MS Thesis Presentation
Title
Approximate Solution of Differential Equations using Legendre Wavelet Collocation Method

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Presentation Contents

Motivation and literature survay

- Differential equations has wide spread applications in science and engineering.
- The mathematical model of many real world problems give rise to linear or nonlinear differential equations.
- There are different techniques available for the solution of differential equation.
- Each technique has its own advantages and limitations.
- Exact solution of differential equations in most cases is not easy particularly when the problem is nonlinear.
- Therefore, researchers are in progress to develop new methods for the solution of differential equations.

Motivation and literature survey

- Recently the use of wavelet analysis for the solution of differential equations is very popular.
- Several wavelet based numerical methods are available in literature for the solution of differential equations.
- These methods include Haar Wavelet Method (HWM), Wavelet Finite Element Method (WFEM), Wavelet Meshless Method (WMM), Wavelet Boundary Element Method (WBEM), Lagrangian Wavelet Method (LWM), Chebyshev Wavelet Method (CWM), and Legendre Wavelet Collocation Method (LWCM)[1, 2, 3, 4].

Legendre Wavelet Collocation Method (LWCM)

- Legendre Wavelet Collocation Method (LWCM) is a wavelet based method which uses Legendre polynomials for the solution of differential equations.

Legendre polynomials and its properties

- The use of orthogonal polynomials in approximation theory has gained much consideration.
- These polynomials can approximate a function.
- The Legendre polynomials $L_n(x)$ satisfies the Legendre differential equation which is given by

$$(1 - x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0 \tag{1}$$

where n is non-negative integer.

Legendre polynomials and its properties

- The Legendre polynomials is given by the following equation [5]

$$L_n(x) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^s (2n-2s)! x^{n-2s}}{2^n s!(n-s)!(n-2s)!}, \tag{2}$$

- Another representation of Legendre polynomials is given by [5]

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \qquad n = 0, 1, 2, 3, \ldots\ldots \tag{3}$$

Formula (3) is known as *Rodrigues* formula.

Legendre polynomials and its properties

- The Legendre polynomials are orthogonal over $[-1, 1]$ i.e.,

$$\int_{-1}^{1} L_n(x)L_m(x)\,dx = \begin{cases} 0, & n \neq m; \\ \frac{2}{2n+1}, & n = m \end{cases} \tag{4}$$

so the set

$$\mu_n(x) = \left\{ \sqrt{n + \frac{1}{2}} L_n(x) \right\},$$

form an orthonormal set.

Wavelet and Legendre wavelet

- **Wavelet** means a small wave. Wavelets are mathematical functions that arise in mathematics, quantum physics and engineering.
- **Legendre wavelets** are constructed from Legendre polynomial. Legendre wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments, k and n are positive integers, m is the order of Legendre polynomials. The Legendre wavelet [7] is defined on $[0,1]$ by

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}\; 2^{\frac{k}{2}}\; L_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leqslant x \leqslant \frac{n}{2^{k-1}}; \\ 0, & \text{Otherwise} \end{cases} \tag{5}$$

Wavelet and Legendre wavelet

where $n = 1, 2, 3........2^{k-1}$, $m = 0, 1, 2, 3, ...M - 1$, the coefficients $\sqrt{m + \frac{1}{2}}$ are for orthonormality. M determines the order or level of the wavelet. $L_m(x)$ are the Legendre polynomials of order m which are defined on the interval [-1, 1] and are given by the following recurrence relations,

$$L_0(x) = 1,$$
$$L_1(x) = x,$$

$$L_{m+1}(x) = \left(\frac{2m+1}{m+1}\right) x \, L_m(x) - \left(\frac{m}{m+1}\right) L_{m-1}(x), \; m = 1, 2, 3...$$

- The Legendre wavelet form an orthonormal basis for $L^2(R)$ [8, 7, 9].

Description of LWCM

Consider the following IBVP of order n

$$u^{(n)}(x) = f(x, u(x), u'(x), ..., u^{(n-1)}(x)), \qquad a \leqslant x \leqslant b, \tag{6}$$

subject to the following initial and boundary conditions

Case (i)

$$u^{(j-1)}(a) = A_j, \qquad j = 1, 2, 3, ..., n. \tag{7}$$

Case (ii)

$$u^{(j-1)}(a) = A_j, \quad u^{(j-1)}(b) = B_j, \qquad j = 1, 2, 3, ..., \frac{n}{2}. \tag{8}$$

Case (iii)

$$u^{(j-2)}(a) = A_j, \quad u^{(j-2)}(b) = B_j, \qquad j = 2, 4, 6, ..., n. \tag{9}$$

Description of LWCM

Where $n = 2, 4, 6, 8, 10, 12,$ A_j and B_j are known constants while $u(x)$ is unknown function to be determined. According to LWCM

$$u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} s_{n,m}\psi_{n,m}(x) = S^T\psi(x), \tag{10}$$

where S and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$S = \left[s_{1,0}, ..., s_{1,M-1}, s_{2,0}, ..., s_{2,M-1}, ..., s_{2^{k-1},0}, ..., s_{2^{k-1},M-1}\right]^T,$$

$$\psi(x) = \left[\psi_{1,0}, ..., \psi_{1,M-1}, \psi_{2,0}, ..., \psi_{2,M-1}, ..., \psi_{2^{k-1},0}, ..., \psi_{2^{k-1},M-1}\right]^T.$$

