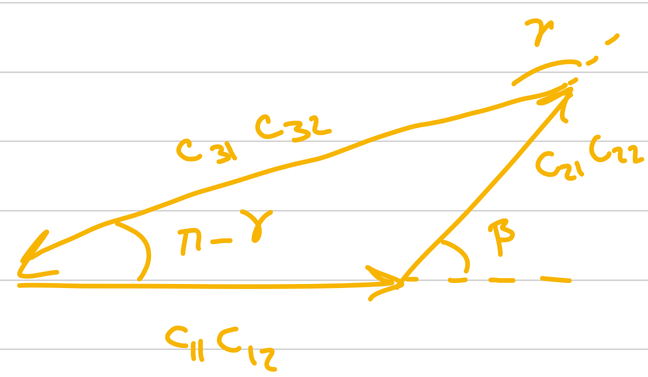
$N = 3$

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22}e^{i\beta} & C_{23}e^{i\varphi} \\ C_{31} & C_{32}e^{i\gamma} & C_{33}e^{i\delta} \end{pmatrix}$$

$$\sum_{\mu} C^*_{\mu n} C_{\mu m} = \delta_{nm}$$

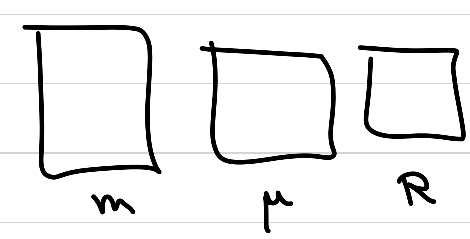
$n, m = 1, 2 \longrightarrow \quad C_{11}C_{12} + C_{21}C_{22}e^{i\beta} + C_{31}C_{32}e^{i\gamma} = 0$



$$|C_{11}C_{12}| \leq |C_{21}C_{22}| + |C_{31}C_{32}|$$

Three SG: $\quad \sum_{\mu} \Gamma_{R\mu} C_{\mu m} = \mathbb{C}_{Rm}$

n, μ, R $\qquad \mathbb{C} = \Gamma C$



Complex Vector Space

$$\sum_{\mu} C^*_{\mu m} C_{\mu n} = \delta_{nm}$$

unit vectors: $|e_m\rangle$, $|e_\mu\rangle \longrightarrow |e_\mu\rangle = \sum_m C_{\mu m} |e_m\rangle$

$$|e_R\rangle = \sum_\mu \Gamma_{R\mu} |e_\mu\rangle \quad , \quad |e_R\rangle = \sum_m \mathbb{C}_{Rm} |e_m\rangle$$

$$\mathbb{C}_{Rm} = \sum_\mu \Gamma_{R\mu} C_{\mu m} \quad \checkmark$$

$$|v\rangle = \sum_m v_m |e_m\rangle \quad ; \quad P_{\mu m} = |\langle e_m | e_\mu\rangle|^2$$

$$\underbrace{\text{if } N \text{ is Pime} \longrightarrow N+1}\ : \quad |e_m\rangle = \frac{1}{\sqrt{N}} \sum_\mu |e_\mu\rangle\, e^{2\pi i \frac{\mu m}{N}}$$

$$C_{\mu m} = \langle e_m | e_\mu\rangle \Rightarrow C^*_{\mu m} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} \mu m}$$

Pure state determination : $|v\rangle = \sum_m v_m |e_m\rangle = \sum_\mu v_\mu |e_\mu\rangle$

$$|v_m|^2 = |\langle e_m | v\rangle|^2 \quad ; \quad |v_\mu|^2 = |\langle e_\mu | v\rangle|^2$$

$\downarrow$ r_m $\qquad$ $\downarrow$ ϕ_m

$N-1$ $\qquad\qquad$ $N-1$

$$\sum_m |v_m|^2 = \sum_\mu |v_\mu|^2 = 1$$

$2(N-1)$ Eq.

$$v_m = r_m e^{i\phi_m}$$

$\uparrow$

$$|v_\mu|^2 = \left|\sum_m C^*_{\mu m} r_m e^{i\phi_m}\right|^2 \longrightarrow \text{Some inequality}$$

