An Introduction to Brownian Motion

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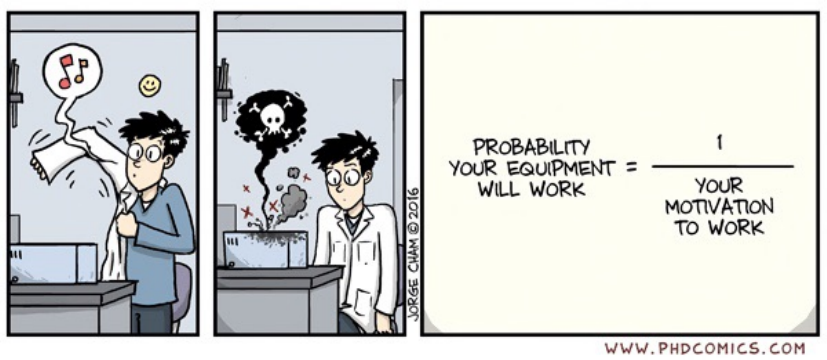
Intro

Randomness-Probability



Probability Basics

Probability Space

The theory of **Probability** uses the general measure theoritic framework with some modifications, which includes

- **A Sample space**(Ω) or set of all possible outcome of a Random experiment.
- **The event set** $\mathcal{F}$: The amount of information available as a result of experiment and the collection of all subsets of possible interest.
- The function $A \mapsto \mathbb{P}(A)$ defined on all elements $A \in \mathcal{F}$
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ and $\mathbb{P}(\Omega) = 1$

The two properties that makes probability different from general measure theory are Independence and Conditional expectation.

Random Variables

Definition

A Mapping $\mathbb{X} : \Omega \to \mathbb{S}$ between two measurable spaces $(\Omega, \mathcal{F})$ and $(\mathbb{S}, \mathcal{S})$ is said to be measurable and $\mathbb{X}$ is called a $\mathbb{S}$ valued random variable if

$$\mathbb{X}^{-1}(B) := \{\omega : \mathbb{X}(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{S}$$

Depending on the nature of $\mathbb{S}$

- For $\mathbb{S} = \mathbb{R}^d$: Random Variable/Vector
- For $\mathbb{S} =$General metric space eg. $\mathrm{C}[0, \infty)$: Random process

Definition

The law of a real valued Random Variable $\mathbb{X}$ is the probability measure on $(\mathbb{R}, \mathcal{B})$ such that $\mathcal{P}_{\mathbb{X}}(B) := \mathbb{P}(\{\omega : \mathbb{X}(\omega) \in B\})$ for any Borel set B.

Distribution function

The distribution function of a real valued random variable $\mathbb{X}$ is

$$\mathsf{F}_{\mathbb{X}}(\alpha) = \mathbb{P}(\{\omega : \mathbb{X}(\omega) \leq \alpha\}) = \mathcal{P}_{\mathbb{X}}((-\infty, \alpha])$$

And the distribution function uniquely determines the law $\mathcal{P}_{\mathbb{X}}$ of $\mathbb{X}$. Since the collection $\pi(\mathbb{R}) = \{(-\infty, b] : b \in \mathbb{R}\}$ generates $\mathcal{B}$.

Expectation

One of the most useful concept in probability is the Mathematical Expectation of a Random variable.

- For simple random variables $\varphi = \sum_{i=1}^{n} c_i \mathbb{I}_{A_i}$ we have $\mathbb{E}[\varphi] = \sum_{i=1}^{n} c_i \mathbb{P}(A_i)$
- For random variables $\mathbb{X} \geq 0$ we have $\mathbb{E}[\mathbb{X}] = \sup\{\mathbb{E}[Y], Y \leq \mathbb{X}\}$ where Y is a simple random variable
- For arbitrary real value random variable we have $\mathbb{E}[\mathbb{X}] = \mathbb{E}[\mathbb{X}_+] - \mathbb{E}[\mathbb{X}_-]$

