Stochastic Control Systems

Linear Quadratic Gaussian (LQG) Design of an Aircraft Lateral Control System

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Abstract

This project presents the design of a Linear Quadratic Gaussian (LQG) controller for the lateral dynamics of an aircraft. The controller is designed to achieve robust dynamic performance in the presence of process and measurement noise. All computations, including the determination of state feedback and Kalman filter gains, were carried out using MATLAB. According to the separation principle, the LQG regulator design procedure is accomplished in two stages: first designing the Kalman filter and then designing the control feedback.

Introduction

Control Objective

The lateral tracking control system, shown in Figure 1, provides coordinated turns by commanding the bank angle $\phi(t)$ to follow a desired reference while maintaining the sideslip angle $\beta(t)$ at zero. The system has two control channels with input

$$u = \begin{bmatrix} u_\phi \\ u_\beta \end{bmatrix},$$

and reference

$$r = \begin{bmatrix} r_\phi \\ r_\beta \end{bmatrix}.$$

The tracking error is

$$e = \begin{bmatrix} e_\phi \\ e_\beta \end{bmatrix} = \begin{bmatrix} r_\phi - \phi \\ r_\beta - \beta \end{bmatrix}.$$

Negative feedback is applied according to standard LQG design principles.

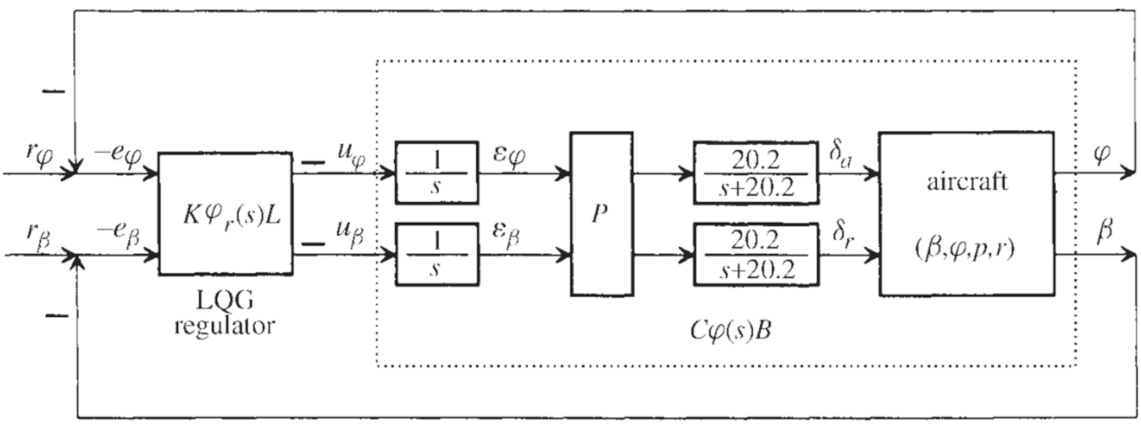

Figure 1: Aircraft turn coordinator control system [2]

Aircraft Model

To obtain the basic aircraft dynamics, the nonlinear model of the aircraft was linearized around a nominal flight condition: a velocity of $VT = 502$ ft/s, an altitude of 0 ft, a dynamic pressure of 300 psf, and a center of gravity (cg) at 0.35c. The state variables

