M.SC PROJECT WORK

PROJECT WORK ON ::

CLASSIFICATION THEOREM OF CLOSED SURFACES

DEPARTMENT OF PURE MATHEMATICS

UNIVERSITY OF CALCUTTA

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UNDER THE GUIDENCE OF

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DECLARATION

I do hereby declare that Mr. Sohan Das and Mr. Supriyo Maity has done this DISSERTATION under my guidence and his project report entitled "CLASSIFICATION THEOREM OF CLOSED SURFACES" has been submitted for partial fulfilment of the dissertation in M.Sc 4^{th} semester for the year 2021 at the Department of Pure Mathematics, University of Calcutta.

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In preparing project it requires a high ammout of expertise and knowledge. It is also requires a clear idea about the procedure and the right steps which works in the background.

We convey our sincere gratutide to our teacher for their overall co-operation and support without which our project would not be possible.

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To complete our project work we were helped by many persons who remained in background and helped us to complete our task successfully.

We are very much thankful to those persons.

Date :

Signature of the Students

Abstract

In this project, we study an important theorem named Classification Theorem of closed surfaces. It mainly classifies the closed surfaces and helps us to study some unknown surfaces with the help of some standard surfaces. We will approach towards the proof in a unique way.

As we are dealing with the closed connected surfaces so at first we will be familiar with all the topological definitions related to our proof. On the next step we will see the different examples of sufaces and will know some important results of topology. We will also construct some standard surfaces using quotient topology which will create a strong visualization so that we can easily step into the third step. The third step is the most vital step in our proof. We will define polygon, polyhedron and will see how by identification of edges, polygons transform to different polyhedrons. Then we will visualize and classify surfaces through Graph theoretic approach. In the last step we will see a relation between the graphical approach and the topology which leads to the completion of our proof.

Contents

Classification of Closed surfaces

1.1 Introduction

The classification theorem is a beautiful example of geometric topology. Although it was discovered in the last century, yet it manages to convey the spirit of present day research. This course is concerned with a special class of topological spaces called surfaces. Common examples of surfaces are the sphere and the cylinder; less common, though probably still familiar, are the torus and the Möbius band. Other surfaces, such as the projective plane and the Klein bottle, may be unfamiliar, but they crop up in many places in mathematics. Our aim is to classify surfaces – that is, to produce criteria that allow us to determine whether two given surfaces are homeomorphic or not. We are here mostly concerned about the surfaces which are *closed* and *connected.* We will specially use graph theoretic approach to classify the closed connected surfaces but topology is inevitible part of it. So we will follow the scheme written in abstract section to prove the theorem known as Classification Theorem.

2.1 Topological Approach

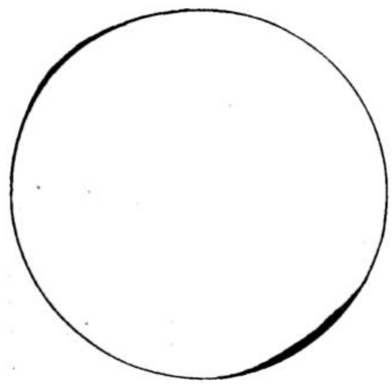
2.1.1 Definitions

Surface **::**

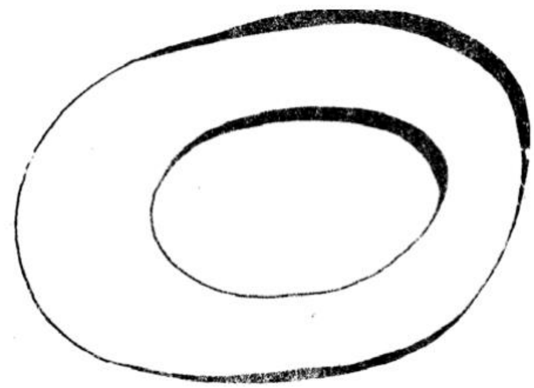
A topological space S is said to be surface if S is Hausdorff, 2nd Countable, Connected and for each point $x \in S$ there exists an open set U in S such that $x \in U$ and U is homeomorphic to an open subset of $\mathbb{R}^2$.

2.1.2 Examples of Surfaces

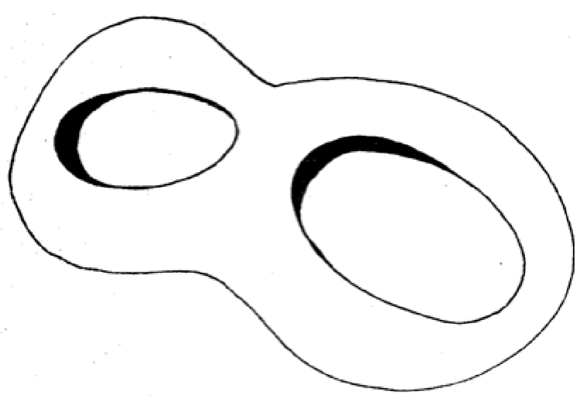
Before starting the theorem let us look at a few examples of surfaces. All the following pictures are of surfaces in 3-dimensions. In example 1 by the word “sphere” we mean just the surface of the sphere, and not the inside. In fact in all the examples we mean just the surface and not the solid inside. We introduce here several familiar examples such as Sphere, Pretzel, Sphere with two or three handles etc. We also illustrate how these surfaces can be constructed from a polygon by identifying edges in our work later.



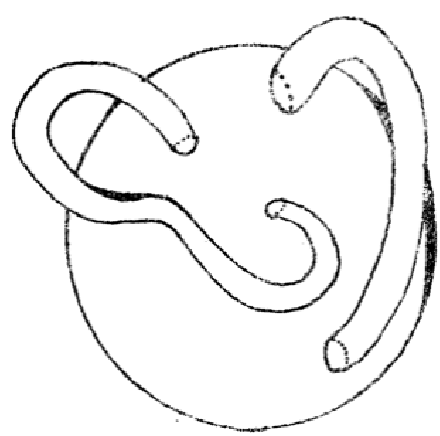
1. Sphere.



