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ABSTRACT

A good abstract is concise, readable, and quantitative. The length should be approximately one paragraph, two at the most, or approximately from 200 to 400 words. Explain in one line why the project is important and summarize the major results. The final sentences explain the major implications of your work. Modern scientific style prefers the active voice. Abstracts are often an exception, but only if the passive voice reduces the total number of letters and words.

Do not repeat information that is in the title. Be explicit. Use numbers and quantifiable information where appropriate. Compose the abstract after you have read your report for the last time. Consider answering these questions to direct the content of the abstract: 1. What did you do? 2. Why did you do it? Which basic question were you trying to answer? 3. How did you do it? State methods. 4. What did you learn? State major results. 5. Why does it matter, what is the significance of your work? Identify one significant implication.

LIST OF SYMBOLS OR ABBREVIATIONS

α	The first letter
ω	The last letter
$\zeta(s)$	The Riemann Zeta function

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2.1 Introduction

Let $(\mathcal{M}, \rho)$ be a metric space.

Lemma 1. ***(Regularity of Probability measures).*** *Every probability measure μ on $(\mathcal{M}, \mathcal{B})$ is regular, i.e. for any Borel set A and any $\epsilon > 0$ one can find an open set G and a closed set F such that $F \subset A \subset G$, and $\mu(G/F) < \epsilon$*

One has to look for a class of **measure determining functions** $\mathcal{F}$ on $(\mathcal{M}, \mathcal{B})$

$$\mu f := \int f d\mu = \nu f := f d\nu, \forall f \in \mathcal{F} \implies \mu = \nu \tag{2.1}$$

Definition 2.

Lemma 3. *$\{I_F | F$ closed(open)$\}$ is a probability measure determining class.*

Proof. Given a Borel set A one can construct two sequences of closed sets $F_{in} \subset A$ for $i \in \{1, 2\}$ such that $\mathbf{P}(F_{in}) \to \mathbf{P}_i(A)$. Also we have $\mathbf{P}(F_{1n} \cup F_{2n}) \to \mathbf{P}_i(A)$. Which implies $\mathbf{P}_1(A) = \mathbf{P}_2(A)$. □

Lemma 4. $C_b(\mathcal{M})$ *:The class of all bounded continuous functions on $\mathcal{M}$ is a measure determining class.*

Lemma 5. $C_{bu}(\mathcal{M})$ *:The class of all bounded uniformly continuous functions on* $\mathcal{M}$ *is a measure determining class.*

Definition 6. (Tight) A probability measure $\mathbf{P}$ on $(\mathcal{M}, \mathcal{B})$ is called tight if for any $\epsilon > 0$, there exist a compact set $K \subset \mathcal{M}$ such that $\mathbf{P}(K) > (1 - \epsilon)$

That is the measure is concentrated around a compact set.

Lemma 7. *Probability measures on* ***polish*** *spaces are tight.*

Proof. Let $\{x_1, x_2, \ldots\}$ be a countable dense set of $\mathcal{M}$. And let $\mathcal{R}_{ik} = B(x_i, 1/k)$. Then $\cup_{i \geq 1} \mathcal{R}_{ik}$ covers $\mathcal{M}$ for any $k \geq 1$. We chose n_k such that $\mathbf{P}(\cup_{1 \leq i \leq n_k} \mathcal{R}_{ik}) > 1 - \frac{\epsilon}{2^k}$. Now setting $A = \cap_{k \geq 1} (\cup_{1 \leq i \leq n_k} \mathcal{R}_{ik})$.

$$\begin{aligned}
\mathbf{P}(A^c) &= \mathbf{P}(\cup_{k \geq 1} (\cup_{1 \leq i \leq n_k} \mathcal{R}_{ik})^c) \\
&\leq \sum_{k \geq 1} \mathbf{P}(\cup_{1 \leq i \leq n_k} \mathcal{R}_{ik})^c) \\
&\leq \sum_{k \geq 1} \frac{\epsilon}{2^k} = \epsilon
\end{aligned}$$

Hence $\mathbf{P}(A) > 1 - \epsilon$. incomplete. □

2.2 Weak Convergence

Let $\mathcal{P}(\mathcal{M})$ denote the set of all probability measures on $\mathcal{M}$. We want to measure the closeness between probability measures.

Definition 8. Let $\mathbf{P}, \mathbf{P}_n \in \mathcal{P}(\mathcal{M}), n \geq 1$. We say that $\mathbf{P}_n$ converges **weakly** to $\mathbf{P}$ if $\mathbf{P}_n f \to \mathbf{P} f, \forall f \in C_b(\mathcal{M})$ (denoted by $\mathbf{P}_n \xrightarrow{w} \mathbf{P}$).

Before proceeding further we define the following function.

$$\left(1 - \frac{\rho(x, F)}{\epsilon}\right)^+ = f_\epsilon(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \notin F_\epsilon \end{cases} \tag{2.2}$$

Where $F \subset \mathcal{M}$ is a closed set and $F_\epsilon = \{y \in \mathcal{M} | \rho(y, F) < \epsilon\}$. And it satisfies $I_F(x) \leq f_\epsilon(x) \leq I_{F_\epsilon}(x)$. Now We have the following characterizations given by the lemma.

Lemma 9. ***(Portmanteau)***. *The following are equivalent.*

1. $\mathbf{P}_n \xrightarrow{w} \mathbf{P}$

2. $\mathbf{P}_n f \to \mathbf{P} f, \forall f \in C_{ub}(\mathcal{M})$.

3. $\limsup_{n\to\infty} \mathbf{P}_n(F) \le \mathbf{P}(F), \forall F$ *closed.*

4. $\liminf_{n\to\infty} \mathbf{P}_n(G) \ge \mathbf{P}(G), \forall G$ *open.*

5. $\mathbf{P}_n(A) \to \mathbf{P}(A), \forall A$ *such that* $\mathbf{P}(\partial A) = 0$

Proof.

- (1) $\implies$ (2) Follows from the definition.

- (2) $\implies$ (3) For any $\epsilon > 0$,

$$\limsup \mathbf{P}_n(F) \le \limsup \mathbf{P}_n(f_\epsilon) = \lim \mathbf{P}_n f_\epsilon = \mathbf{P} f_\epsilon \le \mathbf{P}(F_\epsilon) \tag{2.3}$$

- (3) $\iff$ (4)

- (5) $\implies$ (1)

$\square$

We now state a lemma similar to subsequential convergence.

