Proofs

Markov :-

$$t \cdot \mathbb{1}_{X \geq t} \leq X \Rightarrow E[t \cdot \mathbb{1}_{X \geq t}] \leq E[X]$$

$$\Rightarrow t\, E[\mathbb{1}_{X \geq t}] \leq E[X]$$

$$\Rightarrow \mathbb{P}(X \geq t) \leq t^{-1} E[X]$$

$\mathbb{P}(X \geq \alpha E[X]) \leq 1/\alpha$

Paley - Zygmund :-

Cauchy Schwarz

$$E[X]^2 = E[X \cdot \mathbb{1}_{X>0}]^2 \leq E[X^2]\, E[\mathbb{1}_{X>0}]$$

$$= E[X^2]\, \mathbb{P}(X>0)$$

$\mu = E[X]$

$$E[X \cdot \mathbb{1}_{X>\alpha\mu}]^2 \leq E[X^2]\, \mathbb{P}(X>\alpha\mu)$$

Easier to calculate moments than probabilities (eg. Mean, Variances)

$$\mu = E[X \cdot \mathbb{1}_{X>\alpha\mu}] + E[X\, \mathbb{1}_{X<\alpha\mu}]$$

$$\leq E[X \cdot \mathbb{1}_{X>\alpha\mu}] + \alpha\mu$$

$$\Rightarrow E[X \cdot \mathbb{1}_{X>\alpha\mu}] \geq (1-\alpha)\mu$$

$$\Rightarrow \boxed{\mathbb{P}(X>\alpha\mu) \geq (1-\alpha)^2 \mu^2 / (E[X^2])}$$

BC Lemmas

$$\limsup A_n = \inf_{K} \sup_{n>K} A_n$$

$$= \lim_{m\to\infty} \bigcup_{n=m}^{\infty} A_n$$

$$\liminf A_n = \sup_{K} \inf_{n>K} A_n = \lim_{m\to\infty} \bigcap_{n=m}^{\infty} A_n$$

BC-1 $\rightarrow$ Subadditivity

BC-2 $\rightarrow$ Easy.

<u>Zero one - Laws</u>

<u>Proof</u>:- Define $\mathcal{T}_n := \sigma(\cup_{k>n} \mathcal{F}_k)$ $\boxed{\mathcal{T} = \cap_n \mathcal{T}_n}$

$\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ and $\mathcal{T}_n$ are independent.

$\mathcal{T} \subseteq \mathcal{T}_n \Rightarrow \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \mathcal{T}$ are independent

$\Rightarrow \mathcal{T}, \sigma(\cup_n \mathcal{F}_n)$ are independent

$\mathcal{T} \subseteq \sigma(\cup_n \mathcal{F}_n) \Rightarrow \mathcal{T}$ is independent of itself

$A \in \mathcal{T}$ $\quad P(A)^2 = P(A)P(A) \Rightarrow P(A) = 0, 1$

Proof:- Fix $\theta > 0$ "Markov"

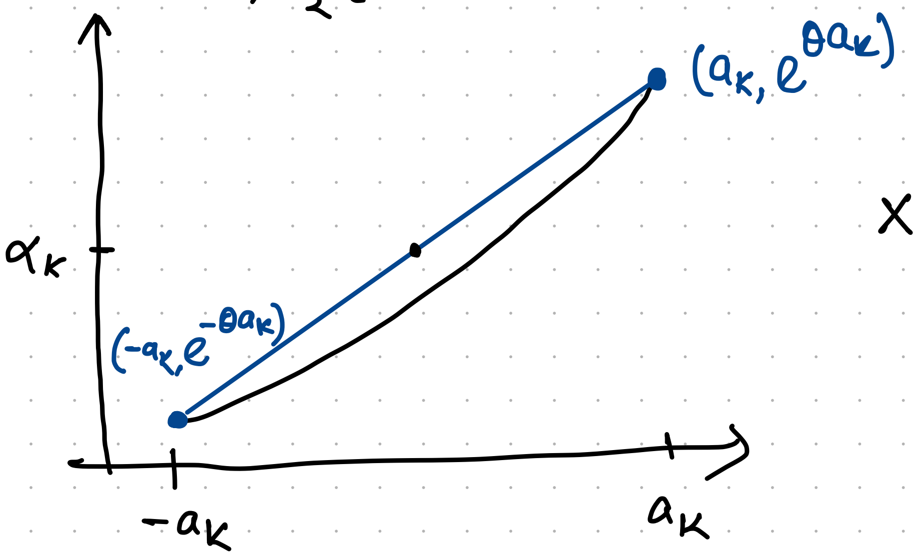
$$P\{S > tA\} = P\{e^{\theta S} > e^{\theta t A}\} \le e^{-\theta t A} E[e^{\theta S}]$$