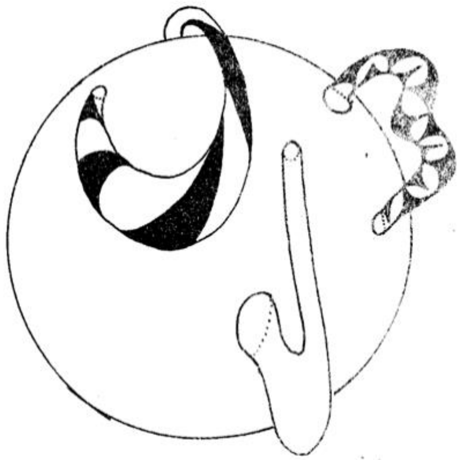
2. Torus



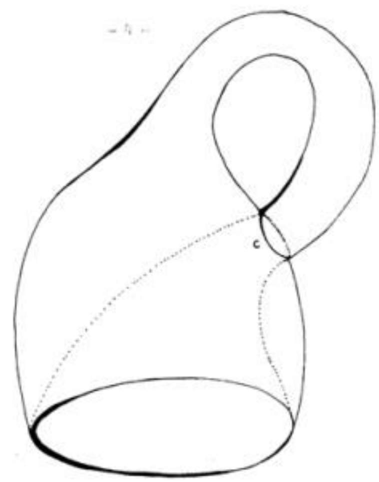
3. Pretzel.



4. Sphere with two handles sewn on.



5. Sphere with three handles sewn on.



6. Klein bottle

Notice that all the examples above have three properties in common

(i) connected

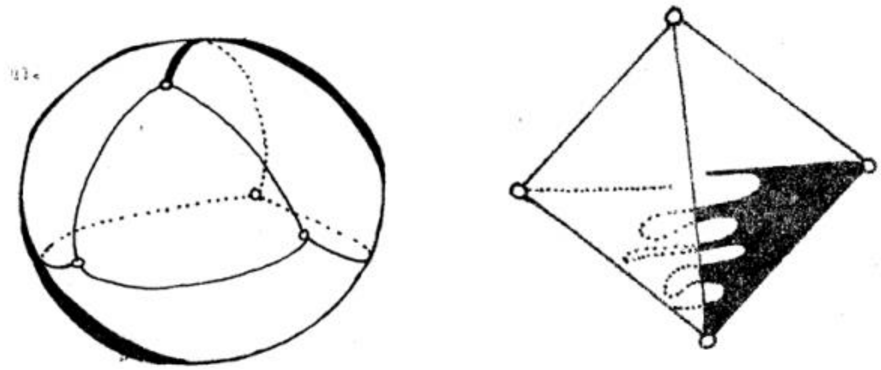
(ii) closed

(iii) triangulable

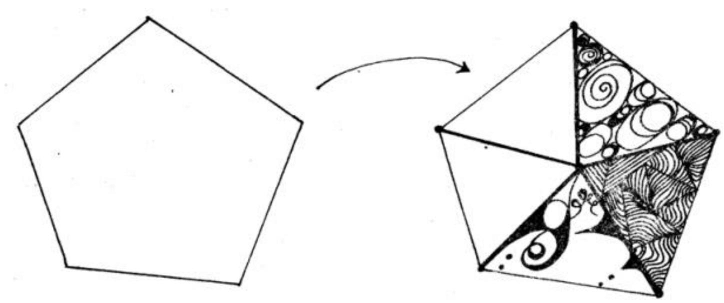
(i) ***Connected*** means that the surface is all in one piece. An equivalent definition is that any two points of the surface can be joined by a path in the surface.

(ii) ***Closed*** means there is no boundary or rim. An example of surface that is not closed is Möbius strip which is formed by taking a strip and sewing the ends together in the “wrong way”. We will also see it’s construction in our later work.

(iii) ***Triangulable*** means that we can chop the surface up into a finite number of vertices, edges and faces. Of course if the surface is curved then the edges and faces also have to be curved, but we can make a model in which they appear straight. For example we can triangulate the sphere with 4 vertices, 6 curved edges, and 4 curved triangles, so that the corresponding straight model is a tetrahedron.



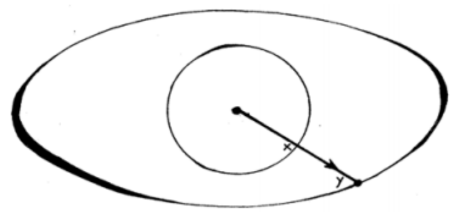
There are lots of ways to chop up a sphere and we could use a million tiny triangles if we wanted to; the main thing is that it can be done in some way. Sometimes it is easier first to chop up a surface into polygons rather than triangles. If we can chop a surface into polygons, then we can also chop it into triangles by putting an extra vertex into each polygon:



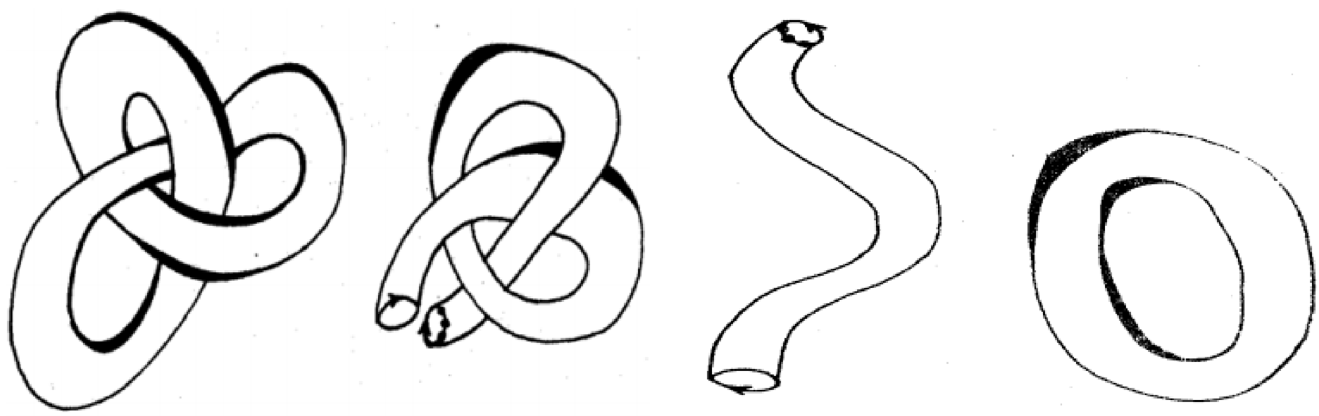
We call a chopping up into triangles a triangulation, and any triangulation of the surfaces has two properties:
(1) Any edge is the edge of exactly two triangles.
(2) Any vertex, v, is the vertex of at least three triangles, and all triangles having v as vertex fit round into a cycle

Our intuition tells us correctly that any surface can be triangulated, but we omit the proof here and so we shall be content to assume triangulability. The great advantage of triangulation is that it reduces our task of classification to a finite combinatorial problem, combination of edges of polygons. Therefore from now on we shall assume that all our surfaces are *connected, closed* and *triangulable*.