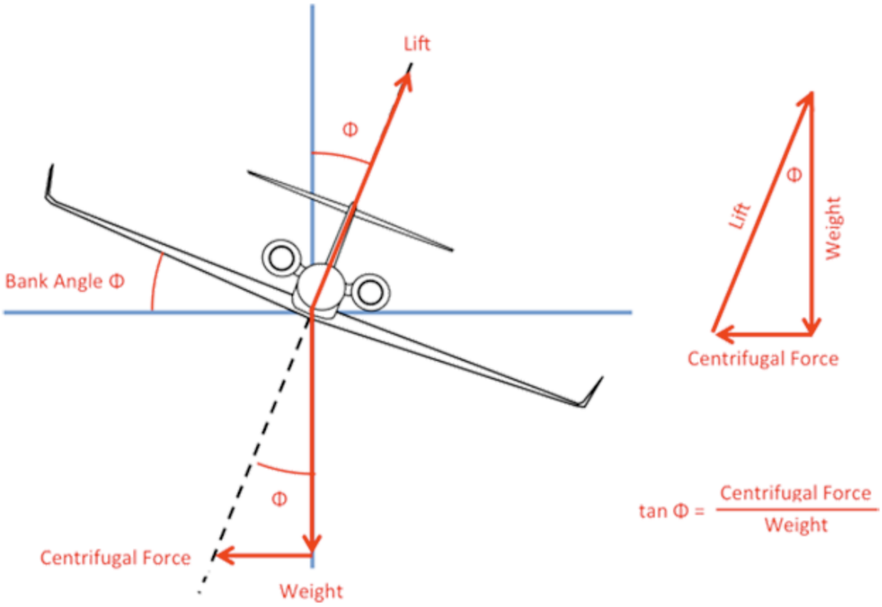

Figure 2: Bank angle response

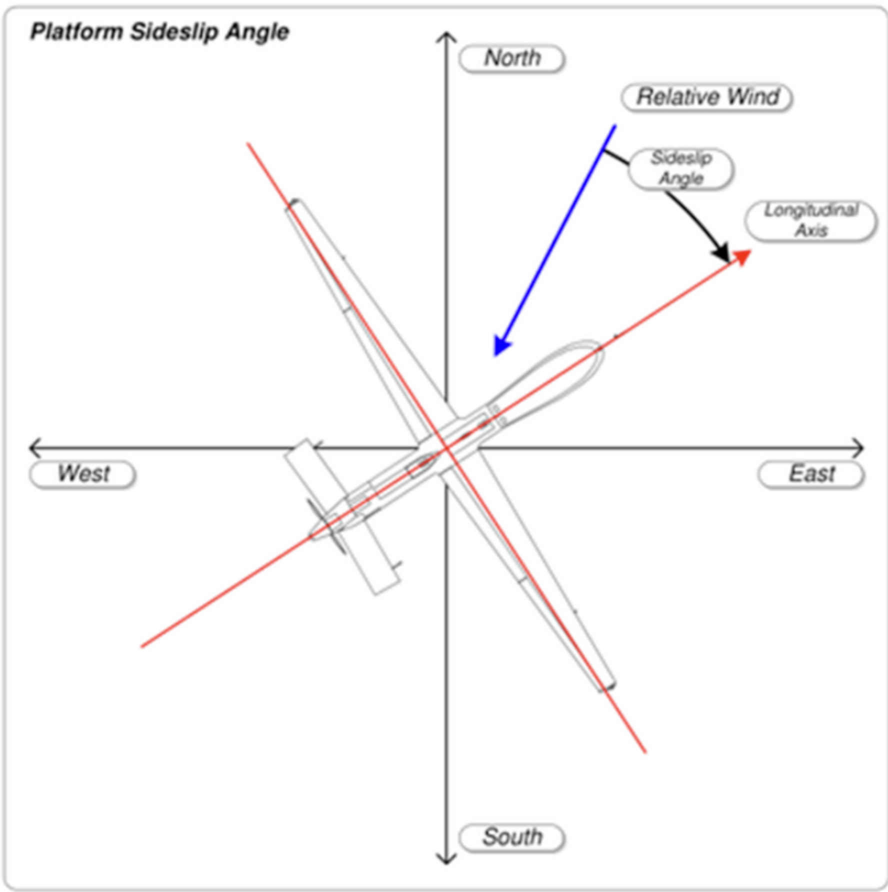

Figure 3: Sideslip angle response

retained from this linearization are the sideslip angle β, bank angle ϕ, roll rate p, and yaw rate r.

