# Computational Neuroscience

*Session 3: Mathematical Foundations (2)*

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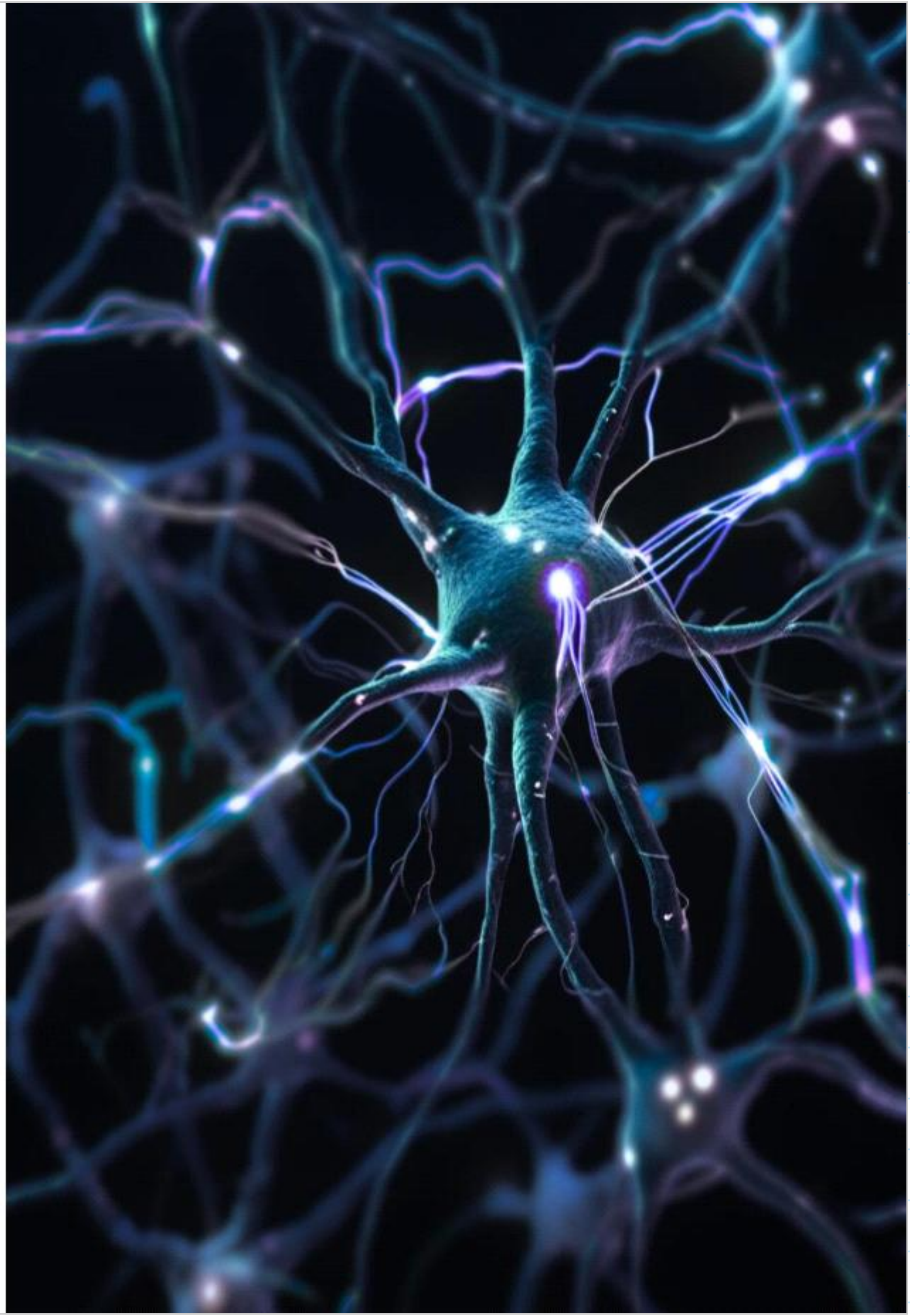
August 13, 2025