$$= e^{-\theta t A} E\left[\prod_{k=1}^{n} e^{\theta X_k}\right]$$

$x \mapsto e^{\theta x}$ is convex on $[-a_k, a_k]$

$$\boxed{x \mapsto \frac{a_k - x}{2a_k} e^{-\theta a_k} + \frac{x + a_k}{2a_k} e^{\theta a_k}}$$

$$\Rightarrow \frac{1}{2}\left(e^{\theta a_k} + e^{-\theta a_k}\right) + \frac{x}{2a_k}\left(e^{\theta a_k} - e^{-\theta a_k}\right)$$



$X_k \in [-a_k, a_k]$

$$e^{\theta X_k} \le \alpha_k + \beta_k X_k$$

$$\alpha_k = \frac{1}{2}\left(e^{\theta a_k} + e^{-\theta a_k}\right)$$

$$\beta_k = \frac{1}{2a_k}\left(e^{\theta a_k} - e^{-\theta a_k}\right)$$

$$\Rightarrow \boxed{P\{S \ge tA\} \le e^{-\theta t A} E\left[\prod_{k=1}^{n} (\alpha_k + \beta_k X_k)\right] = e^{-\theta t A} \prod_{k=1}^{n} \alpha_k}$$

$E[X_k] = 0 \ \forall k$

$$\mathbb{P}\{S \geq tA\} \leq e^{-\theta t A} \prod_{k=1}^{n} \alpha_k$$

Wish to simplify bound over θ

Consequenty $\alpha_k \leq e^{\theta^2 a_k^2/2}$ (Follows from simple step)

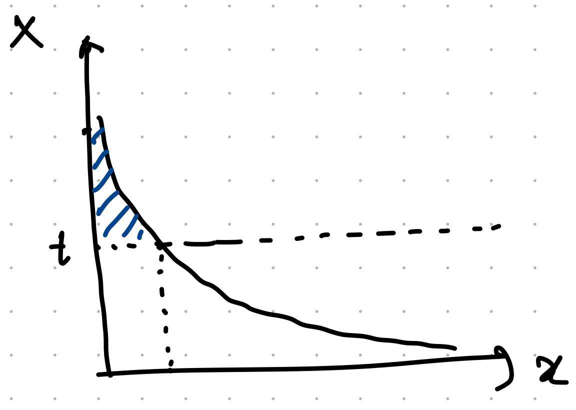
$$\Rightarrow \prod_{k=1}^{n} \alpha_k \leq e^{\theta^2 A^2/2}$$

$$\Rightarrow \mathbb{P}\{S \geq tA\} \leq e^{\underline{-\theta t A + \theta^2 A^2/2}}$$

↳ minimized if $\theta = t/A$

$$\leq \underline{e^{-t^2/2}}$$ Looks like Gaussian ∎

Kolmogorov's maximal inequality:-



$X \sim \exp(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \end{cases} \qquad \lambda = 1$$

$$\mathbb{P}(X_k > t) = e^{-t}$$

$$\mathbb{P}\left(\max_{k<n} X_k > t\right) = 1 - (1 - e^{-t})^n$$

Proof of Maximal inequality.