Definition

We say that a random variable $\mathbb{X}(\omega)$ has a probability density function $f_{\mathbb{X}}$ if and only if its distribution function $\mathbb{F}_{\mathbb{X}}$ can be expressed as

$$\mathbb{F}_{\mathbb{X}}(\alpha) = \int_{-\infty}^{\alpha} f_{\mathbb{X}}(x)dx, \qquad \forall \alpha \in \mathbb{R} \tag{1}$$

A density function $f_{\mathbb{X}}$ must be integrable, Lebesgue almost everywhere nonnegative function with $\int_{-\infty}^{\infty} f_{\mathbb{X}}(x)dx = 1$

	Markov	Chebyshev
$\psi_*(A)\mathbb{P}(\mathbb{X} \in A) \leq \mathbf{E}\psi(X)$ $\psi_*(A) = \inf\{\psi(y) : y \in A\}$	$\psi_*(x) = x_+ \quad A = [a, \infty)$ $\mathbb{P}(X \geq a) \leq \frac{E[X_+]}{a}$	$\psi(x) = \|x\|^q$ $\mathbb{P}(\|Y - EY\| \geq a) \leq \frac{Var(Y)}{a^2}$
Jensen	Cauchy-Schwarz	
$\mathbb{E}[g(\mathbb{X})] \geq g(\mathbb{E}[\mathbb{X}])$	$\mathbb{E}\|\mathbb{X}\mathbb{Y}\| \leq \sqrt{\mathbb{E}[\mathbb{X}^2]}\sqrt{\mathbb{E}[\mathbb{Y}^2]}$	

Definition

A collection of events $\mathcal{A}_\alpha \subseteq \mathcal{F}$ with $\alpha \in \mathcal{I}$(possibly an infinite index set) are $\mathbb{P}$-mutually independent if for any $L < \infty$ and distinct $\alpha_1, \alpha_2, \ldots, \alpha_L \in \mathcal{I}$

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_L) = \prod_{k=1}^{L} \mathbb{P}(A_k) \qquad \forall A_k \in \mathcal{A}_{\alpha_k}$$

The definition of independence in terms of random variables are also equivalent.

The asymptotics of a sequence of random variables is another key aspect of probability theory. We have a sequence $\{\mathbb{X}_n\}$ in $(\Omega, \mathcal{F}, \mathbb{P})$

- **In Probability:** A sequence of random variables converges in probability or $\mathbb{X}_n \xrightarrow{P} X$ if $\lim_{n\to\infty} \mathbb{P}(\{\omega : |\mathbb{X}_n(\omega) - \mathbb{X}(\omega)| > \epsilon\}) = 0$
- **Almost sure:** A sequence of random variables converges in almost sure sense or $\mathbb{X}_n \xrightarrow{a.s.} \mathbb{X}$ if $\mathbb{P}(\omega : \lim \mathbb{X}_n(\omega) = \mathbb{X}(\omega)) = 1$
- **In Distribution:** A sequence of random variables converges in distribution or $\mathbb{X}_n \xrightarrow{D} \mathbb{X}$ if the corresponding distribution functions Converge.

Lemmas

Let A_n be events on a common probability space.

- If $\sum_n \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$
- If A_n are independent and $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\limsup A_n) = 1$

Law of Large Numbers

Let $\mathbb{X}_1, \mathbb{X}_2, \ldots$ be sequence of i.i.d random variables with $\mathbb{E}[\mathbb{X}_1] < \infty$

Weak Law

Then $\mathbb{S}_n/n = \frac{\mathbb{X}_1 + \ldots + \mathbb{X}_n}{n}$ converges to $\mathbb{E}[\mathbb{X}_1]$ in probability

Strong Law

If $\mathbb{X}_1, \mathbb{X}_2, \ldots$ are pairwise independent then $\mathbb{S}_n/n = \frac{\mathbb{X}_1 + \ldots + \mathbb{X}_n}{n}$ converges almost sure to $\mathbb{E}[\mathbb{X}_1]$

Central Limit Theorem

Lindeberg's CLT

The distribution of $\mathbb{S}_n = \frac{1}{\sqrt{n\sigma}}(\sum_{k=1}^{n} \mathbb{X}_k - n\mu)$ approaches the standard normal distribution in the limit $n \to \infty$ where $\{\mathbb{X}_k\}$ are **independent** and **identically** distributed with finite variance.