# CN-Session 3-August 13

Dynamical system

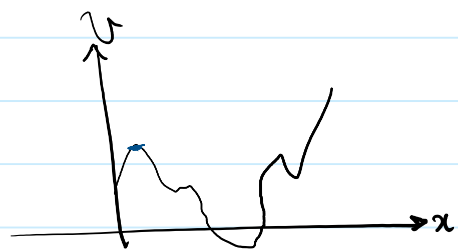
- 1) State space $X(t)$
- 2) $\phi(X, t)$
- Time $(t)$
  - پیوسته $\rightarrow t \in \mathbb{R} \rightarrow \frac{dx}{dt} = f(x)$
  - گسسته $\rightarrow n \in \mathbb{Z} \rightarrow x_{n+1} = f(x_n)$
- Fixed point $(x^*)$
  - Attractor: Stable $\rightarrow f'(x^*) < 0$
  - Unstable $\rightarrow f'(x^*) > 0$
  - Semi-Stable $\rightarrow f'(x^*) = 0 \rightarrow$ مرتبه بالاتر بسط تیلور

$$x(t) = x^* + \epsilon(t) \xrightarrow{d/dt} \frac{dx}{dt} = 0 + \frac{d\epsilon}{dt} \longrightarrow \frac{d\epsilon}{dt} = \frac{dx}{dt} = f(x) = f(x^* + \epsilon(t))$$

$$\rightarrow f(x) = f(x^*) + \epsilon f'(x^*) + O(\epsilon^2) \xrightarrow{\simeq} \frac{d\epsilon}{dt} = f'(x^*)\epsilon \xrightarrow{ODE} \epsilon(t) = \epsilon(0) e^{f'(x^*)t}$$

$$\frac{dx}{dt} = -\frac{dV}{dx}$$

$V$ → potential



# Jacobian Matrix & Eigenvalues

$X(t) \longrightarrow \vec{X} = \{x_1, x_2, \dots, x_n\}, \; x_n \in \mathbb{R}^n$

$F(X, t; \lambda) \longrightarrow F = \{f_1, f_2, \dots, f_n\} \Longrightarrow$ Jacobian Matrix : $J(x) = \dfrac{\partial F}{\partial X} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}, & \dfrac{\partial f_1}{\partial x_2}, & \dots, & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1}, & & & \\ \vdots & & & \\ \dfrac{\partial f_n}{\partial x_1}, & \dfrac{\partial f_n}{\partial x_2}, & \dots, & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix}$

$\dfrac{\partial X}{\partial t} = \underline{\underline{F(X, t; \lambda)}}$

$x^*$ is a fixed point $\longrightarrow f(x^*, t; \lambda) = 0$

$J(X = x^*) = J^* \xrightarrow[\{\mu_i\}]{\text{eigenvalue}} |J^* - \mu I| = 0 \longrightarrow$ Complex Number , جزء حقیقی : $R(\mu_i)$

1) $\forall \mu_i$ , $R(\mu_i) < 0 \longrightarrow$ Stable

2) $\exists \mu_i$ , $R(\mu_i) > 0 \longrightarrow$ Unstable

3) Mixed Signs $\longrightarrow$ Stable in some direction / Unstable other . . . . $\longrightarrow$ Saddle Point

4) $R(\mu_i) = 0 \longrightarrow$ زتا , نوسانی or Bifurcation

# Example (FHN Model)

FitzHugh-Nagumo (FHN) model) Spike-Rest Dynamics $\rightarrow \begin{cases} \dot{v} = v - \frac{v^3}{3} - w + I_{ext} \\ \dot{w} = a(v + b - cw) \end{cases}$

Membrane potential

external input

$(v^*, w^*) \rightarrow \begin{cases} v^* - \frac{(v^*)^3}{3} - w^* + I_{ext} = 0 \\ w^* = \frac{v^* + b}{c} \end{cases} \longrightarrow$ Numerical $\longrightarrow J^* \longrightarrow \{\mu_i, \mu_j\}$

1) $R(\mu_i), R(\mu_j) < 0 \longrightarrow$ Stable $\rightarrow$ Keeps being in Resting mode

2) $R(\mu_i) > 0$ or $\underline{R(\mu_j) > 0}$ $\rightarrow$ Unstable $\rightarrow$ Unstable $\rightarrow$ Spiking mode

3) $R(\mu_i)$ or $R(\mu_j) \longrightarrow$ Hopf Bifurcation

$I_{ext} \downarrow \longrightarrow \mu_i, \mu_j < 0 \longrightarrow$ Stable Resting

$I_{ext} = I^c_{ext} \longrightarrow \mu_i, \mu_j \longrightarrow R(\mu) = 0 \longrightarrow$ Spiking

Complex Conjugate

# Attractors

A set of states towards which a dynamical systems evolves over time from a Wide Range of Initial Condition

