BROWNIAN MOTION AND STOCHASTIC CALCULUS

A REPORT

submitted in partial fulfillment of the requirements

for the award of the dual degree of

Bachelor of Science-Master of Science

in

MATHEMATICS

by

SOHAM PYNE

(16195)

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL
BHOPAL - 462066

May 2021

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान भोपाल
Indian Institute of Science Education and Research
Bhopal
(Estb. By MHRD, Govt. of India)

CERTIFICATE

This is to certify that **Soham Pyne**, BS-MS (Dual Degree) student in the Department of Mathematics, has completed bonafide work on the thesis entitled **'Brownian Motion and Stochastic Calculus'** under my supervision and guidance.

May 2021 **Dr. Dheeraj Kulkarni**
IISER Bhopal

Committee Member	**Signature**	**Date**
Dr. Dheeraj Kulkarni		
Dr. Prahlad Vaidyanathan		
Dr. Rahul Garg		

ACADEMIC INTEGRITY AND COPYRIGHT DISCLAIMER

I hereby declare that this project report is my own work and due acknowledgement has been made wherever the work described is based on the findings of other investigators. This report has not been accepted for the award of any other degree or diploma at IISER Bhopal or any other educational institution. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

I certify that all copyrighted material incorporated into this document is in compliance with the Indian Copyright (Amendment) Act (2012) and that I have received written permission from the copyright owners for my use of their work, which is beyond the scope of the law. I agree to indemnify and safeguard IISER Bhopal from any claims that may arise from any copyright violation.

May 2021 **Soham Pyne**

IISER Bhopal

ACKNOWLEDGEMENT

I want to thank my guide Dr. Dheeraj Kulkarni for giving me the idea of pursuing the project and being a constant support throughout the process. I would like to mention Prof Siva Athreya and Dr. D. Yogeshwaran for developing my interest in probability through their well crafted courses. I would like to thank Dr. Kartick Adhikari and Prof. Manjunath Krishnapur for taking my interest further. I would like to thank Dr. Rahul Garg and Dr. Prahlad Vaidyanathan for being in my evaluation commitee. I also like to thank all the faculty members in the math department for teaching what so ever mathematics I could grasp. Finally I also want to thank countless professors across the globe whose freely available succinct lecture notes made the readings pleasurable.

This project was conceived when the world was beginning to feel the brunt of the raging pandemic and was mostly done in a socially distanced way. I want to acknowledge the pandemic Covid-19 in both positive and negative way for improving the online material access while restricting physical movements. While being socially distant from college especially in final year is not a pleasurable experience for a college student. Nevertheless gaining access to classrooms of some great teachers alleviated some of that pain.

I want to thank the IISER-B Mathematics department especially acknowledging my fellow batchmates and helpful seniors for their support. I would like to thank all my batchmates, juniors and seniors across departments. Since this marks the end of this chapter it seems appropriate to cherish the memories made with my closest friends who are witness to all my mischiefs. Although I do not think they will read my thesis. Finally I thank my family members for everything.

Soham Pyne

ABSTRACT

Brownian motion is one of the example of a canonical random object. The presentation of the thesis aims to justify that by specifying some of the important properties. The seemingly simple definition of Brownian motion gives rise to some of the most striking results. Brownian paths are irregular but they show self similar behaviour at all scales. The techniques of stochastic calculus coupled with the theory of brownian motion is useful in dealing with a lot of real world problems involving randomness. The connection to partial differential equations with Brownian motion is also surprising as it helps in understanding classical problems in a probabilistic setting.

The chapter 1 starts with a brief introduction and some constructions of the process. Chapter 2 deals with the defining properties of Brownian motion. The chapters 3 and 4 develops some of the tools of stochastic calculus for the analysis. We end with a particular connection to partial differential equations using techniques of stochastic calculus in chapter 5.

LIST OF SYMBOLS OR ABBREVIATIONS

Ω	The probaility space
$\mathcal{F}$	Sigma algebra
$\mathbf{P}$	Probability
$\mathcal{N}$	Standard normal distribution
$\mathbf{E}$	Expectation
$\langle M, M \rangle$	Quadratic variation of M
$\mathcal{B}$	The Borel sigma algebra
$s \wedge t$	$\min\{s, t\}$
∂U	The boundary of U
$\mathcal{C}^2$	Twice continuously differentiable functions
$\mathcal{C}^\infty$	Infinitely differentiable functions
$\mathbb{H}^2$	The space of square integrable continuous bounded martingales.

Chapter 1

Introduction

The central object of our study will be Brownian motion.

1.1 Brief history of the problem

The history of the problem dates back to centuries. It is said to be started by inspection of coal dust in alchohol by **Ingenhouz**(1785) which was reverified by British botanist **Robert Brown**(1827) in his careful study of pollen grains. **Joseph Fourier** saw some connection between Brownian motion and Heat equation. Gradually **Einstien and Smoluchowski** independently made some advancements in developing the subject which led to a proof of existence of molecules. Then came **Bacchalier**(1900) and his **"Theory of speculation"** or the first account of now prevalent theory of mathematical finance. In late 1920's **Wiener** provided us with the sample path properties of Brownian motion. Eventually **Donsker** Came up with the idea that Brownian motion is one of the example of **Universal** scaling limit of multitude of microscopic processes.

1.2 Brownian motion

At the outset we set out to define our object of study.

Definition 1. (Brownian motion). The standard one dimensional Brownian motion is a collection of random variables $B = (B_t)_{t\in[0,\infty)}$ on a common probability space$(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the following properties.

- **(Independent increments).** Increments over two disjoint intervals are disjoint.
- **(Gaussian).** $B_t - B_s \sim \mathcal{N}(0, t-s)$ distributed for any $s < t$.
- **(Path continuity).** $t \mapsto B_t$ is continuous $\mathbf{P}$-$a.e.\omega$.

This is equivalent to saying that B is a $C[0,\infty)$ valued random variable such that $B_t, t \geq 0$, are jointly Gaussian with mean zero and covariance $\mathbf{E}[B_tB_s] = \mathbf{E}[(B_s - B_s + B_t))B_s] = \mathbf{E}[B_s^2] + \mathbf{E}[B_t]\mathbf{E}[B_t - B_s] = s$ where $s < t$.

1.3 Existence

We can show the existence of Brownian motion in mutiple ways.

- **Brownian motion as a Gaussian process**
- **Series construction of Brownian motion**

1.3.1 Gaussian process

We take a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and sequence of random variable X_t bounded in L^2. The reason behind the consideration is that L^2 is the only Banach space which is also a Hilbert space. Hence it admits an inner product structure which in turn admits useful geometric considerations.

Definition 2. A real valued random variable Y is called a Gaussian random variable if X has a density

$$p_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \tag{1.1}$$

We consider an Euclidean space $\mathbb{R}^n$ having an inner product $\langle u, v\rangle$ and $X_1, \ldots, X_n$ as independent Gaussian Variable.

Definition 3. (Gaussian vector). An n-dimensional Gaussian vector X is a $\mathbb{R}^n$ valued random variable such that for every $u \in S$ the inner product $\langle u, X \rangle$ is a Gaussian variable.

Imagining the inner product as standard n-dimensional inner product then a random vector $(X_1, \ldots, X_n) \in \mathbb{R}^n$ is a centered gaussian vector if any linear combination of $X_i^{\prime}s$ is a centered gaussian random variable. This definition of Gaussian vectors is independent of basis. We can consider an orthonormal basis for the euclidean space $\mathbb{R}^n$ given by $(e_1, \ldots, e_n)$. And we define following quantities.

- $X = \sum_{i=1}^{n} X_i e_i$
- $\mathbf{E}[X] := \sum_{i=1}^{n} \mathbf{E}[X_i] e_i$
- $\operatorname{var}(\langle u, X \rangle) = \sum_{i,j=1}^{n} u_i u_j \operatorname{cov}(X_i, X_j) \qquad u \in \mathbb{R}^n$

A Gaussian vector is completely characterized by its mean and covariance matrix. We note this two results without proof.

Proposition 4. *The law of X is uniquely characterized by the mean $\mathbf{E}[X]$ and the covariance matrix $\Sigma = \mathbf{E}[(X - \mathbf{E}[X])(X - \mathbf{E}[X])^t] \in \mathbb{R}^{n \times n}$.*

Lemma 5. *A sequence $\{X_i\}$ of Gaussian random variables are independent if and only if covariance matrix $\sum$ is diagonal.*

We consider the space of all random variables with finite second moment. The L^2 space add some geometry to the probability space as it is an **Hilbert space** with the following inner product structure.

$$\langle X, Y \rangle_{L^2(\Omega, \mathcal{F}, \mathbf{P})} = \int_{\Omega} X(\omega) Y(\omega) d\mathbf{P} = \mathbf{E}[XY] \tag{1.2}$$

Definition 6. (Centered Gaussian space) A centered gaussian space is a closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ which contains only centered gaussian variables.

1.3.2 Brownian motion as a Gaussian process

We start be defining.

Proposition 7. *A stochastic process $(X_t)_{t\geq 0}$ is said to be a centered Gaussian process if for all integer $k \in \mathbb{N}$ the vector $(X_{t_1}, \ldots, X_{t_k})$ is a Gaussian vector. The covariance of $(X_t)_{t\geq 0}$ is given by $\sum_X(s,t) = \mathbf{E}[X_s X_t]$.*

Proposition 8. *A Brownian motion $(B_t)_{t\geq 0}$ is a centered Gaussian process with covariance given by*

$$\sum_B(s,t) = \mathbf{E}[B_s B_t] = t \wedge s \tag{1.3}$$

Proof.

- Choose a sequence of deterministic times $t_1 < \ldots < t_k$ and constants $c_1, \ldots, c_k \in \mathbb{R}$.
- The expression $c_1 B_{t_1} + \ldots + c_k B_{t_k}$ can be expressed as linear combination of $B_{t_1}, (B_{t_2} - B_{t_1}), \ldots, (B_{t_k} - B_{t_{k-1}})$ each of which are independent. Hence is a Gaussian vector.
- To check the covariance we have for $s < t$

$$\mathbf{E}[B_s B_t] = s \tag{1.4}$$

□

1.3.3 Wiener Isometry and series construction

Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a centered Gaussian process $(W_t)_{t\geq 0}$ having covariance $\mathbf{E}[W_s W_t] = s \wedge t$ such that they are bounded in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. And we consider the following inner product structure for $f, g \in L^2[0,1]$

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx \tag{1.5}$$

We see that the indicator functions $\mathbf{1}_{[0,t]} \in L^2[0,1]$ satisfy

$$\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle = s \wedge t \tag{1.6}$$

This property of indicator functions in $L^2[0,1]$ led Wiener to have the insight about a linear isometry between $L^2[0,1]$ and $L^2(\Omega, \mathcal{F}, \mathbf{P})$ which maps each indicator function $\mathbf{1}_{[0,t]} \in L^2[0,1]$ to a random variable $W_t \in L^2(\Omega, \mathcal{F}, \mathbf{P})$.

Theorem 9. ***(Wiener's Isometry).*** *Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a sequence of Wiener processes $(W_t)_{t\geq 0}$ bounded in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Then for every non empty interval $J \subseteq [0, \infty)$ the mapping $\mathbf{1}_{[s,t]} \mapsto W_t - W_s$ extends to a linear isometry*

$$I_W : L^2(J) \rightarrow L^2(\Omega, \mathcal{F}, \mathbf{P}) \tag{1.7}$$

For every function $\varphi \in L^2(J)$, the random variable $I_W(\varphi)$ is centered gaussian.