Lyapunov's Theorem

$\{\mathbb{X}_k\}$ are **independent** with finite variance satisfying : There exist $q > 2$

$$\lim_{n\to\infty} \frac{1}{\sqrt{\sigma_n}} \sum_{k=1}^{n} \mathbb{E}(|\mathbb{X}_k - \mathbb{E}\mathbb{X}_k|^q) = 0$$

Then $\frac{1}{\sqrt{\sigma_n}}(\mathbb{S}_n - \mathbb{E}\mathbb{S}_n)$ approaches the standard normal distribution in the limit $n \to \infty$

Kolmogorov's three series theorem

Suppose $\{\mathbb{X}_k\}$ are independent random variables. Let $c > 0$ be a constant and $\mathbb{X}_n^c = \mathbb{X}_n \mathbb{I}_{|\mathbb{X}_n| \leq c}$ be the corresponding truncated random variables and consider the three series

$$\sum_n \mathbb{P}(|\mathbb{X}_n| > c) \qquad \sum_n \mathbb{E}\mathbb{X}_n^c \qquad \sum_n Var(\mathbb{X}_n^c)$$

Then,the random series $\sum_n \mathbb{X}_n$ converges almost surely if and only if for some $c > 0$ all three series converge.

Stochastic Processes

Stochastic Process

Given $(\Omega, \mathcal{F}, \mathbb{P})$ a stochastic process(S.P)$\{\mathbb{X}_t\}_{t\in\mathbb{T}}$ is a collection of random variables indexed by $\mathbb{N}$ or $[0, \infty)$ e.t.c.

Depending on the nature of the indexing set the stochastic processes are classified into

- **Discrete parameter S.P:** Finite/Countable index set. Eg. Markov chain, Martingales, Poisson process,Branching process etc.
- **Continuous parameter S.P:** Uncountable index set. Eg. Wiener Process, Gaussian process etc.

Sample path

Definition

Given a stochastic process $\{\mathbb{X}_t : t \in \mathbb{T}\}$ where the index set is an interval in $\mathbb{R}$. The function $t \mapsto \mathbb{X}_t(\omega)$ is called the **sample function/path/realization/trajectory** of the S.P. at $\omega \in \Omega$

Finite dimensional distributions:

The finite dimensional distributions of a S.P. with uncountable index set we mean the collection of probability measures $\mu_{t_1,t_2,\dots,t_n}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and distinct $t_k \in \mathbb{T}, \forall k \in \{1, 2, \dots, n\}$ such that

$$\mu_{t_1,t_2,\dots,t_n}(B) = \mathbb{P}((\mathbb{X}_{t_1}, \mathbb{X}_{t_2}, \dots, \mathbb{X}_{t_n}) \in B)$$

For any Borel subset B of $\mathcal{B}(\mathbb{R}^n)$

Definition

A collection of finite dimensional distributions is consistent if for any $B_k \in \mathcal{B}$, distinct $t_k \in \mathbb{T}$ and finite n,

- $\mu_{t_1,\dots,t_n}(B_1 \times \ldots \times B_n) = \mu_{t_{\pi(1)},\dots,t_{\pi(n)}}(B_{\pi(1)} \times \ldots \times B_{\pi(n)})$ for any permutation π of $\{1,\dots,n\}$
- $\mu_{t_1,\dots,t_n}(B_1 \times \ldots \times B_{n-1}) = \mu_{t_1,\dots,t_n}(B_1 \times \ldots \times B_{n-1} \times \mathbb{R})$

Description of space

- $\mathbb{R}^{\mathbb{T}} : \{x(t) : \mathbb{T} \mapsto \mathbb{R}\}$ set of all functions
- A **finite dimensional measurable rectangle** in $\mathbb{R}^{\mathbb{T}}$ is of the form

$$\{x(.) : x(t_i) \in B_i, i \in 1, \ldots, n\} : t_i \in \mathbb{T}$$

- The cylindrical σ-algebra $\mathcal{B}^{\mathbb{T}}$ generated by the collection of all finite dimensional measurable rectangles.