Homeomorphism **::**
A function f : (X, τ) → (Y, σ) is called a homeomorphism if f is one-one, onto, continuous and f^{-1} is continuous.
Example (i) A sphere is homeomorphic to an ellipsoid by radial projection x → y.



Example (ii) Suppose that during a deformation of a surface X we made a cut, and later sewed the cut up again exactly as it was before, then the result would be homeomorphic to X. The following four pictures show, for example, that a knotted torus is homeomorphic to an unknotted torus.



1. Knotted torus
2. Cut (the arrows show the direction of cut)
3. Unknot the cylinder
4. Sew up again.

<u>Surface with boundary and without boundary</u> **::**

Let S = (X, τ) be a surface.

A **boundary point** of S is a point of X for which every neighbourhood contains a half-disc like neighbourhood.

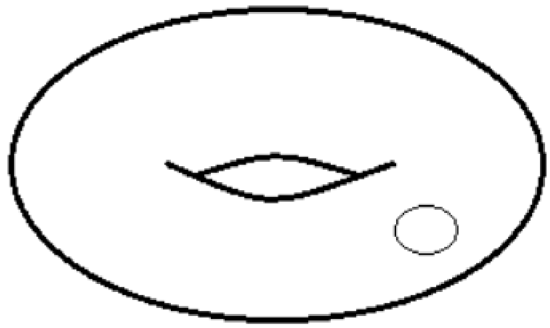
The **boundary** of S is the set of all boundary points of S.

A **surface with boundary** is a surface whose boundary is non-empty.

A **surface without boundary** is a surface with no boundary points. This surface without boundary are known to be **closed surfaces**.

Examples of surfaces without boundary are a *sphere* and a *torus*.

Examples of surfaces with boundary are *surfaces with holes* two such example is *sphere with three holes* or *torus with one holes.*



Torus with 1 hole

<u>Orientable and non Orientable surfaces</u> **::**

The idea of orientability is another fundamental concept that we need for the study of surfaces. To illustrate the underlying idea, we consider two familiar surfaces – a cylinder and a Möbius band. We can distinguish between a cylinder and a Möbius band by noticing that every cylinder has an 'inside' and an 'outside'. That is there are some basic differences between their orientation.

An ***orientable*** surface is one where consistent 'orientation' can be assigned over the entire surface; a ***non-orientable*** surface is one where this cannot be done.

More simply we call a surface ***orientable*** if it does not contain a Möbius strip and we call it ***non-orientable*** if it contains a Möbius strip.

Sphere, Torus all are examples of an orientable surfaces and Möbius strip, Klein Bottle are examples of non-orientable surfaces.

<u>Quotient Topology</u> **::**

Let (X, τ) be a topological space and Y is a set. Let f: X → Y be a surjective mapping. Then the collection τ_f = {U ⊆ Y : f^{-1}(U) ∈ τ} is a topology on Y . This topology τ_f on Y is called the quotient topology or identification topology on Y induced by f or relative to f. Here, we say that (Y,τ_f) is a quotient space.

2.1.3 Some Standard Examples of surfaces and their definition using Quotient Topology

(A) Cylinder ::

We are going to construct a cylinder by gluing together two opposite edges of a rectangular sheet. We take a closed rectangle ABB' A' in the plane and identify the opposite edges AB and A'B', as shown in figure A. This means that:

- we imagine A and A' to be the same point (which we'll label A);
- we imagine B and B' to be the same point (which we'll label B);
- we match up all corresponding pairs of points in between (such as P and P') as shown.

The direction of the identification is given by the order of the letters labelling an edge: labelling an edge AB implies a direction from A to B; similarly A'B' implies a direction from A' to B'. Note that we have drawn arrowheads on the edges AB and A'B' of the rectangle to emphasise that they are to be identified in the same direction. We also obtain a cylinder if we identify the edges AA' and BB'. The below figure clarifies much more how a rectangular disk with the identification of edges gets transformed into a *Cylinder*. Distinctly it is **not a closed surface** as it contains boundary.

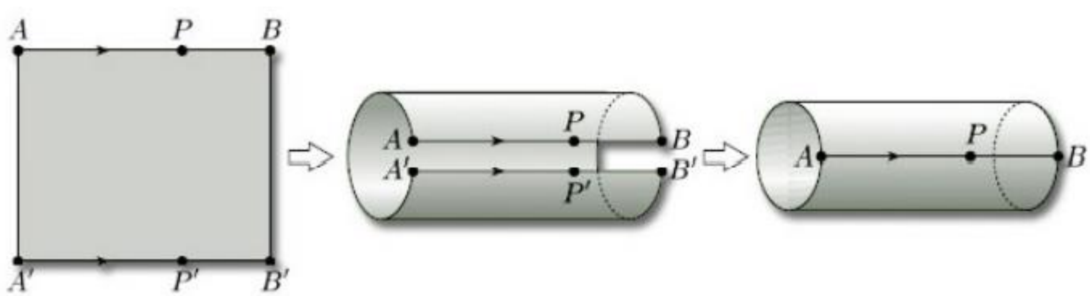


Figure A : Making a cylinder from a rectangle

(B) Möbius band ::

Let us now identify two opposite edges of a rectangle in opposing directions. We start with a rectangle, as before, but this time the edges to be identified have their arrowheads pointing in opposite directions. This means that we cannot glue the edges directly, but have to twist one of them before gluing. More formally, we take a closed rectangle ABA'B' in the plane and identify the opposite edges AB and A'B', as shown in Figure B. This means that:

- we imagine A and A' to be the same point;
- we imagine B and B' to be the same point;
- we pair up all corresponding points in between (such as P and P'), taking note of the directions of the arrowheads.

The below figure clarifies much more how a rectangular sheet with the identification of edges gets transformed into a *Möbius band*. Distinctly it is also **not a closed surface** as it contains boundary.

