The Calculus of Uncertainty: Stochastic Processes & Their Mathematical Universe

Hooman Zare

SUT

July 28, 2025

Random Variable: Formal Definition

Definition

A **random variable** X is a measurable function from a probability space $(\Omega, \mathcal{F}, P)$ to a measurable space $(E, \mathcal{E})$:

$$X : \Omega \to E$$

where $\mathcal{F}$ is a σ-algebra on Ω, and $\mathcal{E}$ is a σ-algebra on E.

Key Implications:

- $\forall B \in \mathcal{E},\ X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$
- $P(X \in B) = P(\{\omega : X(\omega) \in B\})$ is well-defined

σ-Algebra: Information Structure

Definition

A **σ-algebra** $\mathcal{F}$ is a collection of subsets of Ω satisfying:

1. $\Omega \in \mathcal{F}$
2. Closed under complements
3. Closed under countable unions

Borel σ-Algebra:

- Generated by open sets in $\mathbb{R}^n$: $\mathcal{B}(\mathbb{R}^n)$
- Elements are **Borel sets** (intervals, points, countable unions/intersections)

Interpretation: $\mathcal{F}$ encodes "knowable events" in a system.

Stochastic Process: Dynamic Randomness

Definition

A **stochastic process** is a collection of random variables $\{X_t\}_{t \in T}$ indexed by time t, where:

$$X_t : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$$

(T: time set, S: state space)

Key Aspects:

- *Path-space view*: $\omega \mapsto t \mapsto X_t(\omega)$ (a random function)
- *Measure-theoretic view*: $X : \Omega \times T \to S$ (joint measurability)

Examples:

- Brownian motion: $S = \mathbb{R}$, $T = [0, \infty)$
- Poisson process: $S = \mathbb{N}_0$, $T = \mathbb{R}^+$

Filtration: Evolution of Information

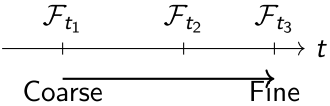
Definition

A **filtration** $\{\mathcal{F}_t\}_{t \geq 0}$ is a family of σ-algebras satisfying:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \text{for} \quad s \leq t$$

Interpretation:

- $\mathcal{F}_t$ = "Information available up to time t"
- X_t is $\mathcal{F}_t$-**adapted** if X_t is $\mathcal{F}_t$-measurable (values knowable at t)



Filtering: Estimating X_t given $\mathcal{F}_s$ $(s \leq t)$

Interdisciplinary Intuition (Part 1/2)

Field	Ω **(Sample Space)**	$\mathcal{F}_t$ **(Filtration)**	**Process**
Biology	All possible mRNA trajectories ω = expression history	Microscopy data up to t Knows if $\{\text{gene ON}\} \in \mathcal{F}_t$	Gene expression X_t (telegraph model)
Economics	All possible market histories ω = price/order evolution	Order book events up to t Knows if $\{\text{bid} > K\} \in \mathcal{F}_t$	Asset price S_t (geometric BM)
Physics	All possible quantum records ω = measurement sequence	Detector outcomes up to t Knows if $\{\text{spin } \uparrow\} \in \mathcal{F}_t$	Quantum state ψ_t (SSE)

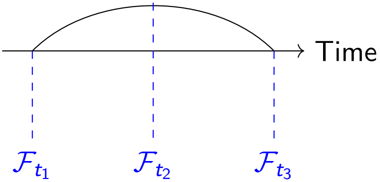
Interdisciplinary Intuition (Part 2/2)

