$\Lambda^{-1} - \Theta \Delta \Theta^T = \text{diag}((s\Sigma_i^{-1} + \Gamma_i^{-1})) - \Theta \Delta \Theta^T$

α-STABLE DIFFUSSIONS:

Extreme Diffussion.

#1.

$\Theta \Delta \Theta^T = \text{diag}((s\Sigma_i^{-1} + \Gamma_i^{-1})) - \Theta \Delta \Theta^T$

The main goal of this notebook lies in developing k-stable diffusions. It is based on the following observation: a lot of what makes diffusion models tractable is that the normal distribution is stable.

That is, (from Wikipedia):

Let X_1 and X_2 be independent realizations of a random variable X. Then X is said to be stable if for any constants $a > 0$ and $b > 0$ the random variable $aX_1 + bX_2$ has the same distribution as $cX + d$ for some constants $c > 0$ and d. The distribution is said to be strictly stable if it holds for $d = 0$.

The normal, Cauchy and Lévy distributions all have the same property.

The probability density function for a general stable distribution cannot be written analytically, (Why?). A random variable that is stable has the following characteristic function, in general,

$$\psi(t; \alpha, \beta, c, \mu) = \exp(it\mu - |ct|^\alpha (1 - i\beta \operatorname{sign}(t) \phi))$$

$$\phi = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ -\frac{2}{\pi} \log|t| & \alpha = 1 \end{cases}$$

$\alpha \in (0, 2]$. shape — ~~assymetry~~ concentration.

$\beta \in [-1, 1]$ shape — assymetry

$c \in (0, \infty)$ scale

$\mu \in (-\infty, \infty)$ location

Properties

- All stable distributions are infinitely divisible. (↗ if can be expressed as the sum of an arbitrary # of iid r.v.'s.)
- With the exception of the normal distribution ($\alpha = 2$), stable distributions are leptokurtic and heavy-tailed distributions
- Closure under convolution.
  - ↳ Can be seen from multiplying chars. function.

α-stable distributions generalize the CLT!

Generalized CLT.

A non-degenerate random variable Z is α-stable for some $0 < \alpha \leq 2$ if and only if there is an independent, identically distributed sequence of random variables $X_1, X_2, \ldots$ and constants $a_n > 0$, $b_n \in \mathbb{R}$ with

$$a_n (X_1 + \ldots + X_n) - b_n \xrightarrow{d.} Z$$

$\Omega \wedge \Theta' = \text{diag}((s\Sigma_i^{-1} + \Gamma_i^{-1})) - \Theta \Delta \Theta^T$

Cauchy Distribution.

A relatively small number of α-stable distributions have a representation in terms of simple functions. Apart from the normal distribution, ($\alpha = 2$), the Cauchy distribution is the most studied. Here are a few properties (Wikipedia):

1. If $X \sim \text{Cauchy}(x_0, \gamma)$ then $kX + l \sim \text{Cauchy}(x_0 k + l, \gamma|k|)$.

2. If $X \sim \text{Cauchy}(x_0, \gamma_0)$ and $Y \sim \text{Cauchy}(x_1, \gamma_1)$ are independent, then $X + Y \sim \text{Cauchy}(x_0 + x_1, \gamma_0 + \gamma_1)$ and
$X - Y \sim \text{Cauchy}(x_0 - x_1, \gamma_0 + \gamma_1)$.

3. If $X \sim \text{Cauchy}(0, \gamma)$ then $\frac{1}{X} \sim \text{Cauchy}(0, \frac{1}{\gamma})$.

4. Has univariate pdf

$$f(x) = \frac{1}{\pi\gamma\left[1 + \left(\frac{x - x_0}{\gamma}\right)^2\right]} \qquad X \sim \text{Cauchy}(x_0, \gamma)$$

Has multivariate pdf:

$$f(x; \mu, \Sigma, k) = \frac{\Gamma\left(\frac{1+k}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\pi^{k/2}|\Sigma|^{\frac{1}{2}}\left[1 + (x-\mu)^T\Sigma^{-1}(x-\mu)\right]^{\frac{1+k}{2}}}$$

Note: $\Sigma = I$ does not imply that the r.v.'s are independent.