Definition

A set $A \subseteq \mathbb{R}^{\mathbb{T}}$ has a countable representation if

$$A = \{x(.) \in \mathbb{R}^{\mathbb{T}} : (x(t_1), x(t_n) \ldots) \in D\}$$

Where $D \in \mathcal{B}_c$ which is a product σ-algebra on countable product spaces. and $\mathbb{Q} = \{t_k\} \subseteq \mathbb{T}$. The set $\mathbb{Q}$ is called the countable base of the countable representation $(\mathbb{Q}, B)$ of A

Let $\mathcal{F}^{\mathbb{X}} = \sigma(\mathbb{X}_t, t \in \mathbb{T})$

Theorem

For any **consistent collection of finite dimensional distributions** (f.d.d) there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $\omega \mapsto \{\mathbb{X}_t(\omega), t \in \mathbb{T}\}$ on it, whose f.d.d are in agreement with the given collection. Further the restriction of the probability measure $\mathbb{P}$ to the σ-algebra $\mathcal{F}^{\mathbb{X}}$ is uniquely determined by the specified f.d.d.

Definition:

The Law or distriution of a S.P. is the probability measure $\mathcal{P}_{\mathbb{X}}$ on $\mathcal{B}^{\mathbb{T}}$ such that for all $A \in \mathcal{B}^{\mathbb{T}}$

$$\mathcal{P}_{\mathbb{X}}(A) = \mathbb{P}(\{\omega : \mathbb{X}_{(.)}(\omega) \in A\})$$

We have seen that the f.d.d **uniquely determine the law of any S.P.** and also provide the **probability of any event in** $\mathcal{F}^{\mathbb{X}}$.

So we want **most events of interest** about a S.P. must be in th $\mathcal{F}^{\mathbb{X}}$ that is **mapped via a sample function** to an element of $\mathcal{B}^{\mathbb{T}}$

Let $\gamma \in \mathbb{R}$ and $\mathbb{I} = [a, b)$ for some $a < b$

- $A_\gamma = \{x \in \mathbb{R}^{\mathbb{I}} : x(t) \leq \gamma \;\; \forall t \in \mathbb{I}\}$
- $C(\mathbb{I}) = \{x \in \mathbb{R}^{\mathbb{I}} : t \mapsto x(t) \text{ is continuous on } \mathbb{I}\}$
- All linear functions,all polynomials, all constants, all non decreasing functions, all function of bounded variation, all differentiable functions, all analytic functions are not an element of $\mathcal{B}^{\mathbb{I}}$

- The construction of f.d.d determines the Law or distributions of S.P on $\mathcal{B}^{\mathbb{T}}$ whose preimage is $\mathcal{F}^{\mathbb{X}} = \sigma(\mathbb{X}_t, t \in \mathbb{T})$
- But as mentioned above $\mathcal{F}^{\mathbb{X}}$ is inadequate as far as properties of the sample functions $t \mapsto \mathbb{X}_t(\omega)$ are concerned.

Continuous and separable modifications

Version

Two S.P. $\{\mathbb{X}_t\}$ and $\{\mathbb{Y}_t\}$ are called versions of one another if they have same f.d.d.

Modifications

A S.P. $\{\mathbb{X}_t\}$ is called a modification of $\{\mathbb{Y}_t\}$ if for all $t \in \mathbb{T}$ we have $\mathbb{P}(\mathbb{X}_t = \mathbb{Y}_t) = 1$

Indistinguishable

Two S.P. are called indistinguishable if $\{\omega : \mathbb{X}_t(\omega) \neq \mathbb{Y}_t(\omega)\}$ for some $t \in \mathbb{T}$ is a $\mathbb{P}$-null set or $\mathbb{P}(\mathbb{X}_t = \mathbb{Y}_t; \forall t \in \mathbb{T}) = 1$

