Addition of angular momentum.

Suppose we have two independent angular momentum operators $\vec{J}_1$ and $\vec{J}_2$ of a system. The operators $\vec{J}_1$ and $\vec{J}_2$ refer to particles 1 and 2 of a two particle system, or they might refer to orbital angular momentum and spin angular momentum of a single particle. Since J_{1i} and J_{2i}, $i = 1,2,3$ are angular momentum operators, ~~they~~ they satisfy the following commutation relations

$$[J_{1i}, J_{1j}] = i\,\epsilon_{ijk}\,J_{1k} \qquad (1)$$

$$[J_{2i}, J_{2j}] = i\,\epsilon_{ijk}\,J_{2k}. \qquad (2)$$

Further, since J_{1i} are independent of J_{2i}, we also have

$$[J_{1i}, J_{2j}] = 0.$$

Next, we define the operator $\vec{J}$ as

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

called the total angular momentum of the system.

It is important to realize that $\vec{J}$ satisfies the angular momentum commutation relations:

$$[J_i, J_j] = [J_{1i} + J_{2i}, J_{1j} + J_{2j}]$$

$$= [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}]$$

$$= i\epsilon_{ijk} J_{1k} + i\epsilon_{ijk} J_{2k}$$

$$= i\epsilon_{ijk}(J_{1k} + J_{2k})$$

$$= i\epsilon_{ijk} J_k .$$

Next, to describe the angular momentum states of the system we need a basis set of states. The basis states are eigenkets of a complete set of commutating observables (CSCO). One such CSCO is

$$J_1^2, J_2^2, J_{1z}, J_{2z} .$$

The simultaneous eigenvectors of this set of operators are written as $|j_1 j_2 m_1 m_2\rangle$. Thus

$$J_1^2 |j_1 j_2 m_1 m_2\rangle = j_1(j_1+1)\hbar^2 |j_1 j_2 m_1 m_2\rangle$$

$$J_2^2 |j_1 j_2 m_1 m_2\rangle = j_2(j_2+1)\hbar^2 |j_1 j_2 m_1 m_2\rangle \tag{4}$$

$$J_{1z} |j_1 j_2 m_1 m_2\rangle = m_1 \hbar |j_1 j_2 m_1 m_2\rangle$$

$$J_{2z} |j_1 j_2 m_1 m_2\rangle = m_2 \hbar |j_1 j_2 m_1 m_2\rangle .$$

Since the operator sets $\{J_1^2, J_{1z}\}$ and $\{J_2^2, J_{2z}\}$ are independent of each other, we can also write

$$|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle ,$$

where $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$ are eigenkets of $\{J_1^2, J_{1z}\}$ and $\{J_2^2, J_{2z}\}$ respectively.

Now, the complete set of commutating observables can be chosen differently. Noting that

$$[J^2, J_1^2] = [J^2, J_2^2] = [J^2, J_z] = 0 ,$$

the set of operators

$$\{J_1^2, J_2^2, J^2 \text{ and } J_z\}$$

is also a complete set of commutating observables.

Therefore, simultaneous eigenkets of this set of operators can also be chosen as a basis. These eigenkets are denoted as

$$|j_1 j_2 \, j m\rangle$$

where

$$J_1^2 |j_1 j_2 \, j m\rangle = j_1(j_1+1)\hbar^2 |j_1 j_2 \, j m\rangle$$

$$J_2^2 |j_1 j_2 \, j m\rangle = j_2(j_2+1)\hbar^2 |j_1 j_2 \, j m\rangle$$

$$J^2 |j_1 j_2 \, j m\rangle = j(j+1)\hbar^2 |j_1 j_2 \, j m\rangle$$

$$J_z |j_1 j_2 \, j m\rangle = m\hbar |j_1 j_2 \, j m\rangle .$$

Since the states $|j_1 j_2 \, j m\rangle$ are the eigenkets of the total angular momentum operators J^2 & J_z, they are also called coupled states. On the other hand, the basis $|j_1 j_2 \, m_1 m_2\rangle$ are called uncoupled states.

We can now state our problem. Given j_1 and j_2, what are the possible values of j? For fixed j_1 and j_2, the two basis sets are related by a unitary transformation

$$|j_1 j_2\, j m\rangle = \sum_{m_1 m_2} \langle j_1 j_2\, m_1 m_2 | j_1 j_2\, j m\rangle\, |j_1 j_2\, m_1 m_2\rangle \qquad \text{---}(6)$$

where we have used the closure relation

$$\sum_{m_1 m_2} |j_1 j_2\, m_1 m_2\rangle\langle j_1 j_2\, m_1 m_2| = 1$$

in the ket space of j_1 and j_2. The coefficients

$$\langle j_1 j_2\, m_1 m_2 | j_1 j_2\, j m\rangle$$

are called the Clebsch-Gordan coefficients. We will write the Clebsch-Gordan (CG) coefficients in short as

$$\langle j_1 j_2\, m_1 m_2 | j m\rangle.$$

The coupled states $|j_1 j_2\, j m\rangle$ are also written as $|j m\rangle$ in short since j_1 and j_2 are fixed.

