes from $|\psi\rangle$ to $|\psi'\rangle$ according to

$$|\psi'\rangle = U_{\hat{n}}(\phi)\,|\psi\rangle \qquad \text{------ (1)}$$

where

$$U_{\hat{n}}(\phi) = e^{-i\vec{J}\cdot\hat{n}\phi/\hbar} \qquad \text{------ (2)}$$

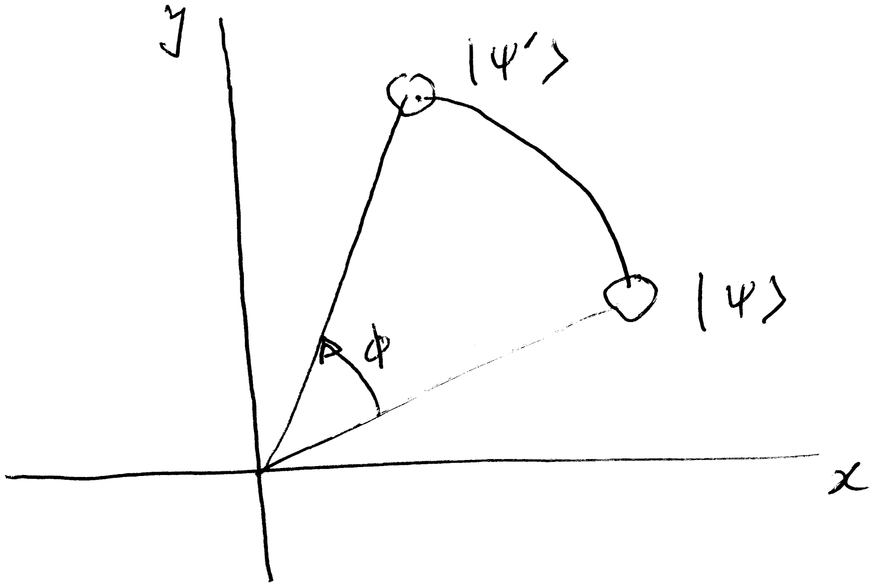
Now, if the system is a spinless particle ($\vec{S} = 0$), then

$$\vec{J} = \vec{L}$$

and the corresponding rotation operator is

$$U_{\hat{n}}(\phi) = e^{-i\vec{L}\cdot\hat{n}\phi/\hbar} \qquad \text{------ (3)}.$$

For simplicity, we will consider rotation about the z-axis and see how the wavefunction changes under such a rotation.



In this case we have

$$|\psi'\rangle = U_z(\phi)\,|\psi\rangle$$

$$= e^{-iL_z\phi/\hbar}\,|\psi\rangle. \quad \cdots\cdots (4)$$

In coordinate representation Eq. (4) is

$$\langle\vec{r}\,|\psi'\rangle = \langle\vec{r}\,|\,e^{-iL_z\phi/\hbar}\,|\psi\rangle \quad \cdots\cdots (5)$$

Now

$$\langle \vec{r} | \hat{L}_z = \langle \vec{r} | (\hat{\vec{r}} \times \hat{\vec{p}})_z$$

$$= \left(\vec{r} \times \frac{\hbar}{i} \vec{\nabla}\right)_z \langle \vec{r} |$$

Therefore, Eq. (5) can be written as

$$\langle \vec{r} | \psi' \rangle = e^{-\frac{i}{\hbar} \phi \left(\vec{r} \times \frac{\hbar}{i} \vec{\nabla}\right)_z} \langle \vec{r} | \psi \rangle$$