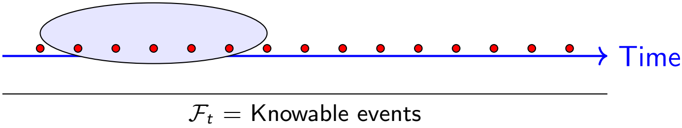
Field	Ω (**Sample Space**)	$\mathcal{F}_t$ (**Filtration**)	**Process**
Ecology	All possible population histories $\omega =$ birth/death/migration	Field surveys up to season t Knows if $\{\text{extinct}\} \in \mathcal{F}_t$	Species count N_t (branching process)
Mathematics	All possible rough paths $\omega = (X, \int dX \otimes dX)$	σ(signature terms) up to ϵ Knows if $\{\mathbb{X} \in A\} \in \mathcal{F}_\epsilon$	$(X_t, \mathbb{X}_{st})$ (fBM enhancement)

Unifying Mathematical Principles:

1. $\Omega =$ Canonical path space $\{\omega : T \to S\}$ (Skorokhod space)
2. $\mathcal{F}_t = \sigma(X_s : s \leq t) =$ Natural filtration (coarsest σ-alg making $\{X_s\}_{s \leq t}$ measurable)
3. Adaptedness: $X_t(\omega) = \omega(t)$ is $\mathcal{F}_t$-measurable
4. Filtering: $\mathbb{E}[X_t | \mathcal{F}_s]$ for $s < t$ (conditional expectation)

Frame Title





Common Theme:

$\mathcal{F}_t = \sigma$(Available observations)

X_t must be $\mathcal{F}_t$-measurable

Moments: Statistical Descriptors

Definition (k-th Moment)

For a random variable X_t at time t:

$$m_k(t) = \mathbb{E}[X_t^k] = \int_{-\infty}^{\infty} x^k p_t(x) dx$$

Key Moments:

- Mean: $\mu(t) = m_1(t)$
- Variance: $\sigma^2(t) = m_2(t) - m_1(t)^2$

Higher Moments:

- Skewness: $\gamma(t) = \frac{m_3(t)}{\sigma^3(t)}$ (asymmetry)
- Kurtosis: $\kappa(t) = \frac{m_4(t)}{\sigma^4(t)}$ (tail heaviness)

Example (Brownian Motion):

- $\mu(t) = 0$, $\sigma^2(t) = t$ (variance grows linearly)
- $\gamma(t) = 0$, $\kappa(t) = 3$ (Gaussian properties)

Ensemble vs. Time Averages

Ensemble Average:

$$\langle X_t \rangle_{\text{ens}} = \frac{1}{N} \sum_{i=1}^{N} X_t^{(i)} \xrightarrow{N\to\infty} \mathbb{E}[X_t]$$

Across identical systems at fixed t

Time Average:

$$\langle X \rangle_{\text{time}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T X_t dt$$

Along single trajectory

Ergodicity: $\langle \cdot \rangle_{\text{ens}} = \langle \cdot \rangle_{\text{time}}$ almost surely

Non-ergodic Examples:

- Glass dynamics: Frozen disorder prevents self-averaging
- Financial markets: Regime shifts (e.g., bull/bear markets)
- Climate systems: Multiple metastable states (e.g., ice ages)

Probability Distributions: Key Concepts

Definition (Cumulative Distribution Function (CDF))

$$F_t(x) = P(X_t \leq x) = \int_{-\infty}^{x} p_t(u)du$$

Definition (Probability Density Function (PDF))

$$p_t(x) = \frac{d}{dx}F_t(x) \quad \text{(1-point density)}$$

Definition (Joint PDF)

$$p_{t_1,\dots,t_n}(x_1,\dots,x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} P(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n)$$

Conditional Distributions & Bayes' Rule

Definition (Conditional PDF)

For X_{t_1} given $X_{t_2} = x_2$ $(t_1 > t_2)$:

$$p(x_1|x_2) = \frac{p_{t_1,t_2}(x_1, x_2)}{p_{t_2}(x_2)}$$

Theorem (Bayes' Rule for Processes)

$$p(x_1|x_2, \ldots, x_r) = \frac{p(x_1, x_2, \ldots, x_r)}{p(x_2, \ldots, x_r)} = \frac{p(x_r|x_1, \ldots, x_{r-1})p(x_1, \ldots, x_{r-1})}{p(x_2, \ldots, x_r)}$$

Filtering Application:

- Estimate hidden state X_t given observations $Y_{1:t}$

$$p(x_t|y_{1:t}) = \frac{p(y_t|x_t)p(x_t|y_{1:t-1})}{p(y_t|y_{1:t-1})}$$

Markov Property & Simplifications

Definition (Markov Process)

$$p(x_t|x_{t-1}, x_{t-2}, \dots) = p(x_t|x_{t-1})$$

Future depends only on present state.