Thus we write Eq. (6) as

$$|jm\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | jm\rangle \, |j_1 j_2 m_1 m_2\rangle . \quad (7)$$

To proceed, we can show that the CG coefficients vanish unless $m = m_1 + m_2$. To show this apply the operator $J_z = J_{1z} + J_{2z}$ to Eq. (7). We have

$$J_z |jm\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | jm\rangle \left(J_{1z} + J_{2z}\right) |j_1 j_2 m_1 m_2\rangle$$

$$\Rightarrow m\hbar \, |jm\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | jm\rangle \, (m_1 + m_2)\hbar \, |j_1 j_2 m_1 m_2\rangle$$

$$\Rightarrow \sum_{m_1 m_2} (m - m_1 - m_2) \langle j_1 j_2 m_1 m_2 | jm\rangle \, |j_1 j_2 m_1 m_2\rangle = 0 .$$

Since the basis states $|j_1 j_2 m_1 m_2\rangle$ are linearly independent, we have

$$(m - m_1 - m_2) \langle j_1 j_2 m_1 m_2 | jm\rangle = 0$$

Hence

$$\boxed{\langle j_1 j_2\, m_1 m_2 | j m\rangle = 0 \quad \text{unless } m = m_1 + m_2} \tag{8}$$

- We are now ready to find the possible values of j for given j_1 and j_2. Without loss of generality we assume that $j_1 \geq j_2$. Now, since $m = m_1 + m_2$, the maximum value of m is

$$m^{max} = m_1^{max} + m_2^{max} = j_1 + j_2 .$$

Since m can take on $(2j+1)$ values $-j, -j+1, \cdots j$, it follows that the maximum possible value of j is also j_1+j_2. Thus there is only one basis state corresponding to $m = m^{max} = j_1+j_2$. This state can be written as either $|j_1 j_2 ; j_1 j_2\rangle$ in the uncoupled $(m_1 m_2)$ basis, or as $|j_1 j_2\ j_1+j_2 , j_1+j_2\rangle$ in the coupled (jm) basis. Since there is only one state corresponding to $m_1 = j_1'$ and $m_2 = j_2$, we have apart from a phase

$$|j_1\, j_2\; j = j_1 + j_2\;\; m = j_1 + j_2\rangle = |j_1\, j_2\;\; m_1 = j_1,\, m_2 = j_2\rangle.$$

Next, consider $m = m^{max} - 1 = j_1 + j_2 - 1$. In the uncoupled $|j_1\, j_2\, m_1\, m_2\rangle$ basis, there are two kets that correspond to this value of m. These two kets are obtained by choosing m_1 and m_2 as follows:

$$m_1 = j_1 \qquad m_2 = j_2 - 1$$

$$m_1 = j_1 - 1 \qquad m_2 = j_2\,.$$

Thus in the $(m_1 m_2)$ basis, the two basis states for $m = j_1 + j_2 - 1$ are

$$|j_1\, j_2\;\, j_1\;\, j_2 - 1\rangle \text{ and } |j_1\, j_2\;\, j_1 - 1,\, j_2\rangle$$

For $m = j_1 + j_2 - 1$, there must be two-fold degeneracy in the basis $|j_1\, j_2\, j\, m\rangle$ as well. Since $m = j_1 + j_2 - 1$ is compatible with either $j = j_1 + j_2$ or with $j = j_1 + j_2 - 1$, the two states in the $|j_1 j_2\, j m\rangle$ basis are identified with

$$j = j_1 + j_2 \quad \text{and} \quad j = j_1 + j_2 - 1 .$$

Next, consider $m = m^{max} - 2 = j_1 + j_2 - 2$. In this case ~~the~~ there are three-fold degeneracy and in the $|j_1 j_2\, m_1 m_2\rangle$ the degeneracy corresponds to

$$m_1 = j_1 \qquad m_2 = j_2 - 2$$

$$m_1 = j_1 - 1 \qquad m_2 = j_2 - 1$$

$$m_1 = j_2 - 2 \qquad m_2 = j_2 .$$

Therefore, there is a three-fold degeneracy in the coupled basis $|j_1 j_2\, j m\rangle$ corresponding to

$$j = j_1 + j_2 ,\ j_1 + j_2 - 1 \ \text{and} \ j_1 + j_2 - 2 .$$
$$= j_1 + j_2 ,\ j_1 + j_2 - 1 ,\ j_1 + j_2 - (d-1) \quad \text{where } d \text{ is the degeneracy.}$$

We can continue in this way, but it is clear that the degeneracy cannot increase indefinitely. Indeed for $m = m^{min} = -j_1 - j_2$ there is again a single ket. The maximum degeneracy is

$(2j_2+1)$-fold as is apparent from the table below.

