Assignment 1

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Chapter 3: Theory ques.

3.8 Given $x(n) = A\cos(\omega_0 n + \phi) + w(n)$ is a RP

where $w(n)$ is white ~~noise~~ Gaussian noise with var $= \sigma_w^2$ & $\mu_w = 0$

(a) A ~~is~~ $\sim N(0, \sigma_A^2)$ & ω_0 & ϕ are constants.

$\Rightarrow r_x(n+k, n) = E(x[n+k]\, x^*[n])$

~~$= E[A\cos(\omega_0 n$~~

Note that $x(n)$ is real valued

$\Rightarrow x^*(n) = x(n)$

$$\Rightarrow r_x(n+k, n) = E[(A\cos(\omega_0 n + \phi) + w(n))(A\cos(\omega_0 n + \omega_0 k + \phi) + w(n+k))]$$

$$= E[A^2\cos(\omega_0 n + \phi)\cos(\omega_0 n + \phi + \omega_0 k)]$$

$$+ E[A\cos(\omega_0 n + \phi)\, w(n+k)]$$

$$+ E[A\cos(\omega_0 n + \phi + \omega_0 k)\, w(n)]$$

$$+ E[w(n+k)\, w(n)]$$

Now, assuming $w(n)$ & $A\cos(\omega_0 n + \phi)$ to be independent we get,

$$r_x(n+k, n) = \mathbb{E}\,\text{Cos}(\omega_0 n + \phi)\,\text{Cos}(\omega_0 n + \phi + \omega_0 k) \cdot \mathbb{E}[A^2] + \mathbb{E}[A\,\text{Cos}(\omega_0 n + \phi)] \cdot \underbrace{\cancel{\mathbb{E}[w(n+k)]}}_{0} + \underbrace{\cancel{\mathbb{E}[w(n)]}}_{0} \cdot \mathbb{E}[A\,\text{Cos}(\omega_0 n + \phi + \omega_0 k)] + \mathbb{E}[w(n)\,w(n+k)]$$

$$\therefore\ r_x(n+k, n) = \sigma_A^2 \cdot \text{Cos}(\omega_0 n + \phi)\,\text{Cos}(\omega_0 n + \phi + \omega_0 k) + \mathbb{E}[w(n)]\,\mathbb{E}[w(n+k)]$$

($\because$ $w(n)$ is ~~gaussian noise~~)

$$\therefore\ r_x(n+k, n) = \begin{cases} \sigma_A^2\,\text{Cos}(\omega_0 n + \phi)\,\text{Cos}(\omega_0 n + \phi) + \underbrace{\mathbb{E}[w^2(n)]}_{\sigma_\omega^2} & \text{if } k = 0 \\ \sigma_A^2\,\text{Cos}(\omega_0 n + \phi)\,\text{Cos}(\omega_0 n + \phi + \omega_0 k) + \underbrace{\mathbb{E}[w(n)]}_{0}\,\mathbb{E}[w(n+k)] & \text{if } k \neq 0 \end{cases}$$

$$\therefore\ r_x(n+k, n) = \begin{cases} \dfrac{\sigma_A^2}{2}\left(1 + \text{Cos}(2\omega_0 n + 2\phi)\right) + \sigma_\omega^2 & ;\ \text{if } k = 0 \\ \dfrac{\sigma_A^2}{2}\left[\text{Cos}(\omega_0 k) + \text{Cos}(2\omega_0 n + 2\phi + \omega_0 k)\right] & \text{otherwise} \end{cases}$$

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$$\therefore \boxed{r_x(n+k, n) = \frac{\sigma_A^2}{2}\left(\cos(\omega_0 k) + \cos(2\omega_0 n + 2\phi + \omega_0 k)\right) + \sigma_w^2 \cdot \delta(k)}$$

$$\therefore \cancel{S_x(\omega) = \frac{\sigma_A^2}{4}\left[2\delta \cos(k) + \delta(\right.}$$

$$S_x(\omega) = \text{DTFT}\{r_x(n+k, n)\}$$

$\therefore$ ~~$S_x(\omega)$~~ = But since $x(n)$ is not a WSS process, its PSD ~~is n~~ doesn't exist!

