Stochastic Optimal Control

How to navigate with noise

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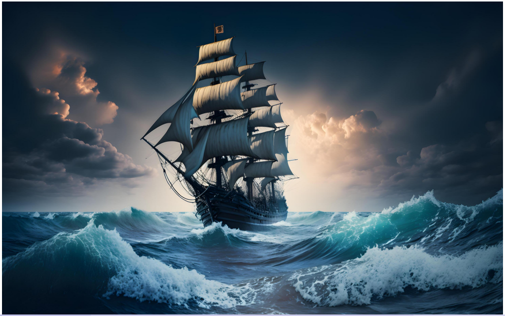
Motivation & Backstory

- Examples: autopilot in gusts, central bank under shocks, robotic navigation with noisy sensors.
- Core operational question: choose an adaptive policy to **minimize expected cumulative cost**.
- Two parallel historical strategies:
  1. *Dynamic Programming* (Bellman) $\rightarrow$ HJB PDE.
  2. *Variational / Maximum Principle* (Pontryagin) $\rightarrow$ adjoint BSDEs.



Problem statement — dynamics and cost (precise)

Probability space: $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ with m-dim Brownian motion W_t.

State (SDE):

$$\boxed{dX_s = b(s, X_s, u_s)\,ds + \sigma(s, X_s, u_s)\,dW_s, \qquad X_t = x.}$$

Cost functional:

$$J(t,x;u) = \mathbb{E}\Big[\int_t^T f(s, X_s, u_s)\,ds + g(X_T)\Big],$$

$$\boxed{V(t,x) := \inf_{u\in\mathcal{A}} J(t,x;u)}$$

Definitions:

- $b : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n$ (drift).
- $\sigma : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^{n\times m}$ (diffusion).
- f running cost, g terminal cost.
- $U \subset \mathbb{R}^k$ control set; $\mathcal{A}$ admissible controls (progressively measurable,

Admissible controls vs Policies (Intuition)

- **Open-loop control:** any adapted process u_t — may depend on other randomness.
- **Policy / feedback:** mapping $\pi : [0, T] \times \mathbb{R}^n \to U$, implement $u_t = \pi(t, X_t)$.

Why policies? Markov policies are attractive: closed-loop stability, implementability, DPP often yields Markov optimal policies.

placeholder_policy.png

Dynamic Programming Principle (DPP)

Theorem (DPP (informal))

For any stopping time τ with $t \le \tau \le T$,

$$V(t,x) = \inf_{u \in \mathcal{A}} \mathbb{E}\Big[\int_t^\tau f(s, X_s^{t,x,u}, u_s)\, ds + V(\tau, X_\tau^{t,x,u}) \Big].$$

Intuition: optimal control splits: optimize on $[t, \tau]$ and then act optimally from τ onward. **Technical remarks:** requires measurability and

concatenation properties of admissible controls; see Fleming–Soner.

Itô's formula (reminder)

If $\phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ then along SDE $dX_s = b\,ds + \sigma\,dW_s$,

$$d\phi(s, X_s) = \big(\phi_t + \mathcal{L}\phi\big)(s, X_s)\,ds + (\nabla_x \phi)^\top \sigma\,dW_s,$$

where

$$\mathcal{L}\phi = b \cdot \nabla_x \phi + \tfrac{1}{2}\,\mathrm{tr}(\sigma\sigma^\top D_x^2 \phi).$$

Note: the stochastic integral has zero expectation (martingale) under standard integrability.

Derivation: DPP $\Rightarrow$ HJB (step 1)

Start: for small $h > 0$ using DPP,

$$V(t,x) = \inf_{u \in \mathcal{A}} \mathbb{E}\Big[\int_t^{t+h} f(s, X_s, u_s)\, ds + V(t+h, X_{t+h}) \Big].$$

Apply Itô to $V(t+h, X_{t+h})$:

$$V(t+h, X_{t+h}) - V(t,x) = \int_t^{t+h} \big(V_t + \mathcal{L}^{u_s} V\big)(s, X_s)\, ds + \int_t^{t+h} (\nabla_x V)^\top \sigma\, dW_s.$$

Take expectation: the martingale term disappears.

Derivation: DPP $\Rightarrow$ HJB (step 2)

Divide by h, let $h \downarrow 0$, and use continuity to obtain (for classical V):

$$0 = \inf_{u \in U} \{ f(t,x,u) + V_t(t,x) + \mathcal{L}^u V(t,x) \}.$$

Rearranged (backward PDE):

$$\boxed{-V_t(t,x) = \inf_{u \in U} \left\{ f + b \cdot \nabla_x V + \tfrac{1}{2} \operatorname{tr}(\sigma \sigma^\top D_x^2 V) \right\}.}$$

This is the Hamilton–Jacobi–Bellman (HJB) PDE.