Table 1: Allowed values of m and $(m_1\, m_2)$ for $j_1 = 2$ and $j_2 = 1$.

$j_1 = 2,\ j_2 = 1$							
m	3	2	1	0	-1	-2	-3
$(m_1\, m_2)$	(2 1)	(1 1) (2 0)	(0 1) (1 0) (2 -1)	(-1 1) (0 0) (1 -1)	(-2 1) (-1 0) (0 -1)	(-2 0) (-1 -1)	(-2 -1)
No. of states (degeneracy)	1	2	3	3	3	2	1
j	3	(3,2)	(3,2,1)	(3,2,1)	(3,2,1)	(3,2)	3

The $(2j_2+1)$-fold degeneracy must be associated with

$$j = j_1+j_2,\ j_1+j_2-1,\ \ldots,\ j_1+j_2-(2j_2+1-1)$$

i.e.,

$$j = j_1+j_2,\ j_1+j_2-1,\ \ldots\ j_1-j_2.$$

If we lift the restriction $j_1 \geq j_2$, we can write

$$\boxed{j = j_1+j_2,\ j_1+j_2-1,\ \cdots\ |j_1-j_2|} \quad \cdots \quad (9)$$

Number of basis vectors for given j_1 and j_2

For given j_1 and j_2, the basis vectors are either

$$|j_1 j_2 m_1 m_2\rangle, \quad m_1 = -j_1, -j_1+1, \cdots m_1$$
$$m_2 = -j_2, -j_2+1, \cdots m_2$$

($m_1 m_2$ basis)

or

$$|j_1 j_2 j m\rangle \quad j = j_1 \oplus j_2 = j_1+j_2,\ j_1+j_2-1, \cdots |j_1-j_2|$$
$$m = -j\ -j+1, \cdots j$$

(jm basis).

The dimension of the vector space for given j_1 and j_2 must be the same no matter which basis set we use. In the $(m_1 m_2)$ basis, the number of basis vectors (i.e., the dimension of the space) is

$$N = (2j_1+1)(2j_2+1).$$

If we do the counting in the $(j\,m)$ basis, the number of basis vectors is

$$N = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1)$$

which is the same as the number of basis vectors in the $(m_1 m_2)$ basis.

Examples of angular momentum addition.

Example 1: Addition of two spins 1/2

Consider a two-particle system where each particle has spin 1/2, i.e., $s_1 = s_2 = 1/2$. The basis of spin states of the system may be written (in the notation $|s_1 s_2; m_1 m_2\rangle = |s_1 m_1\rangle |s_2 m_2\rangle$) as

$$|\tfrac{1}{2}\tfrac{1}{2}; \tfrac{1}{2}\tfrac{1}{2}\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = \alpha(1)\alpha(2)$$

$$|\tfrac{1}{2}\tfrac{1}{2}; \tfrac{1}{2}\,-\tfrac{1}{2}\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2 = \alpha(1)\beta(2)$$

$$|\tfrac{1}{2}\tfrac{1}{2}; -\tfrac{1}{2}\tfrac{1}{2}\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = \beta(1)\alpha(2)$$

$$|\tfrac{1}{2}\tfrac{1}{2}; -\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2 = \beta(1)\beta(2)$$

where α is the short hand notation for spin-up state $|\tfrac{1}{2}\tfrac{1}{2}\rangle$ and β the spin-down state $|\tfrac{1}{2}\,-1/2\rangle$.

Now, the allowed values of the total spin quantum number s of the system are given by

$$s = s_1 \oplus s_2 = \tfrac{1}{2} \oplus \tfrac{1}{2} = 1, 0 .$$

The basis states could also be chosen as the vectors $|s_1 s_2 ; s m\rangle$ which are eigenstates of $\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}^2, \hat{S}_z\}$. There are four such coupled basis states corresponding to $s=1$, $m = 1, 0, -1$ and $s=0$, $m=0$.