Lemma 10. $\mathbf{P}_n \xrightarrow{w} \mathbf{P}$ *if and only if every subsequence of* $\mathbf{P}_n$ *has a convergent subsequence.*

Proof. Let $f \in C_b(\mathcal{M})$. Consider the real sequence $\{\mathbf{P}_n f\}$ and a particular subsequence $\{\mathbf{P}_{n_k} f\}$. Now from the hypothesis $\{\mathbf{P}_{n_k}\}$ has a convergent subsequence $\{\mathbf{P}_{n_{k_l}}\}$ that converges weakly to $\mathbf{P}$. Hence $\mathbf{P}_{n_{k_l}} f$ converges to $\mathbf{P} f$. So any subsequence $\{\mathbf{P}_{n_k} f\}$ has a subsequence $\{\mathbf{P}_{n_{k_l}} f\}$ that converges to $\mathbf{P} f$. It follows that $\mathbf{P}_n f \to \mathbf{P} f$. $\square$

We see that the weak convergence of the metric depends on the metric through the topology. i.e two different metric generating same topology does not affect the weak convergence.

2.2.1 Convergence of Empirical measures

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $X_1, X_2, \ldots$ be i.i.d. S-valued sequence of random variables with values in metric space (S, d). Let μ be the law of X_i on S.

Definition 11. A random empirical measure μ_n on any $A \in \mathcal{B}(S)$

$$\mu_n(A)(\omega) = \frac{1}{n} \sum_{i=1}^{n} I(X_i(\omega) \in A), A \in \mathcal{B}(S) \tag{2.4}$$

We have by Strong law of large numbers,for any $f \in C_b(S)$

$$\int f d\mu_n = \frac{1}{n} \sum_{i=1}^{n} f(X_i) \xrightarrow{a.s.} \mathbf{E} f(X_1) = \int f d\mu \tag{2.5}$$

Since the set $C_b(S)$ can be uncountable, the set where the above convergence does not hold may not be trivial. But we have the following theorem.

Lemma 12. *If (S, d) is separable then there exists a metric ρ on S such that (S, ρ) is totally bounded and ρ and d define the same topology, i.e. $\rho(s_n, s) \to 0$ if and only if $d(s_n, s) \to 0$.*

Theorem 13. ***(Varadharanjan)*** *Let (S, d) be a separable metric space. Then μ_n converges to μ weakly almost surely,*

$$\mathbf{P}\left(\omega : \mu_n(\cdot)(\omega) \to \mu \text{ weakly}\right) = 1 \tag{2.6}$$

Proof. □

We now define two metrics on the set of all probability measures on (S, d).

Definition 14. (Levy-Prohorov distance). Given two probability distributions $\mathbf{P}, \mathbf{Q}$ on S then

$$\rho(\mathbf{P}, \mathbf{Q}) = \inf\{\epsilon > 0 : \mathbf{P}(A) \leq \mathbf{Q}(A^{\epsilon}) + \epsilon, \forall A \in \mathcal{B}\} \tag{2.7}$$

Definition 15.

$$\beta(\mathbf{P}, \mathbf{Q}) = \sup\{|\int f d\mathbf{P} - \int f d\mathbf{Q}| : ||f|| \leq 1\} \tag{2.8}$$

2.3 The mapping theorem and Skorohod's representation

push forward measure The mapping theorem gives us a method for constructing new measures from a given measure.

Theorem 16. ***(Mapping theorem).*** *Let $h : \mathcal{M}_1 \to \mathcal{M}_2$ be such that $\mathbf{P}(D_h) = 0$ where D_h is the set of discontinuity points of h. Then $\mathbf{P}_n \xrightarrow{w} \mathbf{P}$ implies $\mathbf{P}_n \circ h^{-1} \xrightarrow{w} \mathbf{P} \circ h^{-1}$.*

Proof. □

2.4 Relative compactness and Prohorov's theorem.

intro

Definition 17. (Relative compactness) A sequence of probability measures $\{\mathbf{P}_n\}$ is called relatively compact if any subsequence has a further subsequence that converge weakly.

To show the relative compactness of a family of probability distributions, we define a sufficient condition called **tightness** of a family.

Definition 18. A family Π of probability distribution is tight if for any $\epsilon > 0$ there exists a compact set K_ϵ such that $\mathbf{P}(K_\epsilon) > 1 - \epsilon, \forall \mathbf{P} \in \Pi$.

The following theorem By Prohorov

Theorem 19. ***(Prohorov)*** *If a family Π is tight then it is relatively compact.*

2.5 Weak convergence in $C[0,1]$

Lemma 20. *Let $\mathbf{P}_n$ be a sequence of Borel probability measures on $C[0,1]$. Then $\{\mathbf{P}_n\}$ is tight if and only if the following two conditions hold.*

- *For all $\eta > 0, \exists a \geq 0$ such that, for all $n \geq 1$, one has*

$$\mathbf{P}_n(|x(0)| \geq a) \leq \eta \tag{2.9}$$

- *For all eta $> 0, \epsilon > 0, \exists \delta \in (0,1), n_0 \geq 1$ such that,$\forall n \geq n_0$, one has*

$$\mathbf{P}_n(w_x(\delta) \geq \epsilon) \leq \eta \tag{2.10}$$

This means that tightness along with convergence of finite dimensional distributions imply weak convergence. As the lemma says.

Lemma 21. *let $\mathbf{P}_n$ be a sequence of Borel probability measures on $C[0,1]$. Suppose*

- *$\mathbf{P} \circ \pi^{-1}_{t_1,\ldots,t_k}$ converges weakly for any $t_1, \ldots, t_k \in [0,1], k \geq 1$.*
- *for all $\epsilon > 0$*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbf{P}_n(w_x(\delta) \geq \epsilon) = 0 \tag{2.11}$$

Then there is a Borel probability measure $\mathbf{P}$ on $C[0,1]$ such that $\mathbf{P}_n \xrightarrow{w} \mathbf{P}$.