HJB — Anatomy and meaning

- $-V_t$: how the minimal future cost changes moving backward in time.
- f: immediate running cost (infinitesimal).
- $b \cdot \nabla_x V$: deterministic drift's first-order effect on continuation value.
- $\frac{1}{2}\operatorname{tr}(\sigma\sigma^\top D_x^2 V)$: diffusion's second-order (variance) effect — curvature matters.
- $\inf_{u \in U}\{\cdots\}$: choose control that minimizes immediate + infinitesimal expected *change* in value.

Assumptions: what we used and why

1. **Regularity of V:** $V \in C^{1,2}$ to apply Itô pointwise.
2. **SDE well-posedness:** b, σ Lipschitz in x (uniform in u) $\Rightarrow$ unique strong solution.
3. **Integrability:** growth conditions to justify martingale expectation $=$ 0 and dominated convergence.
4. **Control set:** U measurable; for pointwise minimization require continuity in u or measurable-selection.

Remark: if V not $C^{1,2}$ we use viscosity-solution framework (Part 3/4).

Next steps

1. Classical Verification Theorem — statement and detailed proof.
2. Existence of optimal controls — measurable selection, relaxed controls (Young measures).
3. Linear-Quadratic-Gaussian (LQG) problem: scalar matrix, Riccati ODE and ARE.

Verification Theorem (classical) — statement

Theorem (Verification – classical)

Suppose the following hold:

1. $V \in C^{1,2}([0,T] \times \mathbb{R}^n)$ *solves HJB:*

$$-V_t(t,x) = \inf_{u \in U}\{f + \mathcal{L}^u V\}, \qquad V(T,x) = g(x).$$

2. *For each* (t,x) *there exists a measurable selector* $\hat{u}(t,x) \in U$ *achieving the infimum.*
3. *The closed-loop SDE under* $\hat{u}(s,X_s) =: \hat{u}(s,X_s^{t,x})$ *has a unique strong solution.*

Then the feedback $\hat{u}(s,X_s)$ *is optimal and* $V(t,x) = J(t,x;\hat{u})$.

Verification — proof (step 1)

Goal: show $J(t,x;\hat{u}) = V(t,x)$ and $J(t,x;u) \geq V(t,x)$ for any $u \in \mathcal{A}$.

Step 1: Ito on $V(s, X_s^{\hat{u}})$:

$$dV(s, X_s^{\hat{u}}) = \big(V_s + \mathcal{L}^{\hat{u}} V\big)(s, X_s^{\hat{u}})\, ds + (\nabla_x V)^\top \sigma(\cdot)\, dW_s.$$

Use HJB equality at the minimizing control:

$$V_s + \mathcal{L}^{\hat{u}} V + f(\cdot, \hat{u}) = 0.$$

Integrate and take expectations to obtain $V(t,x) = J(t,x;\hat{u})$.

Verification — proof (step 2)

Step 2: For arbitrary $u \in \mathcal{A}$, apply Ito to $V(s, X_s^u)$:

$$V(T, X_T^u) - V(t, x) = \int_t^T \big(V_s + \mathcal{L}^u V\big)(s, X_s^u)\, ds + M_T - M_t.$$

Using HJB (infimum) yields $V_s + \mathcal{L}^u V + f \geq 0$. Taking expectation:

$$V(t, x) \leq \mathbb{E}\Big[\int_t^T f(s, X_s^u, u_s)\, ds + g(X_T^u)\Big] = J(t, x; u).$$

Conclude V is the minimal cost and $\hat{u}$ optimal.

Remarks and caveats

- Smoothness ($C^{1,2}$) is strong — often fails; viscosity theory handles nonsmooth V.
- Existence of measurable minimizer may require compactness/continuity; otherwise use measurable selection theorems.
- The martingale term expectation $= 0$ uses integrability (square-integrable gradients).
- Verification provides a certificate: if you can solve HJB, you can verify optimality.

Existence of optimal controls (I): measurable selection

Idea: if U compact and the Hamiltonian $\mathcal{H}(t,x,u) = f + \mathcal{L}^u V$ is continuous in u, then the argmin set is nonempty and has measurable selections.