We would like to construct the coupled states in terms of the uncoupled states. We simplify our notation and write the coupled states as

$$|s_1 s_2 ; s m\rangle \equiv |s m\rangle = \chi_{s m_s} .$$

Now, the four coupled states are χ_{11}, χ_{10}, χ_{1-1} and χ_{00}. First, consider χ_{11}. For $m=1$, there is only one way m_1 and m_2 can be chosen: $m_1 = 1/2$, $m_2 = 1/2$. Thus

$$\chi_{11} = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 = \alpha(1)\alpha(2). \quad \cdots\cdots \quad (1)$$

Next, to obtain χ_{10}, we obtain the lowering operator

$$S_- = S_{1-} + S_{2-}$$

to the state χ_{11}. We have the general formula

$$S_-|s\,m\rangle = \sqrt{(s+m)(s-m+1)}\;|s\;m-1\rangle$$

similar formulas hold if we apply S_{1-} and S_{2-} to the states $|s_1\,m_1\rangle$ and $|s_2\,m_2\rangle$ respectively. Thus

$$S_-\chi_{11} = \left(S_{1-}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1\right)|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1\left(S_{2-}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2\right)$$

or

$$\sqrt{(1+1)(1-1+1)}\;\chi_{10} = \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\;|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_1\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_2$$

$$+\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_1\,\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\;|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_2\,.$$

or,

$$\sqrt{2}\,\chi_{10} = \left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_1 \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_2 + \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle_1 \left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle_2$$

i.e.,

$$\chi_{10} = \frac{1}{\sqrt{2}}\left(\alpha(1)\beta(2) + \beta(1)\alpha(2)\right) \quad \cdots\cdots (2)$$

Next, for χ_{1-1}, there is again a single possibility for the choice of (m_1, m_2), i.e., $(-1/2, -1/2)$. Thus

$$\chi_{1-1} = \beta(1)\beta(2) \quad \cdots\cdots\cdots\cdots (3)$$

Finally, we have to construct the state $\chi_{sm} = \chi_{00}$. For $m=0$, there are two possibilities for m_1 and m_2:

$$m_1 = \tfrac{1}{2}, \quad m_2 = -\tfrac{1}{2} \quad ; \quad \alpha(1)\beta(2)$$

$$m_1 = -\tfrac{1}{2} \quad m_2 = \tfrac{1}{2} \quad ; \quad \beta(1)\alpha(2).$$

Therefore, χ_{00} like χ_{10} must be a linear combination of $\alpha(1)\beta(2)$ and $\beta(1)\alpha(2)$. The linear combination must be chosen such that χ_{00} is orthogonal to χ_{10} (Eq. 2) and that χ_{00} is normalized. Hence, by inspection we can write

$$\chi_{00} = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2)\right] \quad \cdots\cdots (4)$$

Summarizing, the states of total spin are

$$\left.\begin{aligned} \chi_{11} &= \alpha(1)\alpha(2) \\ \chi_{10} &= \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right] \\ \chi_{1-1} &= \beta(1)\beta(2) \end{aligned}\right\} \text{Triplet, symmetric}$$

and

$$\chi_{00} = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2)\right] \Big\} \begin{array}{l}\text{singlet,}\\ \text{antisymmetric.}\end{array}$$

The three states corresponding to $s=1$ are called triplets and they are symmetric under the interchange of particles 1 and 2, i.e., $1 \leftrightarrow 2$. The singlet state χ_{00} corresponds to $s=0$ and this state is antisymmetric under $1 \leftrightarrow 2$.

Clebsch-Gordan coefficients for addition of two spin-$\frac{1}{2}$'s.

We have

$$|s_1 s_2 ; s m\rangle = \sum_{\substack{m_1 m_2 \\ m_1+m_2=m}}' \langle s_1 s_2 ; m_1 m_2 | s_1 s_2 ; s m\rangle \, | s_1 s_2 ; m_1 m_2\rangle$$

Writing in short,

$$|s m\rangle = \sum_{m_1 m_2}' \langle s_1 s_2 \, m_1 m_2 | s m\rangle \, |s_1 m_1\rangle \, |s_2 m_2\rangle$$

or

$$\chi_{sm} = \sum_{m_1 m_2}' \langle s_1 s_2 \, m_1 m_2 | s m\rangle \, |s_1 m_1\rangle \, |s_2 m_2\rangle .$$

For $s_1 = s_2 = \frac{1}{2}$, we have $s = 1, 0$. Previously we obtained

$$\chi_{11} \equiv |s=1, m=1\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 \, |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = \alpha(1)\alpha(2) .$$

