NCERT 11.9.5

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Question:
Which term of the following sequences:
(a) $2, 2\sqrt{2}, 4 \ldots$ is 128 (b) $\sqrt{3}, 3, 3\sqrt{3} \ldots$ is 729
(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$ is $\frac{1}{19683}$

Answer: (a) Let $x_1(0) = 2$, $r_1 = \frac{2\sqrt{2}}{2} = \sqrt{2}$, then the general term is:

$$x_1(n) = x_1(0)\, r_1^n u[n] \tag{1}$$

where u[0] = 1. Assume n^{th} $(n > 0)$ term is 128:

$$x_1(n) = x_1(0)\, r^n = 128 \tag{2}$$
$$\implies r_1^n = \frac{128}{x_1(0)} \tag{3}$$
$$\implies n = \log_{r_1} \frac{128}{x_1(0)} \tag{4}$$

Using values from Table 1,

$$\implies n = \log_{\sqrt{2}} \frac{128}{2} \tag{5}$$
$$\implies n = \log_{\sqrt{2}} 64 \tag{6}$$
$$\implies n = \log_{\sqrt{2}} \sqrt{2}^{12} \tag{7}$$
$$\therefore n = 12 \tag{8}$$

Thus the 13^{th} term of the G.P $x_1(n)$ is 128.

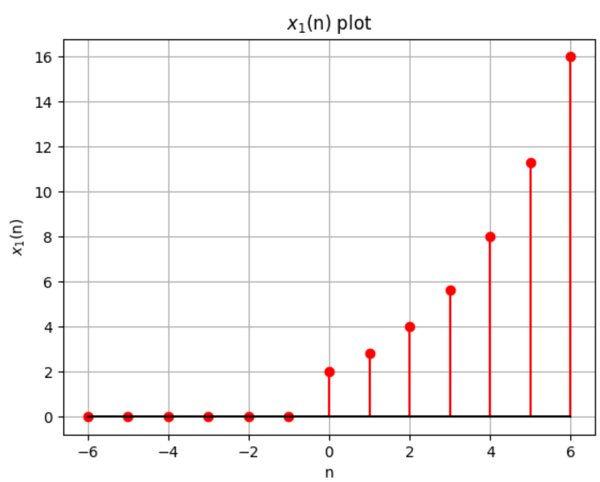

Fig. 1: Plot of $x_1(n)$ from n = -6 to 6

Let $x(n) = x(0)\, r^n u[n]$ be a general sequence then it's Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \tag{9}$$
$$\implies X(z) = \sum_{n=0}^{\infty} x(0)\, r^n z^{-n} \tag{10}$$
$$\implies X(z) = x(0) \sum_{n=0}^{\infty} r^n z^{-n} \tag{11}$$
$$\implies X(z) = x(0) \left(\frac{1}{1 - \frac{r}{z}}\right) \tag{12}$$
$$\therefore X(z) = \frac{x(0)}{1 - rz^{-1}} \forall |z| > |r| \tag{13}$$

$$\therefore X_1(z) = \frac{\sqrt{2}z}{z - \sqrt{2}} \tag{14}$$

with ROC:

$$|z| > \sqrt{2}$$

(b) Let $x_2(0) = \sqrt{3}$, $r_2 = \frac{3}{\sqrt{3}} = \sqrt{3}$, then the general term is:

$$x_2(n) = x_2(0)\, r_2^n u[n] \tag{15}$$

where u[0] = 1. Assume n^{th} $(n > 0)$ term is 729, which gives:

$$x_2(n) = x_2(0)\, r_2^n = 729 \tag{16}$$
$$\implies r_2^n = \frac{729}{x_2(0)} \tag{17}$$
$$\implies n = \log_{r_2} \frac{729}{x_2(0)} \tag{18}$$

Using values from Table 1,

$$\implies n = \log_{\sqrt{3}} \frac{729}{\sqrt{3}} \tag{19}$$
$$\implies n = \log_{\sqrt{3}} \frac{3^6}{\sqrt{3}} \tag{20}$$
$$\implies n = \log_{\sqrt{3}} \sqrt{3}^{11} \tag{21}$$
$$\therefore n = 11 \tag{22}$$

Thus the 12^{th} term of the G.P $x_2(n)$ is 729.