Proposition (informal): If $(t,x) \mapsto \arg\min_{u\in U} \mathcal{H}(t,x,u)$ has closed nonempty values and measurable graph, there exists a measurable selector $\hat{u}(t,x)$.

(Kuratowski–Ryll-Nardzewski measurable selection theorem)

Existence of optimal controls (II): relaxed controls

Problem: minimizing sequences of controls may oscillate; no pointwise limit in U.

Solution: allow controls to be probability measures on U at each time: $\mu_t \in \mathcal{P}(U)$. The dynamics use averages:

$$b^{\mu}(t,x) = \int_U b(t,x,u)\,\mu(du), \quad f^{\mu}(t,x) = \int_U f(t,x,u)\,\mu(du).$$

Theorem (informal): in many settings minimizing relaxed controls exist; under convexity one can recover ordinary controls.

Relaxed controls — intuition

- View a relaxed control as randomized instantaneous action — the law of the control is chosen adaptively.
- Compactness: space of probability measures on compact U is compact (Prokhorov).
- Lower semi-continuity of cost yields existence by direct method in calculus of variations.
- If Hamiltonian is convex in u, the barycenter of the measure is admissible and optimality can be recovered.

Why LQG? A solvable core

- LQG (Linear dynamics + Quadratic costs + Gaussian noise) is the canonical exactly-solvable model.
- It illustrates HJB $\rightarrow$ Riccati reduction, PMP equivalence, and gives explicit feedback.
- Provides intuition about stability, certainty-equivalence, and role of noise.

Scalar LQG: model and cost

Dynamics (scalar):

$$dX_t = aX_t\,dt + bu_t\,dt + \sigma\,dW_t, \qquad X_0 = x.$$

Quadratic cost:

$$J(u) = \mathbb{E}\Big[SX_T^2 + \int_0^T (QX_t^2 + Ru_t^2)\,dt\Big],$$

with constants $Q \geq 0,\ R > 0,\ S \geq 0.$

Ansatz: $V(t,x) = P(t)x^2 + r(t).$

Riccati derivation (scalar) — main steps

- Compute $V_t = P'x^2 + r'$, $V_x = 2Px$, $V_{xx} = 2P$.
- Generator:

$$\mathcal{L}^u V = (ax + bu)2Px + \frac{1}{2}\sigma^2 2P = 2aPx^2 + 2bPxu + P\sigma^2.$$

- HJB minimization in u yields first-order condition $2Ru + 2bPx = 0 \Rightarrow u^* = -\dfrac{bP}{R}x$.
- Riccati ODE:

$$-P' = Q + 2aP - \frac{b^2}{R}P^2, \quad P(T) = S.$$

Matrix LQG and matrix Riccati ODE

Dynamics:

$$dX_t = AX_t\,dt + Bu_t\,dt + \Sigma\,dW_t, \qquad X \in \mathbb{R}^n.$$

Cost:

$$J(u) = \mathbb{E}\Big[X_T^\top S X_T + \int_0^T (X_t^\top Q X_t + u_t^\top R u_t)\,dt\Big],$$

with $Q, S \succeq 0$, $R \succ 0$.

Matrix Riccati ODE:

$$-\dot{P} = Q + A^\top P + PA - PBR^{-1}B^\top P, \quad P(T) = S,$$

and optimal feedback $u^* = -R^{-1}B^\top P(t)X_t$.

LQG intuition and demo placeholders

- $P(t)$ measures marginal cost-per-unit quadratic state: bigger $P \Rightarrow$ stronger control.
- Closed-loop drift becomes $A - BR^{-1}B^{\top}P$; ensure stability.
- Graph in my pc

Next steps

1. Viscosity solutions: motivation, definition, and proof that value function is a viscosity solution (DPP $\Rightarrow$ viscosity).
2. Comparison principle (sketch) and uniqueness; consequences.
3. Numerics: monotone schemes and Barles–Souganidis theorem (semi-Lagrangian).
4. Stochastic Pontryagin Maximum Principle (PMP): full derivation, adjoint BSDE, stationarity, sufficiency, FBSDE.
5. BSDE $\leftrightarrow$ PDE connections (Pardoux–Peng) and limits for fully nonlinear HJB.

Why viscosity solutions?

- HJB is fully nonlinear, second-order; classical $C^{1,2}$ solutions may not exist.
- Value functions often have kinks (nonsmooth) due to control switching, boundaries, or degeneracy.
- Viscosity theory gives:
  - a robust weak solution notion,
  - stability under uniform limits,
  - a comparison principle $\Rightarrow$ uniqueness.