For infinitesimal rotations $d\phi$ we have

$$\psi'(\vec{r}) = \left[1 - \frac{i}{\hbar} d\phi \left(\vec{r} \times \frac{\hbar}{i} \vec{\nabla}\right)_z \right] \psi(\vec{r})$$

$$= \left[1 - d\phi \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \right] \psi(\vec{r})$$

$$= \psi(\vec{r}) - d\phi \, x \frac{\partial \psi(x y z)}{\partial y} + d\phi \, y \frac{\partial \psi(x y z)}{\partial x}$$

$$= \psi(x + y \, d\phi, \; y - x \, d\varphi, \; z)$$

Now

$$R_z(d\phi) = \begin{pmatrix} 1 & -d\phi & 0 \\ d\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \quad R_z^{-1}(d\varphi) = \begin{pmatrix} 1 & d\phi & 0 \\ -d\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

therefore

$$R_z^{-1}(d\varphi)\,\underline{r} = \begin{pmatrix} 1 & d\varphi & 0 \\ -d\varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y\,d\varphi \\ y - x\,d\varphi \\ z \end{pmatrix}$$

Hence

$$\boxed{\psi'(\vec{r}) = \psi(R^{-1}\vec{r})} \quad \cdots\cdots (6)$$

Since $\vec{r}$ is arbitrary in Eq. (6), we can rewrite (6) in the form

$$\boxed{\psi'(R\vec{r}) = \psi(\vec{r})} \quad \cdots\cdots (7)$$

Eq. (7) is intuitively obvious. If the system is rotated, the old wave function at any point $\vec{r}$ must be equal to the ~~rotated~~ new wave function at the rotated point $R\vec{r}$.

We have derived the transformation equation of the wave function (Eq. (6) or (7)) of a spinless particle using the expression rotation operator given in Eq. (3). We could work backwards, i.e., starting from the obvious transformation of the wave function under rotation, i.e., Eq. (6) or (7), we could work out the expression for the rotation operator in the Hilbert space of spinless particles.

Thus, if the system is rotated by an angle φ about the z-axis, the wave function changes from $\psi(\vec{r})$ to $\psi'(\vec{r})$ in such a manner that

$$\psi'(\vec{r}) = \psi(R_z^{-1}(\varphi)\vec{r}), \quad \cdots\cdots (8)$$

We cast Eq. (8) in the form

$$\psi'(\vec{r}) = U_z(\varphi)\psi(\vec{r}) \quad \cdots\cdots (9)$$

where $U_z(\varphi)$ is the rotation operator in the Hilbert space. To express Eq. (8) in the form Eq. (9), it is convenient to consider infinitesimal rotations, Then

$$R_z^{-1}(d\phi) = \begin{pmatrix} 1 & d\phi & 0 \\ -d\varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and Eq. (8) is

$$\psi'(\vec{r}) = \psi(x + y\,d\varphi,\ y - x\,d\varphi,\ z)$$

$$= \psi(x y z) + y\,d\varphi \frac{\partial\psi(x y z)}{\partial x} - x\,d\varphi \frac{\partial\psi(x y z)}{\partial y}$$

$$= \left[1 - d\varphi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi(x y z)$$

$$= \left[1 - \frac{i}{\hbar}d\varphi\ \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi(x y z)$$

$$= \left[1 - \frac{i}{\hbar}d\varphi\left(\vec{r}\times\vec{p}\right)_z\right]\psi(x y z)$$

$$= \left[1 - \frac{i}{\hbar}d\varphi\,\hat{L}_z\right]\psi(x y z). \quad \text{------} \quad (10)$$

For a finite rotation we then have

$$\psi'(\vec{r}) = e^{-\frac{i}{\hbar}\varphi L_z}\psi(\vec{r}). \quad \text{--------} \quad (11)$$

Hence, the rotation operator in the Hilbert space of a spinless particle is

$$U_z(\varphi) = e^{-i\varphi L_z/\hbar} \quad \text{------} \quad (11)$$

- Rotation operator in spin space.