Description of LWCM

Using Eq. (10) in Eq. (6) and Eq. (7) (Case (i)) or Eq. (8) (Case (ii)) or Eq. (9) (Case (iii)) the following equation is obtained

$$S^T\psi^{(n)}(x) = f(x, S^T\psi(x), S^T\psi'(x), ..., S^T\psi^{(n-1)}(x)), \tag{11}$$

and Eq. ((7) (Case (i)) or Eq. (8) (Case (ii)) or Eq. (9) (Case (iii)) will become

$$S^T\psi^{(j-1)}(a) = A_j, \quad j = 1, 2, 3, ..., n. \tag{12}$$

$$S^T\psi^{(j-1)}(a) = A_j, \quad S^T\psi^{(j-1)}(b) = B_j, \quad j = 1, 2, 3, ..., \frac{n}{2}. \tag{13}$$

$$S^T\psi^{(j-2)}(a) = A_j, \quad S^T\psi^{(j-2)}(b) = B_j, \quad j = 2, 4, 6, ..., n. \tag{14}$$

Description of LWCM

From Eq. (10) it is clear that there are total $2^{k-1}M$ unknown constants. To find these unknown constants one need $2^{k-1}M$ equations. Out of these $2^{k-1}M$ equations n number of equations will be obtained from Eq. (12) or (13) or (14). For rest of the equations consider the following collocation points, which are zeros of Chybeshev polynomials [10]

$$x_i = \cos\left[\frac{(2i+1)\pi}{2^k M}\right], \qquad i = 1, 2, 3, 4, ..., 2^{k-1}M - n. \qquad (15)$$

Using these collocation points in Eq. (11) the following equation is obtained

Description of LWCM

$$S^T\psi^{(n)}(x_i) = f(x_i, S^T\psi(x_i), S^T\psi'(x_i), ..., S^T\psi^{(n-1)}(x_i)). \quad (16)$$

Eq. (16) together with Eq. (12) or Eq. (13) or Eq. (14) gives a system of $2^{k-1}M$ equations. Solution of this system of equations will give the unknown matrix S. Using S in Eq. (10) will give the desire solution.

Example 1

Example 1 Consider the nonlinear squeezing flow problem of order four [11]

$$u^{(4)}(x) + R_e u(x) u^{(3)}(x) = 0, \qquad 0 \leqslant x \leqslant 1, \tag{17}$$

with the boundary conditions

$$\begin{aligned} u(0) &= 0, & u''(0) &= 0, \\ u(1) &= 1, & u'(1) &= 0, \end{aligned} \tag{18}$$

where R_e is the Reynold number.
Setting $k = 1$ and $M = 9$ in Eq. (10)

$$u(x) = \sum_{n=1}^{1} \sum_{m=0}^{8} s_{n,m} \psi_{n,m}(x) = S^T \psi(x), \tag{19}$$

Example 1

where S and $\psi(x)$ are given by

$$S = [s_{1,0}, s_{1,1}, ..., s_{1,8}]^T,$$
$$\psi(x) = [\psi_{1,0}, \psi_{1,1}, ..., \psi_{1,8}]^T.$$

Now using Eq. (19) in Eq. (17) and (18) the following equations are obtained

$$S^T\psi^{(4)}(x) + R_e S^t \psi(x) S^T \psi^{(3)}(x) = 0. \tag{20}$$

$$\begin{aligned} S^T\psi(0) &= 0, & S^T\psi''(0) &= 0, \\ S^T\psi(1) &= 1, & S^T\psi'(1) &= 0. \end{aligned} \tag{21}$$

The equations obtained from Eq. (21) in case of $R_e = 1$ are given by

Example 1

$$s_{1,0} - \sqrt{3}s_{1,1} + \sqrt{5}s_{1,2} - \sqrt{7}s_{1,3} + 3s_{1,4} - \sqrt{11}s_{1,5} + \sqrt{13}s_{1,6} - \sqrt{15}s_{1,7} + \sqrt{17}s_{1,8} = 0. \tag{22}$$

$$12\sqrt{5}s_{1,2} - 60\sqrt{7}s_{1,3} + 540s_{1,4} - 420\sqrt{11}s_{1,5} + 840\sqrt{13}s_{1,6} - 1512\sqrt{15}s_{1,7} + 2520\sqrt{17}s_{1,8} = 0. \tag{23}$$

$$s_{1,0} + \sqrt{3}s_{1,1} + \sqrt{5}s_{1,2} + \sqrt{7}s_{1,3} + 3s_{1,4} + \sqrt{11}s_{1,5} + \sqrt{13}s_{1,6} + \sqrt{15}s_{1,7} + \sqrt{17}s_{1,8} = 1. \tag{24}$$

$$2\sqrt{3}s_{1,1} + 6\sqrt{5}s_{1,2} + 12\sqrt{7}s_{1,3} + 60s_{1,4} + 30\sqrt{11}s_{1,5} + 42\sqrt{13}s_{1,6} + 56\sqrt{15}s_{1,7} + 72\sqrt{17}s_{1,8} = 0. \tag{25}$$