Additional states, δ_a and δ_r, are introduced to model the aileron and rudder actuators. Both actuators are approximated by the transfer function $20.2/(s + 20.2)$. The aileron deflection is δ_a, and the rudder deflection is δ_r. Finally, the dynamics are augmented by integrators in each control channel, with the integrator outputs denoted by ϵ_ϕ and ϵ_β.

The resulting state, input, and output vectors are defined as:

$$x = \begin{bmatrix} \beta & \phi & p & r & \delta_a & \delta_r & \epsilon_\phi & \epsilon_\beta \end{bmatrix}^T, \quad u = \begin{bmatrix} u_\phi \\ u_\beta \end{bmatrix}, \quad y = \begin{bmatrix} \phi \\ \beta \end{bmatrix}$$

The state-space matrices A, B, and C are given below:

$$A = \begin{bmatrix} -0.3220 & 0.0640 & 0.0364 & -0.9917 & 0.0003 & 0.0008 & 0 & 0 \\ 0 & 0 & 1 & 0.0037 & 0 & 0 & 0 & 0 \\ -30.6492 & 0 & -3.6784 & 0.6646 & -0.7333 & 0.1315 & 0 & 0 \\ 8.5395 & 0 & -0.0254 & -0.4764 & -0.0319 & -0.0620 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20.2 & 0 & -0.01 & -5.47 \\ 0 & 0 & 0 & 0 & 0 & -20.2 & -0.168 & 51.71 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T, \quad C = \begin{bmatrix} 0 & 57.2958 & 0 & 0 & 0 & 0 & 0 & 0 \\ 57.2958 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The error vector $e(t)$ is defined as the difference between the reference signals $(r_\phi(t), r_\beta(t))$ and the state variables $(\phi(t), \beta(t))$:

$$e(t) = \begin{bmatrix} e_\phi(t) \\ e_\beta(t) \end{bmatrix} = \begin{bmatrix} r_\phi(t) - \phi(t) \\ r_\beta(t) - \beta(t) \end{bmatrix}$$

The complete linear time-invariant (LTI) state-space model, including process noise w and measurement noise v, is:

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ y &= Cx + v \end{aligned}$$

Target Feedback Loop Design

To recover the target loop transfer function, the Kalman filter is designed first. For the case where the process noise and measurement noise are correlated, the Kalman filter equations are driven by a modified Kalman gain. The noise covariance is defined as:

$$E \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w^T(k) & v^T(k) \end{bmatrix} = E \begin{bmatrix} w(k)w^T(k) & w(k)v^T(k) \\ v(k)w^T(k) & v(k)v^T(k) \end{bmatrix} = \begin{bmatrix} Q_n & N_n \\ N_n^T & R_n \end{bmatrix}$$

The modified Kalman gain L and the state error covariance matrix differential equation are given by:

$$\begin{aligned} L &= (PC^T + GN^T)R_n^{-1} \\ \dot{P} &= AP + PA^T + GQ_nG^T - LR_nL^T \end{aligned}$$

The Kalman filter design equations were solved using MATLAB with the following process noise covariance matrix Q_n:

$$Q_n = \mathrm{diag}([0.1, 0.1, 0.1, 0.1, 0.1, 0, 0, 1, 1])$$

Subsequently, the Linear Quadratic Regulator (LQR) design problem was also solved using MATLAB. The weighting matrices were set to $Q = C'C$ and $R = \rho^2 I$. Different feedback gains, K, were obtained by varying the values of $r_\epsilon = \rho^2$. A close match with the target loop gain was achieved for $r_\epsilon = 10^{-11}$.