(b) $\phi \sim U[-\pi, \pi]$ & A, ω_0 are constants.

$$\Rightarrow r_x(n+k, n) = \mathbb{E}[x(n+k)\, x^*(n)]$$

Now, $x^*(n) = x(n)$ $\because$ real RP.

$$\therefore r_x(n+k, n) = \mathbb{E}[A^2 \cos(\omega_0 n + \phi)\cos(\omega_0 n + \phi + \omega_0 k)]$$

$$+ \mathbb{E}[A \cos(\omega_0 n + \phi)\, w(n+k)] +$$

$$\mathbb{E}[A \cos(\omega_0 n + \phi + \omega_0 k)\, w(n)]$$

$$+ \mathbb{E}[w(n) \cdot w(n+k)]$$

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$$\therefore r_x(n+k,n) = \frac{A^2}{2}\mathbb{E}[\cos(\omega_0 k)]$$

$$+ \frac{A^2}{2}\mathbb{E}[\cos(2\omega_0 n + 2\phi + \omega_0 k)]$$

$$+ \cancel{\mathbb{E}[w(n+k)]}^{\to 0}\, \mathbb{E}[A\cos(\omega_0 n + \phi)]$$

$$+ \cancel{\mathbb{E}[w(n)]}^{\to 0}\, \mathbb{E}[A\cos(\omega_0 n + \phi + \omega_0 k)]$$

$$+ \sigma_w^2\,\delta(k)$$

$$\therefore r_x(n+k,n) = \frac{A^2}{2}\cos(\omega_0 k) + \sigma_w^2\,\delta(k)$$

$$+ \frac{A^2}{2}\underbrace{\int_{-\pi}^{\pi}\frac{1}{2\pi}\cdot\cos(2\omega_0 n + 2\phi + \omega_0 k)\,d\phi}_{0}$$

$$\therefore \boxed{r_x(n+k,n) = \frac{A^2}{2}\cos(\omega_0 k) + \sigma_w^2\,\delta(k)}$$

$$\Rightarrow \boxed{r_x(k) = \frac{A^2}{2}\cos(\omega_0 k) + \sigma_w^2\,\delta(k)}$$

↳ WSS process

$$\therefore \boxed{S_x(\omega) = \frac{A^2}{4}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right) + \sigma_w^2}$$

↳ PSD

(c) $w_0 \sim U[w_0-\Delta, w_0+\Delta]$ & A & ϕ are constant

$\therefore$ ~~Amithin~~

$$r_x(n+k,n) = \frac{A^2}{2}E[\cos(w_0k)]$$

$$+ \frac{A^2}{2}E[\cos(2w_0n+2\phi+w_0k)]$$

$$+ \cancelto{0}{E[w(n+k)]}\, E[A\cos(w_0n+\phi)]$$

$$+ \cancelto{0}{E[w(n)]}\, E[A\cos(2w_0n+2\phi+w_0k)]$$

$$+ \sigma_w^2\,\delta(k)$$

$$\therefore r_x(n+k,n) = \frac{A^2}{2}\int_{w_0-\Delta}^{w_0+\Delta}\frac{1}{2\Delta}\cdot\cos(w_0k)\,dw_0$$

$$+ \frac{A^2}{2}\int_{w_0-\Delta}^{w_0+\Delta}\frac{1}{2\Delta}\cos(w_0(2n+k)+2\phi)\,dw_0$$

$$+ \delta(k)\cdot\sigma_w^2$$

$$\therefore r_x(n+k,n) = \frac{A^2}{4\Delta k}\left(\sin((w_0+\Delta)k) - \sin((w_0-\Delta)k)\right)$$

$$+ \frac{A^2}{4\Delta(2n+k)}\left(\sin((w_0+\Delta)(2n+k)+2\phi) - \sin((w_0-\Delta)(2n+k)+2\phi)\right)$$

$$+ \sigma_w^2\,\delta(k)$$

$$r_x(n+k,n) = \frac{A^2}{2\Delta K}\cos(\omega_0 k)\sin(\Delta k) + \frac{A^2}{2\Delta(2n+k)}\cos(\omega_0(2n+k)+2\phi)\cdot\sin(\Delta(2n+k)) + \sigma_w^2\,\delta(k)$$

Auto correlation

$\Rightarrow$ x is not WSS

$\Rightarrow$ PSD doesn't exist.