Definition: viscosity sub-/supersolution

Let $F(t, x, p, X)$ denote the PDE operator. For HJB,

$$-V_t + F(t, x, \nabla V, D^2 V) = 0, \qquad F(t, x, p, X) = \sup_{u \in U} \{-f - b \cdot p - \tfrac{1}{2} \operatorname{tr}(\sigma\sigma^\top X)\}$$

Definition

V is a viscosity *subsolution* if whenever $\phi \in C^{1,2}$ and $V - \phi$ has a local maximum at $(\bar{t}, \bar{x})$,

$$-\phi_t(\bar{t}, \bar{x}) + F(\bar{t}, \bar{x}, \nabla\phi, D^2\phi) \leq 0.$$

Analogous definition for *supersolution* (with local minimum and ≥ 0). If both, V is a viscosity solution.

Proof sketch: DPP $\Rightarrow$ V is viscosity subsolution

Idea: take test function ϕ touching V from above at (t_0, x_0). Use DPP on a short interval $[t_0, t_0 + h]$, replace V by ϕ (since $\phi \geq V$ nearby), apply Itô to ϕ, divide by h, let $h \downarrow 0$ and obtain the inequality.

Key points:

- Need local maximum and control of exit times from neighborhood.
- Martingale term expectation disappears.
- Justifies the viscosity inequality (no classical derivatives of V required).

DPP $\Rightarrow V$ is viscosity supersolution (sketch)

Similar argument but use ε-optimal controls and test functions touching from below. Carefully build concatenation using ε-optimal controls to get the opposite inequality.

Comparison principle — statement and sketch

Theorem (Comparison (informal))

Let u be bounded upper-semicontinuous viscosity subsolution and w bounded lower-semicontinuous viscosity supersolution of the HJB. Under standard structure/continuity conditions, $u \le w$ on $[0, T] \times \mathbb{R}^n$.

Sketch of proof:

- Consider $\Phi_\varepsilon(t, x, s, y) = u(t, x) - w(s, y) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\varepsilon}$.
- Let maximum occur at $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ and use Crandall–Ishii lemma to obtain jets.
- Use sub- and supersolution inequalities and send $\varepsilon \downarrow 0$.

(Technical details in Crandall–Ishii–Lions.)

Consequences

- By DPP we know V is a viscosity solution; comparison gives uniqueness $\Rightarrow$ V is the unique viscosity solution.
- Numerical methods that converge to the unique viscosity solution are meaningful.
- Comparison requires structural conditions — e.g., continuity of coefficients, properness (degenerate ellipticity).

Barles–Souganidis (1991, informal): a numerical scheme that is *consistent*, *stable*, and *monotone* converges uniformly (on compacts) to the unique viscosity solution of the PDE.

Implications for control:

- Semi-Lagrangian schemes are monotone consistent for HJB (good for diffusion).
- Finite-difference schemes must be constructed carefully (upwind, monotone interpolation).

Semi-Lagrangian scheme (idea)

For small Δt,

$$V(t,x) \approx \min_{u\in U} \Big\{ f(t,x,u)\Delta t + \mathbb{E}\big[V(t+\Delta t, X_{t+\Delta t}^{t,x,u}) \big] \Big\}.$$

Approximate expectation by quadrature and use interpolation for the off-grid points. This yields a monotone update and (by Barles–Souganidis) converges to viscosity solution.

Transition to stochastic PMP

- Viscosity theory gives existence/uniqueness of V even when nonsmooth.
- PMP yields necessary conditions via adjoint processes (BSDEs) — constructive and useful for high-dimensional or model-based control.
- We now derive the stochastic maximum principle via first variations.

PMP setup: first variation (notation)

Fix an admissible control u^* and corresponding state X^*. Consider a variation $u^\varepsilon = u^* + \varepsilon\delta u$ (admissible). Let X^ε be the perturbed state. Define the variational state

$$Y_t := \frac{d}{d\varepsilon} X_t^\varepsilon \bigg|_{\varepsilon=0}.$$

The linearized (variational) SDE will involve derivatives $b_x, b_u, \sigma_x, \sigma_u$ evaluated at (t, X_t^*, u_t^*).

Variational equation for Y

Under smoothness,

$$dY_t = \big(b_x(t)Y_t + b_u(t)\delta u_t\big)\,dt + \big(\sigma_x(t)Y_t + \sigma_u(t)\delta u_t\big)\,dW_t, \qquad Y_0 = 0.$$

(Shorthand $b_x(t) = \partial_x b(t, X_t^*, u_t^*)$, etc.)