Stopping times

Fix n and let $\tau = \inf\{k \leq n : |S_n| > t\}$

$$\mathbb{P}\left(\max_{k\leq n} |S_k| > t\right) = \mathbb{P}(|S_\tau| > t) \leq t^{-2} \underline{E[S_\tau^2]}$$

[Chebyshev inequality]

need to control this part

$$
\begin{aligned}
E[S_n^2] &= E[(S_\tau + (S_n - S_\tau))^2] \\
&= E[S_\tau^2] + E[(S_n - S_\tau)^2] + 2E[S_\tau(S_n - S_\tau)] \\
&\geq E[S_\tau^2] + 2E[S_\tau(S_n - S_\tau)]
\end{aligned}
$$

Evaluating II'nd term by splitting as per value of τ

$S_n - S_\tau = 0$ if $\tau = n$

(martingale property)

$$\text{Also, } E[S_\tau(S_n - S_\tau)] = \sum_{k=1}^{n-1} E[1_{\tau=k} S_k (S_n - S_k)]$$

$$= \sum_{k=1}^{n-1} E[1_{\tau=k} S_k] E[S_n - S_k] \quad \text{(independence)}$$

$$= 0$$

$S_k 1_{\tau=k}$ depends only on $x_1, \ldots, x_k$

$S_n - S_k$ depends on $x_{k+1}, \ldots, x_n$

$$\mathbb{P}\left(\max_{k\leq n} |S_k| > t\right) \leq t^{-2} E[S_n^2]$$

Fix an integer $t \geq 1$

$X_{t,k}$ = indicator that the k'th book is not picked in first t days

$$\mathbb{P}(T_n > t) = \mathbb{P}(S_{t,n} \geq 1) \qquad S_{t,n} = \sum_{k=1}^{n} X_{t,k}$$

(No. of books that have not been picked in first t days)

$$E[X_{t,k}] = \sum X_{t,k}\, \mathbb{P}(X_{t,k})$$
$$= 1.\left(1-\tfrac{1}{n}\right)^t + 0.\left(1-\left(1-\tfrac{1}{n}\right)^t\right)$$
$$= \left(1-\tfrac{1}{n}\right)^t$$

$$E[X_{t,k}\, X_{t,l}] = \left(1-\tfrac{2}{n}\right)^t \qquad k \neq l$$

$$E[S_{t,n}] = n\left(1-\tfrac{1}{n}\right)^t$$

Two useful inequalities

- $1-x \leq e^{-x} \quad \forall x$
- $1-x \geq e^{-x-x^2} \quad |x| < \tfrac{1}{2}$

$$n e^{-t/n - t^2/n} \leq n\left(1-\tfrac{1}{n}\right)^t \leq n e^{-t/n} \qquad n > 2$$

$$E[S^2_{t,n}] = n\left(1-\tfrac{1}{n}\right)^t + n(n-1)\left(1-\tfrac{2}{n}\right)^t \leq n e^{-t/n} + n(n-1)\, e^{-2t/n}$$

Set $t = n\log n + n\theta_n$

$$\mathbb{P}(T_n > n\log n + n\theta_n) = \mathbb{P}(S_{t,n} \geq 1)$$
$$\leq E[S_{t,n}] \leq n e^{-\frac{n\log n + n\theta_n}{n}}$$
$$\leq e^{-\theta_n} = o(1)$$

Set $t = n\log n - n\theta_n \quad (\theta_n < \log n)$

$$E[S^2_{t,n}] \leq e^{\theta_n} + e^{2\theta_n} \qquad E[S_{t,n}] \geq e^{\theta_n - \frac{\log n - \theta_n}{n}}$$

$$\mathbb{P}(T_n > n\log n - n\theta_n) = \mathbb{P}(S_{t,n} \geq 1) \geq \frac{E[S_{t,n}]^2}{E[S_{t,n}^2]} \geq \frac{e^{2\theta_n - 2\frac{\log n - \theta_n}{n}}}{e^{\theta_n} + e^{2\theta_n}}$$

$$= 1 - o(1)$$

as $n \to \infty$

$$\mathbb{P}(|T_n - n\log_n| > n\theta_n) \longrightarrow 0 \quad \text{as } \theta_n \to \infty$$

Branching process

$Z_n :=$ G.W Process $\quad Z_n :=$ No of individual in n'th gen.

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \xi_2^{n+1} + \cdots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases}$$

$$p_k = \mathbb{P}(\xi_i^n = k) \qquad \mu = E\,\xi_i^m \in (0, \infty)$$

Want to calculate extinction probability

\# $\mu \leq 1$ $\quad \mathbb{P}(\xi_i^m = 1) < 1$ then $Z_n = 0$ $\forall n$ sufficiently large

\# $\mu > 1$ $\quad \mathbb{P}(Z_n = 0 \text{ for some } n) = \rho$, the sol^n $\varphi(\rho) = \rho$ which is called extinction probability.