2.6 Wiener measure and Donsker's theorem

Let ζ_i be i.i.d. with zero mean and variance σ^2 on some common probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Let $S_n = \sum_{i=1}^n \zeta_i, S_0 = 0$. We Consider a random function $X_n : (\Omega, \mathcal{A}, \mathbf{P}) \to (C[0,1], \mathcal{B}(C[0,1]))$ defined as follows

$$X_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} + (nt - \lfloor nt \rfloor)\frac{\zeta_{\lfloor nt \rfloor + 1}}{\sigma\sqrt{n}} \tag{2.12}$$

We will show that $\{\mathbf{P}_n = \mathbf{P}_n \circ X_n^{-1}\}$ is tight and $\mathbf{P}_n \circ \pi^{-1}_{t_1,\ldots,t_k}$ converges weakly to a certain k-variate Gaussian, for any $t_1, \ldots, t_k \in [0,1], k \geq 1$.

Theorem 22. ***(Existence of Wiener measure and Donsker's theorem)*** *There exist on $C[0,1]$, a Borel probability measure W such that, for any $t_1, \ldots, t_k \in [0,1], k \geq 1$, one has $W \circ \pi^{-1}_{t_1,\ldots,t_k} \equiv \mathcal{N}(0, \Sigma^{(t_1,\ldots,t_k)}_{ij})$, where $\Sigma^{(t_1,\ldots,t_k)}_{ij} = t_i \wedge t_j$. Moreover, $\mathbf{P}_n \xrightarrow{w} W$*

Proof. □

A random function B on $C[0,1]$, whose distribution is the Wiener measure, is called a **standard Brownian motion** on $[0,1]$.

Corollary 23. *Let $h : C[0,1] \to \mathbb{R}$ be such that $W(D_h) = 0$. Then $h(X_n) \xrightarrow{d} h(B)$.*

Table 2.1: Nonlinear Model Results

Case	Method #1	Method #2	Method #3
1	50	837	970
2	47	877	230
3	31	25	415
4	35	144	2356
5	45	300	556

2.7 Properties of Brownian motion

We have seen the # of constructions of Brownian motion. In this section we are going to discuss some of the defining properties of Brownian motion which makes it one of the canonical random objects. We start with recalling the definition.

Definition 24. (Brownian motion). The standard Brownian motion is a collection of random variables $B = (B_t)_{t\in[0,\infty)}$ on a common probability space$(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the following properties.

- **(Independent increments).** Increments over two disjoint intervals are disjoint.
- **(Gaussian).** $B_t - B_s \sim \mathcal{N}(0, t-s)$ distributed for any $s < t$.

- **(Path continuity).** $t \mapsto B_t$ is continuous $\mathbf{P}$-$a.e.\omega$.

This is equivalent to saying that B is a $C[0,\infty)$ valued random variable such that $B_t, t \geq 0$, are jointly Gaussian with mean zero and covariance $\mathbf{E}[B_tB_s] = \mathbf{E}[(B_s - B_s + B_t))B_s] = \mathbf{E}[B_s^2] + \mathbf{E}[B_t]\mathbf{E}[B_t - B_s] = s$ where $s < t$.

2.7.1 Invariance properties

The sample paths of Brownian motion satisfies a number of invariance properties which plays a role in showing interesting geometric structures in all scales of many natural random sets.

Lemma 25. ***(Scaling invariance).*** *Let $B = (B_t)_{t\geq 0}$ is a standard Brownian motion and let $a > 0$. Then the process defined by $X_t = \frac{1}{a}B_{a^2t}$ is also a standard Brownian motion.*

Proof. Path continuity and stationary independent increments remain unchanged by scaling and $X_t - X_s = \frac{1}{a}(B_{a^2t} - B_{a^2s})$ has mean zero and variance $\frac{1}{a^2}(a^2t \wedge a^2s) = s \wedge t$, same as standard Brownian motion. □

Theorem 26. ***(Time inversion).*** *Let $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. Then*

$$X_t = \begin{cases} 0 & \mathit{for} t = 0 \\ tB_{1/t} & \mathit{for} t > 0 \end{cases} \tag{2.13}$$

is a standard Brownian motion.

Lemma 27. ***(Time reversal symmetry).*** *The process defined by $X_t = B_{(1-t)} - B_1$ for $0 \leq t \leq 1$ is a standard Brownian motion on $[0,1]$.*

Lemma 28. ***(Time shift symmetry).*** *Given $t_0 \geq 0$. The process defined by $X_t = B_{t+t_0} - B_{t_0}$ is a standard Brownian motion.*

Lemma 29. ***(Reflection symmetry).*** *The process defined by $X_t = -B_t$ is a standard Brownian motion.*

Lemma 30. ***(Rotational invariance).***

2.7.2 Continuity properties of Brownian motion

intro We begin our discussion for Brownian motion defined on a compact interval, which implies the uniform continuous nature of the sample paths. Now to quantify the uniform continuity we start with the following definition.

Definition 31. (Modulus of continuity).

$$\limsup_{h \downarrow 0} \sup_{0 \le t \le 1-h} \frac{|B_{(t+h)} - B_{(t)}|}{\varphi(h)} \le 1 \tag{2.14}$$

It turns out that there is a nonrandom modulus of continuity for Brownian motion.

Theorem 32. *There exists a constant $C > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $0 \le t \le 1 - h$,*

$$|B_{(t+h)} - B_t| \le C\sqrt{h \log(1/h)} \tag{2.15}$$

The lower bound follows from.

Theorem 33. *For every constant $c < \sqrt{2}$, almost surely, for every $\epsilon > 0$ there exist $0 < h < \epsilon$ and $t \in [0, a - h]$ with*

$$|B_{(t+h)} - B_{(t)}| \ge c\sqrt{h \log(1/h)} \tag{2.16}$$

We note the striking result by Paul Levy

Theorem 34. ***(Levy's modulus of continuity).***

$$\limsup_{h \downarrow 0} \sup_{0 \le t \le 1-h} \frac{|B_{(t+h)} - B_{(t)}|}{\sqrt{2h \log(1/h)}} = 1 \tag{2.17}$$

almost surely.

Definition 35. A function $f : [0, \infty) \to \mathbb{R}$ is said to be locally α-Holder continuous at $x \ge 0$, if there exist $\epsilon > 0$ and $c > 0$ such that

$$|f(x) - f(y)| \le c|x - y|^{\alpha} \tag{2.18}$$

for all $y \geq 0$ with $|y - x| < \epsilon$. Here $\alpha > 0$ is called the Holder exponent and $c > 0$ is called the Holder constant.