Consider a particle with only spin degrees of freedom, i.e., the particle's spatial degrees of freedom are suppressed. In this case $\vec{L} = 0$, and, therefore, $\vec{J} = \vec{S}$ where $\vec{S}$ is the spin angular momentum operator of the particle. The rotation operator in the spin Hilbert space is then

$$^{(S)}U_{\hat{n}}(\varphi) = e^{-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\varphi} \quad \text{------} \quad (13)$$

Consider now a rotation by a finite angle φ about the z-axis. If the ket of a spin-½ particle before rotation is $|\alpha\rangle$, the ket after rotation is given by:

$$|\alpha\rangle_R = {}^{(S)}U_z(\varphi)\,|\alpha\rangle$$

$$= e^{-\frac{i}{\hbar}S_z\varphi}\,|\alpha\rangle \,. \quad \cdots\cdots\cdots (14)$$

To show that the operator (13) really rotates the physical system, let us look at its effects on $\langle S_x\rangle$. Under the rotation about the z-axis, this expectation value changes to

$$\langle S_x\rangle \to \langle S_x\rangle' = {}_R\langle\alpha|\,S_x\,|\alpha\rangle_R$$

$$= \langle\alpha|\,U_z^{+}(\varphi)\,S_x\,U_z(\varphi)\,|\alpha\rangle \,.$$

We must therefore compute

$$U_z^{+}(\varphi)\,S_x\,U_z(\varphi) = e^{iS_z\varphi/\hbar}\,S_x\,e^{-iS_z\varphi/\hbar} \,.$$

Using the identity

$$e^{A}\,B\,e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots$$

we have

$$e^{iS_z\varphi/\hbar}\, S_x\, e^{-iS_z\varphi/\hbar}$$

$$= S_x + \left(\frac{i\varphi}{\hbar}\right)\underbrace{[S_z, S_x]}_{i\hbar S_y} + \frac{1}{2!}\left(\frac{i\varphi}{\hbar}\right)^2 \underbrace{[S_z, \underbrace{[S_z, S_x]}_{i\hbar S_y}]}_{\hbar^2 S_x}$$

$$+ \frac{1}{3!}\left(\frac{i\varphi}{\hbar}\right)^3 \underbrace{[S_z, \underbrace{[S_z, [S_z, S_x]]}_{\hbar^2 S_x}]}_{i\hbar^3 S_y} + \cdots$$

$$= S_x\left(1 - \frac{\varphi^2}{2!} + \cdots\right) - S_y\left(\varphi - \frac{\varphi^3}{3!} + \cdots\right)$$

$$= S_x \cos\varphi - S_y \sin\varphi$$

i.e.,

$$e^{iS_z\varphi/\hbar}\, S_x\, e^{-iS_z\varphi/\hbar} = S_x\cos\varphi - S_y \sin\varphi$$

Thus

$$\langle S_x \rangle' = \langle S_x \rangle \cos\varphi - \langle S_y \rangle \sin\varphi ,$$

Similarly

$$\langle S_y \rangle' = \langle S_x \rangle \sin\varphi + \langle S_y \rangle \cos\varphi \qquad \text{---- (15).}$$

$$\langle S_z \rangle' = \langle S_z \rangle$$

This shows that the 'rotation' operator (13) when applied to the state ket does ~~no~~ rotate the expectation value of $\vec{S}$ around the z-axis by an angle φ. In other words, the expectation values of the spin operator behaves as though it were a classical vector.

Up to now we have dealt with the expectation values of the spin operator in the rotated and unrotated spin states. Now let us look at the effect of the rotation operator $U_z(\varphi)$ on a general spin state $|\alpha\rangle$. For a spin-½ particle we write

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle \quad \text{- - - - (16)}$$

where $|\pm\rangle$ are the eigenstates of S_z with eigenvalues $\pm\frac{1}{2}\hbar$. Now

$$e^{-iS_z\varphi/\hbar}|\alpha\rangle = e^{-i\phi/2}|+\rangle\langle+|\alpha\rangle + e^{i\varphi/2}|-\rangle\langle-|\alpha\rangle.$$

The appearance of the half-angle $\phi/2$ here has an extremely interesting consequence. Let us consider a rotation by an angle 2π. We then have

$$|\alpha\rangle \xrightarrow[R_z(2\pi)]{} |\alpha\rangle_R = -|\alpha\rangle, \quad \text{- - - - - (17)}$$

so that the ket for the 360° rotated state differs from the original ket by a minus sign. We would a 720° ($\varphi = 4\pi$) rotation to get back to the same state with plus sign. Notice that this minus sign disappears from the expectation values of $\vec{S}$ because $\vec{S}$ is sandwiched between $\langle\alpha|$ and $|\alpha\rangle$, both of which change sign.

