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ADA Homework 1

2) Consider an instance of the Stable Matching problem in which there exists a man m & a woman w such that m is ranked first on the priority list of w & w is ranked first on the priority list of m.
Prove/disprove : In every stable matching for this instance, m is matched with w.

Ans) Claim : If m & w rank each other first, then every stable matching pairs m with w.

Gale & Shapely method (Men-proposing) :-
m proposes to w first; since w ranks m first, she accepts & will never drop him.

Proof (by Contradiction):
Assume there exists a stable matching M in which m is not matched with w. Then, in M, let m be matched with w_1, where $w_1 \neq w$ & w be matched with some m_1, where $m_1 \neq m$.

- Because w is m's first choice, m prefers w over w_1.
- Because m is w's first choice, w prefers m over m_1.

∴ m & w both prefer each other to their current partners in M. Hence (m,w) is contradicting the assumption that M is stable.
Thus in every stable matching, m must be paired with w.

1) Consider a stable roommate problem involving 4 students (A, B, C, D) : Each student ranks the other in a strict order of preference. A matching involves forming 2 students pair, & it is considered stable if no two separated (not matched) student would prefer each other over their current roommates.
Prove / disprove : A stable matching always exists in this scenario.

Ans) Claim: "For any preferences of A, B, C, D a stable matching (2 disjoint pairs) always exists".

Disproof by Counter example:
Lets assume ~~a~~ preference lists for A, B, C & D so that every possible pairing is blocked.

- A : B > C > D
- B : C > A > D
- C : A > B > D
- D : A > B > C

Now with 4 people there are exactly 3 possible pairings:
1) {A-B, C-D}
  - Check pair (B, C) (they are not roomates).
    - B prefers C to A
    - C prefers B to D

  So {AB, CD} is not stable

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2.) {A-C, B-D}

- Check pair (A, B)
  - A prefers B to C
  - B prefers A to D
- So {AC, BD} is not stable

3.) {A-D, B-C}

- Check pair (A, C)
  - A prefers C to D
  - C prefers A to B
- So {AD, BC} is not stable

∴ All the possible matches have a blocking pair, no stable matching exists for these preferences.

"No stable matching exists in some cases because in the roommates problem with only one set, preference cycles can arise. These cycles make it ~~possible~~ impossible to avoid ~~the~~ blocking pair. Unlike the bipartite case (men vs women) wher Gale-Shapely ensures stability, the roommates case does not always have a stable matching."

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3) Consider the stable matching problem with n women & n men (where $n \geq 2$).
Prove/disprove: It is not possible for any preference lists, that the Gale-Shapely algorithm with men proposing can match every woman to her least preferred man.

Ans) Claim: It is not possible for any preference lists, that the Gale-Shapely algorithm with men proposing can match every woman to her least preferred man.

Disprove by Counterexample:
Let

$m_1 : w_1 > w_2$ $\quad$ $m_2 : w_2 > w_1$

$w_1 : m_2 > m_1$ $\quad$ $w_2 : m_1 > m_2$

• Run algo: $\{(m_1, w_1)\ (m_2, w_2)\}$

Here, w_1 gets m_1 gets w_1 (m_1 is least),
w_2 gets m_2 (her least).
m_1 prefers w_1 } — stable
m_2 prefers w_2

∴ The Claim is false.

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~~∴ G-S algo can never produce an outcome where every woman is paired with their least preferred when men proposing.~~

4) There are six students in a class: Harry, Ron, Hermoine, Luna, Neville & Ginny. This class requires them to pair up & work on pair ~~prg~~ programming. Each has preferences over who they want to be paired with. The preferences are:

Harry: Hermoine > Neville > Ron > Ginny > Luna
Ron: Ginny > Neville > Hermoine > Harry > Luna
Hermoine: Ron > Neville > Ginny > Harry > Luna
Luna: Harry > Ron > Ginny > Hermoina > Neville
Neville: Harry > Ron > Hermoine > Ginny > Luna
Ginny: Neville > Harry > Hermoine > Ron > Luna

A given matching (of students into pairs) is not stable if there are 2 potential partners who are not currently paired but prefers each other their respective current partners.
Prove/disprove: No matching is stable for the preferences given above.

