NCERT 11.9.5

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Question:
Which term of the following sequences:
(a) $2, 2\sqrt{2}, 4 \dots$ is 128 (b) $\sqrt{3}, 3, 3\sqrt{3} \dots$ is 729
(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \dots$ is $\frac{1}{19683}$
Answer: (a)

Parameter	Explaination	Values
r	Common ratio of G.P	$\sqrt{2}$
a_n	Sequence	$2, 2\sqrt{2}, 4 \dots$
$x_a(n)$	Function equivalent of sequence a_n	—
$X_a(n)$	Z-transform of a_n	—

Table 1: Varibles used in (a)

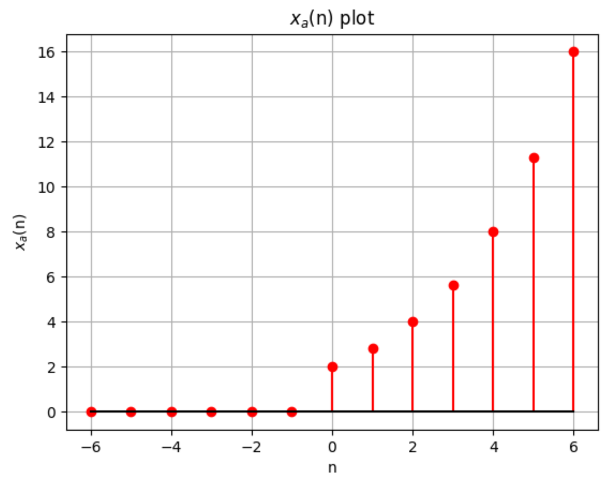

Fig. 1: Plot of x_a(n) from n = -6 to 6

Let $a_1 = 2$, $a_2 = 2\sqrt{2}$, $a_3 = 4$.
Since, $\frac{a_2}{a_1} = \frac{a_3}{a_2}$, the sequence a_1, a_2, a_3 is a G.P Series. Let $r = \frac{a_2}{a_2} = \sqrt{2}$, then the general term is $a_n = a_1 r^{n-1}$.
Assume n^{th} term is 128, which gives:

$$a_n = a_1 r^{n-1} = 128 \tag{1}$$

$$\implies r^{n-1} = \frac{128}{a_1} \tag{2}$$

$$\implies n - 1 = \log_r \frac{128}{a_1} \tag{3}$$

Substituting Values,

$$\implies n - 1 = \log_{\sqrt{2}} \frac{128}{2} \tag{4}$$

$$\implies n - 1 = \log_{\sqrt{2}} 64 \tag{5}$$

$$\implies n - 1 = \log_{\sqrt{2}} \sqrt{2}^{12} \tag{6}$$

$$\implies n - 1 = 12 \tag{7}$$

$$\therefore n = 13 \tag{8}$$

Thus the 13^{th} term of the G.P a_n is 128.

$$x_a(n) = a_1 r^n u[n] \tag{9}$$

where $u[0] = 1$. Taking the Z - transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \tag{10}$$

$$\implies X(z) = \sum_{n=0}^{\infty} a_1 r^n z^{-n} \tag{11}$$

$$\implies X(z) = a_1 \sum_{n=0}^{\infty} r^n z^{-n} \tag{12}$$

$$\implies X(z) = \frac{a_1}{r}\left(\frac{1}{1 - \frac{r}{z}}\right) \tag{13}$$

$$\therefore X(z) = \frac{a_1 z}{r(z - r)} \forall |z| > |r| \tag{14}$$

$$\therefore X_a(z) = \frac{\sqrt{2} z}{z - \sqrt{2}} \forall |z| > \sqrt{2} \tag{15}$$

with ROC:

$$|z| > \sqrt{2}$$

(b)

Parameter	Explaination	Values
r	Common ratio of G.P	$\sqrt{3}$
b_n	Sequence	$\sqrt{3}, 3, 3\sqrt{3}\ldots$
$x_b(n)$	Function equivalent of sequence b_n	—
$X_b(n)$	Z-transform of b_n	—

Table 2: Varibles used in (b)

Let $b_1 = \sqrt{3}$, $b_2 = 3$, $b_3 = 3\sqrt{3}$.
Since $\frac{b_2}{b_1} = \frac{b_3}{b_2}$, the sequence b_1, b_2, b_3 is a G.P Series.
Let $r = \frac{b_2}{b_2} = \sqrt{3}$, then the general term is $b_n = b_1 r^{n-1}$.
Assume n^{th} term is 729, which gives:

$$b_n = b_1 r^{n-1} = 729 \tag{16}$$

$$\implies r^{n-1} = \frac{729}{b_1} \tag{17}$$

$$\implies n - 1 = \log_r \frac{729}{b_1} \tag{18}$$

Substituting Values,

$$\implies n - 1 = \log_{\sqrt{3}} \frac{729}{\sqrt{3}} \tag{19}$$

$$\implies n - 1 = \log_{\sqrt{3}} \frac{3^6}{\sqrt{3}} \tag{20}$$

$$\implies n - 1 = \log_{\sqrt{3}} \sqrt{3}^{11} \tag{21}$$

$$\implies n - 1 = 11 \tag{22}$$

$$\therefore n = 12 \tag{23}$$

Thus the 12^{th} term of the G.P b_n is 729.