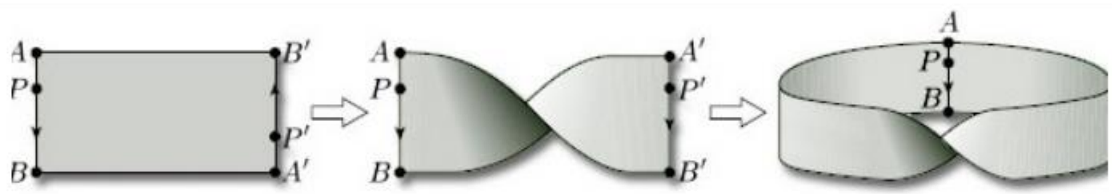


Figure B : Making a Möbius band from a rectangle

(C) Torus ::

Let us now return to the cylinder we obtained in Figure A. We now bend the cylinder round and glue its ends in such a way that the points A and B are identified. Furthermore, bending as shown in Figure C corresponds to identifying the edges AA' and BB' of the original rectangle in the same direction, as indicated in the figure. We obtain a *Torus*. It is an example of **closed surface.**

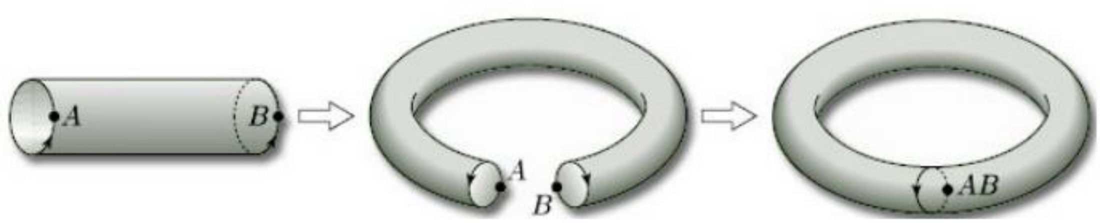


Figure C : Making a torus from a cylinder

We can also directly construct Torus by the identification of the opposite edges of a rectangle which is shown in below figure D.

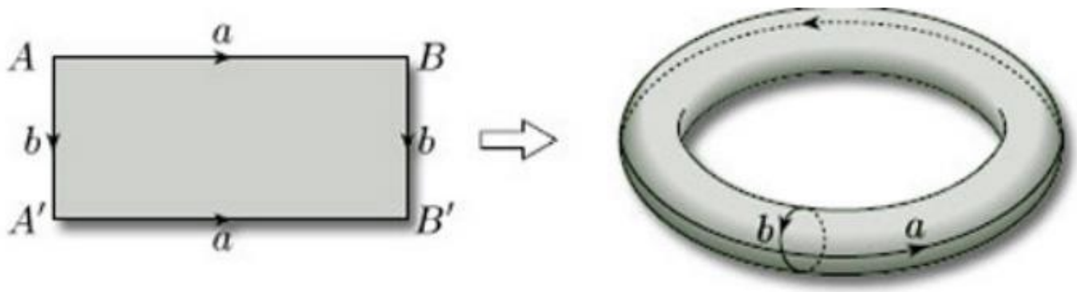


Figure D : Making a torus from a rectangle

(D) Sphere ::

Let us consider a unit closed disk and consider two distint points on the boundary as shown in the below figure E which gives rise two distinct edges. If we identify these two edges along the direction indicated in the figure E we will get a *Sphere*. Distinctly, it is a **closed surface** as it does not contain any boundary.

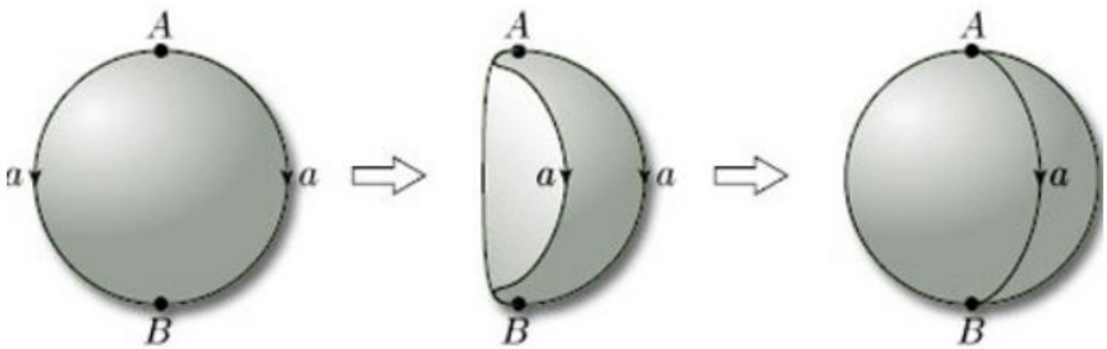


Figure E : Making a sphere from a disc

We can even start with more than one plane figure. For example, a sphere can be formed from two discs by identifying edges as shown in Figure F below.

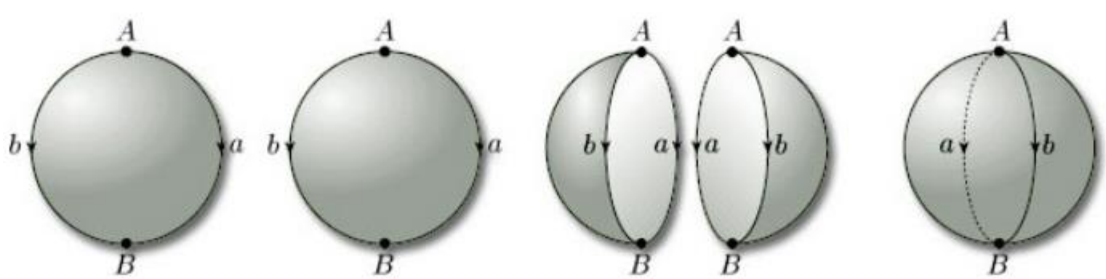


Figure F : Making a sphere from two discs

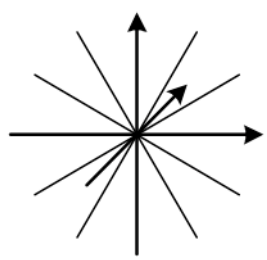
(E) <u>Real Projective Plane</u> ::

We give here three different definitions of *Real Projective Plane* ($\mathbb{R}P^2$) which also indicates it's construction using quotient topology. Though the definitions seems different but are equivalent definitions.