5. Has univariate chf:

$$\varphi_X(t) = E[e^{itX}] = e^{itx_0 t - \gamma|t|}.$$

and multivariate chf

$$\varphi_X(t) = e^{it X_0(t) - \gamma(t)}$$

$X_0(t), \gamma(t)$: real functions of degree one s.t. $x_0(at) = a x_0(t)$, $\gamma(at) = |a|\gamma(t)$.

6. The KL-divergence between two Cauchy distributions has a closed form.

$$KL(P_{x_{0,1},\gamma_1} : P_{x_{0,2},\gamma_2}) = \log\frac{(\gamma_1 + \gamma_2)^2 + (x_{0,1} - x_{0,2})^2}{4\gamma_1\gamma_2}$$

Motivating Question:

DDPM's algorithm for training is:

1. Repeat
2. $X_0 \sim q(x_0)$ $\longmapsto$ sample from data.
3. $t \sim \text{Unif}(\{1, \dots, T\})$. $\mapsto$ T is # of timesteps.
4. $\epsilon \sim N(0, I)$.
5. Take gradient step on.

$$\nabla_\theta \| \epsilon - \epsilon_\theta(x_t, t) \|^2 \longmapsto \nabla_\theta \| \epsilon - \epsilon_\theta(\sqrt{\bar{\alpha}_t} x_0 + \sqrt{1-\bar{\alpha}_t}\epsilon, t) \|^2$$

6. Until converged

A crucial step, (5), ~~(cut)~~ is dependent upon the stability of the normal, because for some schedule of β_t's,

$$X_t = \sqrt{\bar{\alpha}_t} X_0 + \sqrt{1-\bar{\alpha}_t}\,\epsilon$$

Does the Cauchy distribution, with its stability and existing KL divergence determine a viable diffusion process with different or desirable (extremes?) properties

From the review and there are a few things that worry me:

(1) - stability might not be enough? Maybe conjugacy too is needed? Maybe conjugacy is related to stability, right?

(2) - Will it be enough to exchange μ and σ to x_0 and λ?

(3) - Ho's DDPM (2019?) claims that for β_t small, the reverse process is also Gaussian. He cites Sohl-Dickenstein (2015)? for this claim, who in turn cites Feller. (an old article)
How does this claim hold for the Cauchy case?

$\Lambda^{-1} - \hat{\Lambda} \hat{\Theta}^T = \text{diag}((s\Sigma_i^{-1} + \Gamma_i^{-1})) - \Theta \Delta \Theta^T$

With all that in mind, the simplest litmus test for this idea might be assuming univariate distributions (bivariate top), and run with it.

A. Gaussian Diffusion Process gradually transforms data X into random noise by adding increasing amounts of Gaussian noise at each timestep t.

Bishop's New book

has a section on diffusion, and although I find it lacking, it does have one stellar problem, that I hope will help: problem 20.7, where he asks you to show that the reverse $q(z_{t-1} | z_t)$ is approximately Gaussian when β_t is small. If I can reproduce this, then I might be able to reproduce it for the Cauchy case.

Anyway, Bishop's diffusion:

Forward Encoder:

$$z_1 = \sqrt{1-\beta_1} X + \sqrt{\beta_1} \epsilon_1, \qquad \epsilon_1 \sim N(\epsilon_1 | 0, I).$$

then $q(z_1 | X) = N(z_1 | \sqrt{1-\beta_1} X, \beta_1 I)$. $\beta_1 < 1$ is the variance.