3.10

$$x[n] \rightarrow \boxed{\begin{matrix}\text{LSI}\\ h[n]\end{matrix}} \rightarrow y[n]$$

$$h[n] = \delta(n) + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$$

& $r_x(k) = \left(\frac{1}{2}\right)^{|k|}$ & $\mu_x = 0$

(a) we have $y[n] = x[n] * h[n]$

$\therefore$ ~~$E[y^2[n]]$~~ $\sigma_y^2 = E[y^2(n)] - E^2[y(n)]$,

$\therefore E[y[n]] = E\left[\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right]$

$$= \sum_{k=-\infty}^{\infty} h[k]\,\cancel{E[x[n-k]]}^{\,0}$$

$= 0$

Now,

$$E[y^2[n]] = E\left[\left(\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right)\left(\sum_{l=-\infty}^{\infty} h[l]\, x[n-l]\right)\right]$$

$$\therefore E[y^2[n]] = E\left[\left(\sum_{k=-\infty}^{\infty} h^2[k]\, x^2[n-k]\right) + \left(2\sum_{l}\sum_{k\neq l} h[k]\, h[l]\, x[n-k]\, x[n-l]\right)\right]$$

$$\therefore E[y^2[n]] = \sum_{k=-\infty}^{\infty} h^2[k]\, E[x^2[n-k]] + 2\sum_{l}\sum_{k\neq l} h[k]\, h[l]\, E[x[n-k]\, x[n-l]]$$

$$\therefore E[y^2[n]] = \sum_{k=-\infty}^{\infty} h^2[k]\, r_x(0) + 2\sum_{l}\sum_{k\neq l} h[k]\, h[l]\, r_x(l-k)$$

$$\therefore E[y^2[n]] = \left(\frac{1}{2}\right)^0 \left(h^2(0) + h^2(1) + h^2(2)\right) + 2\left[h^2[0]\, r_x(0) + h[0]\, h[1]\, r_x(-1) + h[0]\, h[2]\, r_x(-2)\right.$$

$$+ h[1]h[0]r_x(1) + h^2[1]r_x(0)$$

$$+ h[1]h[2]r_x(-1)$$

$\because h[n]$ is 0 otherwise

$$+ h[2]h[0]r_x(2) + h[2]h[1]r_x(1)$$

$$+ h^2[2]r_x(0) \Big]$$

$$\therefore E[y^2[n]] = \left(1 + \frac{1}{4} + \frac{1}{16}\right)$$

$$+ 2\left(1 + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right)$$

$$\therefore E[y^2[n]] = \frac{21 + (16 + 4 + 1 + 4 + 4 + 1 + 1 + 1 + 1)2}{16}$$

$$\therefore E[y^2[n]] = \frac{87}{16} = 5.43\cancel{4}85$$

$$\Rightarrow \sigma_y^2 = E[y^2[n]] - \overset{0}{\cancel{E[y[n]]}}$$

$$\Rightarrow \boxed{\sigma_y^2 = 5.4385}$$

(b) $y(n) = x(n) * h(n)$

we know that,

$$r_y(k) = r_x(k) * h(k) * h^*(-k)$$

Now let $g(k) = h(k) * h^*(-k)$

Here $h^*(k) = h(k)$

$$\Rightarrow g(k) = h(k) * h^*(-k)$$

$$\Rightarrow g(k) = \sum_{i=0}^{2} h(i)\, h(k+i)$$

$$= h(0)\, h(k) + h(1)\, h(k+1) + h(2)\, h(k+2)$$

$\Rightarrow g(0) = 1 \quad , \quad g(1) = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} = \frac{5}{8}$

$g(2) = \frac{1}{4} \quad , \quad g(-1) = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} = \frac{5}{8}$

$g(-2) = \frac{1}{4}$

otherwise $g(k) = 0$.

$$\Rightarrow r_y(k) = \sum_{i=-2}^{2} g(i)\, r_x(k-i)$$

$$\Rightarrow r_y(k) = \frac{1}{4}\cdot\left(\frac{1}{2}\right)^{|k+2|} + \frac{5}{8}\left(\frac{1}{2}\right)^{|k+1|} + \left(\frac{1}{2}\right)^{|k|} + \frac{5}{8}\left(\frac{1}{2}\right)^{|k-1|} + \frac{1}{4}\left(\frac{1}{2}\right)^{|k-2|}$$

3.11 first order AR ~~system~~ process ?

$$y(n) = a y(n-1) + w(n)$$

where $|a| < 1$, & $w(n) \to$ white noise RP with $\mu_w = 0$ & σ_w^2.