Before we give the definitions we consider : For any space X, if ρ be an equivalance relation on X and φ: $X \to X/\rho$ be the canonical map then $(X/\rho \, , \tau_\varphi)$ becomes the quotient space, where τ_φ is the quotient topology on X/ρ.

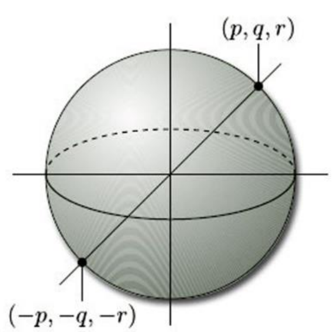
Now we produce the definitions below:

① Let us consider $\mathbb{R}^3_* = \mathbb{R}^3 - \{0\}$. We now define an equivalence relation ρ on $\mathbb{R}^3_*$ by $x\rho y$ iff $y = \lambda x$ for some $\lambda \in \mathbb{R} - \{0\}$ i.e. the equivalence class $\rho[x] = \{\, \lambda x : \lambda \in \mathbb{R} - \{0\} \,\}, \forall x \in \mathbb{R}^3_*$. Then the space $(\mathbb{R}^3_*/\rho \, , \tau_\varphi)$ becomes the surface *Real Projective Plane* ($\mathbb{R}P^2$).



Identifying an origin passing line in $\mathbb{R}^3_*$ gives $\mathbb{R}P^2$

② Let us consider the *unit sphere* S^2. Now if we define an equivalence relation ρ on S^2 by $x\rho y$ iff $y = -x$ holds i.e. the equivalence class $\rho[x] = \{\, x, -x \,\}, \forall x \in S^2$. Then the space $(S^2/\rho \, , \tau_\varphi)$ becomes the surface $\mathbb{R}P^2$.



Identifying all antipodal points of S^2 gives $\mathbb{R}P^2$

③ Now let us consider unit closed disk D^2. If we now define an equivalence relation ρ *on* D^2 by *:*

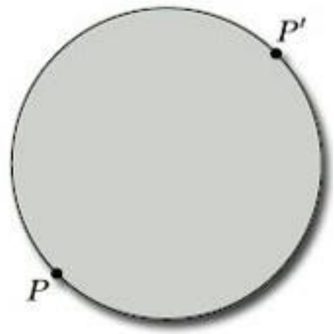
for $x, y \in \partial D^2$ $\qquad x\rho y$ iff $y = -x$.

for $x, y \in D^2 - \partial D^2$ $\qquad x\rho x$ only, where ∂D^2 is the boundary of the disk D^2.

i.e. the equivalence class $\rho[x] = \{ x, -x \}, \forall x \in \partial D^2$

& $\rho[x] = \{x\}, \forall x \in D^2 - \partial D^2$.

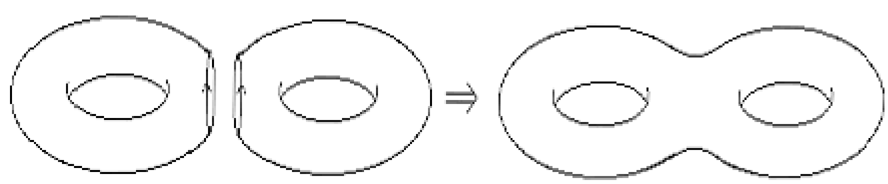
Then the space ($\mathbb{R}^3_*/\rho$, τ_φ) becomes the surface *Real Projective Plane* ($\mathbb{R}P^2$). Which is an example of **closed surface.**



Identifying all P & P' lying exactly opposite on boundary gives $\mathbb{R}P^2$

<u>Connected Sum</u> ::

Given two surfaces S_1 and S_2.The **connected sum** of S_1 and S_2 denoted by S_1 # S_2 is constructed by removing a disc from each and then joining them along the boundaries of the holes.



The connected sum of two torus

What classification means ?

There are infinitely many closed connected surfaces around us. So distinctly it is an impossible task to study all the surfaces individually. Classification Theorem gives us the opportunity to study all those surfaces having a look over some standard surfaces. We classify surfaces by finding a list of standard surfaces and proving that every surface is homeomorphic to one of those standard ones, that is homeomorphism is an equivalence relation on the set of all surfaces, and we list the equivalence classes. Like many results in topology the classification theorem has a remarkable simplicity for the following reason. Homeomorphic surfaces can be drastically different, that is every equivalence class contains huge number of surfaces and we have to study a very few of them, and the list is easy to handle.

2.1.4 Classification Theorem

Classification Theorem

Any closed connected surface is homeomorphic to exactly one of the following surfaces **:** *A sphere*; **or** *a finite connected sum of torus*; **or** *a finite connected sum of real projective planes*.

We are interested to give the proof of this theorem on the basis of graphical approach rather than Topological approach. On that approach we will consider only the edges of a two dimensional planer figure i.e. polygons and will combine the edges to construct a polyhedron. In this combination process we will think as if we are applying quotient topoloy to it. we will also see later that our construced polyhedron gives rise to a surface as we are thinking of after using quotient topology in constuction of that polyhedron.

Now we will elaborate about what the above theorem actullay says.

Discussion on the Classification Theorem **::**

In the statement of the Classification theorem we come across three surfaces : *Sphere* (S^2), *Torus* (π^2), *Real Projective Plane* ($\mathbb{R}P^2$). Among all these three, Sphere and Torus are orientable sufaces and Real projective plane is non-orientable. Finite sum of torus is also orientable and finite sum of real projective plane is non-orientable.

Hence while applying classification theorem in studying a closed surface if we see any surface is orientable then it must be homeomorphic to either sphere or finite connected sum of torus. Again if we see the surface is non-orientable then it must be homeomorphic to finite connected sum of real projective planes.

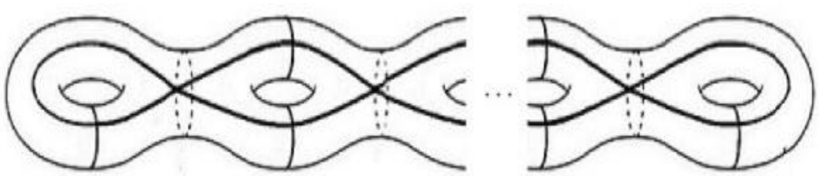
That is for non-orientable surface it is $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots (k - sums)$, k is an positive integer.