More generally, $z_t = \sqrt{1-\beta_t} z_{t-1} + \sqrt{\beta_t} \epsilon_t, \qquad \epsilon_t \sim N(\epsilon_t | 0, I).$

$$q(z_t | z_{t-1}) = N(z_t | \sqrt{1-\beta_t} z_{t-1}, \beta_t I).$$

$$\beta_1 < \beta_2 < \dots < \beta_T$$

Diffusion Kernel.

$$q(z_1, \dots, z_t | x) = q(z_1 | x) \prod_{\tau=2}^{t} q(z_\tau | z_{\tau-1}).$$

One can marginalize over intermediate variables and find that:

$$q(z_t | x) = \mathcal{N}(z_t | \sqrt{\alpha_t}\, x, (1-\alpha_t) I)$$

$$\alpha_t = \prod_{\tau=1}^{t} (1 - \beta_\tau),$$

or $z_t = \sqrt{\alpha_t}\, x + \sqrt{1-\alpha_t}\, \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(\epsilon_t | 0, I).$

and as $T \to \infty$, we have

$$q(z_T | x) = \mathcal{N}(z_T | 0, I).$$

and $\Rightarrow$ $\underbrace{q(z_T)}_{\text{indep. of } x} = \mathcal{N}(z_T | 0, I)$

Conditional distribution.

We are trying to learn to undo the noise process, and so, we want $q(z_{t-1} | z_t)$:

$$q(z_{t-1} | z_t) = \frac{q(z_t | z_{t-1})\, q(z_{t-1})}{q(z_t)}.$$

and

$$q(z_{t-1}) = \int q(z_{t-1} | x)\, p(x)\, dx.$$

where $q(z_{t-1} | x) = \mathcal{N}(z_{t-1} | \sqrt{\alpha_{t-1}}\, x, (1-\alpha_{t-1}) I)$

Hence $q(z_{t-1} | z_t)$ is intractable, so we condition on x.

The proof is left to myself in the future ↓

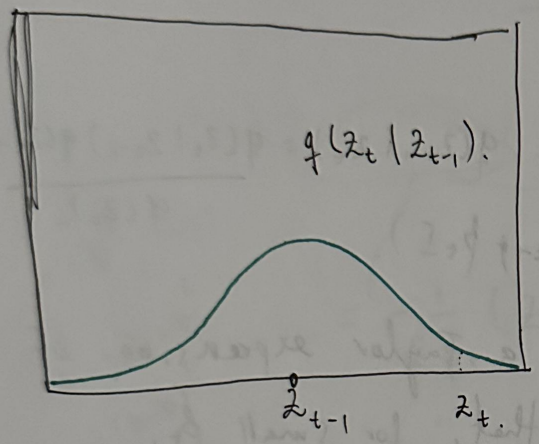
$$q(z_{t-1} | z_t, x) = \frac{q(z_t | z_{t-1}, x)\, q(z_{t-1} | x)}{q(z_t | x)}.$$

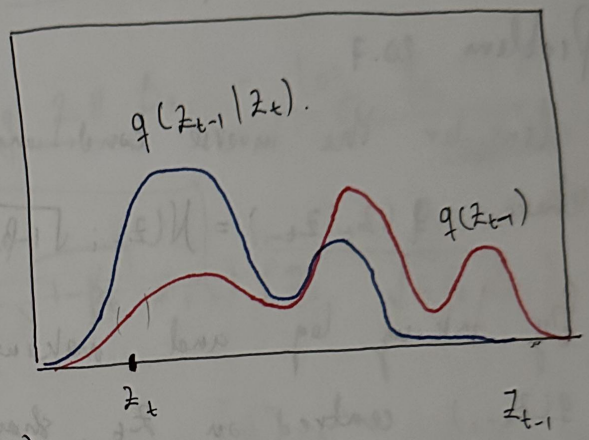
$$\hookrightarrow q(z_{t-1} | z_t, x) = \mathcal{N}\left(z_{t-1} \middle| \frac{(1-\alpha_{t-1})\sqrt{1-\beta_t}\, z_t + \sqrt{\alpha_{t-1}}\, \beta_t x}{1-\alpha_t}, \beta_t \frac{(1-\alpha_{t-1})}{1-\alpha_t} I\right)$$

$\Sigma^T = \mathrm{diag}((S\Sigma_i^{-1} + \Gamma_i^{-1})) - \Sigma \Delta \Sigma^T$

Reverse Decoder.