Example

Example

Consider the probability space$([0,1], \mathcal{B}_{[0,1]}, \mathbb{P})$ an uncountable index set $\mathbb{T} = [0,1]$ and consider

$$\{\mathbb{X}_t(\omega) \equiv 0; t, \omega \in [0,1]\} \qquad \mathbb{Y}_t = \begin{cases} 0 & t \neq \omega \\ 1 & t = \omega \end{cases}$$

- $\mathbb{Y}_t$ is a modification of $\mathbb{X}$ since every $t \in \mathbb{T}$ we have

$$\mathbb{P}[\mathbb{X}_t = \mathbb{Y}_t] = \mathbb{P}(\mathbb{T} \neq t) = 1$$

- But on the other hand they are distinguishable

$$\mathbb{P}(\mathbb{X}_t = \mathbb{Y}_t; \forall t \geq 0) = 0$$

Kolmogorov Centsov continuity theorem

Theorem

Suppose $\{\mathbb{X}_t : t \in \mathbb{T}\}$ is a S.P. indexed by $\mathbb{T}$ which is a compact interval. If there exist positive constants α, β and a finite c such that

$$\mathbb{E}[|\mathbb{X}_t - \mathbb{X}_s|^{\alpha}] \leq c \, \|t - s\|^{1+\beta} \qquad \forall s, t \in \mathbb{T}$$

then there **exist a continuous modification** to the S.P. $\{\tilde{\mathbb{X}}_t : t \in \mathbb{T}\}$ of $\mathbb{X}$ which is locally Holder continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$

$$\mathbb{P}\Big[\omega : \sup_{0<t-s<h(\omega); s,t \in \mathbb{T}} \frac{|\tilde{\mathbb{X}}_t(\omega) - \tilde{\mathbb{X}}_s(\omega)|}{|t-s|^{\gamma}} \leq \delta\Big] = 1$$

Here $h(\omega)$ is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.

Brownian Motion

- Imagine a spherical particle inside water.
- Particle is small but observable under a microscope which means particle is much larger that the size of water molecules.
- Now the suspended particle gets agitated by the water molecules and hence gets pushed around.
- Each collision affects the particle very slightly but the number of collisions in any time interval is very large.
- Hence the total displacement of the particle in an interval of time is a sum of large number of random and mutually independent small displacements.
- Now let $\mathbb{B}_t$ be the displacement of the x-coordinate of the particle.

Definition

A collection of random variables $\mathbb{B} = (\mathbb{B}_t)_{t\geq 0}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfying the following properties.

- For any $n \geq 1$ and any $0 = t_0 < t_1 < \ldots < t_n$, the random variables $\mathbb{B}_{t_k} - \mathbb{B}_{t_{k-1}}$ are independent
- For any $s < t$ the distribution of $\mathbb{B}_t - \mathbb{B}_s$ is $\mathcal{N}(0, t-s)$
- For $a.e.\omega \in \Omega$, the function $t \mapsto \mathbb{B}_t(\omega)$ is continuous

Lemma

The S.P. $\mathbb{W}_t$ satisfies Kolmogorov Centsov's continuity criterion

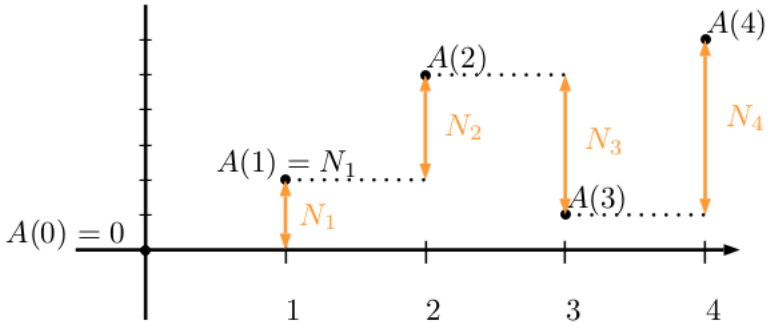
- $\mathbb{W}_t - \mathbb{W}_s \sim \mathcal{N}(0, t-s)$ for all $t > 0, s < t$
- $\mathbb{E}((\mathbb{W}_t - \mathbb{W}_s)^2) = t - s$
- $\sqrt{t-s}(\mathbb{W}_1 - \mathbb{W}_0) \sim \mathcal{N}(0, t-s)$
- $\mathbb{E}((\mathbb{W}_t - \mathbb{W}_s)^4) = \mathbb{E}(\sqrt{t-s}\mathbb{W}_1)^4 = (t-s)^2\mathbb{E}(\mathbb{W}_1^4) < \infty$

