

NCERT 11.9.5

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Question:

Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4, \dots$ is 128 (b) $\sqrt{3}, 3, 3\sqrt{3}, \dots$ is 729

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$ is $\frac{1}{19683}$

Answer: (a)

Parameter	Explanation	Values
r	Common ratio of G.P	$\sqrt{2}$
a_n	Sequence	$2, 2\sqrt{2}, 4, \dots$
$x_a(n)$	Function equivalent of sequence a_n	—
$X_a(n)$	Z-transform of a_n	—

Table 1: Variables used in (a)

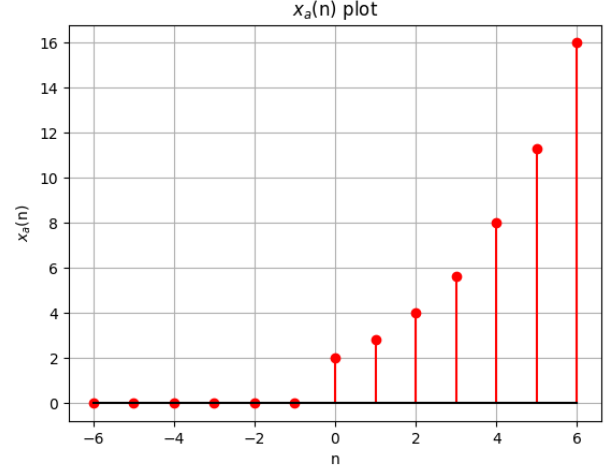


Fig. 1: Plot of $x_a(n)$ from $n = -6$ to 6

Let $a_1 = 2$, $a_2 = 2\sqrt{2}$, $a_3 = 4$.

Since, $\frac{a_2}{a_1} = \frac{a_3}{a_2}$, the sequence a_1, a_2, a_3 is a G.P Series. Let $r = \frac{a_2}{a_1} = \sqrt{2}$, then the general term is $a_n = a_1 r^{n-1}$.

Assume n^{th} term is 128, which gives:

$$a_n = a_1 r^{n-1} = 128 \quad (1)$$

$$\Rightarrow r^{n-1} = \frac{128}{a_1} \quad (2)$$

$$\Rightarrow n - 1 = \log_r \frac{128}{a_1} \quad (3)$$

Substituting Values,

$$\Rightarrow n - 1 = \log_{\sqrt{2}} \frac{128}{2} \quad (4)$$

$$\Rightarrow n - 1 = \log_{\sqrt{2}} 64 \quad (5)$$

$$\Rightarrow n - 1 = \log_{\sqrt{2}} \sqrt{2}^{12} \quad (6)$$

$$\Rightarrow n - 1 = 12 \quad (7)$$

$$\therefore n = 13 \quad (8)$$

Thus the 13th term of the G.P a_n is 128.

$$x_a(n) = a_1 r^n u[n] \quad (9)$$

where $u[0] = 1$. Taking the Z - transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \quad (10)$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} a_1 r^n z^{-n} \quad (11)$$

$$\Rightarrow X(z) = a_1 \sum_{n=0}^{\infty} r^n z^{-n} \quad (12)$$

$$\Rightarrow X(z) = \frac{a_1}{r} \left(\frac{1}{1 - \frac{r}{z}} \right) \quad (13)$$

$$\therefore X(z) = \frac{a_1 z}{r(z - r)} \quad \forall |z| > |r| \quad (14)$$

$$\therefore X_a(z) = \frac{\sqrt{2}z}{z - \sqrt{2}} \quad \forall |z| > \sqrt{2} \quad (15)$$

with ROC:

$$|z| > \sqrt{2}$$

(b)

Parameter	Explanation	Values
r	Common ratio of G.P	$\sqrt{3}$
b_n	Sequence	$\sqrt{3}, 3, 3\sqrt{3} \dots$
$x_b(n)$	Function equivalent of sequence b_n	—
$X_b(n)$	Z-transform of b_n	—

Table 2: Variables used in (b)

Let $b_1 = \sqrt{3}$, $b_2 = 3$, $b_3 = 3\sqrt{3}$.
 Since $\frac{b_2}{b_1} = \frac{b_3}{b_2}$, the sequence b_1, b_2, b_3 is a G.P Series.
 Let $r = \frac{b_2}{b_1} = \sqrt{3}$, then the general term is $b_n = b_1 r^{n-1}$.