Controller Design

The controller design is based on the following standard LQR formulation. The cost function to be minimized is:

$$J(t_0) = \frac{1}{2}E\left[x^T(t_0)S(t_0)x(t_0)\right] + \frac{1}{2}E\left[\int \left\|R^{-1}B^T Sx + u\right\|^2 dt\right] + \frac{1}{2}\text{trace}\int SGQG^T dt$$

This simplifies to the deterministic cost function:

$$\implies J^*(t_0) = \frac{1}{2}E\left[x^T(t_0)S(t_0)x(t_0)\right] + \frac{1}{2}\text{trace}\int SGQG^T dt$$

From this, the optimal feedback gain K and the control input $u(t)$ are derived:

$$\implies K = R^{-1}B^T S(t) \implies u(t) = -K(t)\hat{x}(t)$$

Here, S is the solution to the continuous-time algebraic Riccati equation:

$$\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q$$

Two methods are employed to select the Kalman filter gain (L) and the optimal controller gain (K). These methods are detailed below.

Method 1: Steady-State Gains

This method utilizes the steady-state feedback gain, K_∞, and the steady-state Kalman gain, L_∞. These are computed by solving the continuous-time Algebraic Riccati Equation (ARE), which is the case where the differential Riccati equation equals zero. This approach is suitable for linear time-invariant (LTI) systems operating over an infinite horizon.

The steady-state solutions are found by solving the following equations:

$$AP + PA^T + GQ_nG^T - LR_nL^T = 0 \implies L_\infty = (P_\infty C^T + GN^T)R_n^{-1}$$
$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \implies K_\infty = R^{-1}B^T S_\infty$$

MATLAB Implementation The `icare` function is used to solve the ARE and find the constant, steady-state gains.

```
% Calculate steady-state Kalman filter gain L
[p,L,~] = icare(A',C',G*Qp*G',Rp,[],[],[]);
L = L';

% Calculate steady-state LQR controller gain k
[s,k,~] = icare(A,B,Q,R,[],[],[]);
```

Method 2: Time-Varying Gains (Online Calculation)

This method involves solving the Differential Riccati Equation (DRE) online to calculate the time-varying gains $K(t)$ and $L(t)$. This is optimal for finite-horizon problems or systems where dynamics may change over time.

MATLAB Implementation An ordinary differential equation (ODE) solver, such as `ode23`, is used to integrate the DRE over discrete time steps.

```matlab
% --- Kalman Filter Gain L(t) ---
% Reshape the covariance matrix p for the ODE solver
pinit = reshape(p, [1, 64]);

% Integrate the DRE for p over one time step
[~, pp] = ode23(@pfunc, [t t+step_size], pinit);

% Reshape the result back into an 8x8 matrix
p = reshape(pp(end,:), [8, 8]);

% Calculate the time-varying Kalman gain L
L = (p*C' + G*N') * inv(Rp);

% --- LQR Controller Gain k(t) ---
% Reshape the Riccati solution matrix s for the ODE solver
sinit = reshape(s, [1, 64]);

% Integrate the DRE for s over one time step
[~, ss] = ode23(@sfunc, [t t+step_size], sinit);

% Reshape the result back into an 8x8 matrix
s = reshape(ss(end,:), [8, 8]);

% Calculate the time-varying controller gain k
k = inv(R) * B' * s;
```

While the first method is simpler to implement, the second method was chosen for the final implementation as it provides an optimal solution by calculating time-varying gains.

Simulation and Results

The closed-loop performance of the LQG controller was evaluated using numerical simulations. The main performance metrics include the bank angle response, sideslip angle response, and control surface deflections.

