Foundations of Stochastic Analysis

Measure Theory, Probability, and Limit Theorems

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Infinite Coin Toss Experiment

- **Sample space:** $\Omega = \{0,1\}^{\mathbb{N}}$ (all infinite binary sequences)
- Interpretation:

$$\omega = (\omega_1, \omega_2, \dots), \quad \omega_i \in \{0,1\}$$
$$1 = \text{Heads}, \quad 0 = \text{Tails}$$

- Define two fundamental functions:

$$X_n(\omega) = \omega_n \quad \text{(Outcome of } n\text{-th toss)}$$
$$S_n(\omega) = \sum_{k=1}^{n} \omega_k \quad \text{(Number of heads in first } n \text{ tosses)}$$

0	0	1	1	1	$\cdots$
toss 1	toss 2	toss 3	toss 4	toss 5	

Finite-dimensional Cylinder Sets

Definition

Set $A \subseteq \Omega$ is **finite-dimensional** if $\exists n \in \mathbb{N}$ and $B \subseteq \{0,1\}^n$ such that:

$$A = \{\omega \in \Omega : (\omega_1, \ldots, \omega_n) \in B\} = B \times \{0,1\} \times \{0,1\} \times \cdots$$

Example

$A = \{\omega : \omega_1 = 1, \omega_3 = 0\}$ corresponds to:

$$B = \{(1, x, 0) : x \in \{0,1\}\} \subseteq \{0,1\}^3$$

Field of Cylinder Sets

$\mathcal{F}_* = \{A \subseteq \Omega : A \text{ is finite-dimensional}\}$ forms a field (algebra) but not a σ-algebra.

Probability Measure on Cylinder Sets

Define $P : \mathcal{F}_* \to [0, 1]$:

$$P(A) = \frac{|B|}{2^n} \quad \text{where } A \text{ corresponds to } B \subseteq \{0, 1\}^n$$

Example (Probability of heads on n-th toss)

$$A = \{\omega : X_n(\omega) = 1\} \implies B = \{(x_1, \ldots, x_n) \in \{0, 1\}^n : x_n = 1\}$$

$$|B| = 2^{n-1} \implies P(A) = \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

Example (Probability of k heads in n tosses)

$$A = \{\omega : S_n(\omega) = k\} \implies B = \left\{x \in \{0, 1\}^n : \sum_{i=1}^{n} x_i = k\right\}$$

$$|B| = \binom{n}{k} \implies P(A) = \frac{\binom{n}{k}}{2^n}$$

Law of Large Numbers - Motivation

Theorem (Strong Law of Large Numbers)

For fair coin tosses:

$$P\left(\left\{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \frac{1}{2}\right\}\right) = 1$$

Problem: The set $C = \{\omega : \lim_{n\to\infty} S_n(\omega)/n = 1/2\}$ is not in $\mathcal{F}_*$ (requires infinite-dimensional specification).
Solution: Extend P to $\sigma(\mathcal{F}_*)$ using measure theory.

Measure Theoretic Foundation

Definition (σ-algebra)

A collection $\mathcal{F} \subseteq 2^\Omega$ is a σ-**algebra** if:

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. $A_1, A_2, \cdots \in \mathcal{F} \implies \bigcup_{i=1}^\infty A_i \in \mathcal{F}$

Definition (Probability measure)

A function $P : \mathcal{F} \to [0, 1]$ is a **probability measure** if:

1. $P(\Omega) = 1$
2. Countable additivity: For disjoint $\{A_i\}_{i=1}^\infty$,

$$P\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty P(A_i)$$

Carathéodory Extension Theorem

Theorem (Carathéodory)

Let $\mathcal{F}_*$ be a field and $P : \mathcal{F}_* \to [0,1]$ be:

- *Finitely additive*
- *Countably additive on $\mathcal{F}_*$: If $\{A_n\} \subset \mathcal{F}_*$ disjoint with $\bigcup_n A_n \in \mathcal{F}_*$, then $P(\bigcup_n A_n) = \sum_n P(A_n)$*

Then P extends uniquely to a probability measure on $\sigma(\mathcal{F}_*)$.