And for orientable surface it is :

OR **OR** **OR** ...

$\boldsymbol{S^2}$ $\boldsymbol{\pi^2}$ $\boldsymbol{\pi^2 \# \pi^2}$

OR ...



$\boldsymbol{\pi^2 \# \pi^2 \# \pi^2 \# \ldots (k - sums)}$ **.**

Now we produce the proof of classification Theorem in the below sections.

3.1 Graphical Approach

3.1.1 Polygons and Polyhedra

We start by giving some well-known definitions.

Definition 1. The homeomorphic image of a circular disc in three-dimensional space is called a 2-cell or country.
Similarly homeomorphic image of an edge is called a 1-cell and a point is a 0-cell.

Example 1. The interior of a triangle is a homeomorphic image of a circular disc. Hence it is a 2-cell.

Definition 2. A 2-cell which has its circumference divided into r number of arcs and r number of vertices is called a polygon with r sides or simply a r-gon. The arcs are called the sides of a polygon.

Example 2. Square is a 4-gon. Triangle is a 3-gon.

Before we give a definition of a polyhedron, let us look into an example which will help us to understand the procedure we are going to follow to have a construction of a polyhedron. Let us look into the following scheme.

e	d	f
a	c^{-1}	e^{-1}
f^{-1}	b	a^{-1}
c	b^{-1}	d^{-1}

Now we look into the following diagram. After seeing the diagram of figure 3.1, the scheme

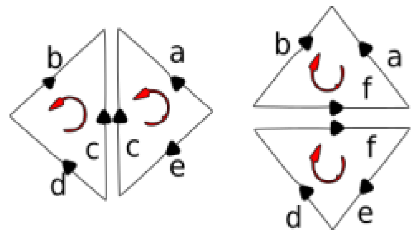


Figure 3.1

above is now understandable. In the scheme, we have a row corresponding to each triangle. Each row consists the name of the edges read along the orientation given in a triangle(denoted by the red arrow). If the arrow in some edge points in the opposite direction of the orientation of the triangle(i.e if the direction of the black arrow is opposite to the direction of the red arrow), then we introduce -1 as index.

With this example in mind, we proceed to give the definition(or the construction) of a Polyhedron.

• Suppose we are given a finite number of polygons such that the total number of sides in the polygons is even.

• We now collect the sides as pairs and denote each pair with a single letter.

• Also we assume that each side has some orientation. We denote this orientation using an arrow on the sides.

• Now we identify each pair of side such that the arrows in the sides points to the same direction just before identification(Here we use Quotient Topology to create some object after identification).

• We call two superimposed sides an edge.

• If after identifying all the pairs of sides we get some connected pseudograph, then we call this a Polyhedron. We give the usual Quotient Topology on it.

Definition 3. Polygons used in the construction of a polyhedron are called *Faces* of the polyhedron.

Definition 4. The information we need to construct a polyhedron such as the polygons,the pairing of sides, the orientation of the sides — are sufficient to identify a polyhedron. We call these informations as the *Plane Representation of the Polyhedron*.

Before giving a compact way to describe a plane representation, we give an important example.

Example 3. Here we have a 2-gon which have only one pair of sides that we denote by 'a'. If we identify the pairs, intitutively it is clear that we will get a polyhedron which looks like "a purse" and is homeomorphic to a sphere.

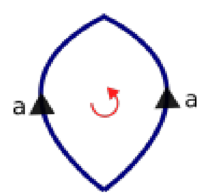


Figure 3.2: Polyhedron homeomorphic to sphere

In figure 3.1, clearly we have a construction of a polyhedron. The scheme we constructed above has all the information regarding the plane representation of the figure 3.1 polyhedron. Thus we propose to make a scheme to present a plane representation. Although we discussed about constructing the scheme in figure 3.1, let us give the proper procedure to construct the scheme corresponding to a plane representation.

• For each polygon we give an orientation to it. (Note that to construct a polyhedron, we do not need any orientation to polygons. Having an orientation in the edges is sufficient.)

• Write down letters of the sides in cyclic manner along the orientation of the polygon.

• If a side ’a’ has a reverse orientation to the orientation of the polygon, then we introduce ’-1’ as the exponent of the letter.

• We use one row of the scheme to describe a polygon.

Example 4. The polyhedron in figure 3.2 has only one polygon. So we only have one row in the scheme. We give an orientation(denoted by the red arrow) to the only polygon. Then the scheme will be:

a	a^{-1}

Now we give some properties of the schemes.

1. Each letter appears exactly twice in the scheme.

2. The set of rows cannot be seperated in disjoint components such that each component satisfy property one. *Otherwise the polyhedron will be disconnected*.

3. We have some operations on scheme applying which the scheme do get changed but the underlying polyhedron remains unchanged.

    (a) Give a cyclic permutation on a row.

    (b) Change exponent of a letter in its both appearance.

    (c) In a row, change all the exponents and reverse the order.

For example consider the scheme in figure 3.1. By applying (a) in the first row, (b) in the letter ’a’ and (c) in the last row, we will have a new scheme which we give below. It is easy to check that the polyhedrons corresponding to both schemes are same.

f	e	d
a^{-1}	c^{-1}	e^{-1}
f^{-1}	b	a
d	b	c^{-1}

3.1.2 Elementary Operations

In this section we introduce some new operations on the scheme which will change the scheme as well as the underlying polyhedron. But they will not change the topological properties i.e upto homeomorphism things will remain unchanged.

(A) <u>(SU1) Subdivision of dimension one</u>

*"An edge **a** is divided into two edges **b** and **c** by introducing a new vertex on **a**"*— this operation on an edge is called Subdivision of dimension one or simply SU1. In the scheme we replace **a** by **bc** and replace $\boldsymbol{a^{-1}}$ by $\boldsymbol{c^{-1}b^{-1}}$.

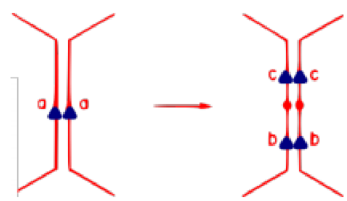


Figure 3.3: Subdivision of Dimension One

(B) <u>(CO1)Composition of Dimension One</u>

This is the reverse process of SU1. In the scheme we replace **ab** and $\boldsymbol{b^{-1}a^{-1}}$ by **c** and $\boldsymbol{c^{-1}}$ respectively.

