Schmidt Decomposition:

$$\psi = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}} + \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}}$$

orthogonal

$$|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 =: \mathcal{H}$$

$$\left.\begin{array}{l} |u\rangle \in \mathcal{H}_1 \\ |v\rangle \in \mathcal{H}_2 \end{array}\right\} |u\rangle \otimes |v\rangle \in \mathcal{H}$$

$$\langle u|u\rangle = \langle v|v\rangle = 1$$

$$M := \langle u \otimes v | \psi \rangle \qquad 0 \leq |M|^2 \leq 1$$

$$\exists\, u, v : \quad |M|^2 \to \max$$

$$|u'\rangle \in \mathcal{H}_1, \quad \langle u'|u\rangle = 0$$

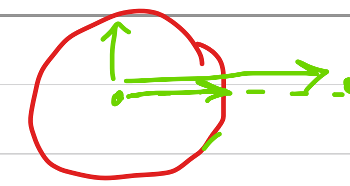
$$\| u + \epsilon u' \|^2 = \underbrace{\|u\|^2}_{1} + \epsilon^2 \underbrace{\|u'\|^2}_{1} + \cancel{2\mathrm{Re}\{\epsilon \langle u|u'\rangle\}} = 1 + \mathcal{O}(\epsilon^2)$$

$$\langle (u + \epsilon u') \otimes v | \psi \rangle = \underbrace{\langle u \otimes v | \psi \rangle}_{M} + \epsilon \langle u' \otimes v | \psi \rangle$$

$$|\langle (u + \epsilon u') \otimes v | \psi \rangle|^2 = |M|^2 + + \underbrace{2\mathrm{Re}\{ M \epsilon \langle u' \otimes v | \psi \rangle \}}_{0} + \mathcal{O}(\epsilon^2)$$

$|M|^2$

Phase of v : $\langle u' \otimes v | \psi \rangle = 0$



$$|\psi'\rangle = |\psi\rangle - M \, |u\rangle \otimes |v\rangle \Rightarrow \langle u \otimes v | \psi \rangle = 0$$

$$|\psi'\rangle \in \mathcal{H}'_u \otimes \mathcal{H}'_v \subset \mathcal{H}_1 \otimes \mathcal{H}_2$$

$\mathcal{H}'_u = \{ u' \mid \langle u' | u \rangle = 0 \}$

$\mathcal{H}'_v = \{ v' \mid \langle v' | v \rangle = 0 \}$

$$|\psi\rangle \to |\psi'\rangle \to |\psi''\rangle \to \cdots \to 0$$

$$|\psi\rangle - \sum_j M_j \, |u_j\rangle \otimes |v_j\rangle = 0 \longrightarrow \boxed{|\psi\rangle = \sum_j M_j \, |u_j\rangle \otimes |v_j\rangle}$$

$$|\psi\rangle = \sum_{i,j} a_{ij} \, |e_i\rangle \otimes |f_j\rangle$$

$$a_{ij} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

a_{ij} : $\underbrace{|e_i\rangle}\otimes\underbrace{|f_j\rangle}$

a_{ji} : $|e_j\rangle \otimes |f_i\rangle$

SVD : Singular Value decomposition

$$\hat{A} = \hat{U} \, \underset{\text{diagonal}}{\underset{\Downarrow}{\hat{\sigma}}} \, V^\dagger$$

$U^\dagger = U^{-1}, \; V^\dagger = V^{-1}$

$$AA^\dagger = U \sigma \underbrace{V^\dagger V}_{\mathbb{1}} \overset{\sigma}{\overset{\|}{\sigma^\dagger}} U^\dagger$$

$$= U \, \sigma^2 \, U^\dagger$$

$$|\psi\rangle = \sum_{i,j} a_{ij} \, |e_i\rangle \otimes |f_j\rangle = \sum_{i,j} (U \sigma V^\dagger)_{ij} \, |e_i\rangle \otimes |f_j\rangle$$

$|u_k\rangle := \sum_i U_{ik} |e_i\rangle$, $|v_k\rangle := \sum_j V_{jk} |f_j\rangle$

$$|\psi\rangle = \sum_k \sigma_k |u_k\rangle \otimes |v_k\rangle$$

$$|\psi\rangle = \sum_j M_j \, |u_j\rangle \otimes |v_j\rangle$$

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,j} M_i M_j^* \, |u_i\rangle\langle u_j| \otimes |v_i\rangle\langle v_j|$$

$$\rho_1 = \mathrm{Tr}_2[\rho] = \sum_{i,j} M_i M_j^* \, |u_i\rangle\langle u_j| \underbrace{\sum_n \underbrace{\langle n|v_i\rangle\langle v_j|n\rangle}_{\langle v_j|n\rangle\langle n|v_i\rangle}}_{\langle v_j|\underbrace{\sum_n |n\rangle\langle n|}_{\mathbb{1}}|v_i\rangle = \langle v_j|v_i\rangle = \delta_{ij}}$$

$$\boxed{\rho_1 = \sum_j |M_j|^2 \, |u_j\rangle\langle u_j|}$$

diagonal !

$$\boxed{\rho_2 = \sum_j |M_j|^2 \, |v_j\rangle\langle v_j|}$$

$$P_i \longrightarrow |\psi_i\rangle \quad : \quad \rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|$$

$$|n\rangle \otimes |m\rangle \quad \perp \quad |m\rangle \otimes |n\rangle$$

unphysical

Physical states : $\rightarrow \quad |n\rangle \otimes |n\rangle \quad , \quad \dfrac{|n\rangle \otimes |m\rangle + |m\rangle \otimes |n\rangle}{\sqrt{2}}$

$$\rightarrow \quad \frac{|n\rangle \otimes |m\rangle - |m\rangle \otimes |n\rangle}{\sqrt{2}}$$

$|n\rangle \otimes |m\rangle$

$$\sqrt{2}\,\frac{0+0}{2}$$

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathrm{Span}\left\{ |n\rangle \otimes |n\rangle , \frac{|n\rangle \otimes |m\rangle + |m\rangle \otimes |n\rangle}{\sqrt{2}} , \frac{|n\rangle \otimes |m\rangle - |m\rangle \otimes |n\rangle}{\sqrt{2}} \right\}$$

$$\mathcal{H}^+ = \mathrm{Span}\left\{ |n\rangle \otimes |n\rangle , \frac{|n\rangle \otimes |m\rangle + |m\rangle \otimes |n\rangle}{\sqrt{2}} \right\} \leftarrow \text{Boson}$$

$$\mathcal{H}^- \oplus \mathcal{H}^+ = \mathcal{H}$$

$$\mathcal{H}^- = \mathrm{Span}\left\{ \frac{|n\rangle \otimes |m\rangle - |m\rangle \otimes |n\rangle}{\sqrt{2}} \right\} \leftarrow \text{Fermion}$$

$$\hat{P}\, |n\rangle \otimes |m\rangle = |m\rangle \otimes |n\rangle \qquad [H, \hat{P}] = 0$$