Key lemma: For cylinder sets, P satisfies:

$$A_n \downarrow \emptyset \implies P(A_n) \to 0$$

Proof sketch: Use compactness of $\Omega = \{0,1\}^{\mathbb{N}}$ with product topology. Finite-dimensional cylinders are clopen sets. If $A_n \downarrow \emptyset$ but $\inf P(A_n) > 0$, compactness gives $\omega \in \cap A_n$, contradiction.

Borel-Cantelli and SLLN Setup

Define the critical set:

$$C = \left\{ \omega : \lim_{n \to \infty} \frac{S_n(\omega)}{n} = \frac{1}{2} \right\}$$

Equivalently:

$$C = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ \omega : \left| \frac{S_n(\omega)}{n} - \frac{1}{2} \right| < \frac{1}{k} \right\}$$

Complement:

$$C^c = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \omega : \left| \frac{S_n(\omega)}{n} - \frac{1}{2} \right| \geq \frac{1}{k} \right\}$$

Chebyshev Inequality and Variance

For fixed k, define:

$$A_k = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \omega : \left| \frac{S_n(\omega)}{n} - \frac{1}{2} \right| \geq \frac{1}{k} \right\}$$

By Chebyshev:

$$P\left(\left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \frac{1}{k} \right) \leq k^2 \cdot E\left[\left(\frac{S_n}{n} - \frac{1}{2} \right)^2 \right]$$

Compute variance:

$$Z_i = X_i - \frac{1}{2}, \quad E[Z_i] = 0, \quad \delta(Z_i) = \frac{1}{4}$$

$$\delta(S_n/n) = \delta\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} \delta(X_i) = \frac{1}{4n}$$

Thus:

$$P\left(\left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \frac{1}{k} \right) \leq k^2 \cdot \frac{1}{4n}$$

Completing the Proof

$$\begin{aligned} P(A_k) &= P\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} D_{n,k}\right) \\ &= \lim_{N\to\infty} P\left(\bigcup_{n=N}^{\infty} D_{n,k}\right) \\ &\leq \lim_{N\to\infty} \sum_{n=N}^{\infty} P(D_{n,k}) \\ &\leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{k^2}{4n} = 0 \end{aligned}$$

where $D_{n,k} = \left\{\omega : \left|\frac{S_n(\omega)}{n} - \frac{1}{2}\right| \geq \frac{1}{k}\right\}$.
Thus $P(A_k) = 0$ for all k, so:

$$P(C^c) = P\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} P(A_k) = 0$$

Random Vectors and Measurability

Definition

Let $(\Omega, \mathcal{F}, P)$ probability space, $(\Omega', \mathcal{F}')$ measurable space. A function:

$$X : \Omega \to \Omega'$$

is a **random vector** if X is $\mathcal{F}/\mathcal{F}'$-measurable:

$$\forall A \in \mathcal{F}', \quad X^{-1}(A) \in \mathcal{F}$$

Generated σ-algebra

The σ-algebra generated by X:

$$\sigma(X) = \{X^{-1}(A) : A \in \mathcal{F}'\}$$

is the smallest σ-algebra making X measurable.

Distribution of Random Vectors

Definition

The **distribution** (or law) of X is the measure P_X on $(\Omega', \mathcal{F}')$:

$$P_X(A) = P(X^{-1}(A)) = P(X \in A), \quad \forall A \in \mathcal{F}'$$

Example (Binomial distribution)

For S_n in coin tosses:

$$P_{S_n}(\{k\}) = P(S_n = k) = \binom{n}{k} \frac{1}{2^n}, \quad k = 0, 1, \ldots, n$$

Example (Dyadic representation)

$X : ([0,1), \mathcal{B}, m) \to (\{0,1\}^{\mathbb{N}}, \sigma(\mathcal{F}_*))$ with:

$$x = \sum_{i=1}^{\infty} \frac{X_i(x)}{2^i}, \quad X_i(x) = i\text{-th binary digit}$$