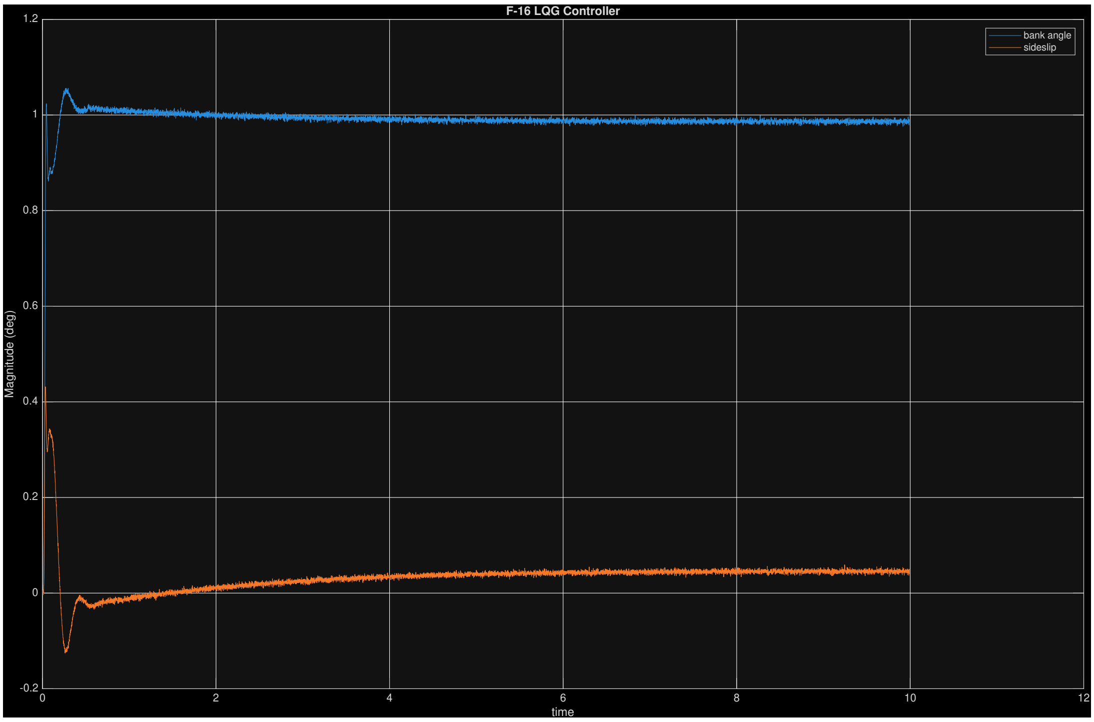

Figure 4: Closed-loop step responses of the LQG regulator

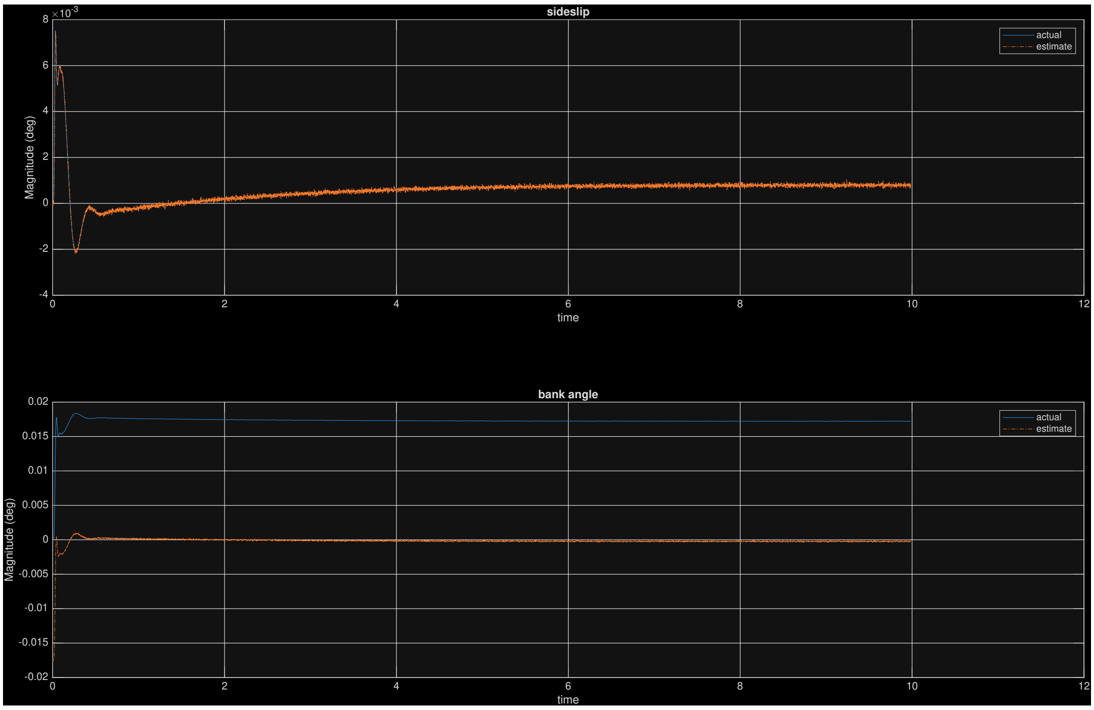

Figure 5: Kalman filter state estimates

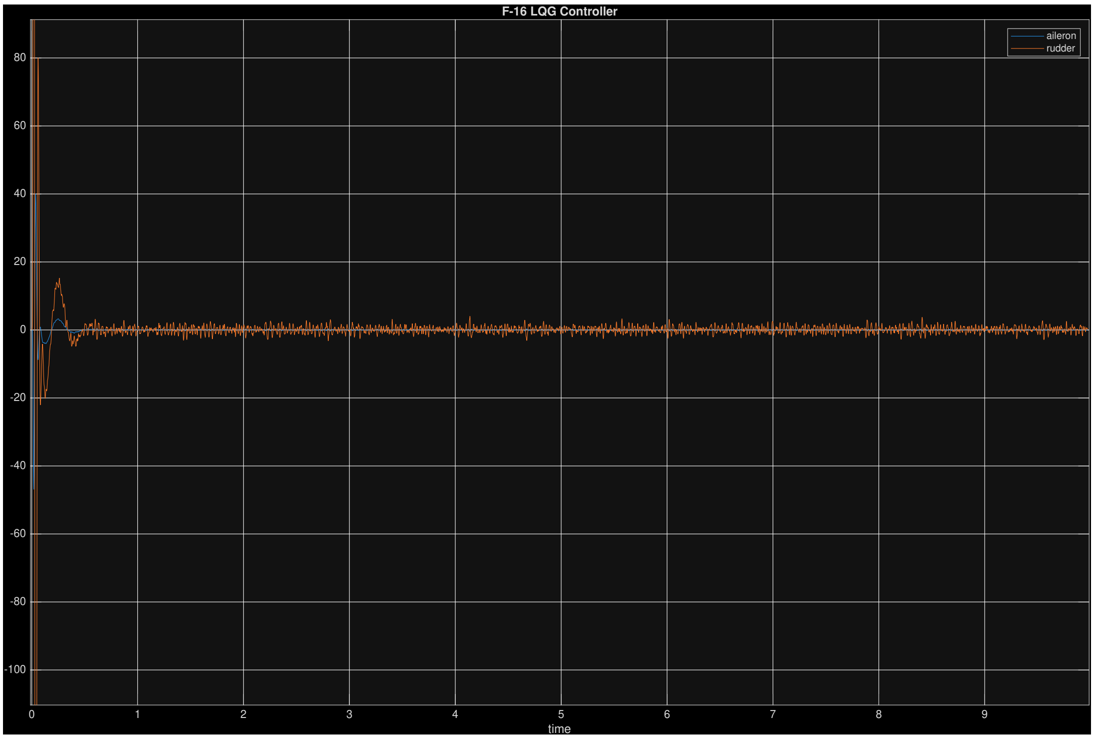

Figure 6: Aileron and rudder deflections

References

1. Lewis, Frank L., Lihua Xie, and Dan Popa. *Optimal and robust estimation: with an introduction to stochastic control theory.* CRC Press, 2017.

2. Stevens, Brian L., Frank L. Lewis, and Eric N. Johnson. *Aircraft control and simulation: dynamics, controls design, and autonomous systems.* John Wiley & Sons, 2015.