(C) <u>(SU2)Subdivision of Dimension Two</u>

"Join two vertices in a polygon by a new edge so that the polygon gets divided into two polygons"—this operation on a polygon is called Subdivision of dimension two or simply SU2. In the scheme if we have a row ***b . . . c d . . . a*** then we replace this row by,

$$\boldsymbol{b \ldots c\, k^{-1}}$$

$$\boldsymbol{k\, d \ldots a}$$

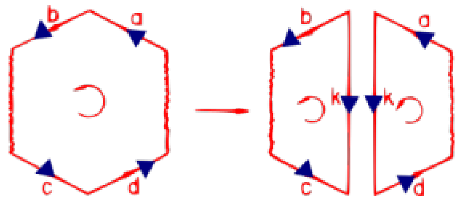


Figure 3.4: Subdivision of Dimension Two

(D) <u>(CO2)Composition of Dimension Two</u>

This is the reverse process of SU2. In scheme if we have,

$$\boldsymbol{a \ldots b\, k^{-1}}$$

$$\boldsymbol{k\, c \ldots d}$$

Then we can replace those two rows by the single row ***a . . . b c . . . d*** Now we introduce an equivalence relation on the set of all polyhedrons.

3.1.3 Elementary Relation

Two polyhedra P and P’ are called *elementarily related* if P can be obtained from P’ by applying a finite sequence of elementary operations on P’.

Definition 5. We call a polyhedron *orientable* if in the scheme of the polyhedron each letter appears in both exponents(One can apply operation (c) in the scheme to get the scheme in proper form).

Remark. It is possible for an orientable polyhedron to have a scheme in which some letter do not have both of the exponents. For example we gave two schemes for the polyhedron in figure 3.1. In the second option letter ‘b’ do not have both exponents. But we cannot conclude that the polyhedron is not orientable. Because if we see the first scheme, it has all letters appearing in both exponents. Hence the polyhedron is orientable. The polyhedron in figure 3.2(which is homeomorphic to sphere) is orientable.

Definition 6. A polyhedron which is not orientable is called *Non-orientable* polyhedron.

3.1.4 Euler Characteristic

We denote the number of vertices, the number of edges and the number of faces of a polyhedron as α_0, α_1 and α_2 respectively. The number of vertices and edges must be counted after identification. For example in figure 3.2, α_0 = 2, α_1= 1, α_2 = 1.

Definition 7. For a polyhedron P, we consider the alternative sum

$E(P) = \alpha_0 - \alpha_1 + \alpha_2$

We define this as the *Euler Characteristic* of the polyhedron P.

Theorem 1

Two elementarily related Polyhedra have same Euler Characteristic.

Proof. The proof is simple once we observe that the Euler Characteristic is invariant under Elementary Operations.
After applying SU1, α_0 increases by 1, α_1 increases by 1 and α_2 remains unchanged. Hence Euler Characteristic remains unchanged under SU1.
Similarly after applying SU2, α_0 remains unchanged, α_1 increases by 1, α_2 increases by 1. Hence Euler Characteristic is invariant under SU2.
This completes the proof as CO1 and CO2 are just reverse process of SU1 and SU2.

Also note that the elementary operations do not effect the orientation of the polygons. Thus we have proved the following theorem.

Theorem 2

Two elementarily related polyhedra have same Euler Characteristic and are both orientable or both non-orientable.

In subsequent sections we prove that the converse of this theorem is also true.

3.1.5 Normal Form of Orientable Surfaces

We have broken apart the set of all polyhedra into disjoint classes in such a way that two polyhedra belong to the same class if and only if they are elementarily related. Now for each class we want to find the simplest polyhedron which will represent the whole class. We call this polyhedron as the **normal form** of the class. We take a number of steps to deduce the normal form of a certain polyhedron from its scheme.

3.1.6 Producing One Row

If the polyhedron has more than one polygon(i.e more than one row in the scheme), we can apply CO2 a finite number of times so that we have only one polygon (i.e only one row in the scheme). So from now we only consider the polyhedra which have one row in the scheme.

3.1.7 Simple Normalaization

Let $Paa^{-1}Q$ be the row in the scheme and assume P and Q are not both empty(Here P and Q are combinations of letters). Then we apply the following combination of elementary operations.

$$Paa^{-1}Q \to \left\{ \begin{array}{l} Pak \\ \\ k^{-1}a^{-1}Q \end{array} \right. \to \left\{ \begin{array}{l} Pc \\ \\ c^{-1}\,Q \end{array} \right. \to PQ$$

So after repetitive use of simple normalization, we will either have the one row as aa^{-1} (which is sphere as in figure 3.2) or we will have the row in such a way that the row do not contain the combination aa^{-1} . In the first case aa^{-1} is the normal form. In the second case we use Handle Normalization. First we claim that, for second case there are letters a, b such that they appear in the following form:

$$\ldots a \ldots b \ldots a^{-1} \ldots b^{-1}$$

The row of the scheme has the form $aPa^{-1}Q$ where P is non-empty otherwise simple normalization will be possible which contradicts the case 2. Now we choose *a* in such a way that P is minimal. Then take any letter *b* in P. Then b^{-1} must belong to Q as P is minimal. Hence our claim is true.

3.1.8 Handle Normalization

By our previous claim the polyhedron in the second case must have the only row as: $PaQbRa^{-1}Sb^{-1}T$. The Handle normalization goes like this:

$$PaQbRa^{-1}Sb^{-1}T \rightarrow \begin{cases} PaQbRc \\ c^{-1}a^{-1}Sb^{-1}T \end{cases} \rightarrow \begin{cases} QbRcPa \\ a^{-1}Sb^{-1}Tc^{-1} \end{cases}$$

$$\rightarrow QbRcPSb^{-1}T\,c^{-1} \rightarrow c^{-1}QbRcPSb^{-1}T \rightarrow \begin{cases} c^{-1}QbRck \\ k^{-1}PSb^{-1}T \end{cases}$$

$$\rightarrow \begin{cases} Rckc^{-1}Qb \\ b^{-1}T\,k^{-1}PS \end{cases} \rightarrow Rckc^{-1}QT\,k^{-1}PS \rightarrow c^{-1}QT\,k^{-1}PSRck$$

$$\rightarrow \begin{cases} c^{-1}QTj \\ j^{-1}k^{-1}PSRck \end{cases} \rightarrow \begin{cases} QT\,jc^{-1} \\ ckj^{-1}k^{-1}PSR \end{cases} \rightarrow QTjkj^{-1}\,k^{-1}PSR$$

$$\rightarrow PSRQTjkj^{-1}k^{-1} \rightarrow PSRQTaba^{-1}\,b^{-1}$$

Now we apply simple normalization or handle normalization whenever applicable to $PSRQT$. After applying several times, we will have the form:

$$a_1b_1a_1^{-1}b_1^{-1}\,a_2b_2a_2^{-1}\,b_2^{-1}\ldots a_pb_pa_p^{-1}\,b_p^{-1} \tag{1.1}$$

We call this form as the *normal form of Orientable surfaces*. Euler Characteristic of aa^{-1} is 2. We denote the polyhedra of this class as S_0. Euler Characteristic of polyhedra with normal form (1.1) is 2 − 2p. We denote the polyhedra of this class as S_p.