Gateaux derivative of the cost

The first variation of cost is

$$\frac{d}{d\varepsilon} J(u^\varepsilon)\Big|_{\varepsilon=0} = \mathbb{E}\Big[\int_0^T \big(f_x(t) \cdot Y_t + f_u(t) \cdot \delta u_t\big)\, dt + g_x(X_T^*) \cdot Y_T \Big].$$

We will remove Y by duality (introduce adjoint p and q).

Adjoint BSDE

Define adjoint pair (p_t, q_t) satisfying backward SDE

$$dp_t = -\big(b_x(t)^\top p_t + \sigma_x(t)^\top q_t + f_x(t)^\top\big)\, dt + q_t\, dW_t, \qquad p_T = g_x(X_T^*).$$

Using integration by parts one can express the Y-terms in the variation through p, q and obtain stationarity condition.

Stationarity condition (PMP)

The first-order optimality condition is (pointwise in time)

$$f_u(t) + b_u(t)^\top p_t + \sigma_u(t)^\top q_t = 0,$$

or equivalently

$$u_t^* = \arg\min_{u \in U} \mathcal{H}(t, X_t^*, u, p_t, q_t),$$

where $\mathcal{H}(t, x, u, p, q) = f + p \cdot b + \mathrm{tr}(q^\top \sigma)$.

Sufficiency under convexity

If the Hamiltonian $\mathcal{H}(t, x, u, p, q)$ is convex in (x, u) (or convex in u with appropriate conditions) and the pair (X^*, p, q) satisfies the FBSDE + stationarity, then u^* is optimal. Convexity turns the necessary condition into a global minimum.

Forward–Backward SDEs (FBSDE) and solvability

The optimality system is a coupled FBSDE:

$$\begin{cases} dX_t = b(t, X_t, u_t)\, dt + \sigma(t, X_t, u_t)\, dW_t, \\ dp_t = -H_x(t, X_t, u_t, p_t, q_t)\, dt + q_t\, dW_t, \\ \text{stationarity } H_u(t, X_t, u_t, p_t, q_t) = 0, \\ X_0 = x, \quad p_T = g_x(X_T). \end{cases}$$

Solvability: existence/uniqueness for nonlinear FBSDEs is delicate; monotonicity (Pardoux–Peng) or contraction arguments are used.

BSDEs $\leftrightarrow$ PDEs (Pardoux–Peng)

For semilinear PDEs of form

$$-Y_t = \mathcal{L}Y + F(t, x, Y, \nabla Y \sigma),$$

solutions can be represented via BSDEs: the pair (Y_t, Z_t) solves a BSDE with driver F. This is the nonlinear Feynman–Kac formula (Pardoux–Peng, 1990).

Caveat: fully nonlinear HJB (control enters diffusion nonlinearly) generally requires second-order BSDEs (2BSDEs) or other techniques.

Final Steps

1. Worked nonlinear example (1D): HJB, PMP, numerical solution.
2. Semi-Lagrangian scheme — algorithm and practical tips.
3. Policy iteration Howard's algorithm (pseudo-code).
4. FBSDE numerical method for PMP (overview).
5. Exercises with solution sketches.
6. Final recommendations and slide bundle export instructions.

Worked example: 1D nonlinear control (problem)

Model:

$$dX_t = u_t\,dt + \sigma\,dW_t, \qquad X_0 = x.$$

Cost:

$$J(u) = \mathbb{E}\Big[\int_0^T \big(x_t^4 + u_t^2\big)\,dt + SX_T^2\Big].$$

HJB (backward):

$$-V_t = \inf_{u\in[-U,U]}\{x^4 + u^2 + uV_x + \tfrac{1}{2}\sigma^2 V_{xx}\}, \qquad V(T,x) = Sx^2.$$

PMP stationarity (formal):

$$2u + V_x = 0 \quad \Rightarrow \quad u_{\text{PMP}}(t,x) = -\tfrac{1}{2}V_x(t,x).$$

Semi-Lagrangian scheme (idea)

$$V(t,x) \approx \min_{u\in U}\Big\{f(x,u)\Delta t + \mathbb{E}\big[V(t+\Delta t, X^{t,x,u}_{t+\Delta t})\big]\Big\}.$$

- compute $\mu = x + u\Delta t$,
- approximate $\mathbb{E}[V(\mu + \sigma\sqrt{\Delta t}Z)]$ with quadrature (e.g., two-point symmetric),
- interpolation for off-grid values must be monotone (linear).