Example 1

Now collocating Eq. (20) at x_i, which are zeros of of chebyshev polynomials, the following equation is obtained

$$S^T\psi^{(4)}(x_i) + R_e S^t\psi(x_i)S^T\psi^{(3)}(x_i) = 0,$$

which gives the following equations

$$\begin{aligned}
&5040s_{1,4} + 48623.66446s_{1,5} + 254635.662s_{1,6} + 959983.48s_{1,7} + 2.915216\times10^{6}s_{1,8}+\\
&(1.000000000s_{1,0} + 1.6794233s_{1,1} + 2.0353392s_{1,2} + 2.1815464s_{1,3}+\\
&2.1493208s_{1,4} + 1.9593666s_{1,5} + 1.6338131s_{1,6} + 1.1990648s_{1,7}+\\
&0.68592302s_{1,8})(120\sqrt{7}s_{1,3} + 2443.431075s_{1,4} + 10393.5823s_{1,5}+\\
&32339.865s_{1,6} + 82161.54s_{1,7} + 180465.8s_{1,8}) = 0.
\end{aligned} \tag{26}$$

Example 1

$$\begin{aligned}&5040s_{1,4} + 36710.42039s_{1,5} + 133424.542s_{1,6} + 311750.48s_{1,7} + 491497.9s_{1,8}+\\&(1.000000000s_{1,0} + 1.26794919s_{1,1} + 0.67942384s_{1,2} - 0.310383928s_{1,3}-\\&1.134526419s_{1,4} - 1.341466309s_{1,5} - 0.820935633s_{1,6} + 0.1438470268s_{1,7}+\\&1.031623756s_{1,8})(120\sqrt{7}s_{1,3} + 1844.768035s_{1,4} + 5325.4908s_{1,5}+\\&9627.518s_{1,6} + 10612.03s_{1,7} + 2791.8s_{1,8}) = 0.\end{aligned} \tag{27}$$

$$\begin{aligned}&5040s_{1,4} + 14320.84528s_{1,5} - 2805.247s_{1,6} - 59472.77s_{1,7} - 58981.2s_{1,8}+\\&(1.000000000s_{1,0} + 0.494630789s_{1,1} - 0.844496220s_{1,2} - 0.979295460s_{1,3}+\\&0.294819829s_{1,4} + 1.149631888s_{1,5} + 0.359054432s_{1,6} - 0.946147563s_{1,7}-\\&0.898606254s_{1,8})(120\sqrt{7}s_{1,3} + 719.649553s_{1,4}-\\&370.56278s_{1,5} - 2728.2515s_{1,6} - 1746.33s_{1,7} + 4647.00s_{1,8}) = 0.\end{aligned} \tag{28}$$

$$\begin{aligned}&5040s_{1,4} - 15844.54765s_{1,5} + 2674.9864s_{1,6} + 57740.682s_{1,7} - 81244.40s_{1,8}+\\&(1.000000000s_{1,0} - 0.547258277s_{1,1} - 0.783192177s_{1,2} + 1.045292682s_{1,3}+\\&0.1327119654s_{1,4} - 1.131717877s_{1,5} + 0.579750401s_{1,6} + 0.767347457s_{1,7}-\\&1.064052468s_{1,8})(120\sqrt{7}s_{1,3} - 796.218478s_{1,4}-\\&141.42273s_{1,5} + 2729.9419s_{1,6} - 2639.641s_{1,7} - 3579.97s_{1,8}) = 0.\end{aligned} \tag{29}$$

Example 1

$$\begin{aligned}&5040s_{1,4} - 15120\sqrt{11}s_{1,5} + 75600\sqrt{13}s_{1,6} - 277200\sqrt{15}s_{1,7} + 831600\sqrt{17}s_{1,8}+\\&(s_{1,0} - \sqrt{3}s_{1,1} + \sqrt{5}s_{1,2} - \sqrt{7}s_{1,3} + 3s_{1,4} - \sqrt{11}s_{1,5} + \sqrt{13}s_{1,6} - \sqrt{15}s_{1,7}+\\&\sqrt{17}s_{1,8})(120\sqrt{7}s_{1,3} - 2520s_{1,4} + 3360\sqrt{11}s_{1,5} - 10080\sqrt{13}s_{1,6} + 25200\sqrt{15}s_{1,7}-\\&55440\sqrt{17}s_{1,8}) = 0.\end{aligned} \tag{30}$$

Solving the system of equations (22-30), the values of unknown constants are obtained which are

$$\begin{aligned}&s_{1,0} = 0.63023, s_{1,1} = 0.30223, s_{1,2} = -0.058734,\\&s_{1,3} = -0.0088964, s_{1,4} = 0.00037135, s_{1,5} = 0,\\&s_{1,6} = 0, s_{1,7} = 0, s_{1,8} = 0.\end{aligned}$$

Using these constants in Eq. (19) the solution can be found. Numerical results and comparison with OHAM for different Reynolds numbers are given in Table 1.