Distribution Functions

Definition

For random variable $X : \Omega \to \mathbb{R}$, the **cumulative distribution function (CDF)** is:

$$F_X(x) = P(X \leq x) = P(\{\omega : X(\omega) \leq x\})$$

Theorem (Properties of CDF)

Any CDF satisfies:

1. *Monotonicity:* $x < y \implies F_X(x) \leq F_X(y)$
2. $\lim_{x\to-\infty} F_X(x) = 0, \quad \lim_{x\to\infty} F_X(x) = 1$
3. *Right-continuity:* $\lim_{y\downarrow x} F_X(y) = F_X(x)$

Conversely, any such function is a CDF for some random variable.

Examples of Distributions

Example (Uniform distribution on [0,1)

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$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Measure is Lebesgue measure on [0,1].

Example (Standard normal distribution)

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

Theorem (Uniqueness of distribution)

If $F_X = F_Y$, then X and Y have the same distribution.

Measure Extension Techniques

Theorem (Dynkin's π-λ Theorem)

Let $\mathcal{P}$ be a π-system (closed under intersection) and $\mathcal{L}$ be a λ-system:

1. $\Omega \in \mathcal{L}$
2. $A \in \mathcal{L} \implies A^c \in \mathcal{L}$
3. *Disjoint* $\{A_n\} \subset \mathcal{L} \implies \bigcup_n A_n \in \mathcal{L}$

If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Application: To show two measures agree on $\sigma(\mathcal{P})$, verify:

1. They agree on a π-system $\mathcal{P}$
2. The collection where they agree is a λ-system

Completing Measure Spaces

Definition

A measure space $(\Omega, \mathcal{F}, \mu)$ is **complete** if:

$$N \in \mathcal{F}, \mu(N) = 0, \quad A \subseteq N \implies A \in \mathcal{F}$$

Theorem (Completion)

Any measure space can be completed:

$$\overline{\mathcal{F}} = \{A \subseteq \Omega : \exists B, C \in \mathcal{F} \text{ with } B \subseteq A \subseteq C, \mu(C \setminus B) = 0\}$$

with $\overline{\mu}(A) = \mu(B) = \mu(C)$.

Example

Lebesgue σ-algebra is the completion of Borel σ-algebra under Lebesgue measure.

Coin Toss vs. Lebesgue Measure

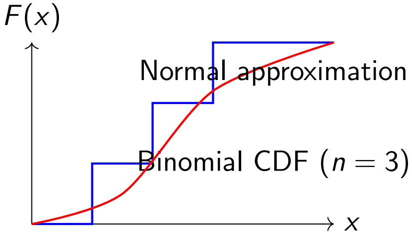
Deep connection:

$$(\{0,1\}^{\mathbb{N}}, \sigma(\mathcal{F}_*), P_{\text{coin}}) \cong ([0,1], \mathcal{B}, m)$$

$$\omega \mapsto x = \sum_{i=1}^{\infty} \frac{\omega_i}{2^i}$$

Consequences:

- SLLN for coin tosses $\iff$ SLLN for Lebesgue-almost every $x \in [0,1]$
- Binomial distribution $\to$ Normal distribution via CLT



Advanced Topics Preview

Further developments:

- **Martingales:** Fair game processes $E[X_{n+1}|\mathcal{F}_n] = X_n$
- **Stochastic integration:** Itô calculus for dW_t (Brownian motion)
- **Ergodic theory:** $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega) = \int f d\mu$
- **Malliavin calculus:** Differentiation in Wiener space

Fundamental sequence

Coin toss $\rightarrow$ Random walk $\rightarrow$ Brownian motion $\rightarrow$ Stochastic calculus

Foundations of Stochastic Analysis

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Measure Theory $\rightarrow$ Probability $\rightarrow$ Limit Theorems $\rightarrow$ Advanced Theory