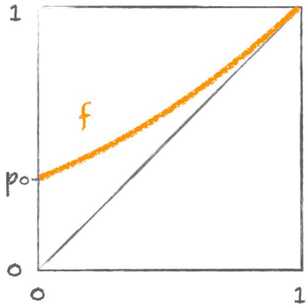
$$\varphi(s) = \sum_{k=0}^{\infty} p_k s^k \qquad p_k = \mathbb{P}(\xi_i^n = k), \quad s \in [0,1]$$

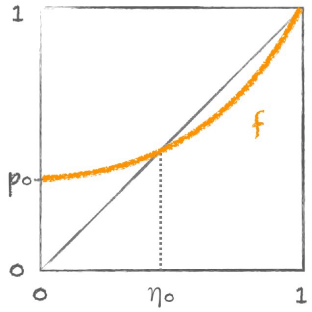
$$f_t(s) := E[s^{Z_t}] = f^{(t)}(s) = \underbrace{f \circ f \circ \cdots \circ f(s)}_{t\text{- iterate of } f \text{ at } s}$$

Extinction Probability

No sol^n other than 1 unique sol^n

$\mu < 1$ $\mu > 1$





$\phi(0) = p_0$

$\phi(1) = 1$

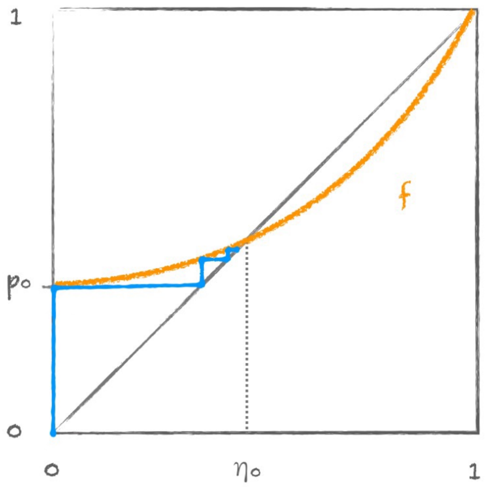


Figure 5.2: Convergence of iterates to a fixed point.

Credits: Notes (Sebastian Roch)

Proof of WLLN :-

Step 1 :- X_i has finite variance σ^2

WLOG $E[X_1]=0$

Chebyshev inequality $\Rightarrow$ $\mathbb{P}\left(\left|\frac{S_n}{n}\right|>\delta\right) \le \frac{1}{n^2\delta^2}\operatorname{Var}(S_n)$

$\operatorname{Var}(S_n) = n\sigma^2$ $\Rightarrow$ $\mathbb{P}\left(\left|\frac{S_n}{n}\right|>\delta\right) \le \frac{1}{n^2\delta^2}(n\sigma^2)$

Independence of X_i's

$$= \frac{\sigma^2}{n\delta^2} \to 0$$

as $n\to\infty$

δ fixed

Step 2 :- Let $E[X_i]<\infty$ not necessarily higher moments

let $E[X_i]=0$

Fixing n $\quad X_k = Y_k + Z_k$ $\qquad Y_k := X_k \mathbb{1}_{|X_k|\le A_n}$

$\qquad Z_k := X_k \mathbb{1}_{|X_k|>A_n}$

$E[Y_k] = -E[Z_k] =: \mu_n$

which depends on A_n and goes to zero as $A_n\to\infty$

$\mu_n \to 0$ as $n\to\infty$

Fix $\delta>0$ Choose n_0 large enough $|\mu_n|<\delta$ for $n\ge n_0$

As $|Y_1|\le A_n$ $\quad \operatorname{Var}(Y_1) \le E[Y_1^2] \le A_n E[|X_1|]$

$$\mathbb{P}\left(\left|\frac{S_n^Y}{n} - \mu_n\right|>\delta\right) \le \frac{\operatorname{Var}(Y_1)}{n\delta^2} \le \frac{A_n E[|X_1|]}{n\delta^2}$$