Therefore

$$\boxed{\langle \tfrac{1}{2}\tfrac{1}{2} ; \tfrac{1}{2}\tfrac{1}{2} | 1 1\rangle = 1} .$$

Also $\chi_{1-1} = |s=1, m=-1\rangle = |\tfrac{1}{2} -\tfrac{1}{2}\rangle_1 \, |\tfrac{1}{2} -\tfrac{1}{2}\rangle_2 = \beta(1)\beta(2)$

So

$$\boxed{\langle \tfrac{1}{2}\tfrac{1}{2} ; -\tfrac{1}{2} -\tfrac{1}{2} | 1 -1\rangle = 1}$$

Next, $\chi_{10} = |s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(\alpha(1)\beta(2) + \beta(1)\alpha(2)\right)$

$\therefore$

$$\boxed{\begin{aligned} \langle \tfrac{1}{2}\,\tfrac{1}{2};\,\tfrac{1}{2}\,-\tfrac{1}{2}|1\,0\rangle &= \frac{1}{\sqrt{2}} \\ \langle \tfrac{1}{2}\,\tfrac{1}{2};\,-\tfrac{1}{2}\,\tfrac{1}{2}|1\,0\rangle &= \frac{1}{\sqrt{2}} \end{aligned}}$$

We have also obtained

$$\chi_{00} = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2)\right]$$

$\therefore$

$$\boxed{\begin{aligned} \langle \tfrac{1}{2}\,\tfrac{1}{2};\,\tfrac{1}{2}\,-\tfrac{1}{2}|0\,0\rangle &= \frac{1}{\sqrt{2}} \\ \langle \tfrac{1}{2}\,\tfrac{1}{2};\,-\tfrac{1}{2}\,\tfrac{1}{2}|0\,0\rangle &= -\frac{1}{\sqrt{2}} \end{aligned}}$$

Example of addition of angular momenta.

Example 2

Suppose an electron in an atom is in the p-state. That is $l=1$ for the electron. The three orbital angular momentum states accessible to the electron are $|l\,m_l\rangle$ with $l=1$ and $m_l = 1, 0, -1$. In the coordinate the orbital angular momentum states are just the spherical harmonics:

$$|l\,m_l\rangle \doteq Y_{l\,m_l}(\theta\,\varphi) .$$

The spin quantum number of the electron is $s=\frac{1}{2}$. The spin states are written as $|s\,m_s\rangle$ or $\chi_{s\,m_s}$. The spin-up states $\chi_{1/2\,1/2}$ are often denoted by α and the spin-down states $\chi_{1/2\,-1/2}$ by β.

Now, the quantum number j for the total angular momentum is

$$j = l \oplus s = 1 \oplus 1/2 = 3/2, 1/2.$$

The coupled states, i.e., the eigenstates of $\{L^2, S^2, J^2, J_z\}$ are written as $|ls; jm\rangle$. Since l and s are given and fixed, the coupled states are simply written as $|jm\rangle$ omitting the quantum numbers l and s.

Sometimes we denote the coupled states with a curly Y. Thus

$$|jm\rangle \equiv \mathcal{Y}_{jm}.$$

Now

$$|jm\rangle = \sum_{m_l, m_s}' \langle ls; m_l m_s | jm\rangle |ls; m_l m_s\rangle$$

$$= \sum_{m_l m_s}' \langle ls; m_l m_s | jm\rangle |l m_l\rangle |s m_s\rangle$$

i.e.,

$$\mathcal{Y}_{jm}(\theta, \phi) = \sum_{m_l, m_s}' \langle ls; m_l m_s | jm\rangle Y_{l m_l}(\theta, \phi) \chi_{s, m_s}.$$

In the present example, the coupled states are

$\mathcal{Y}_{jm}$: $\quad \mathcal{Y}_{3/2\,3/2} \quad \mathcal{Y}_{3/2\,1/2} \quad \mathcal{Y}_{3/2\,-1/2} \quad \mathcal{Y}_{3/2\,-3/2}$

and

$\mathcal{Y}_{1/2\,1/2} \quad \mathcal{Y}_{1/2\,-1/2}$.