Proof. The inner products in two spaces are equal.

$$\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle = \langle W_s, W_t \rangle \tag{1.8}$$

Let H_0 be the set of all finite linear combinations of the indicator functions of intervals in $L^2(J)$. This is a linear and dense subspace of $L^2(J)$. Since every function $f \in L^2(J)$ can be approximated arbitrarily closely using the elements of H_0 in the L^2 metric and I_W is a linear isometry of H_0, so it extends uniquely to a linear isometry of $L^2(J)$.
Now given any $\varphi \in L^2(J)$ there exist a sequence $\varphi_n \in H_0$ such that

$$\varphi_n \xrightarrow{L^2} \varphi \tag{1.9}$$

Since I_W is an isometry of $L^2(J)$ the sequence $I_W(\varphi_n) \xrightarrow{L^2} I_W(\varphi)$. This L^2 convergence implies convergence in distribution and we can use the idea of characteristic functions.

$$\lim_{n\to\infty} \mathbf{E}[\exp i\theta I_W(\varphi_n)] = \mathbf{E}[\exp i\theta I_W(\varphi)] \tag{1.10}$$

The random variables $I_W(\varphi_n)$ has a centered Gaussian distribution by definition which implies its characteristic function is

$$\mathbf{E}[\exp i\theta I_W(\varphi_n)] = \exp\{-\frac{\theta^2\sigma_n^2}{2}\} \tag{1.11}$$

This implies the variances σ_n^2 converges to a nonnegative limit σ^2 and the limit of random variable $I_W(\varphi)$ has the characteristic function

$$\mathbf{E}[\exp\{i\theta I_W(\varphi)\}] = \exp\{\frac{-\theta^2\sigma^2}{2}\} \tag{1.12}$$

This implies that $I_W(\varphi)$ has a centered Gaussian distribution. $\square$

Corollary 10. *Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and an infinite sequence β_n of independent and identically distributed $N(0,1)$ random variables and suppose we have an orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ of $L^2[0,1]$. Then for every $t \in [0,1]$ the infinite series*

$$W_t = \sum_{i=1}^{\infty} \beta_n \langle \mathbf{1}_{[0,t]}, \psi_n \rangle \tag{1.13}$$

converges in L^2. And the resulting stochastic process in a centered gaussian process with covariance $\mathbf{E}[W_t W_s] = s \wedge t$.

We consider one particular example of orthonormal basis for $L^2[0,1]$.

- **Haar basis**

1.3.4 Levy's construction using Haar Basis

This method of construction of Brownian motion was developed by paul levy using a particular orthonormal basis of $L^2[0,1]$. This is called **Haar Basis** and given as follows. For every $t \in [0,1]$ we have $\varphi_0(t) = 1$ and for every integer $n \geq 0$ and every $k \in \{0, 1, \ldots, 2^n - 1\}$

$$\varphi_k^n(t) = 2^{n/2}\mathbf{1}_{[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}})}(t) - 2^{n/2}\mathbf{1}_{[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}})}(t) \tag{1.14}$$

Using this orthonormal basis for $L^2[0,1]$ the inner product has the form

$$\langle \mathbf{1}_{[0,t]}, \varphi_k^n \rangle = \int_0^t \varphi_k^n(s)ds \tag{1.15}$$

Theorem 11. ***(Levy)*** *Suppose we are given with an infinite sequence of independent and identically distributed random variables β_k^n having $N(0,1)$*

distribution, then the infinite series

$$W(t) := \beta_0^0(t) + \sum_{m=1}^{\infty} \sum_{k=0}^{2^m-1} \beta_k^m \int_0^t \varphi_k^n(s)ds \tag{1.16}$$

converges uniformly with probability one for $t \in [0,1]$ to a standard Wiener process.

We begin with a lemma concerning the estimation of standard normal random variable.

Lemma 12. *If Z is a standard normal random variable then for every $x > 0$,*

$$\mathbf{P}(Z > x) \leq \frac{2e^{-x^2/2}}{\sqrt{2\pi}x} \tag{1.17}$$

Proof.

$$\mathbf{P}(Z > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{y^2}{2}} dy \leq \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{xy}{2}} dy = \frac{2e^{-x^2/2}}{\sqrt{2\pi}x} \tag{1.18}$$

□

We now begin our proof of the theorem

Proof. The series defined

$$W(t) := \beta_0^0(t) + \sum_{m=1}^{\infty} \sum_{k=0}^{2^m-1} \beta_k^m \int_0^t \varphi_k^n(s)ds \tag{1.19}$$

in the statement of the theorem is a special case of 10 given by

$$W_t = \sum_{n=1}^{\infty} \beta_n \langle \mathbf{1}_{[0,t]}, \psi_n \rangle \tag{1.20}$$

To show that it indeed defined a Brownian motion we need to show the uniform convergence of the process for $t \in [0,1]$. By definition the functions $\int_0^t \varphi_k^n(s)ds$ has a maximum value $2^{-m/2}$. For uniform convergence we need to show a bound for applying Weierstrass test. Since the random variables $\beta_k^n \sim \mathcal{N}(0,1)$ we can use the estimate derived above to find the following

$$\mathbf{P}(|\beta_k^n| \geq 2^n) \leq \frac{4}{2^{n/4}\sqrt{2\pi}} \exp\{-2^{\frac{n}{2}-1}\} \tag{1.21}$$

Hence the maximum has the upper bound

$$\mathbf{P}(\max_{1\leq k\leq 2^n} |\beta_k^n| \geq 2^{n/4}) \leq 2^n(\frac{4}{2^{n/4}\sqrt{2\pi}} \exp\{-2^{\frac{n}{2}-1}\}) \tag{1.22}$$

Since this bound is summable we can apply Borel-Cantelli lemma to conclude that with probability one

$$\max_k |\beta_k^n| \leq 2^{n/4} \qquad n \geq N \tag{1.23}$$

This implies that the multiple geometric series converges

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^n} |\beta_k^n|/2^{n/2} < \infty \tag{1.24}$$

Hence for all $t \in [0,1]$ the series

$$W(t) := \beta_0^0(t) + \sum_{m=1}^{\infty}\sum_{k=0}^{2^m-1} \beta_k^m \int_0^t \varphi_k^n(s)ds \tag{1.25}$$

converges uniformly to a Wiener process and is continuous with probability one. $\square$

Chapter 2

Properties of Brownian Motion

2.1 Properties of Brownian motion

We have seen the constructions of Brownian motion. In this section we are going to discuss some of the defining properties of Brownian motion which makes it one of the canonical random objects. We start with the invariance properties..

2.1.1 Invariance properties

The sample paths of Brownian motion satisfies a number of invariance properties which plays a role in showing interesting geometric structures in all scales of many natural random sets.

Lemma 13. ***(Scaling invariance).*** *Let $B = (B_t)_{t\geq 0}$ is a standard Brownian motion and let $a > 0$. Then the process defined by $X_t = \frac{1}{a}B_{a^2t}$ is also a standard Brownian motion.*

Proof. We see that properties like path continuity and stationary independent increments remain unchanged by scaling and $X_t - X_s = \frac{1}{a}(B_{a^2t} - B_{a^2s})$ has mean zero and variance $\frac{1}{a^2}(a^2t \wedge a^2s) = s \wedge t$, same as standard Brownian motion. $\square$

Theorem 14. ***(Time inversion).*** *Let $B = (B_t)_{t \geq 0}$ is a standard Brownian motion. Then*

$$X_t = \begin{cases} 0 & for t = 0 \\ tB_{1/t} & for t > 0 \end{cases} \tag{2.1}$$

is a standard Brownian motion.

Proof. We need to look at the finite dimensional distribution $(X_{t_1}, \ldots, X_{t_n})$ of the process X_t. Since brownian motion is a gaussian process the definition X_t is also gaussian. We see that the mean $\mathbf{E}[X_t] = t\mathbf{E}[B_{1/t}] = 0$ and the covariance for time $s, t > 0$

$$\mathbf{E}[X_t X_s] = ts\mathbf{E}[B_{1/t}B_{1/s}] = ts \times (1/s \wedge 1/t) = s \wedge t \tag{2.2}$$

So X_t and B_t are standard gaussian process having same mean and variance. Hence it $X_t \stackrel{d}{=} B_t$. □

Lemma 15. ***(Time reversal symmetry).*** *The process defined by $X_t = B_{(1-t)} - B_1$ for $0 \leq t \leq 1$ is a standard Brownian motion on $[0, 1]$.*

Proof. Clearly X_t is a brownian motion satisfying $X_0 = 0$ and $X_1 = -B_1$. □

Lemma 16. ***(Time shift symmetry).*** *Given $t_0 \geq 0$. The process defined by $X_t = B_{t+t_0} - B_{t_0}$ is a standard Brownian motion.*

Proof. The parts of jointly gaussian and continuity properties follow from the definition. To check covariance we see for $s < t$,

$$\begin{aligned} \mathbf{E}[X_s X_t] &= \mathbf{E}[B_{t+t_0}B_{s+t_0}] - \mathbf{E}[B_{t_0}B_{t+t_0}] - \mathbf{E}[B_{s+t_0}B_{t_0}] + \mathbf{E}[B_{t_0}B_{s_0}] \\ &= (s + t_0) - t_0 - t_0 + t_0 \\ &= s \end{aligned} \tag{2.3}$$

Hence X_t is a standard brownian motion. □

Lemma 17. ***(Reflection symmetry).*** *The process defined by $X_t = -B_t$ is a standard Brownian motion.*

Proof. The properties of jointly gaussian and independence increment follows from definition as

$$X_t - X_s = -(B_t - B_s) \tag{2.4}$$

The continuity property is also obvious. Hence X_t is a standard Brownain motion. $\square$

Lemma 18. ***(Rotational invariance).*** *Let A be an orthogonal $n \times n$ matrix or $AA^T = I_n$ and let B be a standard Brownian motion then for any n-dimensional distribution $B^{(n)} = B_1, \ldots, B_n$ the transformation $AB^{(n)}$ is also a standard n-dimensional standard Brownian motion.*

Proof. We need to look at the finite n-dimensional distributions of B. Since the coordinates of the Brownian motion $B^{(n)}$ are independent, standard normal the density of Brownian motion

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp -\frac{x_i^2}{2} \tag{2.5}$$

The density of AB by the change of density formula is given by

$$f(A^{-1}B^{(n)})|\det(A^{-1})| \tag{2.6}$$

By orthogonality the determinant is 1. So the density of $B^{(n)}$ is invariant under the transformation. $\square$

2.1.2 Continuity properties of Brownian motion

We begin our discussion for Brownian motion defined on a compact interval, which implies the uniform continuous nature of the sample paths. Now to quantify the uniform continuity we start with the following definition.

Definition 19. (Modulus of continuity).

$$\limsup_{h \downarrow 0} \sup_{0 \le t \le 1-h} \frac{|B_{(t+h)} - B_{(t)}|}{\varphi(h)} \le 1 \tag{2.7}$$

It turns out that there is a nonrandom modulus of continuity for Brownian motion.