Example 1

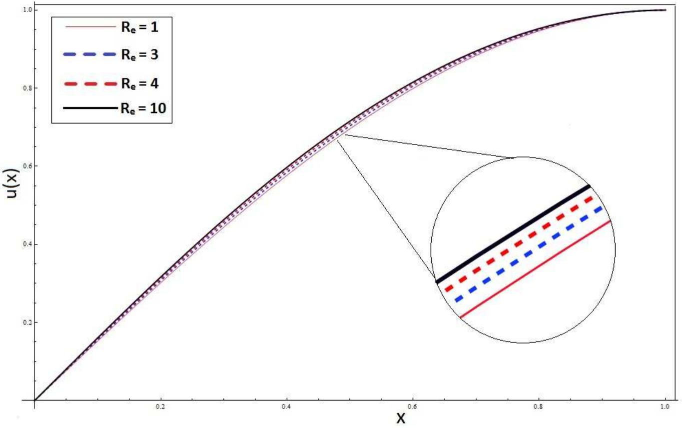

Figure 1: Plot for different values of R_e

Example 1

Table 1: Comparison of LWCM solution with OHAM solution of Example 1 for $k = 1$ and $M = 9$ [11]

x_i	$R_e = 1$		$R_e = 3$		$R_e = 4$		$R_e = 10$	
	OHAM	LWCM	OHAM	LWCM	OHAM	LWCM	OHAM	LWCM
0.00	0.000000	7.4×10^{-5}	0.000000	0.000000	0.000000	-1.1×10^{-4}	0.000000	2.8×10^{-5}
0.15	0.227784	0.227946	0.235947	0.236053	0.239637	0.239530	0.256306	0.252730
0.30	0.444185	0.444507	0.457974	0.457991	0.464078	0.463518	0.489628	0.483890
0.45	0.638166	0.638488	0.653380	0.653183	0.659864	0.658960	0.683059	0.678679
0.60	0.799349	0.799565	0.811634	0.811380	0.816571	0.815828	0.829301	0.829440
0.75	0.918249	0.918365	0.924870	0.924714	0.927305	0.926966	0.929652	0.933226
0.90	0.986419	0.986460	0.987774	0.987701	0.988210	0.988087	0.987436	0.989344

Example 2

Example 2 Consider nonlinear eight order IVP [13]

$$u^{(8)}(x) = e^{-x}u^2(x), \quad 0 < x < 1, \tag{31}$$

with the initial conditions

$$\begin{aligned} u(0) &= 1, u'(0) = 1, \\ u''(0) &= 1, u^{(3)}(0) = 1, \\ u^{(4)}(0) &= 1, u^{(5)}(0) = 1, \\ u^{(6)}(0) &= 1, u^{(7)}(0) = 1. \end{aligned} \tag{32}$$

Exact solution of Eq. (31) is [13]

$$u(x) = e^{x} \tag{33}$$

The given problem was solved for $k = 1$, $M = 12$ with the help of LWCM. The results obtained are given in Table 2.

Example 2

Table 2: Comparison of absolute error of Example 2 for $k = 1$ and $M = 12$.

x_i	**Exact**	**Approximate**	**Absolute error (LWCM)**	**QBS** [13]
0.1	1.1051709180756477	1.1051709180756477	0.000000000000	1.192093×10^{-7}
0.2	1.2214027581601699	1.2214027581602247	5.48450×10^{-14}	1.251698×10^{-5}
0.3	1.3498588075760032	1.3498588075774163	1.41309×10^{-12}	4.529953×10^{-5}
0.4	1.4918246976412703	1.4918246976552993	1.40290×10^{-11}	8.225441×10^{-5}
0.5	1.6487212707001282	1.64872127082562	8.24338×10^{-11}	1.045465×10^{-4}
0.6	1.8221188003905089	1.8221188007377276	3.47219×10^{-10}	9.799004×10^{-5}
0.7	2.0137527074704766	2.0137527086321660	1.16169×10^{-9}	6.628036×10^{-5}
0.8	2.2255409284924680	2.2255409317757855	3.28332×10^{-9}	3.147125×10^{-5}
0.9	2.4596031111569500	2.4596031193152610	8.15831×10^{-9}	1.168251×10^{-5}

Example 3

Example 3 Consider the nonlinear twelfth order BVP[14]

$$u^{(12)}(x) = 2e^x u^2(x) + u^{(3)}(x), \qquad 0 < x < 1, \tag{34}$$

subject to the boundary conditions:

$$u^{(2s)}(0) = 1, \qquad u^{(2s)}(1) = e^{-1}, \qquad s = 0(1)5. \tag{35}$$

Exact solution of Eq. (34) is[14]

$$u(x) = e^{-x}. \tag{36}$$

For $k = 1$ and $M = 18$ the numerical results are given in Table 3

Example 3

Table 3: Comparison of absolute error of Example 3 for $k = 1$ and $M = 18$.

x_i	**Exact**	**Approximate**	**Absolute error (LWCM)**	**DTM** [14]
0	1.0000000000000000	0.9999999999999996	4.44089×10^{-16}	0.000000000
0.1	0.9048374180359595	0.9048374181017839	6.58243×10^{-11}	-1.61×10^{-7}
0.2	0.8187307530779818	0.8187307532028436	1.24862×10^{-10}	-3.07×10^{-7}
0.3	0.7408182206817179	0.7408182208527030	1.70985×10^{-10}	-4.22×10^{-7}
0.4	0.6703200460356393	0.6703200462349629	1.99324×10^{-10}	-4.97×10^{-7}
0.5	0.6065306597126334	0.6065306599193700	2.06737×10^{-10}	-5.22×10^{-7}
0.6	0.5488116360940264	0.5488116362861466	1.92120×10^{-10}	-4.96×10^{-7}
0.7	0.49658530379140947	0.4965853039479183	1.56509×10^{-10}	-4.22×10^{-7}
0.8	0.44932896411722156	0.44932896422019364	1.02972×10^{-10}	-3.07×10^{-7}
0.9	0.4065696597405991	0.40656965977690485	3.63057×10^{-11}	-1.61×10^{-7}
1	0.36787944117144233	0.36787944113400123	3.74411×10^{-11}	-1.11×10^{-16}