Let us first consider $\mathcal{Y}_{3/2\,3/2}$. The only way we can have $m = m_\ell + m_s = 3/2$ is by taking $m_\ell = 1$ and $m_s = 1/2$. So, we must have

$$\mathcal{Y}_{3/2\,3/2} = Y_{11}\,\chi_{1/2\,1/2} \qquad \text{- - - - . (1)}$$

The corresponding CG coefficient is therefore

$$\langle 1\,\tfrac{1}{2}\,;\,1\,\tfrac{1}{2}\,|\,\tfrac{3}{2}\,\tfrac{3}{2}\rangle = 1 .$$

$\underline{\underline{\mathcal{Y}_{3/2\,\frac{1}{2}}}}$

We get the state $\mathcal{Y}_{3/2\,1/2}$ by applying the lowering operator

$$J_- = L_- + S_-$$

to the state $\mathcal{Y}_{3/2\,3/2}$ in Eq. (1). We obtain

$$J_- \mathcal{Y}_{3/2\,3/2} = (L_- Y_{11})\chi_{1/2\,1/2} + Y_{11}(S_- \chi_{1/2\,1/2})$$

or

$$\sqrt{(3/2+3/2)(3/2-3/2+1)}\,\mathcal{Y}_{3/2\,1/2}$$

$$= \sqrt{(1+1)(1-1+1)}\; Y_{10}\,\chi_{1/2\,1/2} + Y_{11}\sqrt{(1/2+1/2)(1/2-1/2+1)}\,\chi_{1/2\,-1/2}$$

or $$\sqrt{3}\,\mathcal{Y}_{3/2\,1/2} = \sqrt{2}\,Y_{10}\,\chi_{1/2\,1/2} + Y_{11}\,\chi_{1/2\,-1/2}$$

or $$\boxed{\mathcal{Y}_{3/2\,1/2} = \sqrt{\frac{2}{3}}\,Y_{10}\,\chi_{1/2\,1/2} + \frac{1}{\sqrt{3}}\,Y_{11}\,\chi_{1/2\,-1/2},} \qquad (2)$$

The corresponding CG coefficients are easily read off from this expression:

$$\langle 1\,1/2;\,0\,1/2\,|\,3/2\,1/2\rangle = \sqrt{\frac{2}{3}}$$

$$\langle 1\,1/2;\,1\,-1/2\,|\,3/2\,1/2\rangle = \frac{1}{\sqrt{3}}\,.$$

$\mathcal{Y}_{3/2\,-1/2}$

Next, to ~~ob~~ obtain the state $\mathcal{Y}_{3/2\,-1/2}$ we could apply the lowering operator again to the state $\mathcal{Y}_{3/2\,1/2}$. However, it is easier to write down the state $\mathcal{Y}_{3/2\,-3/2}$ and then to apply the raising operator to this state.

Now, obviously

$$\boxed{\mathcal{Y}_{3/2\,-3/2} = Y_{1\,-1}\,\chi_{1/2\,-1/2}} \quad \cdots\cdots (3)$$

Hence

$$\langle 1\,\tfrac{1}{2};\,-1\,-1/2\,|\,3/2\,-3/2\rangle = 1.$$

Applying the raising operator $J_+ = L_+ + S_+$ to $\mathcal{Y}_{3/2\,-3/2}$ we obtain

$$J_+ \mathcal{Y}_{3/2\,-3/2} = \left(L_+ Y_{1\,-1}\right)\chi_{1/2\,-1/2} + Y_{1\,-1}\left(S_+ \chi_{1/2\,-1/2}\right)$$

For a raising operator we have

$$J_+ |j\,m\rangle = \sqrt{(j-m)(j+m+1)}\;|j\;m+1\rangle .$$

Hence

$$\sqrt{(3/2-3/2)(3/2+3/2+1)}\;\mathcal{Y}_{3/2\,-1/2}$$

$$= \sqrt{(1+1)(1-1+1)}\;Y_{10}\,\chi_{1/2\,-1/2} + Y_{1\,-1}\sqrt{(1/2+1/2)(1/2-1/2+1)}\;\chi_{1/2\,1/2}$$

or $$\sqrt{3}\,\mathcal{Y}_{3/2\,-1/2} = \sqrt{2}\,Y_{10}\,\chi_{1/2\,-1/2} + Y_{1\,-1}\,\chi_{1/2\,1/2}$$

or $$\boxed{\mathcal{Y}_{3/2\,-1/2} = \frac{1}{\sqrt{3}}\,Y_{1\,-1}\,\chi_{1/2\,1/2} + \sqrt{\frac{2}{3}}\,Y_{10}\,\chi_{1/2\,-1/2}} \quad (4)$$