If $n\ge n_0$ then $|\mu_n|<\delta$ if $\left|\frac{1}{n}S_n^Z + \mu_n\right| \ge \delta$, then at least one of $Z_1, Z_2, \ldots, Z_n$ must be non zero

$$\mathbb{P}\left(\left|\frac{S_n^2}{n}+\mu_n\right|>\delta\right)\le n\,\mathbb{P}(Z_1\neq 0)=n\,\mathbb{P}(X_1>A_n)$$

$$X_k=(Y_k-\mu_n)+(Z_k+\mu_n)$$

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right|>2\delta\right)\le \mathbb{P}\left(\left|\frac{S_n^Y}{n}-\mu_n\right|>\delta\right)+\mathbb{P}\left(\left|\frac{S_n^Z}{n}-\mu_n\right|>\delta\right)$$

$$\le \frac{A_n E[|X_1|]}{n\delta^2}+n\,\mathbb{P}(|X_1|>A_n)$$

$$\le \underbrace{\frac{A_n E[|X_1|]}{n\delta^2}}_{\text{This term is less than } \delta}+\underbrace{\frac{n}{A_n}E\left[|X_1|\mathbb{1}_{|X_1|>A_n}\right]}$$

Take $A_n=\dfrac{\delta^3}{E[|X_1|]}\cdot n$

$=\alpha n$

$$\frac{n}{\alpha n}E\left[|X_1|\mathbb{1}_{|X_1|>\alpha n}\right]$$

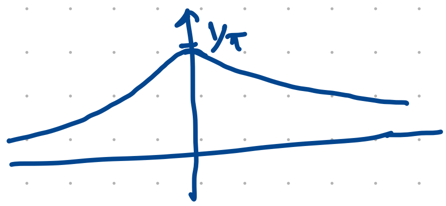
$$\Rightarrow \alpha^{-1}E\left[|X_1|\mathbb{1}_{|X_1|>\alpha n}\right]$$

$$=\frac{E[|X_1|]}{\delta^3}\cdot\underbrace{E\left[|X_1|\mathbb{1}_{|X_1|>\alpha n}\right]}_{\Downarrow}$$

goes to zero as $n\to\infty$

$$\Rightarrow \limsup_{n\to\infty}\mathbb{P}\left(\left|\frac{S_n}{n}\right|>2\delta\right)\le\delta \quad \blacksquare$$

$$f(t)=\begin{cases}\dfrac{1}{\pi(1+t^2)} & -\infty<t<\infty\end{cases}$$



$E[|X|]=\infty$

Weierstrass approximation :-

The set of polynomials is dense in the space of continuous functions.

Pf: $f \in C[0,1]$, $\quad Q_{f,n}(p) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} p^k (1-p)^{(n-k)}$

then, $\| Q_{f,n}(p) - f \| \to 0$ as $n \to \infty$

$$Q_{f,n}(p) = E\left[f\left(S_n/n\right)\right] = \sum_{k=0}^{n} f(k/n)\, \mathbb{P}(S_n = k)$$

For $p \in [0,1]$ $\quad X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Ber}(p)$ random variables

$$\left| E_p\left[f\left(S_n/n\right)\right] - f(p) \right| \leq E_p\left[\left| f\left(S_n/n\right) - f(p) \right|\right]$$

$$= E_p\left[\left| f\left(S_n/n\right) - f(p) \right| \mathbb{1}_{\left|\frac{S_n}{n} - p\right| \leq \delta}\right] +$$

$$E_p\left[\left| f\left(\frac{S_n}{n}\right) - f(p) \right| \mathbb{1}_{\left|\frac{S_n}{n} - p\right| > \delta}\right]$$

$$\leq \omega_f(\delta) + 2\|f\|\, \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \delta\right)$$

$\omega_f(\delta) := \sup\{ |f(x) - f(y)| : |x-y| < \delta \}$

$\mathrm{Var}_p(X_1) = p(1-p)$

$$\mathbb{P}_p\left(\left|\frac{S_n}{n} - p\right| > \delta\right) \leq \frac{p(1-p)}{n\delta^2} \leq 1/4n\delta^2$$

$$\sup_{p \in [0,1]} \left| Q_{f,n}(p) - f(p) \right| \leq \omega_f(\delta) + \frac{\|f\|}{2\delta^2 n}$$

Choosing $\delta > 0$ st $\omega_f(\delta) < \epsilon$, choosing n st $\frac{\|f\|}{2\delta^2 n} < \epsilon$

$$\Rightarrow \| Q_{f,n}(p) - f(p) \| < 2\epsilon \qquad \text{Q.E.D}$$

Monte-Carlo integration:

$f:[a,b] \to \mathbb{R}$ want to integrate

$x_1, \dots, x_n$ are $U([a,b])$ iid $\quad Y_K = f(x_K)$ are also iid

$$E[Y_1] = \int_a^b f(x)\,dx$$

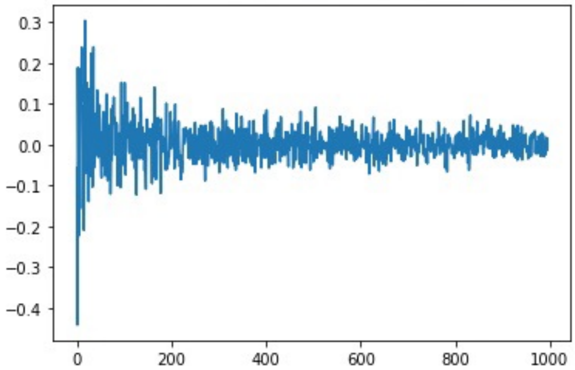
By WLLN $\quad \mathbb{P}\left(\left| \frac{1}{n} \sum_{K=1}^{n} f(x_K) - \int_a^b f(x)\,dx \right| > \delta \right) \to 0$

Random points

Accuracy is bad $\quad \frac{1}{n} \sum_{K=1}^{n} f(x_K)$ has var $O(1/n)$

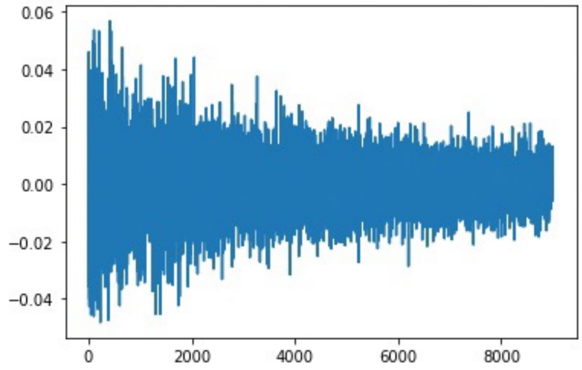
$\Rightarrow sd = O(1/n^{1/2})$

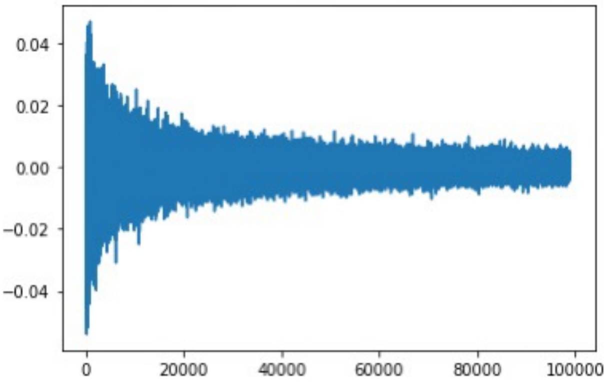
$$\pi = \int_0^1 \frac{4}{1+x^2}\,dx \approx \frac{1}{n} \sum_{K=1}^{n} \frac{4}{1+x_K^2}$$