Consequences:

- Joint PDF factorizes: $p(x_1, \dots, x_T) = p(x_1) \prod_{k=2}^{T} p(x_k|x_{k-1})$
- Conditional PDFs simplify: $p(x_t|x_{1:s}) = p(x_t|x_s)$ for $t > s$

Non-Markovian Example:

- Fractional Brownian motion: $X_t = \int_0^t (t-s)^{H-1/2} dW_s$ with $H \neq 1/2$
- Requires full history: $p(x_t|x_{t-1}, x_{t-2}, \dots) \neq p(x_t|x_{t-1})$

Interdisciplinary Moments & Distributions: Case Studies

Field	Statistical Challenge	Mathematical Tool
Neuroscience (fMRI Dynamics)	Non-Gaussian BOLD signals in epilepsy Kurtosis $\kappa(t) > 3$ detects seizure foci Ω: Neural activity states	Edgeworth expansion: $p(x) = \phi(x)\left[1 + \frac{\kappa-3}{24} He_4(x)\right]$ $He_4(x)$: Hermite polynomial
Economics (Volatility Modeling)	Volatility clustering in asset returns $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ $\mathcal{F}_t$: σ(past returns)	GARCH(1,1) model: $\mathbb{E}[\sigma_t^4] = \frac{\omega^2(1+\alpha\beta)}{(1-\alpha-\beta)(1-\beta^2-2\alpha\beta)}$ (requires $\alpha + \beta < 1$)
Oceanography (Turbulence)	Joint PDF of velocity gradients $p(\partial_x u, \partial_y v)$ in Kolmogorov flow $D^{(k)}$: k-th order diffusion coefficient	Kramers-Moyal expansion: $\frac{\partial p}{\partial t} = \sum_{k=1}^{\infty} \frac{(-\partial)^k}{\partial x^k} D^{(k)} p$ (truncated at $k = 2$ for Fokker-Planck)

Interdisciplinary Filtering & Non-Ergodicity

Field	Statistical Challenge	Mathematical Tool
Epidemiology (Infection Dynamics)	Time-non-ergodic R_t estimation $\langle R_t \rangle_{\text{time}} \neq \langle R_t \rangle_{\text{ens}}$ during interventions λ_t: contact rate process	MCMC for SIR posterior: $p(R_t \mid \Delta I_{1:T}) \propto \prod_{t=1}^{T} \text{Poisson}(\Delta I_t \mid \lambda_t R_{t-1})$ (Hamiltonian Monte Carlo sampling)
Quantum Control (Qubit Readout)	Conditional state discrimination $p(\lvert 0 \rangle \mid I_t, Q_t)$ vs $p(\lvert 1 \rangle \mid I_t, Q_t)$ $\mathcal{F}_t$: σ(IQ-plane history)	Wald sequential test: $\Lambda_n = \prod_{k=1}^{n} \frac{p_1(I_k, Q_k)}{p_0(I_k, Q_k)} \gtrless \eta$ (Optimal stopping time τ)

Bayesian Inference:

$$\underbrace{p(\text{state}|\text{data})}_{\text{posterior}} \propto \underbrace{p(\text{data}|\text{state})}_{\text{likelihood}} \times \underbrace{p(\text{state})}_{\text{prior}}$$

Non-Ergodicity Theorems:

Theorem (Time-Average Ergodicity Breakdown)