Theorem 20. *There exists a constant $C > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $0 \leq t \leq 1 - h$,*

$$|B_{(t+h)} - B_t| \leq C\sqrt{h \log(1/h)} \tag{2.8}$$

Proof. This property is a bit difficult to see from the definition of Brownian motion. So we need to look from the perspective of Levy construction. We briefly mention this again in a slightly different format. We consider a sequence $\{Z_n\}$ of standard normal random variables and define the following process on the dyadic points $\mathcal{D}_n$ on $[0, 1]$.

$$F_0(t) = \begin{cases} Z_1 & t = 1 \\ 0 & t = 0 \\ linear & \text{in between} \end{cases}$$

and in the other dyadic points we define as follows

$$F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t & t \in \mathcal{D}_n / \mathcal{D}_{n-1} \\ 0 & t \in \mathcal{D}_{n-1} \\ linear & \text{in between consequetive points of } \mathcal{D}_n \end{cases}$$

And we can show that this functions define a Brownian motion as follows

$$B_d = \sum_{i=0}^{\infty} F_i(d) \tag{2.9}$$

The proof follows from induction but we just want to look at a particular aspect of the series for our purpose which converges if $c > \sqrt{2\log 2}$.

$$\sum_{n=0}^{\infty} \mathbf{P}(\{\exists d \in \mathcal{D}_n : \text{with } |Z_d| \geq c\sqrt{n}\}) \leq \sum_{n=0}^{\infty} \sum_{d \in \mathcal{D}_n} \mathbf{P}(\{|Z_d| \geq c\sqrt{n}\})$$
$$\leq \sum_{n=0}^{\infty} (2^n + 1) \exp -\frac{c^2 n}{2} \tag{2.10}$$

As the sum is finite Borel cantelli lemma guarantees an N such that for all $n \geq N$ and $d \in \mathcal{D}_n$ we have $|Z_d| < c\sqrt{n}$. This gives the bound on the functions F_n as follows

$$\|F_n\|_\infty < c\sqrt{n} 2^{-n/2} \tag{2.11}$$

Since the functions $F_n(t)$ are piecewise linear it is differentiable almost everywhere except the dyadic points. Hence we can find a bound on the derivatives also.

$$\left\|F_n^{'}\right\|_\infty \leq \frac{2\|F_n\|_\infty}{2^{-n}} \leq 2c\sqrt{n}2^{n/2} \tag{2.12}$$

We have set out to find a bound on $B_{t+h} - B_t$ for the interval $t \in [0, 1]$

$$|B_{t+h} - B_t| \leq \sum_{n=0}^{\infty} |F_n(t+h) - F_n(t)| \leq \sum_{n=0}^{l} \left\|F_n^{'}\right\|_\infty + 2 \sum_{n=l+1}^{\infty} \|F_n\|_\infty \tag{2.13}$$

We can estimate the rightmost expression for $l \geq N$

$$h \sum_{n=0}^{N} \left\|F_n^{'}\right\|_\infty + 2ch \sum_{n=N}^{l} \sqrt{n} 2^{n/2} + 2c \sum_{n=l+1}^{\infty} \sqrt{n} 2^{-n/2} \tag{2.14}$$

We can make the first summand smaller than $\sqrt{h \log(1/h)}$ and choose $l > N$ such that $2^{-l} < h \leq 2^{1-l}$. Using these we can bound all the three summands by a constant multiple of $\sqrt{h \log(1/h)}$. Hence

$$B_{t+h} - B_t \leq C\sqrt{h \log(1/h)} \tag{2.15}$$

$\square$

The lower bound follows from.

Theorem 21. *For every constant $c < \sqrt{2}$, almost surely, for every $\epsilon > 0$ there exist $0 < h < \epsilon$ and $t \in [0, 1-h]$ with*

$$|B_{(t+h)} - B_{(t)}| \geq c\sqrt{h \log(1/h)} \tag{2.16}$$

Proof. We consider the following event for integers $k, n \geq 0$ and $c < \sqrt{2}$

$$E_{k,n} = \{B_{(\frac{k+1}{e^n})} - B_{(\frac{k}{e^n})} > \frac{c\sqrt{n}}{e^{n/2}}\} \tag{2.17}$$

Then using the estimate of standard normal random variable and the invariance of Brownian motion

$$\mathbf{P}(E_{k,n}) = \mathbf{P}(B_{\frac{1}{e^n}} > \frac{c\sqrt{n}}{e^{n/2}}) = \mathbb{P}(B_1 > c\sqrt{n}) \geq \frac{c\sqrt{n}}{c^2 n + 1} \frac{1}{2\pi} \frac{1}{e^{c^2 n/2}} \tag{2.18}$$

Since $c < \sqrt{2}$ we have $e^n \mathbf{P}(\mathbf{E_{k,n}}) \to \infty$ as $n \uparrow \infty$.

$$\mathbf{P}(\bigcap_{k=0}^{\lfloor e^n - 1 \rfloor} E_{k,n}^c) = (1 - \mathbf{P}(E_{k,n})^{e^n} \leq \mathbf{P}(-e^n \mathbf{P}(E_{0,n}) \tag{2.19}$$

Where we use the inequality $1 - x \leq e^{-x}$. we see the right most expression of the above chain of inequalities goes to zero as $n \to \infty$. Hence considering $h = e^{-n}$ we get for any $\epsilon > 0$

$$\mathbf{P}((B_{t+h} - B_t) \leq c\sqrt{h \log(1/h)}) = 0 \tag{2.20}$$

here $h \in (0, \epsilon)$ and $t \in [0, 1-h]$. □

We note the striking result by Paul Levy.

Theorem 22. ***(Levy's modulus of continuity).***

$$\limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \frac{|B_{(t+h)} - B_{(t)}|}{\sqrt{2h \log(1/h)}} = 1 \tag{2.21}$$

almost surely.

Proof. Refer to [5] □

Definition 23. A function $f : [0, \infty) \to \mathbb{R}$ is said to be locally α-Holder continuous at $x \geq 0$, if there exist $\epsilon > 0$ and $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^{\alpha} \tag{2.22}$$

for all $y \geq 0$ with $|y - x| < \epsilon$. Here $\alpha > 0$ is called the Holder exponent and $c > 0$ is called the Holder constant.

As α gets larger the continuity gets stronger. For our function of interest, i.e. Brownian motion the transition from α-Holder continuity to not happens at $\alpha = 1/2$.

Corollary 24. *If $\alpha < 1/2$, then almost surely Brownian motion is everywhere locally α-Holder continuous.*

Proof. We consider a Brownian motion on the intervals $[k, k+1]$ where k is a nonnegative integer and apply the 91 proved above to $\{B_t - B_k | t \in [k, k+1]$. We can find an h for each nonnegative integer k satisfying

$$|B_{t+h} - B_t| \leq C\sqrt{h \log(1/h)} \leq Ch^{\alpha} \tag{2.23}$$

□

We end the subsection by pointing a connection to random walks.

Theorem 25. ***(Law of iterated logarithm for Brownian motion).*** *Suppose $\{B_t : t \geq 0\}$ is a standard Brownian motion. Then, almost surely,*

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t \log\log(t)}} = 1 \qquad \liminf_{t\to\infty} \frac{B(t)}{\sqrt{2t \log\log(t)}} = -1 \tag{2.24}$$

Proof. Refer to [5] □

2.1.3 Zero set of Brownian motion

Brownian motion is a strange curve. It is continuous $a.e[\omega]$.

Definition 26. (Perfect set). A set S is a perfect set if it is closed and each point is a limit point.

Since every point is a limit point of S. S must be infinite and we can construct compact sets around each point. Now using finite intersection property of compact intervals in a metric space we can deduce that the cardinality of S is equal to $\mathbb{R}$. The following proposition is a reminiscent of the familiar **Cantor set**.

Proposition 27. *The zero set of Brownian motion $\mathcal{Z} := \{t \geq 0 : B_t = 0\}$.*

- *It is a perfect set a.e$[\omega]$.*
- *It is unbounded and uncountable.*
- *It has lebesgue measure zero.*

Proof.

- Since Brownian motion is a continuous function almost surely the zero set is closed. To show that zeroes are not isolated we consider the following
$$\tau_q = \inf\{t \geq 0 : B_t = 0\} \tag{2.25}$$
where $q \in [0, \infty)$ is a rational and the stopping time is the first zero with brownian motion starting at the rational time q. The stopping time τ_q is almost surely finite. Using strong markov property for each τ_q we have for each q, almost surely τ_q has no isolated zero from right. As there are countably many rationals almost surely there for all rational q the zero τ_q is not isolated from right. For any $t > 0$ we can chose a sequence of rationals $q_n \uparrow t$ and foreach of these rationals q_n we know that τ_{q_n} is not isolated from right. Hence no $t > 0$ has an isolated zero from left also.
- This follows from properties of perfect set.
- Consider $\mu(\mathcal{Z})$ to be the measure of $\mathcal{Z}$.
$$\mathbf{E}[\mu(\mathcal{Z})] = \mathbf{E}\left(\int_0^\infty \mathbf{1}_{\{B_t=0\}} dt\right) \overset{Fubini}{=} \int_0^\infty \mathbf{P}(B_t = 0) dt = 0 \tag{2.26}$$
This implies $\mu(\mathcal{Z}) = 0$ almost surely.

□

2.2 Markov Property of Brownian motion

The markov property or memoryless property is an important property of Brownian motion and has interesting consequences.

Lemma 28. ***(Weak markov property).*** *For every fixed deterministic time T the process $B_t^T := B_{t+T} - B_t$ is a brownian motion independent of $\mathcal{F}_T$.*

Proof. This follows from the definition since Brownian motion has independence increment property. $\square$

2.2.1 Non differentiability of Brownian motion

Let $(B_t)_{t\geq 0}$ is a Brownian motion. For every $t \geq 0$, we set the filtration

$$\mathcal{F}_t = \sigma(B_s, s \leq t). \tag{2.27}$$

Clearly $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. We also define the σ-algebra providing "infinitesimal glimpse" after time $t = 0$ as

$$\mathcal{F}_{0+} = \bigcap_{s>0} \mathcal{F}_s \tag{2.28}$$

. The first theorem is a $0 - 1$ law.

Theorem 29. ***(Bluementhal's 0-1 Law)*** *The σ-field $\mathcal{F}_{0+}$ is trivial, in the sense that $\mathbf{P}(A) = 0$ or 1 for every $A \in \mathcal{F}_{0+}$.*

Proof. We shall prove that any event $A \in \mathcal{F}_{0+}$ is independent of itself. We consider a sequence of times $0 < t_1 < \ldots < t_n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded continuous function. Then we have by dominated convergence and continuity of brownian paths

$$\mathbf{E}[\mathbf{1}_A f(B_{t_1}, \ldots, B_{t_n})] = \lim_{\epsilon \downarrow 0} \mathbf{E}[\mathbf{1}_A f(B_{t_1} - B_\epsilon, \ldots, B_{t_n} - B_\epsilon)] \tag{2.29}$$

For $t_1 > \epsilon > 0$ the variables $B_{t_1} - B_\epsilon, \ldots, B_{t_n} - B_\epsilon)$ are independent of $\mathcal{F}_\epsilon$ by simple markov property. Hence 1_A can be brought out of the expectation and we have

$$\mathbf{E}[\mathbf{1}_A f(B_{t_1}, \ldots, B_{t_n})] = \mathbf{P}(A)\mathbf{E}[f(B_{t_1}, \ldots, B_{t_n})] \tag{2.30}$$

This implies that any $A \in \mathcal{F}_{0+}$ i independent of $\sigma(B_{t_1}, \ldots, B_{t_n})$ where the times can be arbitrary. Hence $\mathcal{F}_{0+}$ is independent of $\sigma(B_t : t \geq 0)$. Finally $\mathcal{F}_{0+} \subset \sigma(B_t : t \geq 0)$ we have any $A \in \mathcal{F}_{0+}$ is independent of itself. Which implies $\mathbf{P}(A) \in \{0, 1\}$. □

We now state the first proposition showing erratic behaviour of Brownian motion.