But we reeeally want $q(z_{t-1}|z_t)$ because if we are doing generative modeling begining from z_T, we can not assume the ~~existence~~ (knowledge) of X.

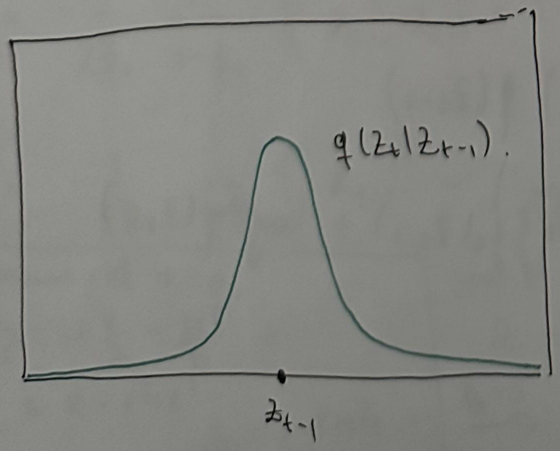


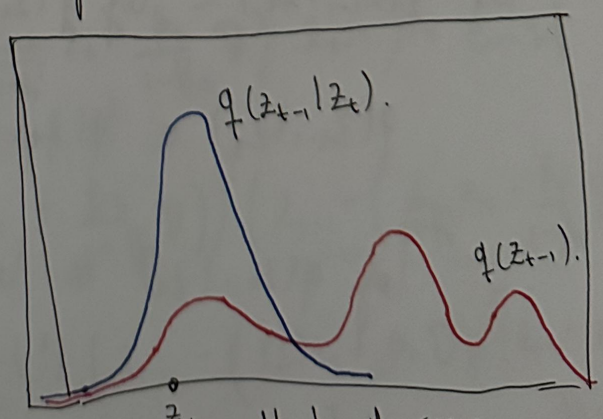


$$q(z_t|z_{t-1}) = \mathcal{N}(z_t | \sqrt{1-\beta_t} z_{t-1}, \beta_t I).$$

$$q(z_{t-1}|z_t) = \frac{q(z_t|z_{t-1})\, q(z_{t-1})}{q(z_t)}.$$

If $q(z_t|z_{t-1})$ is a sufficiently narrow Gaussian then $q(z_{t-1})$ will vary only a small amount over the region in which $q(z_t|z_{t-1})$ has significant mass, and hence $q(z_{t-1}|z_t)$ will be approximately Gaussian.





Maybe the same idea works for Cauchy diffusion?

More formally, $q(z_{t-1} \mid z_t)$ will be approximately Gaussian by making a Taylor expansion of $\ln q(z_{t-1} \mid z_t)$ around the point z_t as a function of z_{t-1}. This also shows that for small variance, the reverse distribution $q(z_t \mid z_{t-1})$ will have a covariance that is close to $\beta_t I$.

Problem 20.7.

Consider the inverse conditional $q(z_{t-1} \mid z_t) = \dfrac{q(z_t \mid z_{t-1})\, q(z_{t-1})}{q(z_t)}$

where $q(z_t \mid z_{t-1}) = \mathcal{N}(z_t ; \sqrt{1-\beta_t}\, z_{t-1}, \beta_t I)$.

By taking log and making a Taylor expansion of $q(z_{t-1})$ centred on z_t, show that, for small β_t, $q(z_{t-1} \mid z_t)$ is approximately Gaussian with mean z_t and covariance $\beta_t I$.

$$\log q(z_{t-1} \mid z_t) = \log q(z_t \mid z_{t-1}) + \log q(z_{t-1}) - \log q(z_t)$$

$$\log q(z_{t-1} \mid z_t) \approx \log q(z_t \mid z_{t-1}) + \log q(z_{t-1}) - \log q(z_t)$$

$$q(z_{t-1} \mid z_t) \propto q(z_t \mid z_{t-1})\, q(z_{t-1})$$

$$\log q(z_{t-1} \mid z_t) \propto \log q(z_t \mid z_{t-1}) + \log q(z_{t-1})$$

$\Sigma^T = \text{diag}((s\Sigma_i^{-1} + \Gamma_i^{-1})) - \textcircled{M} \Delta \textcircled{M}'$