Assume n^{th} term is 729, which gives:

$$b_n = b_1 r^{n-1} = 729 \quad (16)$$

$$\Rightarrow r^{n-1} = \frac{729}{b_1} \quad (17)$$

$$\Rightarrow n - 1 = \log_r \frac{729}{b_1} \quad (18)$$

Substituting Values,

$$\Rightarrow n - 1 = \log_{\sqrt{3}} \frac{729}{\sqrt{3}} \quad (19)$$

$$\Rightarrow n - 1 = \log_{\sqrt{3}} \frac{3^6}{\sqrt{3}} \quad (20)$$

$$\Rightarrow n - 1 = \log_{\sqrt{3}} \sqrt{3}^{11} \quad (21)$$

$$\Rightarrow n - 1 = 11 \quad (22)$$

$$\therefore n = 12 \quad (23)$$

Thus the 12th term of the G.P b_n is 729.

$$x_b(n) = b_1 r^n u[n] \quad (24)$$

By 14, the Z-transform of $x_b(n)$:

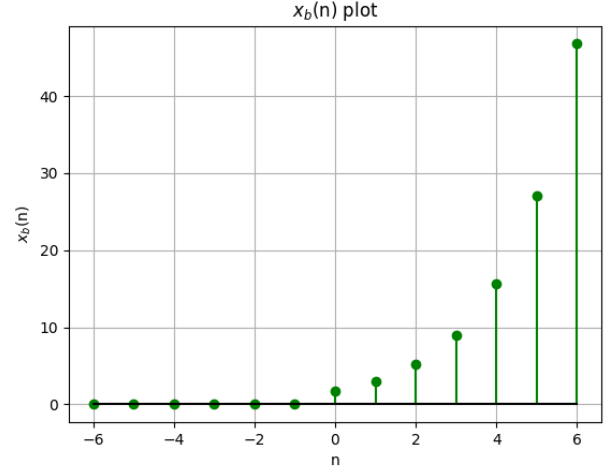
$$X_b(z) = \frac{z}{z - \sqrt{3}} \quad (25)$$

with ROC:

$$|z| > \sqrt{3}$$

(c)

Let $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{9}$, $c_3 = \frac{1}{27}$.
 Since $\frac{c_2}{c_1} = \frac{c_3}{c_2}$, the sequence c_1, c_2, c_3 is a G.P Series.

Fig. 2: Plot of $x_b(n)$ from $n = -6$ to 6

Parameter	Explanation	Values
r	Common ratio of G.P	$\frac{1}{3}$
c_n	Sequence	$\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \dots$
$x_c(n)$	Function equivalent of sequence c_n	—
$X_c(n)$	Z-transform of c_n	—

Table 3: Variables used in (c)

Let $r = \frac{c_2}{c_1} = \frac{1}{3}$, then the general term is $c_n = c_1 r^{n-1}$.
 Assume n^{th} term is $\frac{1}{19683}$, which gives:

$$c_n = c_1 r^{n-1} = \frac{1}{19683} \quad (26)$$

$$\Rightarrow r^{n-1} = \frac{1}{19683 c_1} \quad (27)$$

$$\Rightarrow n - 1 = \log_r \frac{1}{19683 c_1} \quad (28)$$

Substituting Values,

$$\Rightarrow n - 1 = \log_{\frac{1}{3}} \frac{1}{19683 \frac{1}{3}} \quad (29)$$

$$\Rightarrow n - 1 = \log_{\frac{1}{3}} \frac{1}{6561} \quad (30)$$

$$\Rightarrow n - 1 = \log_{\frac{1}{3}} 3^{-8} \quad (31)$$

$$\Rightarrow n - 1 = 8 \quad (32)$$

$$\therefore n = 9 \quad (33)$$

Thus the 9th term of the G.P c_n is $\frac{1}{19683}$.

$$x_c(n) = c_1 r^n u[n] \quad (34)$$

Using the previous result, the Z-transform of $x(n)$:

$$X(z) = \frac{z}{z - \frac{1}{3}} \implies X(z) = \frac{3z}{3z - 1} \quad (35)$$

with ROC:

$$|z| > \frac{1}{3}$$

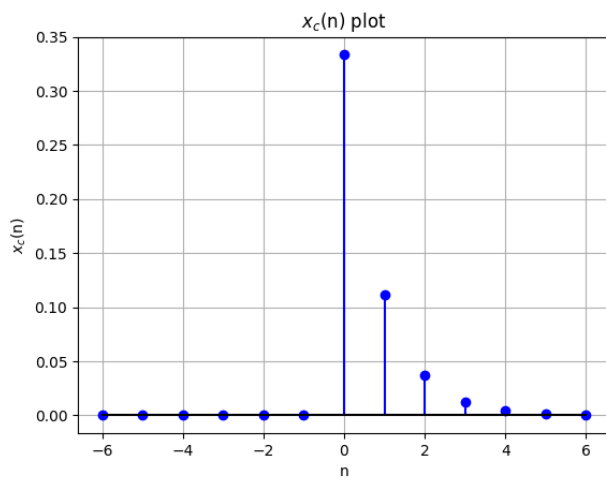


Fig. 3: Plot of $x_c(n)$ from $n = -6$ to 6