Proposition 30. *Almost surely for every $\epsilon > 0$,*

$$\sup_{0 \leq s \leq \epsilon} B_s > 0 \qquad \inf_{0 \leq s \leq \epsilon} B_s < 0 \tag{2.31}$$

For every $a \in \mathbb{R}$, let $T_a = \inf\{t \geq 0 : B_t = a\}$. Then a.s.,$\forall a \in \mathbb{R}, T_a < \infty$. Consequently a.s.

$$\limsup_{t \to \infty} B_t = +\infty \qquad \liminf_{t \to \infty} B_t = -\infty \tag{2.32}$$

Proof. Refer to [3] □

The proposition shows the irregularity as we note.

Corollary 31. *Almost surely, the function $t \mapsto B_t$ is not monotone on any non-trivial interval.*

Proof. We consider a standard brownian motion and suppose there exist an interval $[a, b] \subset [0, \infty)$ where the process is monotone. WLOG we assume it is monotonically increasing, or $B_b \geq B_a$. We can then subdivide the interval in n equal subintervals and since the process is monotonous throughout it will be monotonically increasing in all the sub intervals. Now we know that brownian motion satisfies independence increment property over all the sub intervals. Hence the probability that it is increasing over all the sub intevals is $(\frac{1}{2})^n$. As the number of subintervals $n \to \infty$ we see that the probability goes to zero and almost surely brownian motion is not monotone in any interval. $\square$

Chapter 3

Martingales and Preliminaries of Stochastic calculus

3.1 Introduction

The motivation of probability and in general stochastic processes is to mathematically model random phenomenon and the concept of measurable σ-algebra gives us a framework for that. We begin with the following definition.

Definition 32. (Filtrations). A filtration on $(\Omega, \mathcal{F}, \mathbf{P})$ is a collection of sigma algebras $\{\mathcal{F}_t\}_{0\leq t<\infty}$ of sub-sigma fields of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t < \infty$. We denote $\mathcal{F}_\infty$ by $\mathcal{F}$ and a filtered probability space by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t<\infty}, \mathbf{P})$.

The idea of filtration can be thought as how the σ-algebra evolves as time progresses which is essential in dealing with stochastic processes. In general working with a continuous indexing set requires some care in dealing with measurability issues.

Lemma 33. *The following technical issues are essential in the analysis of stochastic process.*

- ***(Completion).*** *The σ-algebra $\mathcal{F}_0$ contains all $A \subset B$ where $B \in \mathcal{F}_0$ such that $\mathbf{P}(B) = 0$.*

- ***(Right continuous filtration)*** *A filtration $(\mathcal{F}_t)$ is tight continuous if it contains information about the infinitesimal future for every $t \geq 0$.*

$$\mathcal{F}_{t+} := \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t \tag{3.1}$$

Definition 34. (Stochastic process). A stochastic process $\{X_t\}_{t\in\mathcal{I}}$ on $(\Omega, \mathcal{F})$ is a collection of random variables taking values in another measurable space(**state space**) $(S, \mathcal{G})$.

In general the indexing set $\mathcal{I}$ can be discrete(e.g. A finite set or $\mathbb{N}$) or continuous(e.g. $\mathbb{R}, [0,1]$) but we mostly consider the indexing set as the time $t \in [0, \infty)$. In general the indexing representation comes from the structure of the space where the process is taking values. So we want a deeper look at the measurability of the stochastic process.

Definition 35. A stochastic process $X = (X_t)_{t\geq 0}$ with values in $(S, \mathcal{G})$ is said to be measurable if the mapping

$$(\omega, t) \mapsto X_t(\omega) \tag{3.2}$$

defined on $\Omega \times [0, \infty)$ having the product σ-algebra $\mathcal{F} \times \mathcal{B}([0, \infty))$ is measurable.

This is a stronger notion of measurability as we see next.

Definition 36. Let $X = (X_t)_{t\geq 0}$ be a stochastic process with values in $(S, \mathcal{G})$

- **Adapted:** X is adapted if, X_t is $\mathcal{F}_t$ measurable $\forall t \geq 0$.

- **Progressive:** X is progressive if, $\forall t \geq 0$ the mapping

$$(\omega, t) \mapsto X_t(\omega) \tag{3.3}$$

  defined on $\Omega \times [0, t]$ is measurable with respect to $\mathcal{F}_t \times \mathcal{B}([0, t])$.

Having defined a stochastic process one of the thing we are interested is the sample path properties of it.

Definition 37. (Sample path). The sample path of a stochastic process $\{X_t\}$ indexed by $t \in [0, \infty)$ is $t \mapsto X_t(\omega)$ for every $\omega \in \Omega$.

We define three notions of similarity between the paths of a stochastic process.

Definition 38. (Notions of similarity between stochastic processes). Given two stochastic processes X, Y defined on the same probability space $(\Omega, \mathcal{F})$

- **Version:** X and Y are versions of each other if for any integer $n \geq 1$ and $0 \leq t_1 \leq \ldots \leq t_n < \infty$, and for $A \in \mathcal{S}$ we have

$$\mathbf{P}((X_{t_1}, \ldots, X_{t_k}) \in A) = \mathbf{P}((Y_{t_1}, \ldots, Y_{t_k}) \in A) \tag{3.4}$$

- **Modification:** Y is a modification of X if for every $t \geq 0$ we have $\mathbf{P}(X_t = Y_t) = 1$

- **Indistinguishable:** X and Y are indistinguishable if $\{\omega : X_t(\omega) \neq Y_t(\omega)\}$ for some $t \in \mathcal{I}$ is a $\mathbf{P}$-null set.

We consider another concept of right continuous function having left hand limits.

Definition 39. A function f is said to be a right continuous function having left hand limits if for all $t \geq 0$ the following quantities exist.

$$f(t) = \lim_{s \downarrow t} f(s) \qquad f(t) = \lim_{s \uparrow t} f(s) \tag{3.5}$$

3.2 Martingales

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t < \infty}, \mathbf{P})$.

Definition 40. A $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ adapted process $\{X_t\}$ is called

- **(Super Martingale).** $\mathbf{E}[X_t | \mathcal{F}_s] \leq X_s$ for $0 \leq s \leq t < \infty$

- **(Sub Martingale).** $\mathbf{E}[X_t | \mathcal{F}_s] \geq X_s$ for $0 \leq s \leq t < \infty$

- **(Martingale).** $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ for $0 \leq s \leq t < \infty$

Clearly a martingale is both a sub-martingale and super-martingale.

Definition 41. (Continuous Martingales). A process X_t adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a continuous martingale in L^1 if the following holds

- $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ for all $s \leq t$.
- The function $t \mapsto X_t$ is a continuous function $a.e[\omega]$.

Example 42. Brownian motion is a martingale as for $0 \leq s \leq t$

$$\mathbf{E}[B_t|\mathcal{F}_s] = \mathbf{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbf{E}[B_t - B_s] + B_s = B_s \tag{3.6}$$

Definition 43. (Uniformly integrable martingale.). A sequence of random variables $\{X_i\}_{i\in I}$ is uniformly integrable if

$$\lim_{N\to\infty} \sup_{i\in I} \mathbf{E}[|X_i|; |X_i| \geq N] = 0 \tag{3.7}$$

Uniform integrability is the condition that upgrades the convergence in probability to convergence in L^1.

Theorem 44. *The following are equivalent.*

- $X_n \to X$ *in* $L^1(\Omega, \mathcal{F}, \mathbf{P})$
- $X_n \to X$ *in probability and* $(X_n)_{n\geq 1}$ *is uniformly integrable.*

We shall prove a general version of the theorem. To check Uniform integrability criterion we have the two following ways.

Proposition 45. *Let* $\mathcal{X}$ *be a family of random variables bounded in* L^p *for some* $p > 1$. *Then* $\mathcal{X}$ *is uniformly integrable.*

Proof. Since $\mathcal{X}$ is bounded in L^p there exists a constant $M > 0$ satisfying $\mathbf{E}[|X|^p] \leq M$ for all $X \in \mathcal{X}$. Considering a subset $A \in \mathcal{F}$ satisfying $\mathbf{P}(A) \leq \delta$ we have the following by the application of Holder's inequality.

$$\sup_{X\in\mathcal{X}} \mathbf{E}[|X|\mathbf{1}_A] \leq \sup_{X\in\mathcal{X}} \mathbf{E}[|X|^p]^{1/p}\mathbf{E}[\mathbf{1}_A]^{(1-1/p)} \leq M^{1/p}\delta^{1-1/p} \to 0 \tag{3.8}$$

□

Proposition 46. *Let $X \in L^1$. Then the collection of random variables $\mathcal{R} = \{\mathbb{E}(X|\mathcal{G}) : \mathcal{G} \subseteq \mathcal{F}\}$ where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$ is a collection of uniformly integrable random variables.*

Proof. Given $X \in L^1$ implies $\mathbf{E}[|X|] < \infty$. Considering a random variable $Y \in \mathcal{R}$ the application of Markov inequality in first inequality and conditional jensen inequality in second inequality yields.

$$\lim_{t\to\infty} \mathbf{P}(|Y| > t) \leq \lim_{t\to\infty} \frac{\mathbf{E}[|Y|]}{t} \leq \lim_{t\to\infty} \frac{\mathbf{E}[|X|]}{t} = 0 \tag{3.9}$$

Let $Y = \mathbf{E}[X|\mathcal{G}]$ and the event $\{|Y| > M\} \in \mathcal{G}$. and denote

$$\delta(M) = \sup_{Y\in\mathcal{R}} \mathbf{P}(|Y| > M) \tag{3.10}$$

Again using conditional jensen inequality we get

$$\mathbf{E}[|Y|\mathbf{1}_{\{|Y|>M\}}] \leq \mathbf{E}[|X|\mathbf{1}_{\{|Y|>M\}}] \leq \sup_{A:\mathbf{P}(A)\leq\delta(M)} \mathbf{E}[|X|\mathbf{1}_A] \tag{3.11}$$

The bound does not depend on the $Y \in \mathcal{R}$ chosen and it vanishes as M increases to ∞. □

Lemma 47. ***(Martingales under convex function).*** *Suppose $\{X_t\}$ is a $(\mathcal{F}_t)$ martingale then for any convex function φ the transform $\{\varphi(X_t)\}_{t\geq 0}$ is a sub-martingale. If $\{X_t\}$ is a sub-martingale and φ is an increasing and convex function then $\{\varphi(X_t)\}_{t\geq 0}$ is also a sub-martingale.*

We now present some useful inequalities and convergence theorem for dealing with martingales. We shall not present the proofs but instead refer to [3] for details.