(Will this minus sign be observable? See Sakurai).

Matrix representation of the rotation operator in spin space.

The rotation operator in spin space is

$$^{(S)}U_{\hat{n}}(\varphi) = e^{-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\varphi}$$

For a spin-½ particle we can use the eigenkets $|\pm\rangle$ of S_z as basis. Then $\vec{S}$ and $^{(S)}U_{\hat{n}}(\varphi)$ are expressed as 2×2 matrices. We then have

$$\vec{S} \doteq \frac{1}{2}\hbar\vec{\sigma}$$

Therefore

$$e^{-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\varphi} \doteq e^{-i\frac{\vec{\sigma}\cdot\hat{n}}{2}\varphi}$$

The 2×2 matrices ~~at~~ $\exp(-i\,\vec{\sigma}\cdot\hat{n}\varphi/2)$ act on the two-component spinor χ where

$$\chi = \begin{pmatrix} \langle +|\alpha\rangle \\ \langle -|\alpha\rangle \end{pmatrix}.$$

Now

$$^{(S)}U_n(\varphi) \doteq e^{-i\frac{\varphi}{2}\vec{\sigma}\cdot\hat{n}}$$

$$= 1 + \left(-\frac{i\varphi}{2}\right)\vec{\sigma}\cdot\hat{n} + \frac{1}{2!}\left(-\frac{i\varphi}{2}\right)^2(\vec{\sigma}\cdot\hat{n})^2$$

$$+ \dots + \frac{1}{m!}\left(-\frac{i\varphi}{2}\right)^m(\vec{\sigma}\cdot\hat{n})^m + \dots$$

Applying the identity

$$(\vec{\sigma}\cdot\hat{n})^2 = \hat{n}\cdot\hat{n}\,1_{2\times 2} + i\,\sigma\cdot(\hat{n}\times\hat{n})$$

$$= 1_{2\times 2}$$

which leads to

$$\left(\vec{\sigma}.\hat{n}\right)^m = \begin{cases} 1 & \text{if } m \text{ is even} \\ \vec{\sigma}.\hat{n} & \text{if } m \text{ is odd} \end{cases}$$

we get

$$^{(S)}U_{\hat{n}}(\varphi) = \left[1 - \frac{1}{2!}\left(\frac{\varphi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\varphi}{2}\right)^4 - \dots\right] 1_{2\times 2}$$

$$- i\,\vec{\sigma}.\hat{n}\left[\varphi - \frac{1}{3!}\left(\frac{\varphi}{2}\right)^3 + \frac{1}{5!}\left(\frac{\varphi}{2}\right)^5 - \dots\right]$$

Thus

$$\boxed{^{(S)}U_{\hat{n}}(\varphi) = \cos\frac{\varphi}{2}\, 1_{2\times 2} - i\,\vec{\sigma}.\hat{n}\,\sin\frac{\varphi}{2}}$$

Spin-1/2 particles.

Next,

$$\vec{\sigma}.\hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} n_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} n_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} n_z$$

$$= \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

Therefore

$$e^{-i\vec{\sigma}\cdot\hat{n}\,\varphi/2} = \begin{bmatrix} \cos\varphi/2 - i n_z \sin\varphi/2 & (-i n_x - n_y)\sin\varphi/2 \\ (-i n_x + n_y)\sin\varphi/2 & \cos\varphi/2 + i n_z \sin\varphi/2 \end{bmatrix}$$

Note that

$$\left. e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2} \right|_{\varphi = 2\pi} = -1 \quad \text{for any } \hat{n}$$

