NCERT 11.9.3

ee23btech11223 - Soham Prabhakar More

Question: Which term of the following sequences:
(a) $2, 2\sqrt{2}, 4 \ldots$ is 128 (b) $\sqrt{3}, 3, 3\sqrt{3} \ldots$ is 729
(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$ is $\frac{1}{19683}$

Answer: (a) Let $a_1 = 2$, $a_2 = 2\sqrt{2}$, $a_3 = 4$.
Since, $\frac{a_2}{a_1} = \frac{a_3}{a_2}$, the sequence a_1, a_2, a_3 is a G.P Series. Let $r = \frac{a_2}{a_2} = \sqrt{2}$, then the general term is $a_n = a_1 r^{n-1}$.
Assume n^{th} term is 128, which gives:

$$a_n = a_1 r^{n-1} = 128$$
$$\implies r^{n-1} = \frac{128}{a_1}$$
$$\implies n - 1 = \log_r \frac{128}{a_1}$$

Substituting Values,

$$\implies n - 1 = \log_{\sqrt{2}} \frac{128}{2}$$
$$\implies n - 1 = \log_{\sqrt{2}} 64$$
$$\implies n - 1 = \log_{\sqrt{2}} \sqrt{2}^{12}$$
$$\implies n - 1 = 12$$
$$\therefore n = 13$$

Thus the 13^{th} term of the G.P a_n is 128.

$$x(n) = \begin{cases} 0 & n \leq 0 \\ a_1 r^{n-1} & n \geq 1 \end{cases}$$

Taking the Z - transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$
$$\implies X(z) = \sum_{n=1}^{\infty} a_1 r^{n-1} z^{-n}$$
$$\implies X(z) = \frac{a_1}{r} \sum_{n=1}^{\infty} r^n z^{-n}$$
$$\implies X(z) = \frac{a_1}{r} (\sum_{n=0}^{\infty} r^n z^{-n} - 1)$$
$$\implies X(z) = \frac{a_1}{r} (\frac{1}{1 - \frac{r}{z}} - 1)$$
$$\implies X(z) = \frac{a_1}{r} (\frac{z}{z - r} - 1)$$
$$\therefore X(z) = \frac{a_1}{z - r} \forall |z| > |r|$$
$$\therefore X(z) = \frac{2}{z - \sqrt{2}} \forall |z| > \sqrt{2}$$

(b) Let $b_1 = \sqrt{3}$, $b_2 = 3$, $b_3 = 3\sqrt{3}$.
Since $\frac{b_2}{b_1} = \frac{b_3}{b_2}$, the sequence b_1, b_2, b_3 is a G.P Series. Let $r = \frac{b_2}{b_2} = \sqrt{3}$, then the general term is $b_n = b_1 r^{n-1}$.
Assume n^{th} term is 729, which gives:

$$b_n = b_1 r^{n-1} = 729$$
$$\implies r^{n-1} = \frac{729}{b_1}$$
$$\implies n - 1 = \log_r \frac{729}{b_1}$$

Substituting Values,

$$\implies n - 1 = \log_{\sqrt{3}} \frac{729}{\sqrt{3}}$$
$$\implies n - 1 = \log_{\sqrt{3}} \frac{3^6}{\sqrt{3}}$$
$$\implies n - 1 = \log_{\sqrt{3}} \sqrt{3}^{11}$$
$$\implies n - 1 = 11$$
$$\therefore n = 12$$

Thus the 12^{th} term of the G.P b_n is 729.

$$x(n) = \begin{cases} 0 & n \leq 0 \\ b_1 r^{n-1} & n \geq 1 \end{cases}$$

Using the previous result, the Z-transform of x(n):

$$X(z) = \frac{\sqrt{3}}{z - \sqrt{3}} \forall |z| > \sqrt{3}$$

(c) Let $c_1 = \frac{1}{3}, c_2 = \frac{1}{9}, c_3 = \frac{1}{27}$.
Since $\frac{c_2}{c_1} = \frac{c_3}{c_2}$, the sequence c_1, c_2, c_3 is a G.P Series.

Let $r = \frac{c_2}{c_2} = \frac{1}{3}$, then the general term is $c_n = c_1 r^{n-1}$.
Assume n^{th} term is $\frac{1}{19683}$, which gives:

$$\begin{aligned} c_n = c_1 r^{n-1} &= \frac{1}{19683} \\ \implies r^{n-1} &= \frac{1}{19683c_1} \\ \implies n-1 &= \log_r \frac{1}{19683c_1} \end{aligned}$$

Substituting Values,

$$\begin{aligned} \implies n-1 &= \log_{\frac{1}{3}} \frac{1}{19683\frac{1}{3}} \\ \implies n-1 &= \log_{\frac{1}{3}} \frac{1}{6561} \\ \implies n-1 &= \log_{\frac{1}{3}} 3^{-8} \\ \implies n-1 &= 8 \\ \therefore n &= 9 \end{aligned}$$

Thus the 9^{th} term of the G.P c_n is $\frac{1}{19683}$.

$$x(n) = \begin{cases} 0 & n \leq 0 \\ c_1 r^{n-1} & n \geq 1 \end{cases}$$

Using the previous result, the Z-transform of x(n):

$$X(z) = \frac{\frac{1}{3}}{z - \frac{1}{3}} \implies X(z) = \frac{1}{3z-1} \forall |z| > \frac{1}{3}$$