Theorem 48. ***(Doob's Inequalities).*** *$(X_t)_{t\geq 0}$ is a continuous martingale*

- *(**L^p inequality).** X_t bounded in L^p for some $t \geq 0$ and $p > 1$, then*

$$\mathbf{E}[(\sup_{s\leq t} X_s)^p] \leq \frac{p}{p-1}\mathbf{E}[|X_t|^p] \tag{3.12}$$

- ***(Maximal Inequality).*** *Given a* $\lambda > 0$

$$\mathbf{P}(\sup_{[0,t]} |X_s| > \lambda) \leq \frac{1}{\lambda}\mathbf{E}[|X_t|\mathbf{1}(\sup_{s\in[0,t]} |X_s| > \lambda)] \leq \frac{1}{\lambda}\mathbf{E}[|X_t|] \tag{3.13}$$

Theorem 49. *Let* $(X_t)_{t\geq 0}$ *be a martingale adapted to* $(\mathcal{F}_t)$*. Assume* $(X_t) \in L^1$*.*

- $X_t \xrightarrow{a.s.} X_\infty$ *where* $X_\infty \in L^1(\mathcal{F}_\infty)$*.*
- $X_t \xrightarrow{L^1} X_\infty$ *if and only if* X *is uniformly integrable. And we have*

$$\mathbf{E}[X_\infty|\mathcal{F}_s] = X_s \qquad \forall s \in [0,\infty) \tag{3.14}$$

- $X \in L^p$ *for some* $p > 1$ *then* $X_t \xrightarrow{a.s} X_\infty$ *where* $X_\infty \in L^p(\mathcal{F}_\infty)$*.*

3.2.1 Optional Stopping theorems

The concept of stopping time is crucial in analysis of stochastic processes. It gives us a way to say about the properties of the process with the information upto that time.

Definition 50. (Stopping time). A random variable $T : \Omega \to [0,\infty)$ is called a stopping time for the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for any $t \geq 0$, the event $\{T \leq t\} \in \mathcal{F}_t$.

Theorem 51. *Suppose* (X_t) *is a progressive process and let* T *be a finite stopping time. Then the function* $\omega \mapsto X_t^T$ *is* $\mathcal{F}_T$ *measurable.*

Theorem 52. ***(Optional stopping theorem).*** *Given a continuous martingale* $(X_t)_{t\geq 0}$ *and a finite stopping time* T*. Then the stopped process*

$$X_t^T = \begin{cases} X_t & , t \leq T \\ 0 & , t > T \end{cases}$$

is a uniformly integrable martingale and $\mathbf{E}[X_T] = \mathbf{E}[X_0]$*.*

3.3 Local martingales

The idea of local martingale is to localize or stop the process before it becomes too large. This allows us to use the concepts of stopping times and martingale convergence theorems.

Definition 53. (**Local Martingale**). A local martingale is a continuous adapted process M such that there exists a non-decreasing sequence of stopping times $T_n \uparrow \infty$ such that the stopped process

$$M^{T_n} - M_0 = (M_{t\wedge T_n} - M_0)_{t\geq 0} \tag{3.15}$$

is a martingale for each n. The localizing sequence $(T_n)_{n\geq 1}$ is said to reduce M to a martingale. In cases where $M_0 \in L^1$ we can simplify the definition to

$$M^{T_n} = (M_{t\wedge T_n})_{t\geq 0} \tag{3.16}$$

Proposition 54. *The space of continuous local martingales form a vector space.*

Proof. We just show the additive property : Consider two local martingales M_1, M_2 starting at 0 and the localizing sequences $(T^1), (T^2)$ that reduces the local martingales. Then by property of stopping times $T^1 \wedge T^2$ reduces $M_1 + M_2$. The stability under scalar multiplication is obvious. □

Proposition 55. ***(Properties of Continuous local martingales).*** *Let $(M_t)_{t\geq 0}$ is a sequence of continuous local martingales.*

- *A non negative continuous local martingale M bounded in L^1 is a supermartingale.*
- *A bounded continuous local martingale bounded in L^1 by a random variable Z i.e, $|M_t| \leq Z, \forall t \geq 0$ is a uniformly integrable martingale.*

Proof.
- Since $M_0 \in L^1$ we write $M_t = M_0 + N_t$ where N_t is the local martingale satisfying $N_0 = 0$. By definition there exist a sequence of

stopping time (T_n) that reduces N. Then for $s \leq t$ we have

$$N_{s \wedge T_n} = \mathbf{E}[N_{t \wedge T_n} | \mathcal{F}_s] \tag{3.17}$$

Adding M_0 both sides we get

$$M_{s \wedge T_n} = \mathbf{E}[M_{t \wedge T_n} | \mathcal{F}_s] \tag{3.18}$$

Given M is non negative so we can use conditional version of **Fatou's lemma.**

$$\begin{aligned} \liminf_{n \to \infty} \mathbf{E}[M_{t \wedge T_n} | \mathcal{F}_s] &\geq \lim_{n \to \infty} \mathbf{E}[\liminf M_{t \wedge T_n} | \mathcal{F}_s] \\ \mathbf{E}[M_t | \mathcal{F}_s] = M_s &\geq \mathbf{E}[M_t | \mathcal{F}_s] \end{aligned} \tag{3.19}$$

putting expectation both side we get $\mathbf{E}[M_s] \geq \mathbf{E}[\mathbf{E}[M_t | \mathcal{F}_s]] = \mathbf{E}[M_t]$. Putting $s = 0$ we get $\mathbf{E}[M_t] \leq \mathbf{E}[M_0] < \infty$. hence $M_t \in L^1$ and it is a super martingale.

- We already have

$$M_{s \wedge T_n} = \mathbf{E}[M_{t \wedge T_n} | \mathcal{F}_s] \tag{3.20}$$

And given $|M_{t \wedge T_n}| \leq Z$ we can use dominated convergence theorem to get that $M_{t \wedge T_n}$ converges to M_t in L^1. Passing limit in the above equation we get.

$$\begin{aligned} \lim_{n \to \infty} M_{s \wedge T_n} &= \lim_{n \to \infty} \mathbf{E}[M_{t \wedge T_n} | \mathcal{F}_s] \\ M_s &= \mathbf{E}[\lim_{n \to \infty} M_{t \wedge T_n} | \mathcal{F}_s] = \mathbf{E}[M_t | \mathcal{F}_s] \end{aligned} \tag{3.21}$$

Hence bounded continuous local martingale is a martingale.

□

This following example of stopping time for continuous local martingales is useful. The idea is to stop the process before it becomes too large.

Proposition 56. $(M_t)_{t \geq 0}$ *is a continuous local martingale with* $M_0 = 0$ *or*

$M_0 \in L^1$ then

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\} \tag{3.22}$$

reduces M.

Proof. These sequence of times are stopping times. The stopped process M^{T_n} is a continuous local martingale and it is bounded as $M^{T_n} \leq n$. So we have the following bound

$$M^{T_n} \leq n + |M_0| \tag{3.23}$$

Since $M_0 \in L^1$ we use the previous proposition to conclude our result. □

3.3.1 Finite variation processes

We consider a filtered probability space$(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbf{P})$.

Definition 57. (Finite variation process). An adapted process $A = (A_t)_{t\geq 0}$ is called a finite variation process if $a.e[\omega]$ its sample paths

$$t \mapsto A_t(\omega) \tag{3.24}$$

are finite variation functions on $\mathbb{R}_+$.

We can define the stochastic integral with respect to a finite variation process A defined as

$$\int_0^t X_s(\omega) dA_s(\omega) \tag{3.25}$$

We show that the definition indeed makes sense.

Theorem 58. *Let X be a progressively measurable process bounded on $[0, t]$ for any $t \geq 0$ and let A be a finite variation process. Then*

$$\int_0^t X_s(\omega) dA_s(\omega) \tag{3.26}$$

is an adapted finite variation process. On the measure zero set where A is not finite variation we set the integral to be zero.

Proof. Refer to [3] □

For a general account of finite variation processes refer to [2]

3.3.2 Quadratic variation of a Martingale

Suppose we are given with a continuous local martingale $(M_t)_{t\geq 0}$ and we consider an increasing sequence of subdivisions of $[0,t]$ for $t > 0$ given by $0 = t_0^n < t_1^n < \ldots < t_{p_n}^n = t$. The first theorem guarantees the existence of a unique adapted continuous non-decreasing process.

Theorem 59. ***(Existence of quadratic variation).*** *The quadratic variation of $(M_t)_{t\geq 0}$ exists and unique upto indistinguishability which is given by*

$$\langle M, M\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 \tag{3.27}$$

as $p_n \to \infty$ such that $M_t^2 - \langle M, M\rangle_t$ is a continuous local martingale. The convergence happens in probability.

we shall not prove the existence of Quadratic variation instead we refer the reader to [3]. We are more interested in the following example and properties of Quadratic variation.

Example 60. (Quadratic variation of Brownian motion). Brownian motion satisfies the following Quadratic variation

$$\langle B, B\rangle_t = t \tag{3.28}$$

The properties of Continuous local martingales are closely related to its Quadratic variation.

Theorem 61. *Let M be a continuous local martingale such that $M_0 \in L^2$. Then the following are equivalent*

- *M is a true martingale bounded in L^2.*
- $\mathbf{E}[\langle M, M\rangle_\infty] < \infty$.

If these properties hold then $M_t^2 - \langle M, M\rangle_t$ is also a true uniformly integrable martingale.

Proof. • Let $M_0 = 0$. Since $M \in L^2$ we can use Doob's L^2 inequality such that for every $t > 0$,

$$\mathbf{E}\left[\sup_{0\le s\le t} M_s^2\right] \le 4\mathbf{E}[M_t^2] \tag{3.29}$$

since $M_t \in L^2$ we can take $t \to \infty$ to get

$$\mathbf{E}\left[\sup_{0\le s\le t} M_s^2\right] \le 4\sup_{t\ge 0}\mathbf{E}[M_t^2] := K < \infty \tag{3.30}$$

We now consider the sequence of times $T_n = \inf\{t \ge 0\ \langle M, M\rangle_t \ge n\}$. This implies the continuous local martingale $M^2_{t\wedge T_n} - \langle M, M\rangle_{t\wedge T_n}$ is dominated by the following

$$M^2_{t\wedge T_n} - \langle M, M\rangle_{t\wedge T_n} \le \sup_{t\ge 0} M_t^2 + n \tag{3.31}$$

Since a bounded continuous local martingale is an uniformly integrable martingale

$$\mathbf{E}[\langle M, M\rangle_{t\wedge T_n}] = \mathbf{E}[M^2_{t\wedge T_n}] \le \mathbf{E}[\sup_{t\ge 0} M_t^2] \le K \tag{3.32}$$

Using Monotone convergence and letting n and t consequetively to ∞ we get

$$\mathbf{E}[\langle M, M\rangle_\infty] \le K < \infty \tag{3.33}$$

• Now assuming that $\mathbf{E}[\langle M, M\rangle_\infty < \infty$. And consider the sequence of time $S_n = \inf\{t \ge 0 : |M_t| \ge n\}$. This implies the continuous local martingale $M^2_{t\wedge S_n} - \langle M, M\rangle_{t\wedge S_n}$ is dominated as follows

$$M^2_{t\wedge S_n} - \langle M, M\rangle_{t\wedge S_n} \le n^2 + \langle M, M\rangle_\infty \tag{3.34}$$

Again this implies that

$$\mathbf{E}[M^2_{t\wedge S_n}] = \mathbf{E}[\langle M, M\rangle_{t\wedge S_n}] \le \mathbf{E}[\langle M, M\rangle_\infty < \infty \tag{3.35}$$

Letting $n \to \infty$ we get $\mathbf{E}[M_t^2] < \infty$. This implies that $M_t \in L^2$ for $t \geq 0$. We need to show that M_t is a martingale. For $0 \leq t \leq s$

$$\mathbf{E}[M_{s\wedge S_n}|\mathcal{F}_t] = M_{t\wedge S_n} \tag{3.36}$$

Since $M_{s\wedge S_n}$ is an uniformly integrable and the stopped process M^{S_n} is a martingale. Taking $n \to \infty$ gives the desired result.