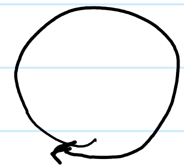
$\phi_t(x_0)$ → Trajectory

$x_0$ → initial Condition

$A \Longrightarrow$
1) if $x(t_0) \in A$, the $x(t) \in A$ ; $(t > t_0)$
2) $\exists U,\ U \subseteq A \mid$ as $t \to \infty$, $\phi_t(x_0) \to A$, for all $x_0 \in U$ → Attraction Condition
3) There be No smaller set that has those 2 above codition

1) fixed point attractor → $\dot{x} = \frac{dx}{dt} = 0$, $x = x^*$

2) limit cycle → closed
→ periodic ( $x(t+T) = x(t)$ ), $T > 0$



3) Strange attractor → fractal like → 1) Bounded
→ 2) Trajectories never repeat

$B(A) = \{x_0 \mid \lim_{t \to \infty} \phi_t(x_0) \in A\}$

initial Condition

Example) HH →

$$\begin{cases} C_m \dfrac{dV}{dt} = I_{ext} - \bar{g}_{Na} m^3 h (V - E_{ion}) - \bar{g}_k n^4 - \bar{g}_L (V - E_L) \\ \dfrac{dm}{dt} = \alpha_m(V)(1-m) - \beta_m(V) m \\ \dfrac{dh}{dt} = \alpha_h(V)(1-h) - \beta_h(V) h \\ \dfrac{dn}{dt} = \alpha_n(V)(1-n) - \beta_n(V) n \end{cases}$$

$\phi(X_0, t; \lambda)$

Control parameters

# Attractors

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1) Stable fixed point $\longrightarrow I_{ext} \downarrow \longrightarrow$ Converges to Resting mode

2) limit cycle : Firing mode

3) Bistability $\longrightarrow$ near threshold $\longrightarrow$ limit cycle $\xrightarrow{I_{ext}^{c}}$
$\longrightarrow$ Fixed point

Basin of Attraction
- 1) Resting : near $V_{rest} \longrightarrow$ Resting
- 2) spiking : firing mode
- 3) $V_{rest} \longrightarrow m, n, h$

Bifurcation : $\lambda > \lambda_c$
1) Saddle-node :
2) Hopf Bifurcation :

Lyapunov Exponent) $\lambda_{max} \longrightarrow \underline{\|\delta x(t)\| \approx \|\delta x(0)\| e^{\lambda_{max} t}}$

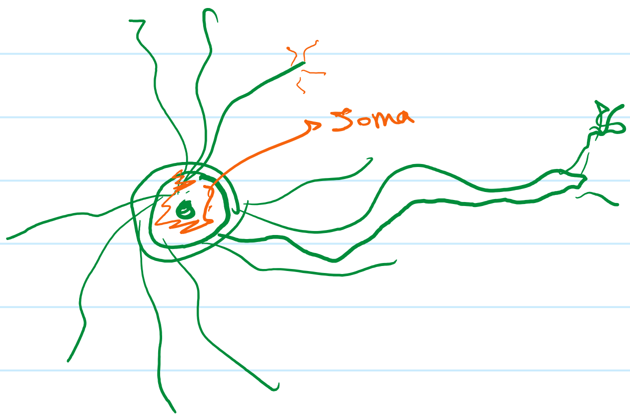
$\lambda_{max}$
- 1) $> 0 \longrightarrow$ Sensitive to initial Condition $\longrightarrow$ *Chaos*
- 2) $= 0 \longrightarrow$ limit cycles
- 3) $< 0 \longrightarrow$ Exponential Convergence to Attractors

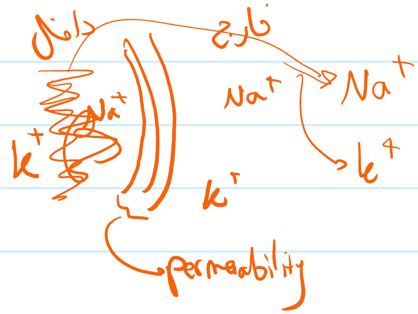
# Pinsky-Rinzel Model

Extended version of Hodgkin-Huxley to a two-compartment model (soma & dendrite): 1) Fast $Na^+$ & $K^+$ gating currents in Soma

2) High-threshold $Ca^{2+}$ and Ca-activated $K^+$ in the dendrites.

3) $[Ca^{2+}]$ Dynamics: $\frac{d[Ca]}{dt} = -\alpha I_{Ca} - \beta[Ca]$




$$\boxed{\frac{d[Ca^{2+}]}{dt} = -\alpha I_{Ca} - \beta[Ca]}$$