Placeholder: show numerical plot of $V(0,x)$ and comparison of HJB control vs PMP control.

Semi-Lagrangian pseudo-code

Inputs: grid x_i, time steps t_n, control grid u_j, terminal $V_i^N = Sx_i^2$.

For $n = N-1 \to 0$:

1. For each i and each u_j compute $\mu = x_i + u_j\Delta t$.
2. Compute quadrature nodes $y_k = \mu + \sigma\sqrt{\Delta t}\, z_k$ and weights w_k.
3. Interpolate V^{n+1} at y_k (monotone interpolation).
4. Set $\mathrm{cost}_j = (x_i^4 + u_j^2)\Delta t + \sum_k w_k V^{n+1}(y_k)$.
5. $V_i^n = \min_j \mathrm{cost}_j$ and store argmin as policy.

Output: V_i^0 and policy table.

Practical tips for semi-Lagrangian

- Use monotone (piecewise linear) interpolation to preserve convergence.
- Control grid: refine near expected optimal region; allow adaptive control discretization.
- Quadrature: 2-point symmetric often sufficient for small Δt; Gauss–Hermite for higher accuracy.
- Domain truncation: choose $x \in [x_{\text{min}}, x_{\text{max}}]$ wide enough so probability of leaving is negligible.
- Time step: CFL not strict for SL, but Δt affects consistency and error.

Policy iteration (Howard) — pseudo-code

Inputs: initial policy $u^0(t_n, x_i)$.

Repeat until convergence:

1. **Policy evaluation:** For fixed u^k, solve linear equation for V^k (backward in time or linear system).
2. **Policy improvement:** For each (t_n, x_i) set

$$u^{k+1}(t_n, x_i) = \arg\min_u \{ f(x_i, u)\Delta t + \mathbb{E}[V^k(t_{n+1}, X^u_{t_{n+1}})] \}.$$

Remarks: often converges faster than value iteration; evaluation step may be linear solve.

FBSDE / PMP numerical approach (overview)

- Solve forward SDE for X with guessed control u.
- Solve backward BSDE for (p, q) given X (e.g., backward regression Monte Carlo).
- Update control via stationarity $u = \arg\min \mathcal{H}(\cdot, p, q)$ (or gradient step).
- Iterate until convergence (policy update loop).

Remarks: regression basis or neural nets are commonly used to represent conditional expectations in high dimension.

Worked numerical experiment (recommended parameters)

Example parameters (recommended):

$$\sigma = 0.8, \quad S = 1.0, \quad T = 0.6, \quad x \in [-3, 3], \quad N_x = 401, \quad N_t = 120, \quad u \in$$

Diagnostics to show:

- $V(0, x)$ plot,
- HJB-derived discrete policy vs PMP $u = -\frac{1}{2} V_x$ plot,
- simulation under both policies: empirical cost comparison.

Placeholder: include earlier computed plots here.

Exercises (with solution sketches)

Derive the scalar Riccati ODE for LQG and show how PMP gives the same feedback via $p_t = 2P(t)X_t$. Implement the semi-Lagrangian scheme for the 1D example; show convergence as $\Delta t, \Delta x \to 0$. Using PMP, set up the FBSDE system for the 1D example and sketch a numerical iteration scheme to solve it.

Solutions (sketches)

Ex 1 (Riccati): (Sketch) plug $V = P(t)x^2 + r(t)$ into HJB, minimize in u, get $-P' = Q + 2aP - (b^2/R)P^2$. PMP: adjoint $p_t = 2P(t)X_t$ satisfies adjoint BSDE; stationarity yields same u^*.

Ex 2 (SL implementation): (Sketch) use grids, interpolation, 2-point quadrature, compute V^n backward; check monotone interpolation and decrease $\Delta t, \Delta x$.

Ex 3 (FBSDE): (Sketch) forward simulate X with guessed u; backward regression to estimate p, q; update u via stationarity; iterate.

Final recommendations and slide bundle

- Combine Parts 1–4 into one Beamer file or keep as four modules.
- Replace placeholder images with plotted PNGs (generated by your code).
- For handout: export as PDF and include code appendix (Jupyter notebook).
- For a 90–120 min lecture pick selections: prefer live derivation + one numerical demo.

References and further reading: Fleming & Soner; Yong & Zhou; Crandall–Lions; Pardoux–Peng; Barles–Souganidis.