As α gets larger the continuity gets stronger. For our function of interest, i.e. Brownian motion the transition from α-Holder continuity to not happens at $\alpha = 1/2$.

Corollary 36. *If $\alpha < 1/2$, then almost surely Brownian motion is everywhere locally α-Holder continuous.*

Proof. □

We end the subsection by the

Theorem 37. ***(Law of iterated logarithm for Brownian motion).*** *Suppose $\{B_t : t \geq 0\}$ is a standard Brownian motion. Then, almost surely,*

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log(t)}} = 1 \qquad \liminf_{t\to\infty} \frac{B(t)}{\sqrt{2t\log\log(t)}} = -1 \tag{2.19}$$

2.7.3 Zero set of Brownian motion

Brownian motion is a strange curve. It is continuous $a.e[\omega]$.

Definition 38. (Perfect set). A set S is a perfect set if it is closed and each point is a limit point.

Since every point is a limit point of S. S must be infinite and we can construct compact sets around each point. Now using finite intersection property of compact intervals in a metric space we can deduce that the cardinality of S is equal to $\mathbb{R}$. The following proposition is a reminiscent of the familiar **Cantor set**.

Proposition 39. *The zero set of Brownian motion $\mathcal{Z} := \{t \geq 0 : B_t = 0\}$.*

- *It is a perfect set $a.e[\omega]$.*
- *It is unbounded and uncountable.*
- *It has lebesgue measure zero.*

2.7.4 Non differentiability of Brownian motion

Let $(B_t)_{t\geq 0}$ is a Brownian motion. For every $t \geq 0$, we set the filtration

$$\mathcal{F}_t = \sigma(B_s, s \leq t). \tag{2.20}$$

Clearly $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. We also define the σ-algebra providing "infinitesimal glimpse" after time $t = 0$ as

$$\mathcal{F}_{0+} = \bigcap_{s>0} \mathcal{F}_s \tag{2.21}$$

. The first theorem is a $0 - 1$ law.

Theorem 40. ***(Bluementhal's 0-1 Law)*** *The σ-field $\mathcal{F}_{0+}$ is trivial, in the sense that $\mathbf{P}(A) = 0$ or 1 for every $A \in \mathcal{F}_{0+}$.*

. We now state the first proposition showing erratic behaviour of Brownian motion.

Proposition 41. *Almost surely for every $\epsilon > 0$,*

$$\sup_{0\leq s\leq \epsilon} B_s > 0 \qquad \inf_{0\leq s\leq \epsilon} B_s < 0 \tag{2.22}$$

For every $a \in \mathbb{R}$, let $T_a = \inf\{t \geq 0 : B_t = a\}$. Then a.s.,$\forall a \in \mathbb{R}, T_a < \infty$. Consequently a.s.

$$\limsup_{t\to\infty} B_t = +\infty \qquad \liminf_{t\to\infty} B_t = -\infty \tag{2.23}$$

The proposition shows the irregularity as we note.

Corollary 42. *Almost surely, the function $t \mapsto B_t$ is not monotone on any non-trivial interval.*

Theorem 43. ***(Paley, Wiener and Zygmund).*** *Almost surely Brownian motion is nowhere differentiable. Further almost surely for all t,*

$$\limsup_{h\downarrow 0} \frac{|B(t+h) - B(t)|}{h} = +\infty \qquad \liminf_{h\downarrow 0} \frac{|B(t+h) - B(t)|}{h} = -\infty \tag{2.24}$$

2.7.5 Markov Property of Brownian motion

2.7.6 Recurrence and transience of Brownian motion

Chapter 3

Conditional Expectation and Martingales

3.1 Introduction

The motivation of probability and in general stochastic processes is to mathematically model random phenomenon and the concept of measurable σ-algebra gives us a framework for that. We begin with the following definition.

Definition 44. (Filtrations). A filtration on $(\Omega, \mathcal{F}, \mathbf{P})$ is a collection of sigma algebras $\{\mathcal{F}_t\}_{0\leq t<\infty}$ of sub-sigma fields of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t < \infty$. We denote $\mathcal{F}_\infty$ by $\mathcal{F}$ and a filtered probability space by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t<\infty}, \mathbf{P})$.

The idea of filtration can be thought as how the σ-algebra evolves as time progresses which is essential in dealing with stochastic processes. In general working with a continuous indexing set requires some care in dealing with measurability issues.

Lemma 45. *The following technical issues are essential in the analysis of stochastic process.*

- ***(Completion).*** *The σ-algebra $\mathcal{F}_0$ contains all $A \subset B$ where $B \in \mathcal{F}_0$ such that $\mathbf{P}(B) = 0$.*

- ***(Right continuous filtration)*** *A filtration* $(\mathcal{F}_t)$ *is tight continuous if it contains information about the infinitesimal future for every* $t \geq 0$.

$$\mathcal{F}_{t+} := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t \tag{3.1}$$

Definition 46. (Stochastic process). A stochastic process $\{X_t\}_{t \in \mathcal{I}}$ on $(\Omega, \mathcal{F})$ is a collection of random variables taking values in another measurable space(**state space**) $(S, \mathcal{G})$.

In general the indexing set $\mathcal{I}$ can be discrete(e.g. A finite set or $\mathbb{N}$) or continuous(e.g. $\mathbb{R}, [0, 1]$) but we mostly consider the indexing set as the time $t \in [0, \infty)$. In general the indexing representation comes from the structure of the space where the process is taking values. So we want a deeper look at the measurability of the stochastic process.

Definition 47. A stochastic process $X = (X_t)_{t \geq 0}$ with values in $(S, \mathcal{G})$ is said to be measurable if the mapping

$$(\omega, t) \mapsto X_t(\omega) \tag{3.2}$$

defined on $\Omega \times [0, \infty)$ having the product σ-algebra $\mathcal{F} \times \mathcal{B}([0, \infty))$ is measurable.

This is a stronger notion of measurability as we see next.

Definition 48. Let $X = (X_t)_{t \geq 0}$ be a stochastic process with values in $(S, \mathcal{G})$

- **Adapted:** X is adapted if, X_t is $\mathcal{F}_t$ measurable $\forall t \geq 0$.
- **Progressive:** X is progressive if, $\forall t \geq 0$ the mapping

$$(\omega, t) \mapsto X_t(\omega) \tag{3.3}$$

  defined on $\Omega \times [0, t]$ is measurable with respect to $\mathcal{F}_t \times \mathcal{B}([0, t])$.