□

We end this section on quadratic variation by stating a **Cauchy schwarz** analogue of quadratic variation.

Theorem 62. ***(Kunita- Watanabe inequality).*** *Let M, N be continuous local martingales and let H, K be two measurable process.*

$$(t, \omega) \mapsto H_t(\omega) \ \ \textit{is} \ \ \mathcal{F} \times \mathcal{B}_{[0,\infty)} \ \ \textit{measurable.} \tag{3.37}$$

then the following inequality holds.

$$\int_0^\infty |H_s K_s||d\langle M, N\rangle_s| \leq \left(\int_0^\infty H_s^2 d\langle M, M\rangle_s\right)^{1/2} \left(\int_0^\infty K_s^2 d\langle N, N\rangle_s\right)^{1/2} \tag{3.38}$$

Proof. We refer to [3] □

We are also interested in asking what is cross variation between two processes.

Definition 63. (Crossvariation). The crossvariation between two continuous local martingales $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ is defined as: for every $t \geq 0$.

$$\langle M, N\rangle_t := \frac{1}{2}(\langle M + N, M + N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t) \tag{3.39}$$

Example 64. If $B, B^{'}$ are two independent $(\mathcal{F}_t)$ adapted brownian motion then for every $t \geq 0$

$$\langle B, B^{'}\rangle_t = 0 \tag{3.40}$$

This gives us a notion similar to orthogonality between processes as we define.

Definition 65. (Orthogonality). Two continuous local martingales M, N are said to be orthogonal if $\langle M, N\rangle_t = 0$ and this is true if and only if MN is a continuous local martingale.

3.3.3 Continuous Semimartingales

Definition 66. A process $X = (X_t)_{t\geq 0}$ is a continuous semimartingale if it can be decomposed uniquely upto indistinguishability in terms of

$$X_t = M_t + A_t \tag{3.41}$$

where M is a continuous local martingale and A is a finite variation process.

Proposition 67. *Given an increasing sequence of subdivision* $0 = t_0^n < \ldots < t_{p_n}^n = t$ *of* $[0, t]$ *for every* $t \geq 0$ *whose mesh tends to* 0*. Then*

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n}) = \langle X, Y\rangle_t \tag{3.42}$$

in probability.

Proof. Refer to [3]. □

Chapter 4

Stochastic calculus

4.1 Introduction

The goal of this chapter is to develop a theory of integration with respect to a semimartingale. We have already seen that a semimartingale X_t can be decomposed uniquely upto indistinguishability as

$$X_t = A_t + M_t \tag{4.1}$$

where A_t is a finite variation process and M_t is a local martingale. We consider a class of processes H_t and try to define

$$(H \cdot X)_t = (H \cdot A)_t + (H \cdot M)_t = \int_0^t H_s dA_s + \int_0^t H_s dM_s \tag{4.2}$$

The first part of the integral is with respect to a finite variation process which correspond to a Lebesgue integral. We shall be brief in our presentation. For a comprehensive coverage we refer the reader to [3] and [4].

4.2 Space of L^2 bounded martingales

Consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. We denote

$$\mathbb{H}^2 = \{\text{Space of all continuous martingales bounded in } L^2 \text{ satisfying } M_0 = 0\} \tag{4.3}$$

We have already seen that if $M \in \mathbb{H}^2$ then $\mathbf{E}[\langle M, M\rangle_\infty] < \infty$. Therefore using Kunita-Watanabe inequality on $M, N \in \mathbb{H}^2$ we get $\mathbf{E}[\langle M, N\rangle_\infty] < \infty$. So we define a scalar product on $\mathbb{H}^2$ as follows.

$$\langle M, N\rangle_{\mathbb{H}^2} = \mathbf{E}[\langle M, N\rangle_\infty] \tag{4.4}$$

The norm associated to this scalar product is

$$\|M\|_{\mathbb{H}^2} = \mathbf{E}[\langle M, N\rangle_\infty]^{1/2} \tag{4.5}$$

Proposition 68. *The space $\mathbb{H}^2$ is a Hilbert space with the associated norm.*

Proof.
- **Completeness**

  Let $(M^n)_n$ be a cauchy sequence in $\mathbb{H}^2$ for this norm. Then we have the following

  $$\lim_{m,n\to\infty} \mathbf{E}[(M^n_\infty - M^m_\infty)^2] = \lim_{m,n\to\infty} \mathbf{E}[\langle M^n - M^m, M^n - M^m\rangle_\infty] = 0 \tag{4.6}$$

  Applying **Doob's L^2 inequality** we get

  $$\lim_{m,n\to\infty} \mathbf{E}[\sup_{t\geq 0}(M^n_t - M^m_t)^2] = 0 \tag{4.7}$$

  We acn extract a sequence (n_k) of integers satisfying

  $$\mathbf{E}\left[\sum_{k=1}^{\infty} \sup_t |M^{n_k}_t - M^{n_k+1}_t|\right] \leq \sum_{k=1}^{\infty} \mathbf{E}\left[\sup_t |M^{n_k}_t - M^{n_k+1}_t|^2\right]^{1/2} < \infty \tag{4.8}$$

This implies almost surely,

$$\sum_{k=1}^{\infty} \sup_t |M_t^{n_k} - M_t^{n_k+1}| < \infty \tag{4.9}$$

Hence the sequence $(M_t^{n_k})$ converges to a limit M uniformly. Since the variables $(M_t^n) \in L^2$ and it admits a subsequence $(M_t^{n_k}$ converging to $M \in L^2$. We conclude $M \in \mathbb{H}^2$.

□

We recall the definition of progressive process.

Definition 69. (Progressive process). A process H_t is called progressive if for all $t \geq 0$ the map

$$(\omega, t) \mapsto H_t(\omega) \tag{4.10}$$

is measurable with respect to $\mathcal{F}_t \times \mathcal{B}([0,t])$.

We now define the following

Definition 70. (Space of Progressive processes). Given an martingale $M \in \mathbb{H}^2$ we define $L^2(M)$ to be space of all progressive processes H satisfying

$$\mathbf{E}\left[\int_0^\infty H_t^2 d\langle M, M\rangle_t\right] < \infty \tag{4.11}$$

Since the quadratic variation $\langle M, M\rangle_t$ has finite variation the integral defined above is a Lebesgue integral. Hence the space $L^2(M)$ is eqivalent to a standard L^2 space defined as

$$L^2(M) = L^2(\Omega \times [0,\infty), \mathcal{P}, \mu) \tag{4.12}$$

Where $\mathcal{P}$ is the σ-algebra generated by sets A such that

$$H_t(\omega) = \mathbf{1}\{\omega, t \in A)\} \tag{4.13}$$

is a progressive processes. And μ is the the measure defined as

$$\mu(A) = \mathbf{E}\left[\int_0^\infty \mathbf{1}\{(\omega, t \in A)\}\right] < \infty \tag{4.14}$$

4.2.1 Stochastic integral as an Isometry

We define an inner product on the space $L^2(M)$ as follows.

$$(H, K)_{L^2(M)} = \mathbf{E}\left[\int_0^\infty H_t K_t d\langle M, M\rangle_t\right] \tag{4.15}$$

The idea is that we define for each $M \in \mathbb{H}^2$, a stochastic integral with respect to M as a L^2 isometry defined below.

$$\mathcal{T}^M : L^2(M) \to \mathbb{H}^2 \tag{4.16}$$

such that $H \mapsto H \cdot M = \int_0^t H_s dM_s$

Definition 71. (Elementary processes). An elementary process is a progressive process of the form

$$H_s(\omega) = \sum_{i=0}^{p-1} H_i(\omega)\mathbf{1}_{(t_i, t_{i+1}]}(s) \tag{4.17}$$

where $0 = t_0 < t_1 < \ldots < t_p$ and for every $i \in \{0, 1, \ldots, p-1\}$ we have H_i as a bounded $\mathcal{F}_{t_i}$ measurable random variable.

It is easy to see that the elementary processes belongs to $L^2(M)$

$$\begin{aligned}
\|H\|^2_{L^2(M)} &= \mathbf{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right] \\
&= \sum_{i=0}^{p-1} \mathbf{E}\left[\int_0^\infty H_i^2 \mathbf{1}_{(t_i, t_{i+1}]}(s) d\langle M, M\rangle_s\right] \\
&\leq C\mathbf{E}\int_0^\infty d\langle M, M\rangle_s = C\mathbf{E}\langle M, M\rangle_\infty < \infty
\end{aligned} \tag{4.18}$$

In fact the set of $\mathcal{E}$ progressive processes forms a dense subset of $L^2(M)$ for every $M \in \mathbb{H}^2$. We now construct the integral $\int_0^t H_s dM_s$ for $M \in \mathbb{H}^2$ and $H \in L^2(M)$ as follows.

Theorem 72. *Let $M \in \mathbb{H}^2$ and $H \in \mathcal{E}$ defined*

$$H_s(\omega) = \sum_{i=0}^{p-1} H_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(s) \tag{4.19}$$

Then

$$(H \cdot M)_t = \sum_{i=0}^{p-1} H_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t}) \tag{4.20}$$

The map $H \mapsto (h \cdot M)$ extends to an isometry from $L^2(M)$ into $\mathbb{H}^2$. Moreover, $H \cdot M$ is the unique martingale in $\mathbb{H}^2$ that satisfies the following property

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle \quad \forall N \in \mathbb{H}^2 \tag{4.21}$$

We shall not proceed further with the construction and properties of stochastic integral rather we refer to [3] and [4] for the extensive coverage.

4.2.2 Ito lemma

Ito's lemma is a stochastic calculus version of the fundamental theorem of calculus. It gives us an explicit formulae for canonical decomposition of semimartingales. We begin our discussion with the one dimensional version.

Theorem 73. ***(One dimensional Ito fomula).*** *Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale taking values almost surely in an open set $D \subset \mathbb{R}, \forall t \geq 0$. And given $F : D \to \mathbb{R}$ is a $\mathcal{C}^2$ function. Then the process $(F(X_t))_{t\geq 0}$ is also a continuous semimartingale, and almost surely for each $t \geq 0$*

$$F(X_t) = F(X_0) + \int_0^t F^{'}(X_s)dX_s + \frac{1}{2}\int_0^t F^{''}(X_s)d\langle X, X\rangle_s \tag{4.22}$$

Since every semimartingale $X = M + V$ has an **unique decomposition**(upto indistinguishability) into a **continuous local martingale** (M)

and a **finite variation process** (V). The integral $\int_0^t F^{'}(X_s)dX_s$ has the decomposition.

$$\int_0^t F^{'}(X_s)dX_s = \int_0^t F^{'}(X_s)dM_s + \int_0^t F^{'}(X_s)dV_s \tag{4.23}$$

Having seen this decomposition we are at a position to realize that the new process $F(X_t)$ has two parts.

1. **Local martingale:** $F(X_0) + \int_0^t F^{'}(X_s)dM_s$
2. **Finite variation:** $\int_0^t F^{'}(X_s)dV_s + \frac{1}{2}\int_0^t F^{''}(X_s)d\langle X, X\rangle_s$

This unique decomposition of the processes is an important corollary of Ito formula.

