NCERT 11.9.5

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Question:
Which term of the following sequences:
(a) $2,2\sqrt{2},4\ldots$ is 128 (b) $\sqrt{3},3,3\sqrt{3}\ldots$ is 729
(c) $\frac{1}{3},\frac{1}{9},\frac{1}{27}\ldots$ is $\frac{1}{19683}$

Answer: (a) Let $x_1(0) = 2$, $x_1(1) = 2\sqrt{2}$, $x_1(2) = 4$. Since, $\frac{x_1(1)}{x_1(0)} = \frac{x_1(2)}{x_1(1)}$, the sequence $x_1(n)$ is a G.P Series. Let $r_1 = \frac{x_1(2)}{x_1(1)} = \sqrt{2}$, then the general term is $x_1(n) = x_1(0)\, r^{n-1}$.
Assume n^{th} term is 128, which gives:

$$x_1(n) = x_1(0)\, r^{n-1} = 128 \tag{1}$$

$$\implies r_1^{n-1} = \frac{128}{x_1(0)} \tag{2}$$

$$\implies n-1 = \log_{r_1} \frac{128}{x_1(0)} \tag{3}$$

Using values from Table 1,

$$\implies n-1 = \log_{\sqrt{2}} \frac{128}{2} \tag{4}$$

$$\implies n-1 = \log_{\sqrt{2}} 64 \tag{5}$$

$$\implies n-1 = \log_{\sqrt{2}} \sqrt{2}^{12} \tag{6}$$

$$\implies n-1 = 12 \tag{7}$$

$$\therefore n = 13 \tag{8}$$

Thus the 13^{th} term of the G.P $x_1(n)$ is 128.

$$x_1(n) = x_1(0) r^n u[n] \tag{9}$$

where $u[0] = 1$.

Let $x(n) = x(0)\, r^n u[n]$ be a general sequence then it's Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \tag{10}$$

$$\implies X(z) = \sum_{n=0}^{\infty} x(0)\, r^n z^{-n} \tag{11}$$

$$\implies X(z) = x(0) \sum_{n=0}^{\infty} r^n z^{-n} \tag{12}$$

$$\implies X(z) = \frac{x(0)}{r}\left(\frac{1}{1-\frac{r}{z}}\right) \tag{13}$$

$$\therefore X(z) = \frac{x(0)\, z}{r(z-r)} \forall |z| > |r| \tag{14}$$

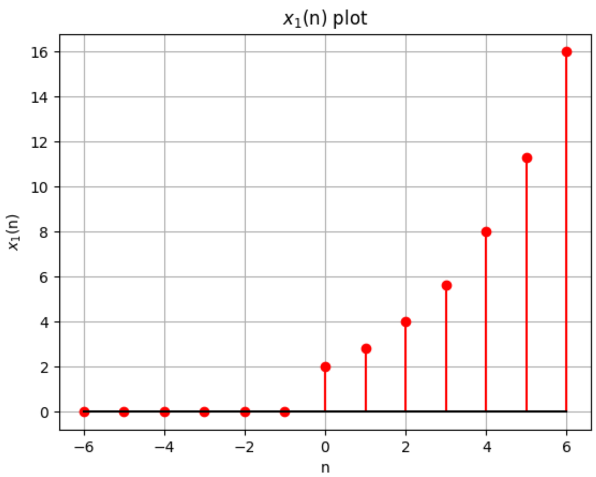

Fig. 1: Plot of $x_1(n)$ from n = -6 to 6

$$\therefore X_1(z) = \frac{\sqrt{2}z}{z - \sqrt{2}} \forall |z| > \sqrt{2} \tag{15}$$

with ROC:

$$|z| > \sqrt{2}$$

(b) Let $x_2(0) = \sqrt{3}$, $x_2(1) = 3$, $x_2(2) = 3\sqrt{3}$. Since $\frac{x_2(1)}{x_2(0)} = \frac{x_2(2)}{x_2(1)}$, the sequence $x_2(n)$ is a G.P Series. Let $r_2 = \frac{x_2(1)}{x_2(1)} = \sqrt{3}$, then the general term is $x_2(n) = x_2(0)\, r^{n-1}$.
Assume n^{th} term is 729, which gives:

$$x_2(n) = x_2(0)\, r^{n-1} = 729 \tag{16}$$

$$\implies r_2^{n-1} = \frac{729}{x_2(0)} \tag{17}$$

$$\implies n-1 = \log_{r_2} \frac{729}{x_2(0)} \tag{18}$$

Using values from Table 1,

$$\implies n-1 = \log_{\sqrt{3}} \frac{729}{\sqrt{3}} \tag{19}$$

$$\implies n-1 = \log_{\sqrt{3}} \frac{3^6}{\sqrt{3}} \tag{20}$$

$$\implies n-1 = \log_{\sqrt{3}} \sqrt{3}^{11} \tag{21}$$

$$\implies n-1 = 11 \tag{22}$$

$$\therefore n = 12 \tag{23}$$

Thus the 12^{th} term of the G.P $x_2(n)$ is 729.