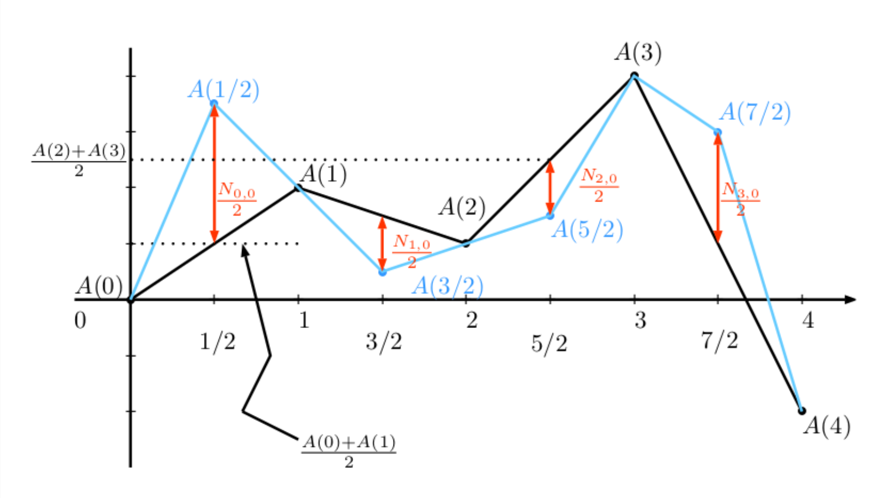
Which implies the process satisfies the properties of **independent increments** and **stationary increments** and the **Normal distribution** and it admits a continuous modification which is nothing but the **Brownian motion.**

Preliminary

- **Existence:** There exist a probability space on which we can define a countable family of centered standard normal random variables: $N_j \sim \mathcal{N}(0,1)$ for all $j \in \mathbb{N}$
- If $N \sim \mathcal{N}(0,1)$ then $aN = \mathcal{N}(0,a^2)$ for all constants a.
- If $X \sim \mathcal{N}(0,\sigma_X^2)$ and $Y = \mathcal{N}(0,\sigma_Y^2)$ and are independent then $X+Y \sim \mathcal{N}(0,\sigma_X^2+\sigma_Y^2)$
- If $N \sim \mathcal{N}(0,\sigma^2)$ and $\tilde{N} \sim \mathcal{N}(0,\sigma^2)$ and are independent then the two random variables $\frac{N+\tilde{N}}{2}$ and $\frac{N-\tilde{N}}{2}$ are independent with common law $\mathcal{N}(0,\sigma^2/2)$

The dyadic intervals are defined as follows. For every $n \in \mathbb{N}$ let $\mathcal{I}_n = \{[j/2^n, (j+1)/2^n]\}$ and let $\mathcal{I} = \cup_{n\in\mathbb{N}}\mathcal{I}_n$





A(3)
A(1/2)
A(7/2)
A(2)+A(3) / 2
A(1)
N0,0 / 2
A(2)
N2,0 / 2
N3,0 / 2
A(5/2)
N1,0 / 2
A(0)
A(3/2)
0
1
2
3
4
1/2
3/2
5/2
7/2
A(4)
A(0)+A(1) / 2

Extension to $[0, \infty)$

- We define for all $n \in \mathbb{N}$, the function $f_n(t)$ to be the linear interpolation of

$$A(0), A(1 \cdot 2^{-n}), \ldots, A(j \cdot 2^{-n}), \ldots \tag{2}$$

- Define

$$V_k = A(\frac{k+1}{2^n}) - A(\frac{k}{2^n}) \quad \forall k \in \{0, \ldots, 2^n - 1\} \tag{3}$$