$$x_b(n) = b_1 r^n u[n] \tag{24}$$

By 14, the Z-transform of x_b(n):

$$X_b(z) = \frac{z}{z - \sqrt{3}} \tag{25}$$

with ROC:

$$|z| > \sqrt{3}$$

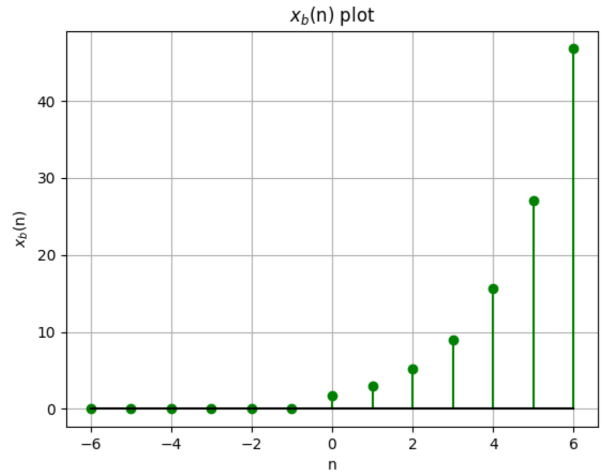

Fig. 2: Plot of x_b(n) from n = -6 to 6

Parameter	Explaination	Values
r	Common ratio of G.P	$\frac{1}{3}$
c_n	Sequence	$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}\ldots$
$x_c(n)$	Function equivalent of sequence c_n	—
$X_c(n)$	Z-transform of c_n	—

Table 3: Varibles used in (c)

(c)
Let $c_1 = \frac{1}{3}, c_2 = \frac{1}{9}, c_3 = \frac{1}{27}$.
Since $\frac{c_2}{c_1} = \frac{c_3}{c_2}$, the sequence c_1, c_2, c_3 is a G.P Series.

Let $r = \frac{c_2}{c_2} = \frac{1}{3}$, then the general term is $c_n = c_1 r^{n-1}$.
Assume n^{th} term is $\frac{1}{19683}$, which gives:

$$c_n = c_1 r^{n-1} = \frac{1}{19683} \tag{26}$$

$$\implies r^{n-1} = \frac{1}{19683 c_1} \tag{27}$$

$$\implies n - 1 = \log_r \frac{1}{19683 c_1} \tag{28}$$

Substituting Values,

$$\implies n - 1 = \log_{\frac{1}{3}} \frac{1}{19683\frac{1}{3}} \tag{29}$$

$$\implies n - 1 = \log_{\frac{1}{3}} \frac{1}{6561} \tag{30}$$

$$\implies n - 1 = \log_{\frac{1}{3}} 3^{-8} \tag{31}$$

$$\implies n - 1 = 8 \tag{32}$$

$$\therefore n = 9 \tag{33}$$

Thus the 9^{th} term of the G.P c_n is $\frac{1}{19683}$.

$$x_c(n) = c_1 r^n u[n] \tag{34}$$

Using the previous result, the Z-transform of x(n):

$$X(z) = \frac{z}{z - \frac{1}{3}} \implies X(z) = \frac{3z}{3z - 1} \tag{35}$$

with ROC:

$$|z| > \frac{1}{3}$$

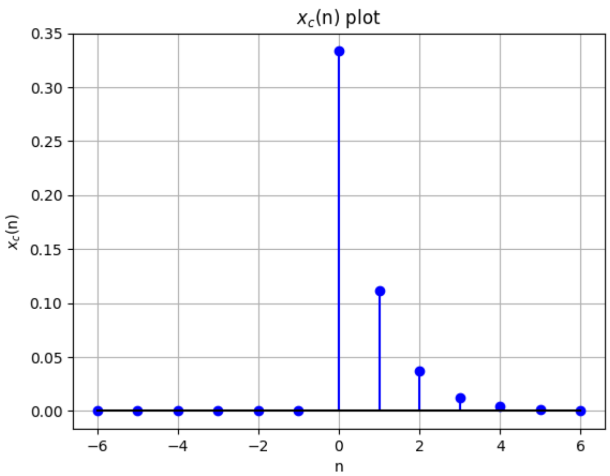

Fig. 3: Plot of x_c(n) from n = -6 to 6