Thus

$$\chi \xrightarrow[2\pi \text{ rotation}]{} -\chi .$$

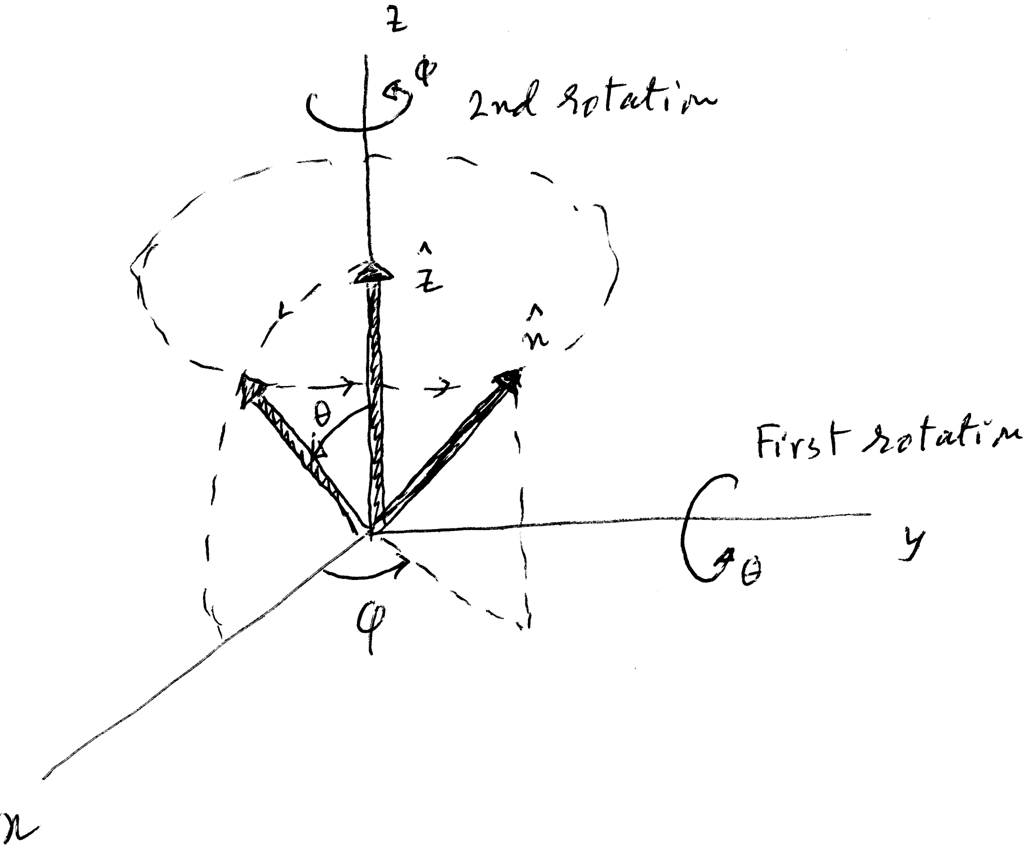
- Eigen spinors of $\vec{\sigma}\cdot\hat{n}$

As an instructive application of the rotation matrix, let us see how we can construct eigenspinors of $\vec{\sigma}\cdot\hat{n}$ with eigenvalues ± 1 where $\hat{n}$ is some unit vector along some specified direction. Our object is to construct $\chi_\pm$ satisfying

$$\vec{\sigma}\cdot\hat{n}\,\chi_\pm = \pm\,\chi_\pm \quad \text{------} (1)$$

Actually, $\chi_\pm$ can be obtained as a straight eigenvalue problem, but here we present an alternative method using the rotation matrix.

Let the polar and azimuthal angle of $\hat{n}$ be θ and ϕ, respectively. Let us start with $\begin{pmatrix}1\\0\end{pmatrix}$, the two-component spinor that represents the spin-up state. Given this, we first rotate about the y-axis by an angle θ, then rotate by an angle φ about the z-axis, as shown in the figure below.