IDEAS OF Right continuity

Having defined a stochastic process one of the thing we are interested is the sample path properties of it.

Definition 49. (Sample path). The sample path of a stochastic process $\{X_t\}$ indexed by $t \in [0, \infty)$ is $t \mapsto X_t(\omega)$ for every $\omega \in \Omega$.

We define three notions of similarity between the paths of a stochastic process.

Definition 50. (Notions of similarity between stochastic processes). Given two stochastic processes X, Y defined on the same probability space $(\Omega, \mathcal{F})$

- **Version:** X and Y are versions of each other if for any integer $n \geq 1$ and $0 \leq t_1 \leq \ldots \leq t_n < \infty$, and for $A \in \mathcal{S}$ we have

$$\mathbf{P}((X_{t_1}, \ldots, X_{t_k}) \in A) = \mathbf{P}((Y_{t_1}, \ldots, Y_{t_k}) \in A) \tag{3.4}$$

- **Modification:** Y is a modification of X if for every $t \geq 0$ we have $\mathbf{P}(X_t = Y_t) = 1$

- **Indistinguishable:** X and Y are indistinguishable if $\{\omega : X_t(\omega) \neq Y_t(\omega)\}$ for some $t \in \mathcal{I}$ is a $\mathbf{P}$-null set.

3.2 Martingales

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t < \infty}, \mathbf{P})$.

Definition 51. A $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ adapted process $\{X_t\}$ is called

- **(Super Martingale).** $\mathbf{E}[X_t | \mathcal{F}_s] \leq X_s$ for $0 \leq s \leq t < \infty$

- **(Sub Martingale).** $\mathbf{E}[X_t | \mathcal{F}_s] \geq X_s$ for $0 \leq s \leq t < \infty$

- **(Martingale).** $\mathbf{E}[X_t | \mathcal{F}_s] = X_s$ for $0 \leq s \leq t < \infty$

Clearly a martingale is both a sub-martingale and super-martingale.

Definition 52. (Continuous Martingales). A process X_t adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ is a continuous martingale in L^1 if the following holds

- $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ for all $s \leq t$.
- The function $t \mapsto X_t$ is a continuous function $a.e[\omega]$.

Example 53. Brownian motion

Definition 54. (Uniformly integrable martingale.).

Theorem 55. *The following are equivalent.*

- $X_n \to X$ *in* $L^1(\Omega, \mathcal{F}, \mathbf{P})$
- $X_n \to X$ *in probability and* $(X_n)_{n\geq 1}$ *is uniformly integrable.*

Proof. □

To check Uniform integrability criterion we have the two following ways.

Proposition 56. *Let* $\mathcal{X}$ *be a family of random variables bounded in* L^p *for some* $p > 1$. *Then* $\mathcal{X}$ *is uniformly integrable.*

Proposition 57. *Let* $X \in L^1$. *Then the collection of random variables* $\{\mathbb{E}(X|\mathcal{G}) : \mathcal{G} \subseteq \mathcal{F}\}$ *where* $\mathcal{G}$ *is a sub-*σ*-field of* $\mathcal{F}$ *is a collection of uniformly integrable random variables.*

Lemma 58. ***(Martingales under convex function).*** *Suppose* $\{X_t\}$ *is a* $(\mathcal{F}_t)$ *martingale then for any convex function* φ *the transform* $\{\varphi(X_t)\}_{t\geq 0}$ *is a sub-martingale. If* $\{X_t\}$ *is a sub-martingale and* φ *is an increasing and convex function then* $\{\varphi(X_t)\}_{t\geq 0}$ *is also a sub-martingale.*

We now present dome useful inequalities for dealing with martingales.

Theorem 59. ***(Doob's Inequalities).*** $(X_t)_{t\geq 0}$ *is a continuous martingale*

- ***(L^p inequality).*** X_t *bounded in* L^p *for some* $t \geq 0$ *and* $p > 1$, *then*

$$\mathbf{E}[(\sup_{s\leq t} X_s)^p] \leq \frac{p}{p-1}\mathbf{E}[|X_t|^p] \tag{3.5}$$

- ***(Maximal Inequality).*** *Given a* $\lambda > 0$

$$\mathbf{P}(\sup_{[0,t]} |X_s| > \lambda) \leq \frac{1}{\lambda}\mathbf{E}[|X_t|\mathbf{1}(\sup_{s\in[0,t]} |X_s| > \lambda)] \leq \frac{1}{\lambda}\mathbf{E}[|X_t|] \tag{3.6}$$

3.2.1 Martingale convergence theorems

Theorem 60. *Let $(X_t)_{t\geq 0}$ be a martingale adapted to $(\mathcal{F}_t)$. Assume $(X_t) \in L^1$.*

- $X_t \xrightarrow{a.s.} X_\infty$ *where* $X_\infty \in L^1(\mathcal{F}_\infty)$.
- $X_t \xrightarrow{L^1} X_\infty$ *if and only if* X *is uniformly integrable. And we have*

$$\mathbf{E}[X_\infty|\mathcal{F}_s] = X_s \qquad \forall s \in [0, \infty) \tag{3.7}$$

- $X \in L^p$ *for some* $p > 1$ *then* $X_t \xrightarrow{a.s} X_\infty$ *where* $X_\infty \in L^p(\mathcal{F}_\infty)$.

3.2.2 Optional Stopping theorems

Intro

Definition 61. (Stopping time). A random variable $T : \Omega \to [0, \infty)$ is called a stopping time for the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for any $t \geq 0$, the event $\{T \leq t\} \in \mathcal{F}_t$.

Theorem 62. *Suppose (X_t) is a progressive process and let T be a finite stopping time. Then the function $\omega \mapsto X_t^T$ is $\mathcal{F}_T$ measurable.*

Theorem 63. ***(Optional stopping theorem).*** *Given a continuous martingale $(X_t)_{t\geq 0}$ and a finite stopping time T. Then the stopped process*

$$X_t^T = \begin{cases} X_t & , t \leq T \\ 0 & , t > T \end{cases}$$

is a uniformly integrable martingale and $\mathbf{E}[X_T] = \mathbf{E}[X_0]$.