For process with aging (e.g., glasses):

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X_s ds \neq \mathbb{E}[X_t] \quad \textit{if} \quad \frac{\partial \mu(t)}{\partial t} \neq 0$$

Theorem (Filtering Fundamental Limit)

Minimum MSE for $X_t|\mathcal{F}_s$ $(s < t)$ bounded by:

$$\mathbb{E}[(X_t - \hat{X}_t)^2] \geq \frac{1}{I_{\mathcal{F}_s}(X_t)} \quad \textit{(Fisher information)}$$

Stationarity: Time-Translation Symmetry

Definition (Strict Stationarity)

A process $\{X_t\}$ is **strictly stationary** if $\forall \tau, t_1, \ldots, t_n$:

$$(X_{t_1}, \ldots, X_{t_n}) \stackrel{d}{=} (X_{t_1+\tau}, \ldots, X_{t_n+\tau})$$

(All finite-dimensional distributions are time-invariant)

Definition (Weak Stationarity)

$\{X_t\}$ is **weakly stationary** if:

1. $\mu(t) = \mu$ (constant mean)
2. $\mathrm{Cov}(X_t, X_s) = C(|t-s|)$ (covariance depends only on lag)

Physical Analogy:

- Equilibrium thermodynamics: Macroscopic observables invariant under time shift
- Crystal lattice: Atomic vibrations statistically identical over time

Non-Stationary Examples:

- Big Bang cosmology (Hubble expansion)
- Neuronal spike trains during learning
- COVID-19 case counts during pandemic waves

Chapman-Kolmogorov Equation: Markov Consistency

Theorem (Chapman-Kolmogorov)

For a Markov process, transition densities satisfy:

$$p(x_3, t_3|x_1, t_1) = \int p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1)dx_2$$

for any $t_1 < t_2 < t_3$*.*

Necessary Markov Condition:

The transition kernel $p(x_{n+1}, t_{n+1}|x_n, t_n)$ must determine all multi-point statistics.

Intuition:

- Quantum path integrals: Sum over intermediate states
- Optics: Huygens' principle (wave propagation via secondary wavelets)

Testable Implication:

Conditional independence: $X_{t_3} \perp X_{t_1}|X_{t_2}$ for $t_1 < t_2 < t_3$

$\Rightarrow p(x_3|x_2, x_1) = p(x_3|x_2)$

Testing Markov Property & Stationarity

Markov Test (Likelihood Ratio):
Compare models via AIC/BIC:

$$\text{AIC} = 2k - 2\ln \hat{L}, \quad \hat{L}_{\text{Markov}} = \prod_{i=2}^{n} p(x_i | x_{i-1})$$

$$\hat{L}_{\text{non-Markov}} = \prod_{i=2}^{n} p(x_i | x_{i-1}, \ldots, x_{i-m})$$

Stationarity Tests:

1. Augmented Dickey-Fuller: Reject unit root H_0 if $|\tau| > c$

$$\Delta X_t = \alpha + \beta t + \gamma X_{t-1} + \sum_{i=1}^{k} \delta_i \Delta X_{t-i} + \epsilon_t$$

2. Kwiatkowski-Phillips-Schmidt-Shin (KPSS): Test stationarity around deterministic trend

Continuous Stochastic Processes: Mathematical Framework

Definition (Continuous Process)

$\{X_t\}$ is **continuous** if $\forall \epsilon > 0$:

$$\lim_{h \to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$$

(Probability of jumps vanishes as $h \to 0$)

Key Examples:

- Wiener process: $dW_t \sim \mathcal{N}(0, dt)$ (nowhere differentiable)
- Ornstein-Uhlenbeck: $dX_t = -\theta X_t dt + \sigma dW_t$ (mean-reverting)

Physical Analogies:

Process	Physics System
Brownian motion	Pollen in water (Einstein)
Langevin equation	Particle in potential + noise
Geometric BM	Stock prices (Black-Scholes)