The desired spin state is then obtained.

$$\chi_+ = e^{-i\sigma_z \phi/2}\, e^{-i\sigma_y \theta/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \left[\cos\frac{\varphi}{2} - i\sigma_z \sin\frac{\phi}{2}\right]\left[\cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or,

$$\chi_+ = \begin{bmatrix} \cos\varphi/2 - i\sin\varphi/2 & 0 \\ 0 & \cos\frac{\varphi}{2} + i\sin\varphi/2 \end{bmatrix} \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{bmatrix} \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix} = \begin{bmatrix} e^{-i\varphi/2}\sin\theta/2 \\ e^{i\varphi/2}\cos\theta/2 \end{bmatrix}.$$

To get χ_- we could apply the same sequence of rotations to $\binom{0}{1}$, or, we could get χ_- from χ_+ simply by noting that $-\hat{n}$ has polar coordinates $(\pi-\theta, \pi+\varphi)$, so that

$$\chi_- = \begin{pmatrix} -i\, e^{-i\varphi/2}\sin\theta/2 \\ i\, e^{i\varphi/2}\cos\theta/2 \end{pmatrix}.$$

Disregarding the overall phase factor i, we can write

$$\chi_- = \begin{pmatrix} -e^{-i\varphi/2}\sin\theta/2 \\ e^{i\varphi/2}\cos\theta/2 \end{pmatrix}.$$

Rotation of two-component spinors

We are now prepared to study the global behaviour of a spin-1/2 particle under rotation. That is, we shall now take into account both the internal and external degrees of freedom of the particle.

Consider a spin-1/2 particle whose state is represented by the ket $|\psi\rangle$ in the state space (Hilbert space) $\mathcal{E} = \mathcal{E}_r \otimes \mathcal{E}_s$. The ket can be represented by the spinor $[\psi](\vec{r})$ having the components

$$\psi_\epsilon(\vec{r}) = \langle \vec{r}, \epsilon | \psi \rangle$$

where $\epsilon = \pm$ represents the two degrees of freedom in the spin space. Thus

$$|\psi\rangle \doteq [\psi](\vec{r}) = \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

If we perform an arbitrary rotation on the particle, its state vector changes to $|\psi'\rangle$ where

$$|\psi'\rangle = U|\psi\rangle$$

with

$$U = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\varphi} = e^{-\frac{i}{\hbar}(\vec{L}+\vec{S})\cdot\hat{n}\varphi}$$

$$= e^{-\frac{i}{\hbar}\vec{L}\cdot\hat{n}\varphi}\, e^{-\frac{i}{\hbar}\vec{S}\cdot\hat{n}\varphi} \qquad (\text{Note}: [\vec{L},\vec{S}]=0)$$

$$= {}^{(r)}U(\varphi) \otimes {}^{(s)}U(\varphi)$$

where

$${}^{(r)}U(\varphi) = e^{-i\vec{L}\cdot\hat{n}\varphi/\hbar}$$

$${}^{(s)}U(\varphi) = e^{-i\vec{S}\cdot\hat{n}\varphi/\hbar}.$$

The rotation operator ${}^{(r)}U(\varphi)$ acts in the space $\mathcal{E}_r$ with basis $\{|\vec{r}\rangle\}$, i.e., the space of external variables, and the operator ${}^{(s)}U(\varphi)$ acts on the spin space $\mathcal{E}_s$ with basis $\{|+\rangle$ and $|-\rangle\}$.

We write the spinor corresponding to the transformed state $|\psi'\rangle$ as

$$[\psi'](\vec{r}) = \begin{pmatrix} \psi'_+(\vec{r}) \\ \psi'_-(\vec{r}) \end{pmatrix}.$$

We will now ~~directly~~ derive a formula which connects the spinor $[\psi'](\vec{r})$ to the spinor $[\psi](\vec{r})$. First, let us write the components $\psi'_\epsilon(\vec{r})$ of the spinor $[\psi'](\vec{r})$ as

$$\psi'_\epsilon(\vec{r}) = \langle \vec{r}\,\epsilon|\psi'\rangle$$
$$= \langle \vec{r}\,\epsilon|U|\psi\rangle.$$