$q(z_{t-1}|z_t)$

$$\log q(z_{t-1}) \approx \log q_{z_{t-1}}(z_t) + (z_{t-1} - z_t)\, D\, q_{z_{t-1}}(z_t)$$

$$\log q(z_{t-1}|z_t) \approx -\frac{1}{2\beta_t}(z_t - \sqrt{1-\beta_t}\, z_{t-1})^T (z_t - \sqrt{1-\beta_t}\, z_{t-1})$$

$$+ \log q_{z_{t-1}}(z_t) + (z_{t-1} - z_t) D q_{z_{t-1}}(z_t)$$

$$- \log 2\pi\beta_t - \log q_{z_t}(z_t)$$

constants wrt z_{t-1}

$a^2 - b^2$

$a^2 - 2ab + b^2$

$= (a-b)^2$

$a = \sqrt{1-\beta_t}\, z_{t-1}$

$$= -\frac{1}{2\beta_t}(z_t - \sqrt{1-\beta_t}\, z_{t-1})^T (z_t - \sqrt{1-\beta_t}\, z_{t-1})$$

$$+ (z_{t-1} - z_t)^T D q_{z_{t-1}}(z_t) + \text{const}$$

$$= -\frac{1}{2\beta_t}\left((1-\beta_t) z_{t-1}^T z_{t-1} - 2\sqrt{1-\beta_t}\, z_t^T z_{t-1}\right) + D^T z_{t-1} + \text{const}$$

$$= -\frac{1}{2\beta_t}\left((1-\beta_t) z_{t-1}^T z_{t-1} - 2\left(\sqrt{1-\beta_t}\, z_t + \beta_t D\right)^T z_{t-1}\right) + \text{const}$$

$$= -\frac{1}{2\beta_t}\frac{1}{1-\beta_t}\left(z_{t-1}^T z_{t-1} - 2\left(\frac{1}{\sqrt{1-\beta_t}} z_t + \frac{\beta_t}{1-\beta_t} D\right)^T z_{t-1}\right) + \text{const}$$

$$= -\frac{1}{2\beta_t}\frac{1}{1-\beta_t}\left(z_{t-1} - \left(\frac{1}{\sqrt{1-\beta_t}} z_t + \frac{\beta_t}{1-\beta_t} D\right)\right)^T \left(z_{t-1} - \left(\frac{1}{\sqrt{1-\beta_t}} z_t + \frac{\beta_t}{1-\beta_t} D\right)\right)$$

Expansions at $x = 0$

$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{x}{2} + \frac{3x^2}{8}$

$\frac{x}{1-x} \approx x + x^2 + x^3$

$\frac{1}{1-x} \approx 1 + x + x^2 + \dots$

Small β_t

$$\approx -\frac{1}{2\beta_t}(1)\left(z_{t-1} - \left((1)z_t + \beta_t D\right)\right)^T \left(z_{t-1} - (z_t + \beta_t D)\right)$$

$\beta_t \to 0$

$$\Longrightarrow -\frac{1}{2\beta_t}(z_{t-1} - z_t)^T (z_{t-1} - z_t) \sim \mathcal{N}(z_t, \beta_t I)$$

Can we apply this to a Cauchy diffusion?

Let's define it:

Forward Encoder:

Consider x given, ~~in~~ in the univariate setting.

$$z_1 = \sqrt{1-\beta_1}\, x + \sqrt{\beta_1}\, \epsilon_1, \quad \epsilon_1 \sim \text{Cauchy}(\epsilon_1 \mid 0, 1). \tag{1}$$

$$\Rightarrow q(z_1 \mid x) = \text{Cauchy}\left(z_1 \mid \sqrt{1-\beta_1}\, x, \sqrt{\beta_1}\, I\right)?$$

From property (1) of the previous page on the Cauchy distribution:

1. If $X \sim \text{Cauchy}(x_0, \gamma)$ then $kX + l \sim \text{Cauchy}(x_0 k + l, \gamma|k|)$.

Hence, if $\epsilon_1 \sim \text{Cauchy}(\epsilon_1 \mid 0, 1)$, then:

$$z_1 \mid x \sim \text{Cauchy}\left(\sqrt{1-\beta_1}\, x, \sqrt{\beta_1}\right). \tag{2}$$

Each successive ~~image~~ point is given by:

(3).

$$z_t = \sqrt{1-\beta_t}\, z_{t-1} + \sqrt{\beta_t}\, \epsilon_t \qquad \epsilon_t \sim \text{Cauchy}(\epsilon_t \mid 0, 1)$$

Where again, by the Markov property and 1.

$$q(z_t \mid z_{t-1}) = \text{Cauchy}\left(z_t \mid \sqrt{1-\beta_t}\, z_{t-1}, \sqrt{\beta_t}\right).$$

Select β_t s.t. $\beta_1 < \beta_2 < \ldots < \beta_T$.

Diffusion Kernel:

Does it follow that

$$q(z_t \mid x) = \text{Cauchy}\left(z_t \mid \sqrt{\alpha_t}\, x, (1-\alpha_t) I\right)$$

$$\text{with} \quad \alpha_t = \prod_{\tau=1}^{t} (1-\beta_\tau). \quad ?$$

$\Sigma_i^{-1} \cdot \Theta^T = \text{diag}((s\Sigma_i^{-1} + \Gamma_i^{-1})) - \Theta\Delta\Theta^T$

We proceed via diffusion.

This clearly holds when $t=1$, and $z_0 = x$.

Assume t holds, that is

$$z_t = \sqrt{\alpha_t}\, x + (1-\alpha_t)\epsilon_t \quad \text{and} \quad q(z_t|x) = \text{Cauchy}(\sqrt{\alpha_t}x, (1-\alpha_t))$$
$$\epsilon_t \sim \text{Cauchy}(\epsilon_t|0,1).$$

Then

$$z_{t+1} = \sqrt{1-\beta_{t+1}}\, z_t + \sqrt{\beta_{t+1}}\, \epsilon_{t+1}$$
$$= \sqrt{1-\beta_{t+1}}\left(\sqrt{\alpha_t}x + (1-\alpha_t)\epsilon_t\right) + \sqrt{\beta_{t+1}}\,\epsilon_{t+1}$$
$$= \sqrt{\alpha_{t+1}}\, x + \sqrt{1-\beta_{t+1}}\,(1-\alpha_t)\epsilon_t + \sqrt{\beta_{t+1}}\,\epsilon_{t+1}$$

$\text{Cauchy}(s|0, \sqrt{\beta_{t+1}})$

$\text{Cauchy}(s|0, \sqrt{1-\beta_{t+1}}(1-\alpha_t)) \rightsquigarrow$ not induction - sum does not work

It is not true. But we can then define Cauchy diffusion

$$z_1 = (1-\beta_1)x + \beta_1\epsilon_1 \qquad \epsilon_1 \sim \text{Cauchy}(\epsilon_1|0,1)$$
$$z_t = (1-\beta_t)z_{t-1} + \beta_t\epsilon_t \qquad \epsilon_t \sim \text{Cauchy}(\epsilon_t|0,1)$$

Now for the diffusion kernel: $z_t|z_{t-1} \sim \text{Cauchy}(z_t|(1-\beta_t)z_{t-1}, \beta_t)$

$t=1$, does $q(z_t|x) = \text{Cauchy}(z_t|\alpha_t x, (1-\alpha_t))$

$\hookrightarrow z_1 = (1-\beta_1)x + \beta_1\epsilon_1 = \alpha_1 x + (1-\alpha_1)\epsilon_1$

For $t+1$, assuming t

$$z_{t+1} = (1-\beta_{t+1})z_t + \beta_{t+1}\epsilon_{t+1}$$
$$= (1-\beta_{t+1})(\alpha_t x + (1-\alpha_t)\epsilon_t) + \beta_{t+1}\epsilon_{t+1}$$
$$= \alpha_{t+1}x + \underbrace{(1-\beta_{t+1})(1-\alpha_t)\epsilon_t}_{*} + \underbrace{\beta_{t+1}\epsilon_{t+1}}_{**}$$

* $(1-\beta_{t+1})(1-\alpha_t)\epsilon_t \sim \text{Cauchy}(s \mid 0, (1-\beta_{t+1})(1-\alpha_t))$.

