GATE 2021 ME 3Q

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Question: The Dirac-delta function ($\delta(t-t_0)$) for $t, t_0 \in \Re$, has the following property

$$\int_a^b \phi(t)\,\delta(t-t_0)\,dt = \begin{cases}\phi(t_0) & a < t_0 < b \\ 0 & otherwise\end{cases} \tag{1}$$

The Laplace Transform of the Dirac-delta function $\delta(t-a)$ for $a > 0$; $\mathcal{L}(\delta(t-a)) = F(s)$ is

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Solution:

Parameter	Description
$F(s)$	Laplace transform of $\delta(t-a)$
$G(f)$	Fourier transform of $\delta(t-a)$
$w_T(t)$	Delta Comb, $\sum_{k=-\infty}^{\infty}\delta(t-kT)$
$W_T(t)$	Fourier transform of $w_T(t)$

TABLE 1: Table of parameters

By (1) and $a > 0$,

$$F(s) = \int_0^{\infty}\delta(t-a)\,e^{-st}dt \tag{2}$$

$$\therefore F(s) = e^{-as} \tag{3}$$

The fourier transform,

$$G(f) = \int_{-\infty}^{\infty}\delta(t-a)\,e^{-2\pi jft}dt \tag{4}$$

$$\therefore G(f) = e^{-j2\pi fa} \tag{5}$$

For a periodic signal the fourier transform is defined as:

$$H(f) = \sum_{k=-\infty}^{\infty} c_k\delta\left(f-\frac{k}{T}\right) \tag{6}$$

where c_k are the fourier series coefficients and T is the period. Thus,

$$W_T(f) = \sum_{k=-\infty}^{\infty} c_k\delta\left(f-\frac{k}{T}\right) \tag{7}$$

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} w_T(t)\,e^{-j2\pi\frac{k}{T}f}dt \tag{8}$$

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\sum_{k=-\infty}^{\infty}\delta(t-kT)\right)e^{-j2\pi\frac{k}{T}f}dt \tag{9}$$

$$\tag{10}$$

$$c_k = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t-kT)\,e^{-j2\pi\frac{k}{T}f}dt \tag{11}$$

$$c_k = \frac{1}{T} \tag{12}$$

$$W_T(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right) \tag{13}$$

$$\therefore W_T(f) = \frac{1}{T}w_{\frac{1}{T}}(f) \tag{14}$$

Thus, the fourier transform of impulse train is another impulse train.