(a) we have

$$y(n) - a y(n-1) = w(n)$$

∴ multiplying with $y^*(n-l)$ on both sides & taking exp.

we get, (s.t $l = \{0, 1\}$)

$$E[y(n) y^*(n-l)] - a E[y(n-1) y^*(n-l)]$$

$$= E[w(n) y^*(n-l)]$$

$$\therefore r_y(l) - a r_y(l-1) = \begin{cases} \sigma_w^2 & \text{if } l = 0 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore r_y(0) - a r_y(-1) = \sigma_w^2 \quad \text{— (i)}$$

$$\& \quad r_y(1) - a r_y(0) = 0 \quad \text{— (ii)}$$

From (ii) we get $r_y(1) = a r_y(0)$ — (iii)

~~using result (iii) in (i) we get,~~

$$\cancel{\frac{r_y(1)}{a} = \sigma_w - a r_y(-1)}$$

(a) $y(n) - a y(n-1) = w(n)$

$\Rightarrow$ taking z transform on both sides we get,

$$Y(z)\left(1 - a z^{-1}\right) = W(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 - a z^{-1}}$$

$\Rightarrow$ using standard inv. z transforms, we can say that,

$$\boxed{h(n) = a^n \cdot u(n)}$$

(where $u(n)$ is unit step f^n)

$\hookrightarrow$ impulse responce of 1^{st} order AR process

(b) we have

$$y(n) = w(n) * h(n)$$

$\therefore$ using the know results we have,

$$r_y(k) = \cancel{r_w(k)}^{\,\sigma_w^2} * h(k) * h^*(-k)$$

$$\therefore\ r_y(k) = \sigma_w^2 \cdot \left(h(k) * h^*(-k) \right)$$

Since $h(k)$ is a real seq^n. we get,

$$r_y(k) = \sigma_w^2 \left(h(k) * h(-k) \right)$$

$$\Rightarrow r_y(k) = \sigma_w^2 \sum_{i=0}^{\infty} h(i)\, h(k+i)$$

$$= \sigma_w^2 \sum_{i=0}^{\infty} (a)^i \cdot (a)^{k+i} u(k+i)$$

$$= \sigma_w^2 \cdot a^{(k)} \sum_{i=0}^{\infty} (a)^{2i}$$

Since $|a| < 1$, the sum converges.

$$\therefore\ \boxed{r_y(k) = \sigma_w^2 \cdot \frac{a^{(k)}}{1-a^2}}$$

(c) $S_y(\omega) = \text{DTFT}\{ r_y(k) \}.$

$$= \sum_{k=-\infty}^{\infty}$$

$$\therefore\ r_y(k) = \begin{cases} \sigma_w^2 \cdot a^k \sum_{i=-k}^{\infty} a^{2i} & \text{if } k < 0 \\ \sigma_w^2\, a^k \sum_{i=0}^{\infty} a^{2i} & \text{otherwise} \end{cases}$$

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$$\therefore \boxed{r_y(k) = \begin{cases} \sigma_w^2 \cdot \dfrac{a^{-k}}{1-a^2} & \text{if } k<0 \\ \sigma_w^2 \dfrac{a^k}{1-a^2} & \text{otherwise} \end{cases}}$$

(c) $S_y(\omega) = \text{DTFT}\{r_y(k)\}$

$$\therefore S_y(\omega) = \sum_{k=-\infty}^{-1} \sigma_w^2 \cdot \frac{a^{-k} \cdot e^{-j\omega k}}{1-a^2} + \sum_{k=0}^{\infty} \sigma_w^2 \frac{a^k}{1-a^2} e^{-j\omega k}$$

$$\therefore \boxed{S_y(\omega) = \frac{\sigma_w^2}{1-a^2}\left[\frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}}\right]}$$

$$\therefore \cancel{S_y(\omega)} = \cancel{\frac{\sigma_w^2}{1+a^2}}$$

3.12 An MA(q) process is as follows :-

$$y(n) = \sum_{k=0}^{q} b(k)\, w(n-k)$$

where $w(n) \rightarrow$ white noise with σ_w^2 & μ_w

(a) we have $y(n) = \sum_{k=0}^{q} b(k)\, w(n-k)$

clearly $y(n) = w(n) * \left(b(n) \left(u(n) - u(n-q-1) \right) \right)$

$$\Rightarrow \boxed{h(n) = b(n)\left[u(n) - u(n-q-1)\right]}$$

(b) $r_y(\cancel{k,l}) = \cancel{\ } r_w(k) * h(k) * h^*(-k)$

$$= \sigma_w^2 \left(h(k) * h^*(-k) \right) \cancel{\delta(k)}$$

$$= \sigma_w^2 \sum_{l=0}^{q} \left(b(l) \cdot b^*(k+l) \cancel{\delta(k)} \cdot \cancel{\delta(k)} \right) \left[u(k+l) - u(k+l-q-1) \right]$$