Application of LWCM to Integral Equations (IEs)

Consider the following nonlinear Voltera-Fredholm IE [15]

$$u(x) = g(x) + \mu_1 \int_0^x k_1(x,t)[u(t)]^{r_1}\, dt + \mu_2 \int_0^1 k_2(x,t)[u(t)]^{r_2}\, dt, \tag{37}$$

where $0 \leqslant x, t \leqslant 1$, r_1 and r_2 are nonnegative integers and μ_1, μ_2 are known constants, $g(x)$ is known function, kernals $k_1(x,t)$ and $k_2(x,t)$ are known, square integrable and have nth order derivatives in the interval $[0, x]$ and $[0, 1]$ respectively whereas $u(x)$ is unknown function. There are two special cases of Eq. (37).

- When $\mu_1 = 0$ and $r_2 = 1$ then it is said to be linear Fredholm IE.
- When $\mu_2 = 0$ and $r_1 = 1$ then Eq. (37) is said to be linear Voltera IE.

Application of LWCM to Integral Equations

The integrations have been evaluated with the help of Gaussian integration formula [16]. In order to use Gaussian integration formula our interval should be $[-1, 1]$ if it is not then one have to make it by some transformation. According to LWCM

$$u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} h_{n,m}\psi_{n,m}(x) = H^T \psi(x). \tag{38}$$

In Eq. (38) $h_{n,m}$ are unknown constants to be determined, H and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$H = [h_{1,0}, ..., h_{1,M-1}, h_{2,0}, ...h_{2,M-1}, ..., h_{2^{k-1},0}, ..., h_{2^{k-1},M-1}]^T,$$

$$\psi(x) = [\psi_{1,0}, ..., \psi_{1,M-1}, \psi_{20}, ..., \psi_{2,M-1}, ..., \psi_{2^{k-1},0}, ..., \psi_{2^{k-1},M-1}]^T.$$

Application of LWCM to Integral Equations

Using Eq. (38) in Eq. (37)

$$H^T\psi(x) = g(x) + \mu_1 \int_0^x k_1(x,t)[H^T\psi(t)]^{r_1}\, dt + \mu_2 \int_0^1 k_2(x,t)[H^T\psi(t)]^{r_2}\, dt. \quad (39)$$

To make the calculation simple let us make the substitutions

$$D_1(x,t) = k_1(x,t)[H^T\psi(t)]^{r_1},$$
$$D_2(x,t) = k_2(x,t)[H^T\psi(t)]^{r_2}.$$

Therefore, Eq. (39) become

$$H^T\psi(x) = g(x) + \mu_1 \int_0^x D_1(x,t)\, dt + \mu_2 \int_0^1 D_2(x,t)\, dt. \quad (40)$$

Application of LWCM to Integral Equations

Now collocating Eq. (40) at the following points

$$x_i = \frac{i-0.5}{2^{k-1}M}, \qquad i = 1, 2, 3, ..., 2^{k-1}M.$$

These collocation points are chosen in such a way that they uniformly divide the interval $[0, 1]$. Therefore, Eq. (40) will become

$$H^T\psi(x_i) = g(x_i) + \mu_1 \int_0^{x_i} D_1(x_i, t)\, dt + \mu_2 \int_0^1 D_2(x_i, t)\, dt. \tag{41}$$

In order to apply Gaussian integration formula transform the intervals $[0, x_i]$ and $[0, 1]$ to $[-1, 1]$ respectively by the following transformations

Application of LWCM to Integral Equations

$$p_1 = \frac{2}{x_i}t - 1,$$
$$p_2 = 2t - 1.$$

So, Eq. (41) will take the form

$$H^T \psi(x_i) = g(x_i) + \mu_1 \frac{x_i}{2} \int_{-1}^{1} D_1 \left(x_i, \frac{x_i}{2}(p_1 + 1) \right) dp_1 + \frac{\mu_2}{2} \int_{-1}^{1} D_2 \left(x_i, \frac{p_2 + 1}{2} \right) dp_2. \quad (42)$$

Now using the Gaussian integration formula

$$H^T \psi(x_i) = g(x_i) + \mu_1 \frac{x_i}{2} \sum_{r=1}^{q_1} w_{1r} D_1 \left(x_i, \frac{x_i}{2}(p_{1r} + 1) \right) + \frac{\mu_2}{2} \sum_{r=1}^{q_2} w_{2r} D_2 \left(x_i, \frac{p_{2r} + 1}{2} \right), \quad (43)$$

Application of LWCM to Integral Equations

where $i = 1, 2, ..., 2^{k-1}M$, p_{1r}, p_{2r} are q_1 and q_2 zeros of Legendre polynomials L_{2q_1-1}, L_{2q_2-1} respectively and w_{1r}, w_{2r} are respective weights [17]. Eq. (43) gives $2^{k-1}M$ equations which can be solved for unknown matrix H. Using H in Eq. (38) will give the solution.