Using the closure (i.e., completeness) relation

$$\sum_{\epsilon'=\pm}\int d^3r'\, |\vec{r}\,'\epsilon'\rangle\langle\vec{r}\,'\epsilon'| = \hat{1}$$

we obtain

$$\psi'_\epsilon(\vec{r}) = \sum_{\epsilon'=\pm}' \int d^3r' \langle \vec{r}\epsilon|U|\vec{r}'\epsilon'\rangle\langle \vec{r}'\epsilon'|\psi\rangle .$$

Now, since the basis vectors $\{|\vec{r}\epsilon\rangle = |\vec{r}\rangle \otimes |\epsilon\rangle\}$ are tensor products, the matrix elements of the operator U in this basis can be decomposed in the following manner:

$$\langle \vec{r}\epsilon|U|\vec{r}'\epsilon'\rangle = \langle r|{}^{(r)}U|\vec{r}'\rangle\langle \epsilon|{}^{(s)}U|\epsilon'\rangle$$

Now,

$${}^{(r)}U|\vec{r}\rangle = |R\vec{r}\rangle$$

Therefore,

$$\langle \vec{r}|{}^{(r)}U^{+} = \langle Rr|$$

or,

$$\langle \vec{r}| = \langle R\vec{r}|{}^{(r)}U \qquad \left(\text{since } {}^{(r)}U \text{ is unitary } {}^{(r)}U^{+} = {}^{(r)}U^{-1}\right)$$

Since $\vec{r}$ is arbitrary

$$\langle R^{-1}\vec{r}| = \langle \vec{r}|{}^{(r)}U$$

Therefore we have

$$\langle \vec{r}|{}^{(r)}U|\vec{r}'\rangle = \langle R^{-1}\vec{r}|\vec{r}'\rangle = \delta(\vec{r}' - R^{-1}\vec{r}).$$

Next, let us call

$$\langle \epsilon |\, {}^{(s)}U \,| \epsilon' \rangle = U^{(1/2)}_{\epsilon \epsilon'}$$

Then

$$\langle \vec{r}\, \epsilon |\, U \,| \vec{r}\,' \epsilon' \rangle = \delta(\vec{r}\,' - R^{-1}\vec{r}\,)\, U^{(1/2)}_{\epsilon\epsilon'}$$

and the transformed ~~wave function~~ spinor is

$$\boxed{\psi'_{\epsilon}(\vec{r}\,) = \sum_{\epsilon'=\pm} U^{(1/2)}_{\epsilon\epsilon'}\, \psi_{\epsilon'}(R^{-1}\vec{r}\,)}\;.$$

Explicitly

$$\begin{pmatrix} \psi'_{+}(\vec{r}\,) \\ \psi'_{-}(\vec{r}\,) \end{pmatrix} = \begin{pmatrix} U^{(1/2)}_{++} & U^{(1/2)}_{+-} \\ U^{(1/2)}_{-+} & U^{(1/2)}_{--} \end{pmatrix} \begin{pmatrix} \psi_{+}(R^{-1}\vec{r}\,) \\ \psi_{-}(R^{-1}\vec{r}\,) \end{pmatrix}.$$

Thus we obtain the following result: each component of the new spinor $[\psi']$ at the point $\vec{r}$ is a linear combination of the two components of the original spinor $[\psi]$ at the point $R^{-1}\vec{r}$. The coefficients of these linear combinations are the elements of the 2×2 matrix which represents ${}^{(s)}U$ in the $\{|+\rangle, |-\rangle\}$ basis.

Spin precession Sakurai

Consider a spin-1/2 particle with its space degrees of freedom suppressed. If the particle is subjected to an external magnetic field, its Hamiltonian can be written as

$$H = -\vec{\mu} \cdot \vec{B} \quad \cdots\cdots \quad (1)$$

where $\vec{\mu}$ is the magnetic moment operator of the spin-1/2 particle, say electron. Since the electron has only spin degrees of freedom, we can write

$$\vec{\mu} = g_s \frac{q_e}{2m_e} \vec{S} \quad \cdots\cdots \quad (2)$$

where $q_e = -e$ is the charge of the electron and $g_s = 2$ is the spin gyromagnetic ratio of the electron. With $g_s = 2$, the Hamiltonian can be written as