$$\therefore \boxed{r_y(k) = \begin{cases} \sigma_w^2 \sum_{l=-k}^{q} \left(b(l) \cdot b^*(k+l) \right) \cancel{\delta(k)} & \text{if } k < 0 \\ \sigma_w^2 \sum_{l=0}^{q-k} \left(b(l) \cdot b^*(k+l) \right) & \text{if } k \geq 0 \ \& \ k \leq q, \\ 0 & \text{otherwise} \end{cases}}$$

P.T.O

(c) $S_y(\omega) = \cancel{\Sigma}\, \sigma_\omega^2 \left| B_q(\overset{\omega}{\cancel{k}}) \right|^2$

where

$b_q(k) = \cancel{\Sigma}\, b(k)\big(u(k) - u(k-1)\big)$

& $\boxed{B_q(\omega) = \text{DTFT}\{ b_q(k) \}.}$

3.25 $\underline{x} = [x(0), x(1), \ldots, x(n_0-1), x(n_0+1), \ldots, x(N)]^T$

(a) $R_x = \mathbb{E}[\underline{x}\,\underline{x}^H]$

Now $\underline{x}^H = (x^T)^*$

$$R_x = \mathbb{E}\begin{bmatrix} x(0)x^*(0) & \cdots & & \cdots & x(0)x^*(N) \\ \vdots & \ddots & & & \vdots \\ & & x(n_0-1)x^*(n_0-1) & & \\ & & & x(n_0+1)x^*(n_0+1) & \\ \vdots & & & \ddots & \vdots \\ x(N)x^*(0) & \cdots & & \cdots & x(N)x^*(N) \end{bmatrix}$$

$(N)\times(N)$

$$\Rightarrow R_x = \begin{bmatrix} r_x(0) & r_x(-1) & \cdots & & r_x(-N) \\ r_x(1) & \ddots & & & \vdots \\ \vdots & & r_x(0) & & \vdots \\ \vdots & & & r_x(0) & \vdots \\ \vdots & & & \ddots & \vdots \\ r_x(N) & \cdots & & \cdots & r_x(0) \end{bmatrix}$$

$(N)\times(N)$

$$\Rightarrow R_x = \begin{bmatrix} r_x(0) & r_x^*(1) & \cdots & & r_x^*(N) \\ r_x(1) & \ddots & & & \vdots \\ \vdots & & r_x(0) & & \vdots \\ \vdots & & & r_x(0) & \vdots \\ r_x(N) & \cdots & & \cdots & r_x(0) \end{bmatrix}$$

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1. ~~I~~ Yes R_x is Toeplitz because all diagonal elements are same & equal to $r_x(0)$

2. Yes R_x is Hermitian because

$R_x = R_x^H$

due to ~~sym~~ pseudo symme

3. $a^H R_x a = a^H \mathbb{E}[\underline{x}\,\underline{x}^H] a$

$= \mathbb{E}[a^H \underline{x}\,\underline{x}^H a]$

$= \mathbb{E}[(a^H \underline{x})(a^H \underline{x})^H]$

$= \mathbb{E}[(a^H \underline{x})(a^H \underline{x})^*]$

$= \mathbb{E}[|a^H \underline{x}|^2]$

$$\boxed{a^H R_x a \geq 0}$$

$\Rightarrow$ Yes it is positive semidefinite

P.T.O

(b) Yes, it is possible to get the auto corr. matrix R_x for

$$\underline{x} = [x(0), x(1), \ldots, x(n_0), \ldots, x(N)]$$

because from the current R_x matrix ~~we~~ one row r_{n_0} & one column c_{n_0} are missing?

$$r_{n_0} = [r_x^*(n_0) \cdots r_x(1) \cdot r_x(0)\; r_x^*(1) \cdots]$$

$$\& \; c_{n_0} = [r_x^*(n_0) \cdots r_x^*(1)\; r_x(0)\; r_x(1) \cdots]^T$$

Since we already know

$$r_x(k) \; \forall \; k$$

we can easily find ~~these~~ these missing row & column

$\Rightarrow$ New R_x can thus be made from original one.

* questions C3.3 & C3.4 done on "matlab" & plots are attached.