1000 points
Errors are plotted

10000 points





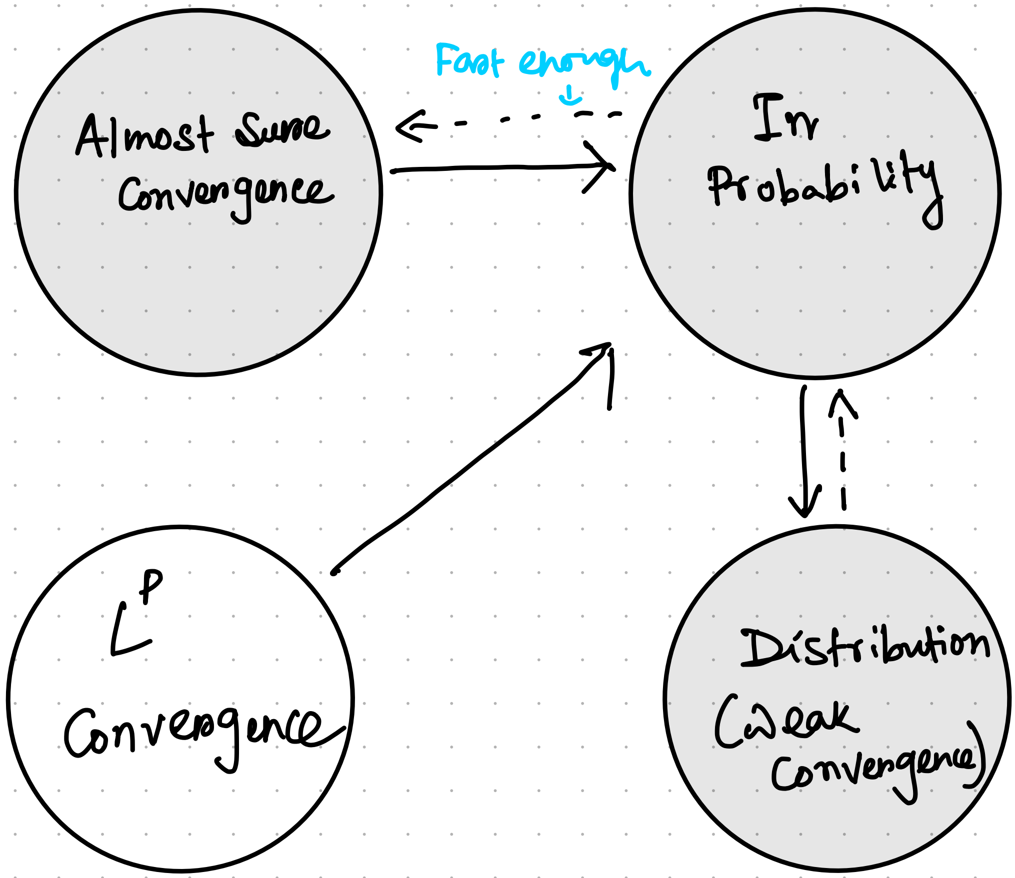
100,000 points

① $X_n \xrightarrow{a.s} X \Rightarrow X_n \xrightarrow{P} X$

② $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$

③ $X_n \xrightarrow{d} X \Rightarrow X_n \xrightarrow{P} X$

constant a.s.



a.s. and in probability

$\mathbb{P}(|X_n - X| > \delta) \to 0$ as $n \to \infty$

1.) $\mathbb{1}_{A_n} \xrightarrow{P} 0 \iff \lim_{n\to\infty} \mathbb{P}(A_n) = 0$

$\mathbb{P}(\omega : \lim X_n(\omega) = X(\omega)) = 1$

1.) $\mathbb{1}_{A_n} \xrightarrow{a.s.} 0 \iff \limsup \mathbb{P}(A_n) = 0$

$\mathbb{P}(\limsup A_n) \geq \limsup \mathbb{P}(A_n)$

a.s $\not\Leftarrow$ P

$P(A_n) = 1/n$

$\lim_{n\to\infty} \mathbb{P}(A_n) = 0$

$\sum_n P(A_n) = \infty$ (BC-2)

$\mathbb{P}(\limsup A_n) = 1$

$X_n = \mathbb{1}_{[0, 1/n]}$ $\quad X_n \xrightarrow{a.s} 0$

$X_n \xrightarrow{\mathbb{P}} 0$

$\mathbb{P}(|X_n| > \delta)$ is not summable for any $\delta > 0$

no rate of convergence is assumed.

Convergence in distributions :-

let $X_n = 1 - X \quad \forall n$ $\qquad X \sim$ binomial

$X_n \xrightarrow{d} X \qquad X_n \overset{P}{\not\to} X$

SLLN

Q: Does WLLN $\overset{?}{\Rightarrow}$ SLLN (i.e $P \overset{?}{\Rightarrow}$ a.s)

A: Possible if $\sum_n \mathbb{P}\left(\left|\frac{S_n}{n} - E[X_1]\right| > \delta\right) < \infty$ (i.e Summable) $\forall\ \delta > 0$

Supposing $Var(X_1) = \sigma^2 < \infty$ we get only a bound of $\frac{\sigma^2}{n\delta^2}$ (chebyshev) $\mathbb{P}\left(\left|S_n/n - E[X_1]\right| > \delta\right) < \sigma^2/n\delta^2$

which is not summable.