Chapter 4

Stochastic calculus

4.1 Introduction

The goal of this chapter is to develop a theory of integration with respect to

4.1.1 Finite variation processes

Suppose we have a function on $0 \leq t \leq T$ having continuous derivative then we have the following.

$$\begin{aligned}
\langle f, f \rangle_t &= \lim_{p_n \to \infty} \sum_{i=0}^{p_n} [f(t_{i+1}) - f(t_i)]^2 \\
&= \lim_{p_n \to \infty} \sum_{i=0}^{p_n} |f^{'}(t_i^*)|^2 (t_{i+1} - t_i)^2 \\
&\leq \sup_i |t_{i+1} - t_i| \int_0^T |f^{'}(t)|^2 dt \\
&= 0
\end{aligned} \tag{4.1}$$

We will show that Brownian motion satisfies the property of having a non zero quadratic variation which gives rise to new properties especially defining stochastic integrals.

4.1.2 Previsible processes

introduction

Definition 64. The previsible σ-algebra $\mathcal{P}$ on $\Omega \times [0, \infty)$ is the σ-algebra generated by the sets of the form $A \times (s, t]$ for $A \in \mathcal{F}_s$. A process $H : \Omega \times [0, \infty) \to \mathbb{R}$ is called a previsible if it is $\mathcal{P}$-measurable.

Proposition 65. *Let X be a cadlag, adapted process. Then $H_t = X_{t-} = \lim_{s \to t-} X_s$ is a previsible process.*

Proof.
$$H_t^n = H_{t_n^-} = \sum_{k=0}^{\infty} H_{\frac{k}{2^n}} \mathbf{1}(\frac{k}{2^n}, \frac{(k+1)}{2^n}] \tag{4.2}$$

and $H_{\frac{k}{2^n}}$ is $\mathcal{F}_{\frac{k}{2^n}}$ measurable, it follows that H_t^n is previsible. Since $t_n^- \uparrow t$ an $n \to \infty$ we have $H_t^n \to H_t$ by left continuity. Therefore, H is a previsble process. $\square$

Proposition 66. *Let A be a cadlag, adapted, finite variation process with associated total variation process V. Let H be a previsible process such that for all $t \geq 0, \omega \in \Omega$, then*

$$\int_0^t |H(\omega, s)| dV(\omega, s) < \infty \tag{4.3}$$

Then the process $(H \cdot A)_t = \int_0^t H_s dA_s$ is a cadlag,adapted and finite variation process.

Proof.
- **cadlag:** We have the following from a simple application of dominated convergence theorem.

$$\begin{aligned}
(H \cdot A)_t &= \int H_s \lim_{r \to t^+} \mathbf{1}(s \in (0, r]) dA_s \\
&= \int \lim_{r \to t^+} H_s \mathbf{1}(s \in (0, r]) dA_s \\
&= \lim_{r \to t^+} (H \cdot A)_r
\end{aligned}$$

$\square$

4.2 Local martingales

intro

Definition 67. (**Local Martingale**). A local martingale is a continuous adapted process M such that there exists a non-decreasing sequence of stopping times $T_n \uparrow \infty$ such that the stopped process

$$M^{T_n} - M_0 = (M_{t \wedge T_n} - M_0)_{t \geq 0} \tag{4.4}$$

is a martingale for each n. The localizing sequence $(T_n)_{n \geq 1}$ is said to reduce M to a martingale. In cases where $M_0 \in L^1$ we can simplify the definition to

$$M^{T_n} = (M_{t \wedge T_n})_{t \geq 0} \tag{4.5}$$

Proposition 68. *The space of continous local martingales form a vector space.*

Proposition 69. ***(Properties of Continuous local martingales).*** *Let $(M_t)_{t \geq 0}$ is a sequence of continuous local martingales.*

- *A non negative continuous local martingale M bounded in L^1 is a supermartingale.*
- *A bounded continuous local martingale bounded in L^1 by a random variable Z i.e, $|M_t| \leq Z, \forall t \geq 0$ is a uniformly integrable martingale.*

This following example of stopping time for continuous local martingales is useful. The idea is to stop the process before it becomes too large.

Proposition 70. *$(M_t)_{t \geq 0}$ is a continuous local martingale with $M_0 = 0$ or $M_0 \in L^1$ then*

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\} \tag{4.6}$$

reduces M.

4.2.1 Quadratic variation of a Martingale

need Suppose we are given with a continuous local martingale $(M_t)_{t\geq 0}$ and we consider an increasing sequence of subdivisions of $[0,t]$ for $t>0$ given by $0=t_0^n<t_1^n<\ldots<t_{p_n}^n=t$. The first theorem guarantees the existence of a unique adapted continuous non-decreasing process.

Theorem 71. ***(Existence of quadratic variation).*** *The quadratic variation of $(M_t)_{t\geq 0}$ exists and unique upto indistinguishability which is given by*

$$\langle M,M\rangle_t=\lim_{n\to\infty}\sum_{i=1}^{p_n}(M_{t_i^n}-M_{t_{i-1}^n})^2 \tag{4.7}$$

as $p_n\to\infty$ such that $M_t^2-\langle M,M\rangle_t$ is a continuous local martingale. The convergence happens in probability.

Example 72. (Quadratic variation of Brownian motion).

We are also interested in asking what is cross variation between two processes.

Definition 73. (Crossvariation). The crossvariation between two continuous local martingales $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ is defined as foe every $t\geq 0$.

$$\langle M,N\rangle_t:=\frac{1}{2}(\langle M+N,M+N\rangle_t-\langle M,M\rangle_t-\langle N,N\rangle_t) \tag{4.8}$$

Proposition 74. ***(Properties of crossvariation).***

Example 75. If $B,B^{'}$ are two independent $(\mathcal{F}_t)$ adapted brownian motion then for every $t\geq 0$

$$\langle B,B^{'}\rangle_t=0 \tag{4.9}$$

This gives us a notion similar to orthogonality between processes as we define.

Definition 76. (Orthogonality). Two continuous local martingales M,N are said to be orthogonal if $\langle M,N\rangle_t=0$ and this is true if and only if MN is a continuous local martingale.

4.2.2 Continuous submartingales

Definition 77. A process $X = (X_t)_{t\geq 0}$ is a continuous semimartingale if it can be decomposed in terms of

$$X_t = M_t + A_t \tag{4.10}$$

where M is aa continuous local martingale and A is a finite variation process.