The corresponding Clebsch-Gordan coefficients are

$$\langle 1\,1/2;\,-1\,1/2 | 3/2\,-1/2\rangle = \frac{1}{\sqrt{3}}$$

$$\langle 1\,1/2;\,0\,-1/2 | 3/2\,-1/2\rangle = \sqrt{2/3}\,.$$

Finally for $j = 1/2$, we have to form the two coupled states $\mathcal{Y}_{1/2\,1/2}$ and $\mathcal{Y}_{1/2\,-1/2}$. For $m = m_l + m_s = 1/2$, the possible ~~choices~~ choices of $(m_l\ m_s)$ are: $(m_l\ m_s) = (0\ 1/2)$ and $(1\ -1/2)$. Thus $\mathcal{Y}_{1/2\,1/2}$ must be a linear combination of states $Y_{10}\chi_{1/2\,1/2}$ and $Y_{11}\chi_{1/2\,-1/2}$. We write

$$\mathcal{Y}_{1/2\,1/2} = c_1 Y_{10}\chi_{1/2\,1/2} + c_2 Y_{11}\chi_{1/2\,-1/2} .$$

The state $\mathcal{Y}_{3/2\,1/2}$ (Eq. 2) is a different linear combination of the same two states ~~of~~ $Y_{10}\chi_{1/2}$ and $Y_{10}\chi_{1/2\,-1/2}$. Now $\mathcal{Y}_{1/2\,1/2}$ must be orthogonal to $\mathcal{Y}_{3/2\,1/2}$ and also should be normalized. Therefore,

$$(\mathcal{Y}_{3/2\,1/2}, \mathcal{Y}_{1/2\,1/2}) = 0$$

i.e., $$\sqrt{\frac{2}{3}}\,c_1 + \frac{1}{\sqrt{3}}\,c_2 = 0 \quad \cdots\cdots (5)$$

and $$(\mathcal{Y}_{1/2\,1/2}, \mathcal{Y}_{1/2\,1/2}) = 1$$

i.e.,

$$|c_1|^2 + |c_2|^2 = 1 \, , \quad \text{------} \quad (6)$$

We can choose c_1 and c_2 to be real with the values

$$c_1 = \frac{1}{\sqrt{3}} \quad \text{and} \quad c_2 = -\sqrt{\frac{2}{3}} \, .$$

Therefore,

$$\boxed{\mathcal{Y}_{\frac{1}{2}\frac{1}{2}} = \frac{1}{\sqrt{3}} Y_{10}\, \chi_{\frac{1}{2}\frac{1}{2}} - \sqrt{\frac{2}{3}}\, Y_{11}\, \chi_{\frac{1}{2},-\frac{1}{2}}} \quad \text{--} \ (7)$$

The corresponding CG coefficients are then

$$\langle 1\, \tfrac{1}{2}\, ;\, 0\, \tfrac{1}{2} \,|\, \tfrac{1}{2}\, \tfrac{1}{2} \rangle = \frac{1}{\sqrt{3}}$$

$$\langle 1\, \tfrac{1}{2}\, ;\, 1\, -\tfrac{1}{2} \,|\, \tfrac{1}{2}\, \tfrac{1}{2} \rangle = -\sqrt{\frac{2}{3}} \, .$$

Note that there is an arbitrariness in the choice of sign of the CG coefficients. We could equally well have chosen $c_1 = -\frac{1}{3}$ and $c_2 = \sqrt{2/3}$. This choice will reverse the sign of the CG coefficients above.

Finally, we have to construct the state $\mathcal{Y}_{\frac{1}{2}-\frac{1}{2}}$. This state can now be obtained by applying the lowering operator to $\mathcal{Y}_{\frac{1}{2}\frac{1}{2}}$. We obtain

$$J_- \mathcal{Y}_{\frac{1}{2}\frac{1}{2}} = \frac{1}{\sqrt{3}}\left(L_- Y_{10}\right)\chi_{\frac{1}{2}\frac{1}{2}} + \frac{1}{\sqrt{3}} Y_{10}\left(S_- \chi_{\frac{1}{2}\frac{1}{2}}\right)$$

$$- \sqrt{\frac{2}{3}}\left(L_- Y_{11}\right)\chi_{\frac{1}{2}-\frac{1}{2}} - \sqrt{\frac{2}{3}} Y_{11}\underbrace{\left(S_- \chi_{\frac{1}{2}-\frac{1}{2}}\right)}_{=0}$$

or

$$\sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)\left(\tfrac{1}{2}-\tfrac{1}{2}+1\right)}\; \mathcal{Y}_{\frac{1}{2}-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{3}}\sqrt{(1+0)(1-0+1)}\; Y_{1-1}\chi_{\frac{1}{2}\frac{1}{2}} + \frac{1}{\sqrt{3}} Y_{10}\sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)\left(\tfrac{1}{2}-\tfrac{1}{2}+1\right)}\;\chi_{\frac{1}{2}-\frac{1}{2}}$$