By eqn 13, the Z-transform of $x_2(n)$:

$$X_2(z) = \frac{z}{z - \sqrt{3}} \tag{23}$$

with ROC:

$$|z| > \sqrt{3}$$

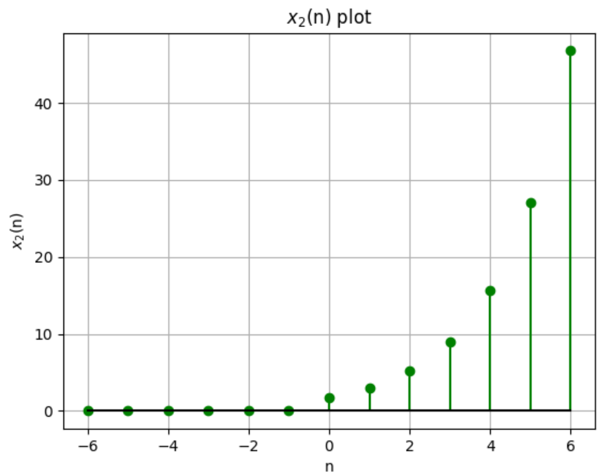

Fig. 2: Plot of $x_2(n)$ from n = -6 to 6

(c) Let $x_3(0) = \frac{1}{3}$, $r_3 = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$, then the general term is:

$$x_3(n) = x_3(0)\, r_3^n u[n] \tag{24}$$

where u[0] = 1. Assume $n^{th}\,(n > 0)$ term is $\frac{1}{19683}$, which gives:

$$x_3(n) = x_3(0)\, r_3^n = \frac{1}{19683} \tag{25}$$

$$\implies r_3^n = \frac{1}{19683 x_3(0)} \tag{26}$$

$$\implies n = \log_{r_3} \frac{1}{19683 x_3(0)} \tag{27}$$

Using values from Table 1,

$$\implies n = \log_{\frac{1}{3}} \frac{1}{19683\frac{1}{3}} \tag{28}$$

$$\implies n = \log_{\frac{1}{3}} \frac{1}{6561} \tag{29}$$

$$\implies n = \log_{\frac{1}{3}} 3^{-8} \tag{30}$$

$$\therefore n = 8 \tag{31}$$

Thus the 9^{th} term of the G.P $x_3(n)$ is $\frac{1}{19683}$.
By eqn 13, the Z-transform of $x_3(n)$:

$$X_3(z) = \frac{z}{z - \frac{1}{3}} \implies X_3(z) = \frac{3z}{3z - 1} \tag{32}$$

with ROC:

$$|z| > \frac{1}{3}$$

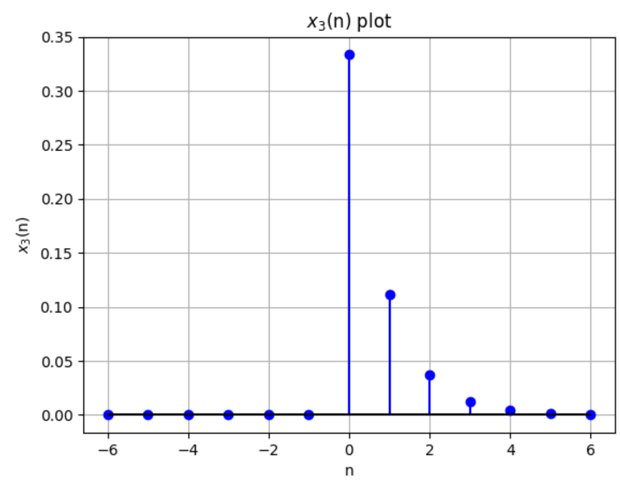

Fig. 3: Plot of $x_3(n)$ from n = -6 to 6

Parameter	**Description**	**Value**
r_i	Common ratio of G.P (a),(b),(c)	$\sqrt{2}, \sqrt{3}, \frac{1}{3}$
$x_i(n)$	Sequence	$x_i(0)\, r_i^n u[n]$
$X_i(z)$	Transform of $x_i(n)$	$\frac{x(0)}{1 - rz^{-1}}$

TABLE 1: Table of parameters