Uncertainty relations: 2D Complementary bases:

$$C_{\mu m} = \frac{1}{\sqrt{2}} e^{-\frac{2\pi i}{2}\mu m} = \frac{1}{\sqrt{2}} (-1)^{\mu m}$$

$$|V_\mu|^2 = \left| \sum_m \frac{1}{\sqrt{2}} (-1)^{\mu m} r_m e^{i\phi_m} \right|^2 = \frac{1}{2} \left| \pm r_1 e^{i\phi_1} + r_2 e^{i\phi_2} \right|^2$$

$$= \frac{1}{2}\left(r_1^2 + r_2^2 \pm 2 r_1 r_2 \cos(\phi_1 - \phi_2) \right) \xrightarrow[\phi_2 = 0]{} \phi_1$$

$$\frac{1}{2}|r_1 - r_2|^2 \leq |V_\mu|^2 \leq \frac{1}{2}|r_1 + r_2|^2$$

$$V_m = \delta_{m1} \longrightarrow |V_\mu|^2 = \frac{1}{2}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow |e_x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e_y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Polarizer: θ $\quad \frac{E_x}{E_y} = \frac{\cos\theta}{\sin\theta}$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \rightarrow \text{Pass}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \rightarrow \text{Fail}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = C_1 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + C_2 \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$C_1 = \alpha \cos\theta + \beta \sin\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}^{\dagger} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$C_2 = -\alpha \sin\theta + \beta \cos\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}^{\dagger} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Calcite X-Y : $|c_1|^2$, $|c_2|^2$

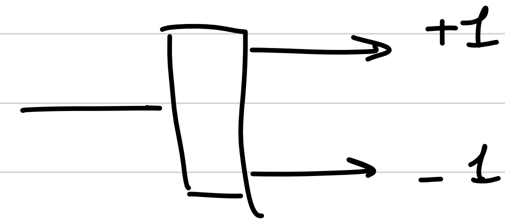
$|\alpha|^2 \sim c_1$

$|\beta|^2 \sim c_2$

Calcite $\pm 45^\circ$: $|c_1'|^2$, $|c_2'|^2$ $\longrightarrow$ $\frac{1}{2}|\alpha\pm\beta|^2$

α/β $\quad |\alpha|^2+|\beta|^2=1$

Observables



$A \longrightarrow a_R$

Prob. of a_R : $|\langle e_R|v\rangle|^2 \longrightarrow \langle A\rangle = \sum_R a_R |\langle e_R|v\rangle|^2 = \sum_R a_R |v_R|^2$

$$\langle A\rangle = \sum_R a_R \Big|\sum_m \langle C_{Rm} e_m|v\rangle\Big|^2 = \sum_R a_R \Big|\sum_m C^*_{Rm} \underbrace{\langle e_m|v\rangle}_{v_m}\Big|^2$$

$$= \sum_{R,m,n} a_R C^*_{Rn} C_{Rm} v_n v^*_m$$

$$\langle A\rangle = \sum_{m,n} v^*_m A_{mn} v_n \qquad A_{mn} = \sum_R C_{Rm} a_R C^*_{Rn}$$

Preperation

Test

Matrices $\begin{cases} C_{\mu m} \rightarrow \text{two bases} \rightarrow \text{unitary} & C^{\dagger} C = \mathbb{1} \\ A_{mn} \rightarrow \text{Single base} \rightarrow \text{Hermitian} & A^{\dagger} = A \end{cases}$

$$A_{\mu\nu} = \sum_{R} \underline{\Gamma_{R\mu}\, a_R\, \Gamma_{R\nu}^{*}}$$

$$A_{mn} = \sum_{R} \mathbb{C}_{Rm}\, a_R\, \mathbb{C}_{Rn}^{*} = \sum_{R} \left(\sum_{\mu} \Gamma_{R\mu} C_{\mu m} \right) a_R \left(\sum_{\nu} \Gamma_{R\nu}^{*} C_{\nu n}^{*} \right)$$

$$\boxed{A_{mn} = \sum_{\mu\nu} C_{\mu m} A_{\mu\nu} C_{\nu n}^{*}}$$