Example 74. Let $X_t = B_t$ which is a continuous martingale. Then using $\langle B, B\rangle_s = s$

$$F(B_t) = F(B_0) + \int_0^t F^{'}(B_s)dB_s + \frac{1}{2}\int_0^t F^{''}(B_s)ds \tag{4.24}$$

Proof. We consider a compact interval $[0, t]$ and an increasing sequence of subdivision given by $0 = t_0^n < \ldots < t_{p_n}^n = t$ whose mesh tends to zero as $p_n \to \infty$. Then writing $F(X_t)$ in a telescopic sum as folllows

$$F(X_t) = F(X_0) + \sum_{i=0}^{p_n - 1} F(X_{t_{i+1}^n}) - F(X_{t_i^n}) \tag{4.25}$$

Since $F \in \mathcal{C}^2$ we can write the second order Taylor theorem with Langrange form of the remainder of $F(X_{t_{i+1}^n}) - F(X_{t_i^n})$.

$$F(X_{t_{i+1}^n}) - F(X_{t_i^n}) = F^{'}(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) + \frac{1}{2}F^{''}(h)(X_{t_{i+1}^n} - X_{t_i^n})^2 \tag{4.26}$$

where $h \in [X_{t_i^n}, X_{t_{i+1}^n}]$. Now summing over all the partitions we get the following convergence **in probability.**

$$\lim_{n\to\infty}\sum_{i=0}^{p_n-1} F^{'}(X_{t_i^n})(X_{t_{i+1}^n}-X_{t_i^n}) = \int_0^t F^{'}(X_s)dX_s \tag{4.27}$$

For the expression $\sum_{i=0}^{p_n-1} F^{''}(h)(X_{t_{i+1}^n}-X_{t_i^n})^2$ we show that the following convergence happens in probability.

$$\left|\sum_{i=0}^{p_n-1} F^{''}(h)(X_{t_{i+1}^n}-X_{t_i^n})^2 - \sum_{i=0}^{p_n-1} F^{''}(X_{t_i^n})(X_{t_{i+1}^n}-X_{t_i^n})^2\right| \xrightarrow{n\to\infty} 0 \tag{4.28}$$

This indeed is the case since

$$\sup_{0\leq i\leq p_n-1} |F^{''}(h)-F^{''}(X_{t_i^n})| \leq \sup_{0\leq i\leq p_n-1}\left(\sup_{x\in[\min(X_{t_i^n},X_{t_{i+1}^n}),\max(X_{t_i^n},X_{t_{i+1}^n})]} |F^{''}(x)-F^{''}(X_{t_i^n})|\right) \tag{4.29}$$

The mesh tends to zero as $n\to\infty$ and uniform continuity over compact interval implies the expression on the right of the above equation goes to zero. Only thing needs to be shown is the fact that

$$\lim_{n\to\infty}\sum_{i=0}^{p_n-1} F^{''}(X_{t_i^n})(X_{t_{i+1}^n}-X_{t_i^n})^2 \xrightarrow{P} \int_0^t F^{''}(X_s)d\langle X,X\rangle_s \tag{4.30}$$

The left hand side can be written as follows

$$\int_{[0,t]} F^{''}(X_s)\mu_n(ds) \tag{4.31}$$

such that

$$\mu_n(dr) = \sum_{i=0}^{p_n-1}(X_{t_{i+1}^n}-X_{t_i^n})^2\delta_{t_i^n}(dr) \tag{4.32}$$

Now consider a dense subset D of $[0,t]$ consisting of all t_i^n and $0\leq i\leq p_n$. By the limit of quadratic varition we get

$$\mu_n([0,r]) = \sum_{i:t_i^n\leq r}(X_{t_{i+1}^n}-X_{t_i^n})^2 \xrightarrow{P} \langle X,X\rangle_r \tag{4.33}$$

By using diagonal argument we can find a subsequence along the subsequence defined above to that for all $r \in D$

$$\mu_n([0,r]) \xrightarrow{a.s} \langle X, X \rangle_r \tag{4.34}$$

By monotonicity this holds for all $r \in [0,t]$. Since the measure on $\langle X, X \rangle$ is continuous it is enough to show that $\mu_n \to d\langle X, X \rangle$ in distribution. Since F'' is bounded on range of X on $[0,t]$.

$$\int_{[0,t]} F''(X_s)\mu_n(ds) \xrightarrow{a.s.} \int_{[0,t]} F''(X_s)\langle X, X \rangle_s \tag{4.35}$$

over the subsequence. $\square$

The general version of Ito formula has a proof similar to the one dimensional version where we need to use the general version of Taylor formula with Langrange form of remainder.

Theorem 75. ***Multi dimensional Ito*** *Let $X^1, \ldots, X^p$ be p continuous semimartingales, and let F be twice differentiable function on $\mathbb{R}^p$. Then for every $t \geq 0$,*

$$\begin{aligned} F(X^1, \ldots, X^p) = F(X_0^1, \ldots, X_0^p) + \sum_{i=1}^{p} \int_0^t \frac{\partial F}{\partial x^i}(X_s^1, \ldots, X_s^p) dX_s^i \\ + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s^1, \ldots, X_s^p) d\langle X^i, X^j \rangle_s \end{aligned} \tag{4.36}$$

Proof. Refer to [3] $\square$

The following example is useful application of Ito's formula used in proving the Levy characterization of Brownian motion.

Example 76. If M is a local martingale, then

$$\mathcal{E} = \exp\left(\lambda M_t - \frac{\lambda^2}{2}\langle M, M \rangle_t\right) \tag{4.37}$$

is also a local martingale for all $\lambda \in \mathbb{C}$.

Proof. Consider the the following continuous semimartingale taking values in $\mathbb{R}^2$.

$$X_t = (M_t, \langle M, M\rangle_t) \tag{4.38}$$

And the following $\mathcal{C}^2$ function $F : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$F(a, b) = \exp(\lambda a - \frac{\lambda^2}{2} b) \tag{4.39}$$

We note that the quadratic variation terms involving the second variable vanish since $\langle M, M\rangle_t$ is a continuous bounded process having no quadratic variation. Applying Ito's formula we get

$$\begin{aligned} F(X_t) = F(X_0) + \int_0^t \lambda f(X_s) dM_s + \int_0^t -\frac{\lambda^2}{2} f(X_s) d\langle M, M\rangle_s \\ + \frac{1}{2}\int_0^t \lambda^2 f(Z_s) d\langle M, M\rangle_s \\ = F(X_0) + \int_0^t \lambda f(X_s) dM_s \end{aligned} \tag{4.40}$$

Which is a local martingale. □

This proof shows that if F is any $\mathcal{C}^2$ function on $\mathbb{R}^2$ satisfying

$$\partial_a F(a, b) + \frac{1}{2}\partial_a^2 F(a, b) = 0 \tag{4.41}$$

Then $(F(M_t, \langle M, M\rangle_t))_{t\geq 0}$ is a local martingale if M is a local martingale.

Theorem 77. ***(Levy's characterization of Brownian motion in $\mathbb{R}^d$)*** *Let $X = (X^1, \ldots, X^d)$ be a d-dimensional continuous local martingale started from the origin such that:*

$$\langle X^i, X^j\rangle = \delta_{ij} t \qquad 1 \leq i, j \leq d \tag{4.42}$$

Then X is a Brownian motion in $\mathbb{R}^d$.

Proof. Consider a vector $(u_1, \ldots, u_d) \in \mathbb{R}^d$. Then $u \cdot X = \sum_{i=1}^d u_i X^i$ is a continuous local martingale with quadratic variation

$$\sum_{j=1}^{d}\sum_{k=1}^{d} u_j u_k \langle X^j, X^k\rangle_t = \|u\|^2 t \tag{4.43}$$

Using the previous example we see that $\exp(iu \cdot X_t + \frac{1}{2}\|u\|^2 t)$ is a complex continuous local martingale bounded on every interval $[0, b]$. So it is a true martingale. Hence for any $0 \le s \le t$,

$$\mathbf{E}[\exp(iu \cdot X_t + \frac{1}{2}\|u\|^2 t)|\mathcal{F}_s] = \exp(iu \cdot X_s + \frac{1}{2}\|u\|^2 s) \tag{4.44}$$

This implies

$$\mathbf{E}[\exp(iu \cdot (X_t - X_s)|\mathcal{F}_s] = \exp(-\frac{1}{2}\|u\|^2 (t-s)) \tag{4.45}$$

using the idea of characteristic functions of random variables we see that the left hand side is the characteristic function of $X_t - X_s$ and the right hand side is the characteristic function of $\mathcal{N}(0, (t-s)I_d)$. This means X has the same finite dimensional distribution as Brownian motion on $\mathbb{R}^d$. □

4.3 Some preliminaries on PDE

We are taking a brief digression to a specific partial differential equation called **Laplace equation**. This has an interesting connection to Brownian motion. This presentation is inspired by [1]

Definition 78. (Laplace equation).

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{4.46}$$

Now it is an easy exercise to see that laplace equation is invariant under rotations. i.e, $v(x) = u(\mathcal{O}x)$ also satisfies laplace equation, where $\mathcal{O}$ is an orthogonal matrix. We remember the fact that Brownian motion also satisfies the rotational invariance property as in lemma 18.

We now provide the way to solve the laplace equation in $U = \mathbb{R}^n$.. Harnessing the radial symmetry of laplace equation we search for the radial slutions in $\mathbb{R}^n$.

$$u(x) = v(r) \tag{4.47}$$

where $r = |x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$. Calculating the following derivatives for $x \neq 0$ and for $i \in \{1, \ldots, n\}$

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} 2x_i = \frac{x_i}{r} \tag{4.48}$$

Using the above we get

$$u_{x_i x_i} = v^{''}(r)\frac{x_i^2}{r^2} + v^{'}(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \tag{4.49}$$

Summing over all i we get

$$\Delta u = v^{''}(r) + \frac{n-1}{r}v^{'}(r) \tag{4.50}$$

This is equivalent to solving the following differential equation.

$$v^{''}(r) + \frac{n-1}{r}v^{'}(r) = 0 \tag{4.51}$$

which is

$$v(r) = \begin{cases} b\log(r) + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3) \end{cases} \tag{4.52}$$

Definition 79. (Harmonic function). A function $u \in \mathcal{C}^2(\Omega)$ is said to be harmonic if it satisfies

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{4.53}$$

where $\Omega \subset \mathbb{R}^n$

This definitions give rise to sub/super harmonic functions which satisfies the $\Delta u \geq 0/\Delta u \leq 0$. Harmonicity gives us an way to calculate the value of u in the following way in any domain $\Omega \subset \mathbb{R}^n$.

Theorem 80. ***(Mean value property).*** *Given $\Omega \subset \mathbb{R}^n$ be a domain, i.e. a connected open set and $\partial\Omega$ be its boundary. Let $u : \Omega \to \mathbb{R}$ be measurable and locally bounded function. Then the following conditions are equivalent.*

1. *u is harmonic.*

2. *Given any ball $\mathcal{B}(x, r) \subset U$, we have*

$$u(x) = \frac{1}{\mathcal{L}(\mathcal{B}(x,r))} \int_{\mathcal{B}(x,r)} u(y)dy \tag{4.54}$$

3. *Given any ball $\mathcal{B}(x, r) \subset U$,*

$$u(x) = \frac{1}{\sigma_{x,r}(\partial\mathcal{B}(x,r))} \int_{\partial\mathcal{B}(x,r)} u(y)d\sigma_{x,r}(y) \tag{4.55}$$

   where $\sigma_{x,r}$ is the surface measure of $\partial\mathcal{B}(x, r)$

The mean value property also has the converse property.