Ans) Claim: No matching is stable for these preferences.
Proof by Contradiction:-
Step 1: 2 unavoidable pairs/points
1. Ron & Ginny: Ron ranks Ginny 1st. If Ron is not paired with Ginny, then to avoid (Ron, Ginny), Ginny must be with someone she prefers more, i.e, one of {Neville, Harry, Hermoine}.
2. Harry & Hermoine: Harry ranks Hermoine 1st & Hermoine ranks Harry above Luna. If Harry is not with Hermoine, the to avoid (Harry, Hermoine) blocking, Hermoine must be with someone she prefers over Hary, i.e, Ron, Neville, Ginny
These two constraints will Collide.
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Step 2. Case 1 - Ron paired with Ginny

Remaining to pair: {Harry, Hermoine, Neville, Luna}.

Consider the only 3 ways to pair these 4:

- (Harry, Hermoine), (Neville, Luna)
  Here Neville prefers Hermoine over Luna, & Hermoine prefers Neville over Harry.
- (Harry, Neville), (Hermoine, Luna)
  Here both Harry & Hermoine prefers each other over their partners.
- (Harry, Luna), (Hermoine, Neville)
  Since Harry prefers Neville over Luna, & Neville prefers Harry over Hermoine.

All 3 fail. So with (Ron, Ginny) fixed, no stable matching exists.

Step 3. Case 2 - Ron not paired with Ginny

Then by step 1, Ginny must be with {Neville, Harry, Hermoine}.

- Ginny with Neville
  Try to avoid (Harry, Hermoine) by pairing Hermoine with Ron. Then (Harry, Neville) blocks because Harry prefers Neville over any leftover partner & Neville prefers Harry over Ginny.
- Ginny with Harry
  Neville ranks Harry 1st & Harry ranks Neville second above Ginny, (Neville, Harry) blocks unless Harry is with Hermoine. But Harry can't be with both Hermoine & Ginny Contradiction.
- Ginny with Hermoine
  Then (Ron, Hermoine) blocks because Ron prefers Hermoine over any other than Ginny, & Hermoine prefers Ron over Ginny.

Both cases lead to contradiction. Therefore, no matching is stable for the given preferences.

Ungraded Problems

1-Ans) Claim : It is possible to have an instance where 2 women have the same best valid partner.

Answer : False

Reason :

- Each woman's best valid partner means her partner in the women-optimal stable matching.
- In that matching, every man is matched to exactly one woman.
- If two woman had the same best valid man, the woman-optimal matching would need to match that man to both which is not possible.

2-Ans) Claim : In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w & w is ranked first on the preference list of m.

Disprove by Counterexample:-

let $n=2$, Men → m1, m2 & women → w1, w2

Pref list:

m1 : w2 > w1

m2 : w1 > w2

w1 : m1 > m2

w2 : m2 > m1

No mutual to pairs (check: no pair where both rank each other first). Stable matchings exists, but none contain a mutual ~~pair~~ top pair, since none exists.

Ungraded

3Ans) Claim: It is possible that Gale-Shapely algo with man-proposing can match every woman to her most preferred man.

Disprove. It is not possible.
The condition implies not mutual tops (no woman's top man ranks her first). The Gale-Shapely algo produces the man-optimal / woman-pessimal stable matching, where women get their worst valid partners. For it to give every woman her most preferred, the matching must be unique.
However, under no mutual tops, constructing such a unique matching where women get tops leads to multiple stables, contradicting uniqueness.
Thus, men-proposing cannot give all women their tops.

4-Ans) Claim: By swapping 2 men in her list, a woman can get a more preferred partner.

Proof. by example
men:
$m_1 : w_1 > w_2 > w_3$
$m_2 : w_1 > w_2 > w_3$
$m_3 : w_2 > w_1 > w_3$

women (true):
$w_1 : m_3 > m_1 > m_2$
$w_2 : m_1 > m_3 > m_2$
$w_3 : m_1 > m_2 > m_3$

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Ungraded

Continued.

4-Ans) • ~~The~~ Truthful run (men-proposing): $(m_1, \omega_1), (m_2, \omega_3), (m_3, \omega_2)$.

ω_1 get m_1 (her 2nd).

• Now ω_1 lies by swapping m_1 & m_2 : $m_3 > m_2 > m_1$

• Run algo again : $(m_1, \omega_2), (m_2, \omega_3), (m_3, \omega_1)$

Now ω_1 gets m_3 (her top).

Hence, a single swap improved her outcome under her true preferences.