Example 4

Example 4 Consider nonlinear Voltera IE [18]

$$u(x) = q(x) + \int_0^x x\tau^2 (u(\tau))^2 \, d\tau, \tag{44}$$

where

$$q(x) = (1 + \frac{11}{9}x + \frac{2}{3}x^2 - \frac{1}{3}x^3 + \frac{2}{9}x^4)\ln(x+1) - \frac{1}{3}(x + x^4)(\ln(x+1))^2 - \frac{11}{9}x^2 + \frac{5}{18}x^3 - \frac{2}{27}x^4.$$

The exact solution of Eq. (44) is given by [18]

$$u(x) = \ln(x+1). \tag{45}$$

Setting $k = 1$ and $M = 6$ in Eq. (38) the following equation is obtained

$$u(x) = \sum_{n=1}^{1} \sum_{m=0}^{5} h_{n,m}\psi_{n,m}(x) = H^T\psi(x), \tag{46}$$

Example 4

where

$$H = [h_{1,0}, h_{1,1}, ..., h_{1,5}]^T,$$

$$\psi(x) = [\psi_{1,0}, \psi_{1,1}, ..., \psi_{1,5}]^T.$$

Using Eq. (46) in Eq. (44) the following equation is obtained

$$H^T\psi(x) = q(x) + \int_0^x x\tau^2 (H^T\psi(\tau))^2 \, d\tau. \tag{47}$$

Now collocating Eq. (47) at the following collocation points

$$x_i = \frac{i - 0.5}{2^{k-1}M}, \qquad i = 1, 2, 3, ..., 2^{k-1}M. \tag{48}$$

Eq. (47) will become

$$H^T\psi(x_i) = q(x_i) + \int_0^{x_i} x_i\tau^2 (H^T\psi(\tau))^2 \, d\tau. \tag{49}$$

Example 4

In order to transform the interval from $[0, x_i]$ to $[-1, 1]$ the following transformation is used

$$P = \frac{2}{x_i}\tau - 1,$$

so Eq. (49) will take the form

$$H^T\psi(x_i) = q(x_i) + \frac{1}{8}\int_{-1}^{1} x_i^4 (P+1)^2 \left(H^T\psi\left(\frac{x_i}{2}(P+1)\right)\right)^2 dP. \tag{50}$$

Now by Gaussian integration formula

$$H^T\psi(x_i) = q(x_i) + \frac{1}{8}\sum_{r=1}^{q} w_r x_i^4 (P_r+1)^2 \left(H^T\psi\left(\frac{x_i}{2}(P_r+1)\right)\right)^2, \tag{51}$$

Eq. (51) gives gives the following system of equations.

Example 4

$$
\begin{aligned}
&-0.0800426+h_{1,0}-1.44338h_{1,1}+1.2112h_{1,2}-0.520576h_{1,3}-0.357928h_{1,4}+1.11562h_{1,5}+\\
&\frac{1}{8}(1.0057277385970295\times 10^{-7}(h_{1,0}-1.71851h_{1,1}+2.18383h_{1,2}-2.52285h_{1,3}+\\
&2.76955h_{1,4}-2.9382h_{1,5})^2-4.916723418537301\times 10^{-6}(h_{1,0}-1.66543h_{1,1}+1.98303h_{1,2}-\\
&2.06418h_{1,3}+1.94306h_{1,4}-1.64786h_{1,5})^2 0.0000274348(h_{1,0}-1.58771h_{1,1}+1.70034h_{1,2}-\\
&1.45685h_{1,3}+0.939001h_{1,4}-0.251869h_{1,5})^2-0.0000546326(h_{1,0}-1.50999h_{1,1}+1.43117h_{1,2}-\\
&0.922766h_{1,3}+0.156219h_{1,4}+0.654381h_{1,5})^2 0.0000415162(h_{1,0}-1.45692h_{1,1}+1.25511h_{1,2}-\\
&0.598298h_{1,3}-0.26431h_{1,4}+1.04243h_{1,5})^2)=0.
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&-0.223103+h_{1,0}-0.866025h_{1,1}-0.279508h_{1,2}+1.15752h_{1,3}-0.867188h_{1,4}-0.297978h_{1,5}+\frac{1}{8}\\
&(-8.14639468263594\times 10^{-6}(h_{1,0}-1.69143h_{1,1}+2.08057h_{1,2}-2.28425h_{1,3}+2.33281h_{1,4}-2.24259\\
&h_{1,5})^2-0.00336281(h_{1,0}-0.906651h_{1,1}-0.198993h_{1,2}+1.1287h_{1,3}-0.972148h_{1,4}-0.119h_{1,5})^2-\\
&0.000398255(h_{1,0}-1.5322h_{1,1}+1.50671h_{1,2}-1.06812h_{1,3}+0.358838h_{1,4}+0.439484h_{1,5})^2-\\
&0.00442524(h_{1,0}-1.06587h_{1,1}+0.152151h_{1,2}+0.90079h_{1,3}-1.25306h_{1,4}+0.631138h_{1,5})^2-\\
&0.00222222(h_{1,0}-1.29904h_{1,1}+0.768648h_{1,2}+0.186029h_{1,3}-1.05029h_{1,4}+1.38098h_{1,5})^2)=0.
\end{aligned}
\tag{53}
$$

