Mathematical Example of Logistic Regression (2 Features, Binary Label)

1. Training Data

We are given a dataset with two features and binary output:

i	$x_1^{(i)}$	$x_2^{(i)}$	$y^{(i)}$
1	1	2	0
2	2	1	0
3	3	4	1
4	4	3	1

Number of samples:

$$m = 4$$

2. Model Initialization (Random Parameters)

We start by selecting parameters randomly:

$$w_1 = 0.2, \quad w_2 = -0.1, \quad b = 0.0$$

$\downarrow$

3. Linear Model (Decision Boundary)

The linear equation is:

$$z^{(i)} = w_1 x_1^{(i)} + w_2 x_2^{(i)} + b$$

Decision boundary equation:

$$0.2x_1 - 0.1x_2 = 0$$

This boundary is **random and incorrect initially.**

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4. Forward Pass (Compute z)

For sample 1:

$$z^{(1)} = 0.2(1) - 0.1(2) = 0$$

For sample 2:

$$z^{(2)} = 0.2(2) - 0.1(1) = 0.3$$

For sample 3:

$$z^{(3)} = 0.2(3) - 0.1(4) = 0.2$$

For sample 4:

$$z^{(4)} = 0.2(4) - 0.1(3) = 0.5$$

5. Apply Sigmoid Function

$$\hat{y}^{(i)} = \sigma(z^{(i)}) = \frac{1}{1 + e^{-z^{(i)}}}$$

$$\hat{y} = [0.50,\ 0.574,\ 0.550,\ 0.622]$$

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6. Classification Based on Probability

Rule:

$$\hat{y} \geq 0.5 \Rightarrow \text{Class } 1$$

i	$\hat{y}$	Predicted Class	True Class
1	0.50	1	0
2	0.574	1	0
3	0.550	1	1
4	0.622	1	1

❌ Poor classification → high loss

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7. Compute Loss (Binary Cross-Entropy)

$$J = -\frac{1}{4} \sum_{i=1}^{4} \left[y^{(i)} \log(\hat{y}^{(i)}) + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right]$$

Loss is **high**, so parameters must be updated.

8. Compute Errors

$$\hat{y} - y = [0.50,\ 0.574,\ -0.450,\ -0.378]$$

9. Gradient Calculation

Gradient w.r.t w_1

$$\frac{\partial J}{\partial w_1} = \frac{1}{4}[(0.50)(1) + (0.574)(2) + (-0.450)(3) + (-0.378)(4)]$$

$$= \frac{1}{4}(-1.21) = -0.3025$$

Gradient w.r.t w_2

$$\frac{\partial J}{\partial w_2} = \frac{1}{4}[(0.50)(2) + (0.574)(1) + (-0.450)(4) + (-0.378)(3)]$$

$$= \frac{1}{4}(-1.36) = -0.34$$

Gradient w.r.t Bias

$$\frac{\partial J}{\partial b} = \frac{1}{4}(0.50 + 0.574 - 0.450 - 0.378) = 0.0615$$

10. Gradient Descent Update

Learning rate:

$$\alpha = 0.1$$

$$w_1 = 0.2 - 0.1(-0.3025) = 0.23025$$

$$w_2 = -0.1 - 0.1(-0.34) = -0.066$$

$$b = 0 - 0.1(0.0615) = -0.00615$$

11. New Decision Boundary

$$0.23025x_1 - 0.066x_2 - 0.00615 = 0$$

📌 Boundary **moved slightly** toward correct separation.