Master Equation: Foundation of Stochastic Dynamics

Definition (General Master Equation)

Rate of change of probability $P_n(t)$ to be in state n:

$$\frac{dP_n}{dt} = \sum_{m \neq n} \Big[\underbrace{W_{n\leftarrow m} P_m(t)}_{\text{gain from } m} - \underbrace{W_{m\leftarrow n} P_n(t)}_{\text{loss to } m} \Big]$$

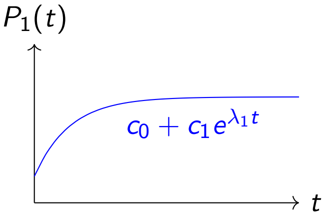
where $W_{n\leftarrow m}$ is the transition rate $m \to n$.

Key Properties:

- Probability conservation: $\sum_n dP_n/dt = 0$
- Irreversibility: $dS/dt \geq 0$ (entropy $S = -k_B \sum P_n \ln P_n$)
- Equilibrium: $dP_n/dt = 0$ when detailed balance holds: $W_{n\leftarrow m} P_m^{eq} = W_{m\leftarrow n} P_n^{eq}$

Physical Analogy:

Master Equation	**Continuity Equation**
Probability flow between states	Mass flow between compartments
$W_{n\leftarrow m}$ = Transition rate	J_{ij} = Flux density
Detailed balance = Zero circulation	Irrotational flow



Matrix Form & Spectral Analysis

$$\frac{d\mathbf{P}}{dt} = \mathbf{QP}, \quad \mathbf{Q} = \begin{pmatrix} -\sum_{j\neq 1} W_{j1} & W_{12} & \cdots \\ W_{21} & -\sum_{j\neq 2} W_{j2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Eigenvalue Decomposition:

$$\mathbf{P}(t) = \sum_k c_k e^{\lambda_k t} \mathbf{v}_k, \quad \mathbf{Q}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

Spectral Properties:

- $\lambda_0 = 0$ (steady state $\mathbf{v}_0$)
- $\mathrm{Re}(\lambda_k) \leq 0$ (decaying modes)
- $|\mathrm{Re}(\lambda_1)| =$ Spectral gap (mixing time scale)

Example (Two-State System):

$$\mathbf{Q} = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix}, \quad \lambda_k = \{0, -(a+b)\}$$

$$\mathbf{v}_0 = \frac{1}{a+b}\begin{pmatrix} b \\ a \end{pmatrix}, \quad \tau_{\text{relax}} = \frac{1}{a+b}$$

Interdisciplinary Applications I: Quantum & Condensed Matter

System	Master Equation & Solution
Fermi Levels (Semiconductors)	$\frac{dP_n}{dt} = \sum_m \Gamma_{nm}[f(\epsilon_m - \mu)P_m - f(\epsilon_n - \mu)P_n]$ Steady state: $P_n^{eq} \propto e^{-(\epsilon_n - \mu)/k_B T}$ *Problem solved*: Carrier distribution in bands
Lindblad Equation (Quantum Optics)	$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$ Eigenvalues: Decay rates of coherence *Problem solved*: Qubit decoherence in cavity
Redfield Relaxation (NMR Spectroscopy)	$\frac{d\sigma_{ab}}{dt} = -i\omega_{ab}\sigma_{ab} - R_{abcd}\sigma_{cd}$ R_{abcd}: Relaxation supermatrix *Problem solved*: T_1, T_2 times from spectral density
Förster/Dexter (Energy Transfer)	$\frac{dP_D}{dt} = -k_{DA}P_D + k_{AD}P_A$ Rates: $k_{DA} \propto J(\text{overlap}) \times \kappa^2/r^6$ (Förster) *Problem solved*: Exciton migration in photosynthesis