Example 4

$$
\begin{aligned}
&-0.347534+1.h_{1,0}-0.288675h_{1,1}-1.02486h_{1,2}+0.630816h_{1,3}+0.822627h_{1,4}-0.90545h_{1,5}+\frac{1}{8}\\
&(-0.000062858(h_{1,0}-1.66434h_{1,1}+1.97896h_{1,2}-2.05512h_{1,3}+1.9273h_{1,4}-1.62436h_{1,5})^2-\\
&0.0341453(h_{1,0}-0.621756h_{1,1}-0.685823h_{1,2}+1.11866h_{1,3}-0.106739h_{1,4}-1.0456h_{1,5})^2-\\
&0.0259476(h_{1,0}-0.356384h_{1,1}-0.976033h_{1,2}+0.75896h_{1,3}+0.672239h_{1,4}-1.03638h_{1,5})^2-\\
&0.0171468(h_{1,0}-1.01036h_{1,1}+0.0232924h_{1,2}+1.00211h_{1,3}-1.18339h_{1,4}+0.368739h_{1,5})^2-\\
&0.00307295(h_{1,0}-1.39897h_{1,1}+1.07009h_{1,2}-0.279792h_{1,3}-0.628327h_{1,4}+1.2905h_{1,5})^2)=0.
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
&-0.454281+1.h_{1,0}+0.288675h_{1,1}-1.02486h_{1,2}-0.630816h_{1,3}+0.822627h_{1,4}+0.90545h_{1,5}+\frac{1}{8}\\
&(-0.000241475(h_{1,0}-1.63726h_{1,1}+1.87899h_{1,2}-1.83532h_{1,3}+1.55184h_{1,4}-1.07856h_{1,5})^2-\\
&0.0658711(h_{1,0}-0.721688h_{1,1}-0.535725h_{1,2}+1.17512h_{1,3}-0.432527h_{1,4}-0.819844h_{1,5})^2-\\
&0.131173(h_{1,0}-0.177639h_{1,1}-1.08275h_{1,2}+0.399886h_{1,3}+1.00812h_{1,4}-0.606775h_{1,5})^2-\\
&0.0996803(h_{1,0}+0.193883h_{1,1}-1.07601h_{1,2}-0.434964h_{1,3}+0.986096h_{1,4}+0.655862h_{1,5})^2-\\
&0.0118051(h_{1,0}-1.26574h_{1,1}+0.673157h_{1,2}+0.318873h_{1,3}-1.13974h_{1,4}+1.33766h_{1,5})^2)=0.
\end{aligned}
\tag{55}
$$

Example 4

$$-0.538033+1.h_{1,0}+0.866025h_{1,1}-0.279508h_{1,2}-1.15752h_{1,3}-0.867188h_{1,4}+\frac{1}{8}(-0.18(h_{1,0}-0.433013h_{1,1}-0.908403h_{1,2}+0.888807h_{1,3}+0.473145h_{1,4}-1.12673h_{1,5})^2-0.000659858(h_{1,0}-1.61018h_{1,1}+1.78065h_{1,2}-1.62469h_{1,3}+1.20529h_{1,4}-0.600405h_{1,5})^2-0.272388(h_{1,0}+0.74415h_{1,1}-0.498913h_{1,2}-1.18051h_{1,3}-0.504399h_{1,4}+0.752635h_{1,5})^2-0.358444(h_{1,0}+0.266479h_{1,1}-1.03864h_{1,2}-0.586492h_{1,3}+0.866062h_{1,4}+0.85332h_{1,5})^2-0.0322586(h_{1,0}-1.1325h_{1,1}+0.315919h_{1,2}+0.745936h_{1,3}-1.2857h_{1,4}+0.924822h_{1,5})^2)+0.297978h_{1,5}=0. \tag{56}$$

$$-0.58468+1.h_{1,0}+1.44338h_{1,1}+1.2112h_{1,2}+0.520576h_{1,3}-0.357928h_{1,4}+\frac{1}{8}(-0.607838(h_{1,0}+1.29442h_{1,1}+0.755247h_{1,2}-0.205117h_{1,3}-1.06413h_{1,4}-1.37684h_{1,5})^2-0.401674(h_{1,0}-0.144338h_{1,1}-1.09474h_{1,2}+0.326891h_{1,3}+1.04751h_{1,4}-0.501533h_{1,5})^2-0.00147249(h_{1,0}-1.58309h_{1,1}+1.68396h_{1,2}-1.42308h_{1,3}+0.886522h_{1,4}-0.185301h_{1,5})^2-0.0719857(h_{1,0}-0.999272h_{1,1}-0.00162743h_{1,2}+1.01946h_{1,3}-1.16545h_{1,4}+0.315657h_{1,5})^2-0.799875(h_{1,0}+0.710597h_{1,1}-0.553485h_{1,2}-1.17143h_{1,3}-0.39672h_{1,4}+0.850888h_{1,5})^2)-1.11562h_{1,5}=0. \tag{57}$$

Example 4

Solving the nonlinear equations (52-57), the unknown constants can found. The values of the unknown constants are

$$h_{1,0} = 0.386297, h_{1,1} = 0.196942, h_{1,2} = -0.0173982,$$
$$h_{1,3} = 0.00201391, h_{1,4} = -0.00025331, h_{1,5} = 0.0000340598.$$

Using these constants in Eq. (46) the solution is obtained. Numerical results and maximum absolute error for $k = 1$ and $M = 6$ are given the Table 4.