** $\beta_{t+1}\epsilon_{t+1} \sim \text{Cauchy}(s' \mid 0, (\beta_{t+1}))$.

* and ** are independent, and so by Cauchy properties

2. If $X \sim \text{Cauchy}(x_0, \gamma_0)$ and $Y \sim \text{Cauchy}(x_1, \gamma_1)$ are independent, then

$$X + Y \sim \text{Cauchy}(x_0 + x_1, \gamma_0 + \gamma_1)$$

$$X - Y \sim \text{Cauchy}(x_0 - x_1, \gamma_0 + \gamma_1).$$

Hence,

$$* + ** \sim \text{Cauchy}(s'' \mid 0, (1-\beta_{t+1})(1-\alpha_t) + \beta_{t+1})$$

$$= \text{Cauchy}(s'' \mid 0, 1 - (1-\beta_{t+1})\alpha_t)$$

$$= \text{Cauchy}(s'' \mid 0, 1 - \alpha_{t+1}).$$

and so, reparameterizing appropriately, we have that

$$q(z_t \mid X) = \text{Cauchy}(z_t \mid \sqrt{\alpha_t} X, 1-\alpha_t).$$

Makes sense.

Now, Reverse Process. What is, if any, the distribution such that approximates $q(z_{t-1} \mid z_t)$ for small β_0?

$= S\, \eta\eta' \cdot \left(\sum_i \frac{1}{\sigma_i^2}\right) \otimes I_p$

$\ldots(\Sigma_i^{-1} + \Gamma_i^{-1})) - \Theta \Delta \Theta^T$

$$q(z_{t-1} \mid z_t) \propto q(z_t \mid z_{t-1})\, q(z_{t-1}).$$

$$= \frac{1}{\pi\beta_t\left[1 + \left(\frac{z_t - (1-\beta_t) z_{t-1}}{\beta_t}\right)^2\right]} \cdot q(z_{t-1}).$$

Taylor

$$q(z_{t-1}) \approx q_{z_{t-1}}(z_t) + (z_{t-1} - z_t)\, D\, q_{z_{t-1}}(z_t).$$

$$+ (z_{t-1} - z_t)^2 \frac{\partial^2}{\partial z_{t-1}^2} q_{z_{t-1}}(z_{t-1}) \Big|_{z_t}$$

Maybe I need a 2nd order approx mod.

$$\approx \frac{q_{z_{t-1}}(z_t) + (z_{t-1} - z_t)\, D\, q_{z_{t-1}}(z_t)}{\pi\beta_t\left[1 + \left(\frac{z_t - (1-\beta_t) z_{t-1}}{\beta_t}\right)^2\right]}$$

$=$

$$\log q(z_{t-1} \mid z_t) = \log q(z_t \mid z_{t-1}) + \log q(z_{t-1}) - \log q(z_t)$$

$$= -\log \pi\beta_t - \log\left(1 + \left(\frac{z_t - (1-\beta_t) z_{t-1}}{\beta_t}\right)^2\right) + \log q(z_{t-1}) + \text{const}.$$

$$\approx -\log \beta_t - \log\left(1 + \left(\frac{z_t - (1-\beta_t) z_{t-1}}{\beta_t}\right)^2\right) + \log q(z_t) + (z_{t-1} - z_t)\, D(z_t) + \text{const}$$

$$= -\log\left(\beta_t^2 + (z_t - (1-\beta_t) z_{t-1})^2\right) + z_{t-1}\, D(z_t) + \text{const}$$

$$= \log\left(\frac{e^{z_{t-1} D(z_t)}}{\beta_t^2 + (z_t - (1-\beta_t) z_{t-1})^2}\right)$$