Interdisciplinary Applications II: Chemistry & Biology

System	Master Equation & Solution
Reaction Kinetics (Chemical Networks)	$\frac{dP_{\vec{n}}}{dt} = \sum_r k_r \left[(\mathbb{E}_r^{s_r} - 1) \prod_i n_i!(n_i - \nu_{ri})! P_{\vec{n}}\right]$ $\mathbb{E}_r$: Step operator for reaction r *Problem solved*: Stochastic oscillations in Br2/NO2
Gene Expression (mRNA Dynamics)	$\frac{dP_m}{dt} = \lambda P_{m-1} + \gamma(m+1)P_{m+1} - (\lambda + \gamma m)P_m$ Solution: $P_m^{ss} = \frac{(\lambda/\gamma)^m e^{-\lambda/\gamma}}{m!}$ (Poisson) *Problem solved*: Transcriptional bursting noise
SIR Model (Epidemiology)	$\frac{dP_{S,I}}{dt} = \beta \mathbb{E}_S^{-1} \mathbb{E}_I^{1} SIP_{S,I} + \gamma \mathbb{E}_I^{-1} \mathbb{E}_R^{1} IP_{S,I} - \cdots$ Mean-field: $\dot{S} = -\beta SI$, $\dot{I} = \beta SI - \gamma I$ *Problem solved*: Critical vaccination threshold
Hidden CTMCs (State Estimation)	$\frac{dP_n}{dt} = \sum_m Q_{nm} P_m, \quad Y_t \sim g(y \mid X_t)$ Filtering: $\pi_t(n) = P(X_t = n \mid Y_{0:t})$ *Problem solved*: Ion channel gating from noisy currents

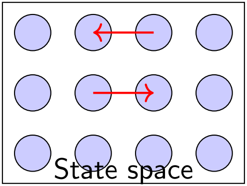
Interdisciplinary Applications III: Cosmology & Control Theory

System	Master Equation & Solution
Density Matrix (Cosmological Inflation)	$\frac{d\rho}{dt} = -i[H,\rho] + \kappa\mathcal{D}[a]\rho$ $\mathcal{D}[a]\rho = a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}$ *Problem solved*: Quantum-to-classical transition of perturbations
Hamilton-Jacobi-Bellman (Optimal Control)	$\frac{\partial V}{\partial t} + \min_u \left[\mathcal{L}V + c(x,u)\right] = 0$ $\mathcal{L}$: Generator of controlled process *Problem solved*: Minimum-fuel spacecraft trajectory
Agent-Based Models (Econophysics)	$\frac{dP_C}{dt} = \mu P_D + \beta\langle k\rangle P_C P_D - \delta P_C$ Mean-field: $\dot{\rho}_C = \mu(1-\rho_C) + \beta\langle k\rangle\rho_C(1-\rho_C) - \delta\rho_C$ *Problem solved*: Phase transitions in opinion dynamics
ABM Calibration (Machine Learning)	$\frac{dP_\theta}{dt} \propto \sum_{\text{paths}}[\ln p_\theta(\text{path}) - \lambda]P_\theta$ Gradient flow on parameter space θ *Problem solved*: Fitting market sentiment parameters

Common Solution Approaches:

- Spectral Methods: Diagonalize $\mathbf{Q}$ for exponential dynamics (e.g., NMR relaxation modes)
- Mean-Field Approximation: $P_n \approx \prod_i \rho_i^{n_i}$ for large systems (e.g., SIR compartmental models)
- Generating Functions: $G(s,t) = \sum_n s^n P_n(t)$ converts to PDEs (e.g., gene expression)
- Monte Carlo: Gillespie algorithm for exact trajectory sampling

Universal Physical Analogies:



- Energy landscapes: $W_{n\leftarrow m} \propto e^{-\Delta E/k_B T}$
- Entropy production: $\frac{d_i S}{dt} = \frac{1}{2}\sum_{m,n} J_{mn} F_{mn} \geq 0$
- Detailed balance $\Leftrightarrow$ equilibrium