$$- \sqrt{\frac{2}{3}}\sqrt{(1+1)(1-1+1)}\; Y_{10}\chi_{\frac{1}{2}-\frac{1}{2}} + 0$$

or

$$\mathcal{Y}_{\frac{1}{2}-\frac{1}{2}} = \sqrt{\frac{2}{3}}\, Y_{1-1}\chi_{\frac{1}{2}\frac{1}{2}} + \frac{1}{\sqrt{3}} Y_{10}\chi_{\frac{1}{2}-\frac{1}{2}}$$

$$- \frac{2}{\sqrt{3}} Y_{10}\chi_{\frac{1}{2}-\frac{1}{2}}$$

or

$$\boxed{\mathcal{Y}_{\frac{1}{2}-\frac{1}{2}} = \sqrt{\frac{2}{3}}\, Y_{1-1}\chi_{\frac{1}{2}\frac{1}{2}} - \frac{1}{\sqrt{3}} Y_{10}\chi_{\frac{1}{2}-\frac{1}{2}}} \quad (8)$$

The corresponding CG coefficients are

$$\langle 1\ \tfrac{1}{2}\ ;\ -1\ \tfrac{1}{2}\ |\ \tfrac{1}{2}\ -\tfrac{1}{2} \rangle = \sqrt{2/3}$$

$$\langle 1\ \tfrac{1}{2}\ ;\ 0\ -\tfrac{1}{2}\ |\ \tfrac{1}{2}\ -\tfrac{1}{2} \rangle = -\ 1/\sqrt{3}\ .$$

Properties of Clebsch - Gordan coefficients.

The Clebsch - Gordan coefficients are written as

$$\langle j_1 j_2\ ;\ m_1 m_2\ |\ j_1 j_2\ j\, m \rangle$$

or, in short

$$\langle j_1 j_2\ ;\ m_1 m_2\ |\ j\, m \rangle .$$

Some of the properties of the CG coefficients are listed below :

1. The CG coefficients are chosen to be real
2. $\langle j_1 j_2\ ;\ m_1 m_2\ |\ j\, m \rangle = 0$ unless $m = m_1 + m_2$
3. $\langle j_1 j_2\ ;\ m_1 m_2\ |\ j\, m \rangle = 0$

   unless $j = j_1 + j_2\ ,\ j_1 + j_2 - 1\ ,\ \ldots\ |j_1 - j_2|\ .$

4. Orthogonality properties of the CG coefficients.

We have

$$|j_1 j_2 ; j m\rangle = \sum_{\substack{m_1, m_2 \\ (m_1+m_2=m)}} |j_1 j_2 ; m_1 m_2\rangle \langle j_1 j_2 ; m_1 m_2 | j m\rangle$$

Since the vectors $|j_1 j_2 ; j m\rangle$ also form an orthonormal basis in the space of j_1 and j_2, we can also write

$$|j_1 j_2 ; m_1 m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j_1 j_2 ; j m\rangle \langle j m | j_1 j_2 ; m_1 m_2\rangle$$

Since CG coefficients are chosen to be real,

$$\langle j m | j_1 j_2 ; m_1 m_2\rangle = \langle j_1 j_2 ; m_1 m_2 | j m\rangle$$

Now, the kets $|j_1 j_2 ; j m\rangle$ (i.e., $|j m\rangle$ in short) are orthonormal, i.e.,

$$\langle j m | j' m'\rangle = \delta_{jj'} \delta_{mm'} \quad \cdots\cdots (1)$$

Using the closure relation

$$\sum_{m_1 m_2}' |j_1 j_2; m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| = 1,$$

Eq. (1) can be written as

$$\sum_{m_1 m_2}' \langle j m | j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2 | j' m'\rangle = \delta_{jj'}\,\delta_{mm'}$$

or

$$\boxed{\sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j m\rangle\langle j_1 j_2; m_1 m_2 | j' m'\rangle = \delta_{jj'}\,\delta_{mm'}}. \quad (2)$$

Similarly, the kets $|j_1 j_2; m_1 m_2\rangle$ are also orthonormal,

i.e.,

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; m_1' m_2'\rangle = \delta_{m_1 m_1'}\,\delta_{m_2 m_2'}$$

Inserting the closure relation

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j m\rangle\langle j m| = 1$$

we have

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} \langle j_1 j_2 ; m_1 m_2 | j m \rangle \langle j m | j_1 j_2 ; m_1' m_2' \rangle$$
$$= \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

Taking the reality of the CG coefficients into account, we can write

$$\boxed{\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} \langle j_1 j_2 ; m_1 m_2 | j m \rangle \langle j_1 j_2 ; m_1' m_2' | j m \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}}$$

— X —

END