Example 4

Table 4: Maximum absolute error of Example 4 for $k = 1$ and $M = 6$.

x_i	**Exact**	**Approximate**	**Absolute error (LWCM)**
0	0.0000000000000000	0.00007806602199808	7.80660×10^{-5}
0.1	0.09531017980432493	0.09530596617842522	4.21363×10^{-6}
0.2	0.18232155679395460	0.1823178676160363	3.68918×10^{-6}
0.3	0.26236426446749106	0.2623660604227691	1.79596×10^{-6}
0.4	0.33647223662121290	0.3364726860211799	4.49400×10^{-7}
0.5	0.40546510810816440	0.4054638973181012	1.21079×10^{-6}
0.6	0.47000362924573563	0.4700040188542996	3.89609×10^{-7}
0.7	0.53062825106217050	0.5306297069541335	1.45589×10^{-6}
0.8	0.58778666490211910	0.5877841098752108	2.55030×10^{-6}
0.9	0.64185388617239470	0.6418510279580474	2.85821×10^{-6}
1	0.69314718055994530	0.6931890737757236	4.18932×10^{-5}

Example 5

Example 5 Consider nonlinear Voltera IDE [19]

$$u'(x) + u(x) = 2\int_0^x \sin(x)(u(\tau))^2\, d\tau + \cos(x) \\ +(1-x)\sin(x) + \cos(x)\sin^2(x), \tag{58}$$

with initial condition

$$u(0) = 0. \tag{59}$$

The exact solution of Eq. (58) is given by [19]

$$u(x) = \sin(x). \tag{60}$$

Applying the same method as discussed above. Numerical results for $k = 1$ and $M = 10$ are given in Table 5.

Example 5

Table 5: Maximum absolute error of Example 5 for $k = 1$ and $M = 10$.

x_i	Exact	Approximate	Absolute error (LWCM)	HWM [19]
0	0.000000000000000	$3.6544389480 \times 10^{-17}$	3.65444×10^{-17}	0.000000000000
0.1	0.0998334166468281	0.0998334166464689	3.59171×10^{-13}	3.7171×10^{-6}
0.2	0.198669330795061	0.198669330812547	1.74867×10^{-11}	1.3100×10^{-5}
0.3	0.295520206661339	0.295520206795646	1.34307×10^{-10}	1.6507×10^{-5}
0.4	0.389418342308650	0.389418342859133	5.50483×10^{-10}	2.4808×10^{-6}
0.5	0.479425538604203	0.479425540226639	1.62244×10^{-9}	3.8089×10^{-5}
0.6	0.564642473395035	0.564642477279204	3.88417×10^{-9}	6.0150×10^{-6}
0.7	0.644217687237691	0.644217695292190	8.05450×10^{-9}	1.1697×10^{-5}
0.8	0.717356090899522	0.717356105933222	1.50337×10^{-8}	6.0064×10^{-6}
0.9	0.783326909627483	0.783326935523101	2.58956×10^{-8}	3.0721×10^{-5}
1	0.841470984807896	0.841471026738621	4.19307×10^{-8}	1.0440×10^{-4}

Example 6

Example 6 Consider the following nonlinear Voltera IDE [19]

$$u'(x) = -3\int_0^x \cos(x-\tau)(u(\tau))^2\,d\tau + 2\sin(x)\cos(x), \tag{61}$$

subject to initial condition

$$u(0) = 1. \tag{62}$$

Exact solution of Eq. (61) is given by [19]

$$u(x) = \cos(x). \tag{63}$$

The numerical results for $k = 1$ and $M = 10$ are given in the Table 6.

Example 6

Table 6: Maximum absolute error of Example 6 for $k = 1$ and $M = 10$.

x_i	Exact	Approximate	Absolute error (LWCM)	HWM[19]
0	1.0000000000000000	1.0000000000000000	0.00000000000	0.0000000000
0.1	0.9950041652780258	0.9950041578295699	7.44846×10^{-9}	7.2362×10^{-5}
0.2	0.9800665778412416	0.9800665486936870	2.91476×10^{-8}	9.4238×10^{-5}
0.3	0.9553364891256060	0.9553364258654512	6.32602×10^{-8}	6.7747×10^{-5}
0.4	0.9210609940028851	0.9210608870448603	1.06958×10^{-7}	2.0992×10^{-6}
0.5	0.8775825618903728	0.8775824051447482	1.56746×10^{-7}	1.0772×10^{-4}
0.6	0.8253356149096782	0.8253354060683267	2.08841×10^{-7}	7.7475×10^{-5}
0.7	0.7648421872844884	0.7648419277128872	2.59572×10^{-7}	8.3647×10^{-5}
0.8	0.6967067093471654	0.6967064036074017	3.05740×10^{-7}	1.1763×10^{-4}
0.9	0.6216099682706644	0.6216096233158043	3.44955×10^{-7}	1.6964×10^{-4}
1	0.5403023058681397	0.5403019300126196	3.75856×10^{-7}	2.2900×10^{-4}

Conclusion

- LWCM has been applied to linear and nonlinear ODEs as well as Integral and IDEs.
- It is observed that the method is very simple in implementation, require less number of collocation points and also require less computational cost.
- The method works very well in case of nonlinearity and converge to solution very fast.
- The accuracy of the method is given interms of absolute error for each problem.

Future work

- This method can be extended for the solution of complex models arising in science and engineering.
- The modification of one dimensional Legendre wavelet to two dimensional Legendre wavelet can be used for the solution of two dimensional IEs and PDEs.
- Fractional differential equations (FDE) can also be solved with the help of Fractional Legendre wavelet.
- Problems with large domain can be solved with the help of extended Legendre wavelet.

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