- $$var(V_k) = \frac{1}{2^n} \tag{4}$$

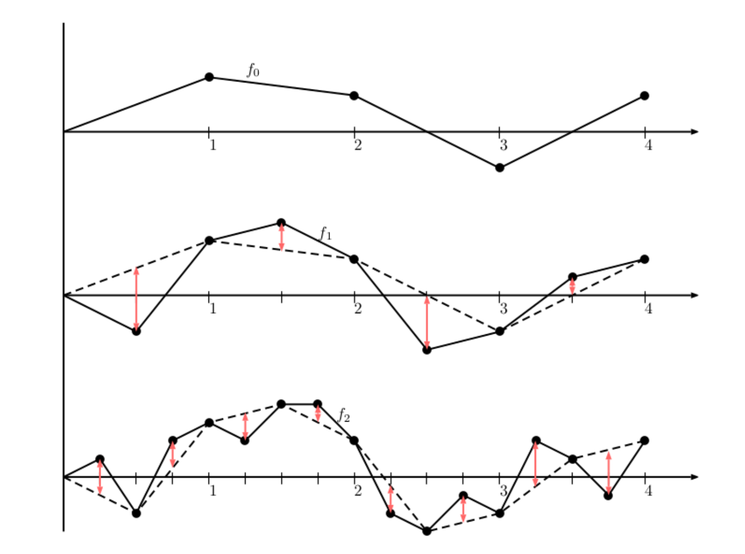
- We have

$$\mathbb{P}(\max_k |V_k| \geq \frac{1}{n^2}) \leq 2^n \mathbb{P}(|V_k| \geq \frac{1}{n^2}) \leq 2^{n+1} \exp\left(-\frac{2^{(n-1)}}{n^4}\right) \tag{5}$$

- So

$$\sum_{n \geq 1} \mathbb{P}(\max_k |V_k| \geq \frac{1}{n^2}) < \infty \tag{6}$$

MODIFICATION TO $[0, \infty)$



- By Borel-Cantelli's Lemma we have

$$\mathbb{P}(\limsup\{\max_{k}|V_k| \geq \frac{1}{n^2}\}) = 0 \tag{7}$$

- Given $t \in [0,1]$ we consider its dyadic expansion $t = \sum_{j=1}^{\infty} \frac{t_j}{2^j}$ for $t_j \in \{0,1\}$ and let $t(n) = \sum_{j=1}^{n} \frac{t_j}{2^j}$
- Then the sequence

$$A(t(n)) = 0 + \sum_{j=1}^{n}[A(t(k) - A(t(k-1))]$$

converges a.s. to some limit since with probability one

$$|A(t(k) - A(t(k-1))| \leq \frac{1}{n^2}$$

is cauchy.

We define the limit to be $\mathbb{Z}_t$ which agrees with $A(t)$ on the dense subset of all dyadic $t \in [0,1]$

If we can prove $\mathbb{Z}_t$ is **sample path continuous** then all **finite dimensional distributions** of $\mathbb{Z}_t$ and $A(t)$ will coincide.

- Take any $t, s \in [0,1]$ such that $|t-s| \leq \frac{1}{2^n}$
- If $t(n) = \frac{k}{2^n}$ and $s(n) = \frac{m}{2^n}$ then $|k-m| \in \{0,1\}$
- And we have $|A(t(n)) - A(s(n))| \leq \frac{1}{n^2}$ for large enough n
- which shows that

$$
\begin{aligned}
|\mathbb{Z}_t - \mathbb{Z}_s| &\leq |\mathbb{Z}_t - A(t(n))| + |A(t(n) - A(s(n)| + |A(s(n)) - \mathbb{Z}_s| \\
&\leq \sum_{l \geq n} \frac{1}{l^2} + \frac{1}{n^2} + \sum_{l \geq n} \frac{1}{l^2} \leq \frac{c}{n}
\end{aligned}
$$

Which proves the continuity of $\mathbb{Z}_t$. We set $\mathbb{Z}_t = 0$ for a measure zero set. **The Process so defined is called the Brownian motion.**

Conclusion

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