As we are at a "Borderline Summability".

Etemadi's version:-

<u>Proof:</u> $X_i = X_i^+ - X_i^-$ then proving it for non neg r.v

Assume X_i are non neg r.v

<u>Step 1</u> Truncate X_i's to produce more tractable version

Define $Y_i := X_i \mathbb{1}_{\{X_i < i\}}$

$\mu := E[X_1]$

$$\sum_{i=1}^{\infty} \mathbb{P}(X_i \neq Y_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(X_i \geq i) = \sum_{i=1}^{\infty} \mathbb{P}(X_1 \geq i) \leq E[X_1] < \infty$$

By BC-1 $\mathbb{P}(X_i \neq Y_i \text{ i.o.}) = 0$ [it suffices to show $n^{-1} \sum_{i=1}^{n} Y_i \xrightarrow{a.s.} \mu$]

Note $E[Y_i] - \mu = E[X_i ; X_i \geq i] \to 0$ as $i \to \infty$ by DCT

$E[X_i ; X_i \geq i] \leq E[X_i] < \infty$

$\therefore\ n^{-1}\sum_{i=1}^{n} E[Y_i] \to \mu$ as $n\to\infty$

Define $Z_n := \frac{1}{n}\sum_{i=1}^{n}(Y_i - E[Y_i])$ Suffices to prove

$Z_n \to 0$ a.s.

Take $\alpha > 1$ $\quad k_n := [\alpha^n]$

For any choice of $\alpha>1$ $Z_{k_n}\to 0$ a.s.

Take $\epsilon>0$ $\quad X_i$'s are pairwise independent

so are Y_i $\quad Cov(Y_i, Y_j) = 0$ for any $i\neq j$

$$\mathbb{P}(|Z_{k_n}|>\epsilon) \le \frac{1}{\epsilon^2 k_n^2}\sum_{i=1}^{k_n} Var(Y_i)$$

$$\sum_{n=1}^{\infty}\mathbb{P}(|Z_{k_n}|>\epsilon) \le \sum_{n=1}^{\infty}\frac{1}{\epsilon^2 k_n^2}\sum_{i=1}^{k_n} Var(Y_i)$$

$$\frac{[\alpha^{n+1}]}{[\alpha^n]}$$

$$= \frac{1}{\epsilon^2}\sum_{i=1}^{\infty} Var(Y_i)\sum_{n:k_n\ge i} 1/k_n^2$$

$\exists\ \beta>1$ depending on α s.t. $k_{n+1}/k_n \ge \beta$ for all n large enough

$$\sum_{n:k_n\ge i}\frac{1}{k_n^2} \le \frac{1}{i^2}\sum_{n=0}^{\infty} 1/\beta^n \le c/i^2$$

$$\sum_{n=1}^{\infty}\mathbb{P}(|Z_{k_n}|>\epsilon) \le \frac{c'}{\epsilon^2}E[X_1] < \infty$$

$$BC.1 \Rightarrow \mathbb{P}(|Z_{k_n}|>\epsilon \text{ i.o.}) = 0$$

$\Rightarrow Z_{k_n}\to 0$ as $n\to\infty$ (we see this subsequence goes to 0 a.s.)

Final step $Z_n \to 0$ a.s.

Define $T_n := \sum_{i=1}^{n} Y_i$ $\quad k_n < m \leq k_{n+1}$

$$\frac{k_n}{k_{n+1}} \frac{T_{k_n}}{k_n} = \frac{T_{k_n}}{k_{n+1}} \leq \frac{T_m}{m} \leq \frac{T_{k_{n+1}}}{k_n} = \frac{T_{k_{n+1}}}{k_{n+1}} \frac{k_{n+1}}{k_n}$$

Let $m \to \infty$ $\quad \frac{k_{n+1}}{k_n} \to \alpha$ $\quad T_{k_n}/k_n \xrightarrow{a.s.} \mu$ (Subsequence we shown above)

$k_n \to \infty$

$$\Rightarrow \frac{\mu}{\alpha} \leq \liminf_{m \to \infty} \frac{T_m}{m} \leq \limsup_{m \to \infty} \frac{T_m}{m} \leq \alpha \mu \quad a.s.$$

$\alpha > 1$ is arbitrary.

This completes the proof.