Definition 8. We define the number p of the polyhedron S_p as the Genus of the respective class.

3.1.9 Normal Form of Non-Orientable Surfaces

Suppose we are given a non-orientable Polyhedron. Firstly we apply Simple Normalization as often as possible. After that we apply CO2 repetitively so that the scheme will have only one row. We also use the notation $P^{-1} = d^{-1}c^{-1}b^{-1}a^{-1}$ for any combination of letters $P = abcd$.

3.1.10 Cross-Cap Normalization

The only row of the scheme must look like $PcQcR$. Then Cross-Cap Normalization goes like this:

$$PcQcR \rightarrow \begin{cases} Pck^{-1} \\ \\ kQcR \end{cases} \rightarrow \begin{cases} P^{-1}kc^{-1} \\ \\ cRkQ \end{cases} \rightarrow P^{-1}kRkQ \rightarrow kRkQP^{-1}$$

$$\rightarrow \begin{cases} kRc \\ \\ c^{-1}kQP^{-1} \end{cases} \rightarrow \begin{cases} Rck \\ \\ k^{-1}cP\,Q^{-1} \end{cases} \rightarrow RccPQ^{-1} \rightarrow PQ^{-1}Rcc$$

3.1.11 Transformation of One Handle into Two Cross Caps

Suppose the only row looks like this: $Rccaba^{-1}b^{-1}$. Then:

$$Rccaba^{-1}b^{-1} \rightarrow a^{-1}b^{-1}Rccab \rightarrow \begin{cases} a^{-1}\,b^{-1}Rck \\ \\ k^{-1}\,cab \end{cases}$$

$$\to \left\{ \begin{array}{l} Rcka^{-1}b^{-1} \\ \\ bk^{-1}\,ca \end{array} \right. \to Rcka^{-1}\,k^{-1}ca \xrightarrow{\text{cross-cap}} Rkak^{-1}acc \xrightarrow{\text{cross-cap}} Rkkccaa$$

Hence by repititve use of *cross-cap* and *transformation of one handle*, we have the normal form as:

$$c_1c_1c_2c_2\ldots c_qc_q \qquad (1.2)$$

Definition 9. We define the polyhedron in 1.2 as the Non-Orientable Polyhedron of genus q and denote by N_q.

Hence we found that any polyhedron is elementarily related to one of the form:

Notation	Normal Form	Euler Characteristic	Orientability
S_0	aa^{-1}	2	YES
S_p	$a_1b_1a_1^{-1}b_1^{-1}\ a_2b_2a_2^{-1}\ b_2^{-1}\ldots a_pb_pa_p^{-1}\ b_p^{-1}$	$2-2p$	YES
N_q	$c_1c_1c_2c_2\ldots c_qc_q$	$2-q$	NO

3.1.12 Conclusion

Hence we are ready to give a short proof of the converse of Theorem 2.

Theorem 2

Two polyhedra are elementarily related if and only if both have same Euler Characteristic and both are either orientable or non-orientable.

Proof. Theorem 1 shows that the condition is necessary.

To proof that the condition is sufficient consider two orientable polyhera P and P' with same Euler Characteristic. But there is only one class of polyhedron for a given value as Euler Characteristic. Hence P and P' are elementarily related. Similarly we can proof the same for non-orientable case also.

Definition 10. Each class of polyhedron is defined as *Closed Surface*. The class S_0 is called as Sphere and the class S_1 is called as Torus.

3.1.13 Generating higher genus surfaces from Sphere by attaching handles

Consider the Polyhedron in figure 3.2. This is sphere. Now apply SU1 to get figure 3.3. We now have four extra edges namely x,y,z,w. We take two new polygons(in grey color) which has the edges x,y,z,w.

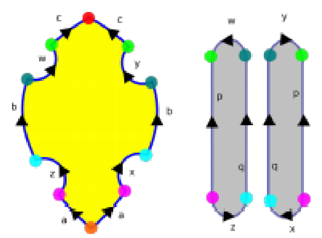


Figure 3.5: Attaching a handle to sphere

After identification(along the arrows), we get a polyhedron whose Euler Charateristic is 0. Hence the polyhedron is Torus. If we identify the edges p,q of the grey polygons, then we get a cylinder like shape. That is why this process is called as attaching a handle. It is very easy to see that by attaching p many handles to S_0, we obtain the surface S_P.

Conclution to the Classification Theorem ::

In Differential Topology, one defines 'Closed Surface' as 2-Dimesional Manifolds without boundary which is already been defined in topological section. We have seen before, the Theorem which *says 'Every Closed Surface admits a triangulation'* i.e any closed surface is homeomorphic to some Polyhedron which has triangular faces. For a elegent proof of this theorem, one can refer to [3].

In this way we proved that any closed surface is homeomorphic to some Polyhedron and hence the Definition 10 of Closed Surface and all other definitions like orientibility of a surface also agrees with the topological version of it. So elementary relation and homeomorphism becomes an equivalent process. Thus permissibility of triangulation of surface becomes a door way between Topology and Graph Theory which enables us to think a surface with the help of some polygons. After triangulating sphere to some number of polygons and identifying edges its normal form becomes aa^{-1} i.e. normal form of S_0. So sphere is in the class S_0. Similarly after triangulating p number of connected sum of torus and q number of connected sum of real projective plane we get the normal form of S_p and $N_q$respectively. Therefore we get one member from each class of S_0, S_p, N_q. Thus classification happens.

This proves our desired **Classification Theorem**.

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