$$x_2(n) = x_2(0) r_2^n u[n] \tag{24}$$

By eqn 14, the Z-transform of $x_b(n)$:

$$X_2(z) = \frac{z}{z - \sqrt{3}} \tag{25}$$

with ROC:

$$|z| > \sqrt{3}$$

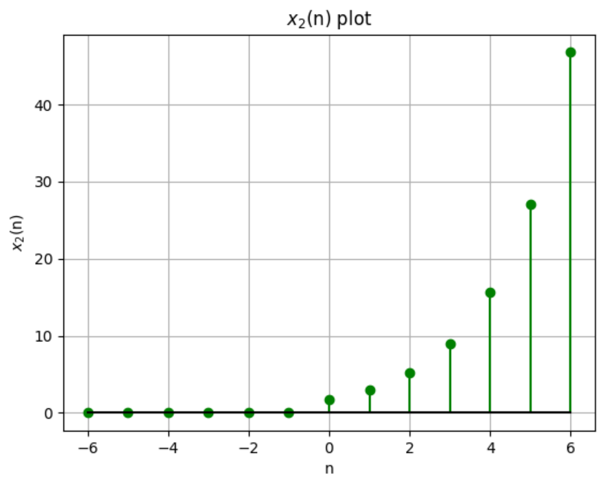

Fig. 2: Plot of $x_b(n)$ from n = -6 to 6

(c) Let $x_3(0) = \frac{1}{3}, x_3(1) = \frac{1}{9}, x_3(2) = \frac{1}{27}$. Since $\frac{x_3(1)}{x_3(0)} = \frac{x_3(2)}{x_3(1)}$, the sequence $x_3(n)$ is a G.P Series. Let $r_3 = \frac{x_3(1)}{x_3(1)} = \frac{1}{3}$, then the general term is $x_3(n) = x_3(0)\, r^{n-1}$.
Assume n^{th} term is $\frac{1}{19683}$, which gives:

$$x_3(n) = x_3(0)\, r_3^{n-1} = \frac{1}{19683} \tag{26}$$

$$\implies r_3^{n-1} = \frac{1}{19683 x_3(0)} \tag{27}$$

$$\implies n - 1 = \log_{r_3} \frac{1}{19683 x_3(0)} \tag{28}$$

Using values from Table 1,

$$\implies n - 1 = \log_{\frac{1}{3}} \frac{1}{19683 \frac{1}{3}} \tag{29}$$

$$\implies n - 1 = \log_{\frac{1}{3}} \frac{1}{6561} \tag{30}$$

$$\implies n - 1 = \log_{\frac{1}{3}} 3^{-8} \tag{31}$$

$$\implies n - 1 = 8 \tag{32}$$

$$\therefore n = 9 \tag{33}$$

Thus the 9^{th} term of the G.P $x_3(n)$ is $\frac{1}{19683}$.

$$x_3(n) = x_3(0) r_3^n u[n] \tag{34}$$

By eqn 14, the Z-transform of $x_3(n)$:

$$X_3(z) = \frac{z}{z - \frac{1}{3}} \implies X_3(z) = \frac{3z}{3z - 1} \tag{35}$$

with ROC:

$$|z| > \frac{1}{3}$$

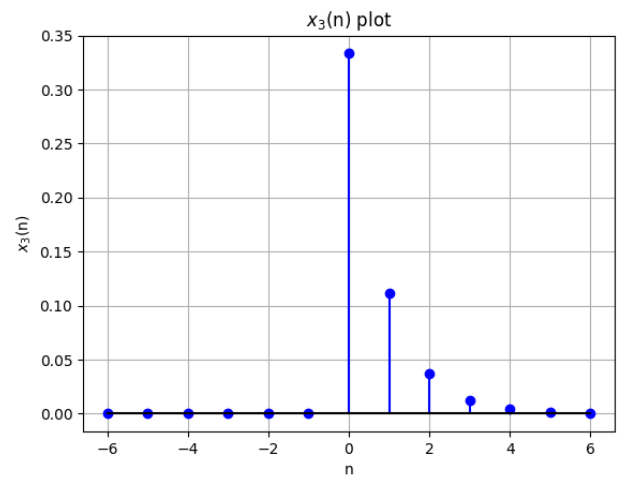

Fig. 3: Plot of $x_3(n)$ from n = -6 to 6

Parameter	**Description**	**Value**
r_1, r_2, r_3	Common ratio of G.P (a),(b),(c)	$\sqrt{2}, \sqrt{3}, \frac{1}{3}$
$x_i(n)$	Sequence	$x_i(0)\, r_i^n u[n]$
$X_i(z)$	Transform of $x_i(n)$	—

TABLE 1: Table of parameters