Proposition 78. *Given an increasing sequence of subdivision* $0 = t_0^n < \ldots < t_{p_n}^n = t$ *of* $[0,t]$ *for every* $t \geq 0$ *whose mesh tends to* 0*. Then*

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n}) = \langle X, Y\rangle_t \tag{4.11}$$

in probability.

Definition 79. (square integrable martingales)

$$\mathcal{M}^2 = \{X = (X_t)_{t\geq 0} \text{ continuous martingale with } \sup_{t\geq 0} \mathbb{E}[X_t^2] < \infty\} \tag{4.12}$$

Using martingale convergence theorem we deduce

$$X_t \xrightarrow{a.s.} X_\infty \qquad \textit{and} \qquad X_t \xrightarrow{L^2} X_\infty \tag{4.13}$$

Theorem 80. *The vector space* $\mathcal{M}^2$ *is complete with respect to the norm*

$$||X||_{\mathcal{M}^2} = \left(\mathbb{E}[X_\infty^2]\right)^{1/2}$$

4.2.3 Ito lemma

Ito's lemma is a stochastic calculus version of the fundamental theorem of calculus. It gives us an explicit formulae for canonical decomposition of semimartingales. We begin our discussion with the one dimensional version.

Theorem 81. ***(One dimensional Ito fomula).*** *Let* $X = (X_t)_{t\geq 0}$ *be a continuous semimartingale taking values almost surely in an open set* $D \subset$

$\mathbb{R}, \forall t \geq 0$. *And given* $F : D \to \mathbb{R}$ *is a* $\mathcal{C}^2$ *function. Then the process* $(F(X_t))_{t\geq 0}$ *is also a continuous semimartingale, and almost surely for each* $t \geq 0$

$$F(X_t) = F(X_0) + \int_0^t F^{'}(X_s)dX_s + \frac{1}{2}\int_0^t F^{''}(X_s)d\langle X, X\rangle_s \tag{4.14}$$

Since every semimartingale $X = M + V$ has an **unique decomposition**(upto indistinguishability) into a **continuous local martingale** (M) and a **finite variation process** (V). The integral $\int_0^t F^{'}(X_s)dX_s$ has the decomposition.

$$\int_0^t F^{'}(X_s)dX_s = \int_0^t F^{'}(X_s)dM_s + \int_0^t F^{'}(X_s)dV_s \tag{4.15}$$

Having seen this decomposition we are at a position to realize that the new process $F(X_t)$ has two parts.

1. **Local martingale:** $F(X_0) + \int_0^t F^{'}(X_s)dM_s$
2. **Finite variation:** $\int_0^t F^{'}(X_s)dV_s + \frac{1}{2}\int_0^t F^{''}(X_s)d\langle X, X\rangle_s$

This unique decomposition of the processes is an important corollary of Ito formula.

Example 82. Let $X_t = B_t$ which is a continuous martingale. Then using $\langle B, B\rangle_s = s$

$$F(B_t) = F(B_0) + \int_0^t F^{'}(B_s)dB_s + \frac{1}{2}\int_0^t F^{''}(B_s)ds \tag{4.16}$$

Proof. □

Theorem 83. ***Multi dimensional Ito*** *Let* $X^1, \ldots, X^p$ *be* p *continuous semimartingales, and let* F *be twice differentiable function on* $\mathbb{R}^p$. *Then for* *every* $t \geq 0$,

$$F(X^1, \ldots, X^p) = F(X_0^1, \ldots, X_0^p) + \sum_{i=1}^{p} \int_0^t \frac{\partial F}{\partial x^i}(X_s^1, \ldots, X_s^p) dX_s^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s^1, \ldots, X_s^p) d\langle X^i, X^j \rangle_s \tag{4.17}$$

Theorem 84. ***(Levy's characterization of Brownian motion in*** $\mathbb{R}^d$***)*** *Let* $X = (X^1, \ldots, X^d)$ *be a d-dimensional continuous local martingale started from the origin such that:*

$$\langle X^i, X^j \rangle = \delta_{ij} t \qquad 1 \leq i, j \leq d \tag{4.18}$$

Then X *is a Brownian motion in* $\mathbb{R}^d$.

4.3 Partial DE

We are taking a brief digression to a specific partial differential equation called **Laplace equation**. This has an interesting connection to Brownian motion. The presentation is inspired by [?]

Definition 85. (Laplace equation).

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{4.19}$$

Now it is an easy exercise to see that laplace equation is invariant under rotations. i.e, $v(x) = u(\mathcal{O}x)$ also satisfies laplace equation, where $\mathcal{O}$ is an orthogonal matrix. We remember the fact that Brownian motion also satisfies the rotational invariance property as in lemma 30.

We now provide the way to solve the laplace equation in $U = \mathbb{R}^n$.. Harnessing the radial symmetry of laplace equation we search for the radial slutions in $\mathbb{R}^n$.

$$u(x) = v(r) \tag{4.20}$$

where $r = |x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$. Calculating the following derivatives for $x \neq 0$ and for $i \in \{1, \ldots, n\}$

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} 2x_i = \frac{x_i}{r} \tag{4.21}$$

Using the above we get

$$u_{x_i x_i} = v^{''}(r)\frac{x_i^2}{r^2} + v^{'}(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \tag{4.22}$$

Summing over all i we get

$$\Delta u = v^{''}(r) + \frac{n-1}{r}v^{'}(r) \tag{4.23}$$

This is equivalent to solving the following differential equation.

$$v^{''}(r) + \frac{n-1}{r}v^{'}(r) = 0 \tag{4.24}$$

which is

$$v(r) = \begin{cases} b\log(r) + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3) \end{cases} \tag{4.25}$$

Definition 86. (Harmonic function). A function $u \in \mathcal{C}^2(\Omega)$ is said to be harmonic if it satisfies

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{4.26}$$

where $\Omega \subset \mathbb{R}^n$

This definitions give rise to sub/super harmonic functions which satisfies the $\Delta u \geq 0/\Delta u \leq 0$. Harmonicity gives us an way to calculate the value of u in the following way in any domain $\Omega \subset \mathbb{R}^n$.