$$H = -\frac{q_e}{m_e} \vec{S} \cdot \vec{B} \quad \cdots\cdots \quad (3)$$

If the direction of $\vec{B}$ is taken as the $\hat{z}$-axis, i.e., if $\vec{B} = \hat{z} B$, the Hamiltonian becomes

$$H = -\frac{q_e B}{m_e} S_z$$

or,

$$H = \frac{eB}{m_e} S_z .$$

$|q_e| = e$ = positive $= 1.6 \times 10^{-19} C$

Defining

$$\omega = \frac{eB}{m_e} \quad \text{-------------} \quad (4)$$

we can write

$$H = \omega S_z . \quad \text{-------------} \quad (5)$$

Now suppose that the electron is in an initial spin-state $|\alpha, t=0\rangle$. We ask what is the state of the system at a later time t. We can find this state by applying the time evolution operator to the initial state :

$$|\alpha, t\rangle = T(t, 0) |\alpha, t=0\rangle \quad \text{-----} \quad (6)$$

where $T(t,0)$ is the time evolution operator given by

$$T(t,0) = e^{-iHt/\hbar} \quad \text{(assuming } H \text{ is time independent)}, \tag{7}$$

In our case H is given by Eq. (5) so that

$$T(t,0) = e^{-i\omega t S_z/\hbar} \quad \text{------} \quad (8).$$

Applying Eq. (8) in Eq. (6) we obtain

$$|\alpha, t\rangle = e^{-i\omega t S_z/\hbar} \, |\alpha, t=0\rangle \quad \text{------} \quad (9).$$

We notice that the time-evolution operator (Eq. (8)) is nothing but the rotation operator in the Hilbert space corresponding to the rotation of the physical system by an angle ωt about the z-axis, i.e., about the direction of the applied magnetic field. This means that the spin precesses around the magnetic field as an axis. To see this let us calculate the expectation values of S_x, S_y and S_z at time t.

$$\langle S_x\rangle_t = \langle\alpha t|S_x|\alpha t\rangle = \langle\alpha\, t=0|\, e^{i\omega t S_z/\hbar}\, S_x\, e^{-i\omega t S_z/\hbar}\, |\alpha\, t=0\rangle$$

$$\langle S_y\rangle_t = \langle\alpha t|S_y|\alpha t\rangle = \langle\alpha\, t=0|\, e^{i\omega t S_z/\hbar}\, S_y\, e^{-i\omega t S_z/\hbar}\, |\alpha\, t=0\rangle$$

$$\langle S_z\rangle_t = \langle\alpha t|S_z|\alpha t\rangle = \langle\alpha\, t=0|\, e^{i\omega t S_z/\hbar}\, S_z\, e^{-i\omega t S_z/\hbar}\, |\alpha\, t=0\rangle.$$

Now, we can easily show the following relations:

$$e^{i\omega t S_z/\hbar}\, S_x\, e^{-i\omega t S_z/\hbar} = S_x \cos\omega t - S_y \sin\omega t$$

$$e^{i\omega t S_z/\hbar}\, S_y\, e^{-i\omega t S_z/\hbar} = S_x \sin\omega t + S_y \cos\omega T$$

$$e^{i\omega t S_z/\hbar}\, S_z\, e^{-i\omega t S_z/\hbar} = S_z .$$

Thus

$$\langle S_x\rangle_t = \langle S_x\rangle_{t=0} \cos\omega t - \langle S_y\rangle_{t=0} \sin\omega t$$

$$\langle S_y\rangle_t = \langle S_x\rangle_{t=0} \sin\omega t + \langle S_y\rangle_{t=0} \cos\omega t \qquad \cdots (10)$$

$$\langle S_z\rangle_t = \langle S_z\rangle_{t=0} .$$

The set of equations (10) shows clearly that the spin of the electron precesses around the z-axis, i.e., around the direction of $\vec{B}$ with an angular frequency ω, i.e., with a time period $\tau = 2\pi/\omega$ (figure below):