Theorem 81. *If $u \in \mathcal{C}^2(U)$ satisfies the mean value property*

$$u(x) = \frac{1}{\sigma_{x,r}(\partial\mathcal{B}(x,r))} \int_{\partial\mathcal{B}(x,r)} u(y)d\sigma_{x,r}(y) \tag{4.56}$$

for each ball $\mathcal{B}(x, r) \subset U$, then u is harmonic.

A direct consequence of mean value property is the maximum principle.

Theorem 82. ***(Maximum principle).*** *A subharmonic function $u : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is a domain.*

1. *If u attains its maximum in Ω, then u is a constant.*

2. *If u is continuous on $\overline{\Omega}$ and Ω is bounded, then*

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x) \tag{4.57}$$

Harmonicity of a function give rise to **regularity** as we see that harmonic functions are infinitely differentiable.

Theorem 83. ***Regualrity*** *If $u \in \mathcal{C}(U)$ satisfies the mean-value-property for each ball $\mathcal{B}(x, r) \subset U$, then*

$$u \in \mathcal{C}^\infty(U) \tag{4.58}$$

This behaviour of u is not necessarily extended to the boundary ∂U.

Chapter 5

Brownian motion and PDE

We have already seen the analytic solution of Dirichlet problem in the previous section. Now we set out to see the probabilistic solutions using properties of Brownian motion and techniques of Stochastic calculus. This presentation is inspired by [3]

5.1 Brownian motion and Harmonic function

Consider a Brownian motion started inside a domain $U \subset \mathbb{R}^d$.

Definition 84. (Hitting time of Boundary). We define the hitting time of the boundary ∂U.

$$\tau = \min\{t \geq 0 : B_t \in \partial U\} \tag{5.1}$$

Proposition 85. *Let f be a harmonic function on a domain U and U' be a bounded subdomain in U such that $\overline{U'} \subset U$, then for every $x \in U'$,*

$$f(x) = \mathbf{E}_x f(B_\tau) \tag{5.2}$$

Proof. Applying Ito's lemma to the stopped process B^τ we get

$$f(B_{t\wedge\tau}) = f(B_0) + \int_0^{t\wedge\tau} \nabla f(B_s) \cdot dB_s \tag{5.3}$$

is a local martingale. Since U' is a bounded subdomain both the functions f and ∇f are bounded in U'. So from the equation above $f(B_{t\wedge\tau})$ is also bounded. Hence $f(B_{t\wedge\tau})$ is a true martingale and we have for $x \in D'$

$$f(x) = \mathbf{E}_x f(B_{t\wedge\tau}) \tag{5.4}$$

Letting $t \to \infty$ and using dominated convergence we get

$$f(x) = \mathbf{E}_x[f(B_\tau)] \tag{5.5}$$

$\square$

Proposition 86. ***(Mean value property).*** *Given ant point $x \in U$ such that $\exists$ a $\overline{\mathcal{B}(x,r)} \subset U$ for some $r > 0$, then $f(x) = \mathbf{E}_x f(B_\tau)$ satisfies*

$$f(x) = \int_{\partial\mathcal{B}(x,r)} f(y)\sigma_{x,r}(dy) \tag{5.6}$$

where $\sigma_{x,r}$ is the uniform measure on the ball $\partial\mathcal{B}(x,r)$.

Proof. Consider the ball $\mathcal{B}(x,r)$ and let

$$\tau_{x,r} = \inf\{t \geq 0 : |B_t - x| = r\} \tag{5.7}$$

Then the distribution $B_{\tau_{x,r}}$ is invariant under the vector isometries. And is the uniform measure $\sigma_{x,r}$ on the boundary $\partial\mathcal{B}(x,r)$. Using the previous proposition we have

$$f(x) = \mathbf{E}_x f(B_{\tau_{x,r}}) = \int f(y)\sigma_{x,r}(dy) \tag{5.8}$$

$\square$

This property characterizes the harmonic functions as we see

Lemma 87. *Let f be a bounded measurable function on U satisfying the mean value property. Then f is harmonic on D.*

Proof. Refer to [3] $\square$

We observe a Brownian motion starting at a point $x \in U$ and wait till the time it first hits the boundary ∂U.

Proposition 88. *Let f be a continuous function on ∂U, and u be the solution of the Dirichlet problem on U satisfying the boundary condition f. Then, for every $x \in U$.*

$$u(x) = \mathbf{E}_x[f(B_\tau)] \tag{5.9}$$

Proof. Let's take a point $x \in U$ such that we can chose a ball of raidus $\epsilon > 0$ $\epsilon < d(x, \partial U)$ and τ be as above starting at x.

$$\tau_\epsilon = \inf\{t > 0 : d(B_t, x) = \epsilon\} \tag{5.10}$$

We have the following conditions for the ball.

$$u(x) = \mathbf{E}_x[u(B_{\tau_\epsilon})] \tag{5.11}$$

As $\epsilon \downarrow 0$ we have $\tau_\epsilon \uparrow \tau$. Let $\tau^{'}$ be an increasing limit of τ_ϵ. Naturally $\tau_\epsilon \leq \tau$ and continuity of sample paths imply $B_{\tau'} \in \partial U$. So using dominated convergence theorem $\mathbf{E}_x[u(B_{\tau_\epsilon})]$ converges to $\mathbf{E}_x[f(B_\tau)]$. □

Theorem 89. *Let f be a bounded, measurable function from the boundary $f : \partial U \to \mathbb{R}$ such that the following*

$$u(x) = \mathbf{E}_x[f(B_\tau)\mathbf{1}(\tau < \infty)], \quad \forall x \in U \tag{5.12}$$

is locally bounded. Then u is a harmonic function.

Proof. To prove the harmonicity of u we can show the mean value property using the strong Markov property. Consider a point $x \in U$ and a ball of radius $\epsilon > 0$ such that $\overline{\mathcal{B}(x, \epsilon)} \subset U$ and let

$$\tau_{x,\epsilon} = \inf\{t > 0 : B(t) \notin \mathcal{B}(x, \epsilon)\} \tag{5.13}$$

Using strong markov property we get

$$u(x) = \mathbf{E}_x[\mathbf{E}_x[f(B_\tau]\mathbf{1}(\tau < \infty)] = \mathbf{E}_x[u(B_{\tau_{x,\epsilon}})]$$
$$= \int_{\partial\mathcal{B}(x,\epsilon)} u(y)\sigma_{x,\delta}(dy) \tag{5.14}$$

Where $\sigma(x,\epsilon)$ is the uniform measure on the boundary $\partial\mathcal{B}(x,\epsilon)$. There fore u satisfies mean value property and also locally bounded hence it is harmonic. □

We have seen that if we are given with a solution u of a dirichlet problem satisfying the boundary condition $f : \partial U \to \mathbb{R}$ then it is given by the probabilistic solution format

$$u(x) = \mathbf{E}_x[f(B(\tau)] \tag{5.15}$$

And for any choice of bounded measurable function on the boundary the probabilistic solution yields a function u that is harmonic in the domain U. But in general it is not true that the solution will extend to the boundary. So we need some conditions on the boundary points.

Definition 90. (Poincare cone condition). A domain U satisfies the poincare cone condition at $x \in \partial U$ if there exist a cone K at x having angle $\alpha > 0$, and a $r > 0$ such that $K \cap \mathcal{B}(x,r) \subset U^c$.

Lemma 91. *The poincare cone condition implies the following: for every $y \in \partial U$ and every $\eta > 0$*

$$\lim_{x\to y, x\in U} \mathbf{P}_x(\tau > \eta) = 0 \tag{5.16}$$

Proof. Refer to [3] □

Theorem 92. ***(Dirichlet problem).*** *The function $u : \overline{U} \mapsto \mathbb{R}$ defined as*

$$u(x) = \mathbf{E}_x[f(B_\tau)\mathbf{1}(\tau < \infty)], \quad \forall x \in U \tag{5.17}$$

satisfies

$$\begin{aligned} \Delta u &= 0 \qquad x \in U \\ u(x) &= f(x) \quad x \in \partial U \end{aligned} \tag{5.18}$$

where each boundary point of U satisfies the poincare cone condition and f is continuous.

Proof. We already know that the probabilistic solution $u(x)$ satisfies the Dirichlet problem inside U. Only thing we need to show is that for every $y \in \partial U$

$$\lim_{x \to y, x \in U} u(x) = g(y) \tag{5.19}$$

Since f is continuous we can find $\delta > 0$ for any given $\epsilon > 0$ satisfying $|f(z) - f(y)| \leq \epsilon/3$ whenever $z \in \partial U$. We can also find an M such that $|f(z)| \leq M$ for every $z \in \partial U$. Then for every $\eta > 0$,

$$\begin{aligned} |u(x) - f(y)| &\leq \mathbf{E}_x[|f(B_\tau) - g(y)|\mathbf{1}_{\tau \leq \eta}] + \mathbf{E}_x[|f(B_\tau) - f(y)|\mathbf{1}_{\tau > \eta}] \\ &\leq \mathbf{E}_x[|f(B_\tau) - f(y)|]\mathbf{1}_{\tau \leq \eta}]\mathbf{1}[\sup\{|B_t - x| \leq \delta/2\}] \\ &+ 2M\mathbf{P}_x\left(\sup_{t \leq \eta} |B_t - x| > \delta/2\right) + 2M\mathbf{P}_x(\tau > \eta) \end{aligned} \tag{5.20}$$

We now bound each of the terms on the right. Let $|y - x| < \delta/2$ then we have $|B_\tau - y| \leq |B_\tau - x| + |y - x| < \delta$. This implies

$$\mathbf{E}_x[|f(B_\tau) - f(y)|]\mathbf{1}_{\tau \leq \eta}]\mathbf{1}[\sup\{|B_t - x| \leq \delta/2\}] \leq \epsilon/3 \tag{5.21}$$

For the second expression $2M\mathbf{P}_x\left(\sup_{t \leq \eta} |B_t - x| > \delta/2\right)$ we can translate x to 0 and this becomes

$$2M\mathbf{P}_x\left(\sup_{t \leq \eta} |B_t - x| > \delta/2\right) = 2M\mathbf{P}_0\left(\sup_{t \leq \eta} |B_t| > \delta/2\right) \tag{5.22}$$

By fixing particular $\eta > 0$ we can make the expression bounded by $\epsilon/3$. And the third expression is also bounded by $\epsilon/3$ which follows from the 91.

□

Chapter 6

Conclusions

The theory of Brownian motion is a well developed subject and over the years it has grown into multiple branches. This theory leads to the study of general theory of stochastic processes as well as theory of diffusions. The advent of stochastic calculus had been in the field of financial mathematics which has grown enormously. The current development in the past two decades using the techniques of stochastic calculus along with complex analysis has given rise to theory of Schramm Loewner Evolution. The contents of the thesis is an attempt to develop the necessary background for the brownian motion and stochastic calculus part of SLE. This is by no means exhaustive. All the mistakes and errors in writing the thesis is credited to the author only.

Bibliography

[1] L.C. Evans and American Mathematical Society. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 1998.

[2] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013.

[3] Jean-Francois Le Gall. *Brownian Motion, Martingales, and Stochastic Calculus*. Springer Publishing Company, Incorporated, 2018.

[4] I. Karatzas, I.K.S. Shreve, S. Shreve, and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics (113) (Book 113). Springer New York, 1991.

[5] Peter Mörters and Yuval Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.