Theorem 87. ***(Mean value property).*** *Given $\Omega \subset \mathbb{R}^n$ be a domain, i.e. a connected open set and $\partial\Omega$ be its boundary. Let $u : \Omega \to \mathbb{R}$ be measurable and locally bounded function. Then the following conditions are equivalent.*

1. *u is harmonic.*

2. *Given any ball $\mathcal{B}(x,r) \subset U$, we have*

$$u(x) = \frac{1}{\mathcal{L}(\mathcal{B}(x,r))} \int_{\mathcal{B}(x,r)} u(y)dy \tag{4.27}$$

3. *Given any ball $\mathcal{B}(x,r) \subset U$,*

$$u(x) = \frac{1}{\sigma_{x,r}(\partial\mathcal{B}(x,r))} \int_{\partial\mathcal{B}(x,r)} u(y)d\sigma_{x,r}(y) \tag{4.28}$$

where $\sigma_{x,r}$ is the surface measure of $\partial\mathcal{B}(x,r)$

The mean value property also has the converse property.

Theorem 88. *If $u \in \mathcal{C}^2(U)$ satisfies the mean value property*

$$u(x) = \frac{1}{\sigma_{x,r}(\partial\mathcal{B}(x,r))} \int_{\partial\mathcal{B}(x,r)} u(y)d\sigma_{x,r}(y) \tag{4.29}$$

for each ball $\mathcal{B}(x,r) \subset U$, then u is harmonic.

A direct consequence of mean value property is the maximum principle.

Theorem 89. ***(Maximum principle).*** *A subharmonic function $u : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is a domain.*

1. *If u attains its maximum in Ω, then u is a constant.*

2. *If u is continuous on $\overline{\Omega}$ and Ω is bounded, then*

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x) \tag{4.30}$$

Harmonicity of a function give rise to **regularity** as we see that harmonic functions are infinitely differentiable.

Theorem 90. ***Regualrity*** *If $u \in \mathcal{C}(U)$ satisfies the mean-value-property for each ball $\mathcal{B}(x,r) \subset U$, then*

$$u \in \mathcal{C}^\infty(U) \tag{4.31}$$

This behaviour of u is not necessarily extended to the boundary ∂U.

4.3.1 Brownian motion and Dirichlet problem

Consider a Brownian motion started inside a domain $U \subset \mathbb{R}^d$.

Definition 91. (Hitting time of Boundary). We define the hitting time of the boundary ∂U.

$$\tau = \min\{t \geq 0 : B_t \in \partial U\} \tag{4.32}$$

We observe a Brownian motion starting at a point $x \in U$ and wait till the time it first hits the boundary ∂U. Then any real valued bounded measurable function f defined on the boundary ∂D satisfies following property.

Theorem 92. *Let f be a bounded, measurable function from the boundary $f : \partial U \to \mathbb{R}$ such that the following*

$$u(x) = \mathbf{E}_x[f(B_\tau)\mathbf{1}(\tau < \infty)], \quad \forall x \in U \tag{4.33}$$

is locally bounded. Then u is a harmonic function.

Proof. Let's take a point $x \in U$ such that we can chose a ball of radius $r < d(x, \partial U)$ and τ be as above starting at x.

$$\tau = \inf\{t > 0 : d(B_t, x) = r\} \tag{4.34}$$

Using tower property of conditional expectation

$$u(x) = \mathbf{E}_x[\mathbf{E}[f(B_t)|\mathcal{F}_\tau]] \tag{4.35}$$

Brownian motion reaching the boundary forgets the past as the application of **Strong Markov property** gives us a Brownian motion starting at B_τ having same distribution.

$$\mathbf{E}[f(B_t)|\mathcal{F}_\tau] = \mathbf{E}_{B_\tau}[f(B_t)] = u(B_\tau) \tag{4.36}$$

□

The previous theorem almost gave a solution to the Dirichlet problem except for the boundary condition. So we need some conditions on the boundary points.

Definition 93. (Poincare cone condition). A domain U satisfies the poincare cone condition at $x \in \partial U$ if there exist a cone K at x having angle $\alpha > 0$, and a $r > 0$ such that $K \cap \mathcal{B}(x, r) \subset U^c$.

Theorem 94. ***(Dirichlet problem).*** *The function $u : \overline{U} \mapsto \mathbb{R}$ defined as*

$$u(x) = \mathbf{E}_x[f(B_\tau)\mathbf{1}(\tau < \infty)], \qquad \forall x \in U \tag{4.37}$$

satisfies

$$\begin{aligned} \Delta u &= 0 & x &\in U \\ u(x) &= f(x) & x &\in \partial U \end{aligned} \tag{4.38}$$

where each boundary point of U satisfies the poincare cone condition.

Example 95. Dirichlet problem in annulus

The existence of the solutions requires more stronger conditions on the boundary points.

Definition 96. (Regularity). A boundary point $x \in \partial U$ is regular if any Brownian motion starting at x leaves the domain U immidiately.

$$\inf\{t \geq 0 : B_t \in U^c\} = 0 \tag{4.39}$$

4.3.2 Conformal invariance

Conformal invariance is a property of Brownian motion that is true only in plane but has given rise to great development in the theory of probability in the last two decades.

Chapter 5

Conclusions

5.1 Observations

Definition 97. Definition is here.
Definition is here.
Definition is here.

Remark 98. Remark is here.
Remark is here.
Remark is here.

Example 99. Example is here.
Example is here.
Example is here.

Theorem 100. *Theorem is here.*
Theorem is here.
Theorem is here.

Proof. Proof is here.
Proof is here.
Proof is here. □

Theorem 101. *Theorem is here.*
Theorem is here.
Theorem is here.

Proof. Proof is here.
Proof is here.
Proof is here. □

5.2 Applications

Definition 102. Definition is here.
Definition is here.
Definition is here.

Remark 103. Remark is here.
Remark is here.
Remark is here.

Example 104. Example is here.
Example is here.
Example is here.

Theorem 105. *Theorem is here.*
Theorem is here.
Theorem is here.

Proof. Proof is here.
Proof is here.
Proof is here. □

Theorem 106. *Theorem is here.*
Theorem is here.
Theorem is here.

Proof. Proof is here.
Proof is here.
Proof is here. □

Appendix A

Appendix

I Basic Definitions

This is the first section of the appendix.

II Additional Theorems

This is the second section of the appendix.

Bibliography