The Calculus of Uncertainty: Stochastic Processes & Their Mathematical Universe

Hooman Zare

SUT

July 25, 2025

Formal Definition & Measure-Theoretic Grounding

Definition

A stochastic process is a collection $\{X_t : t \in T\}$ of random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where:

- T: index set (time/space)
- Ω: sample space
- $\mathcal{F}$: σ-algebra
- $\mathbb{P}$: probability measure

Classification:

$$\text{Time: } T = \begin{cases} \mathbb{Z}^+ & \text{(discrete)} \\ \mathbb{R}^+ & \text{(continuous)} \end{cases}$$

$$\text{State: } S = \begin{cases} \text{countable} & \text{(chain)} \\ \mathbb{R}^d & \text{(field)} \end{cases}$$

Theorem (Kolmogorov Consistency)

Finite-dimensional distributions must satisfy:

$$\mu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \mu_{t_{\sigma(1)},\dots,t_{\sigma(k)}}(A_{\sigma(1)} \times \cdots \times A_{\sigma(k)})$$

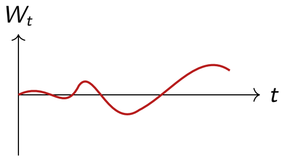
for any permutation σ.

From Bernoulli to Kolmogorov: Foundational Theorems

Landmark Contributions:

- 1713: Bernoulli's LLN
- 1827: Brownian motion (Brown)
- 1905: Einstein's explanation
- 1913: Markov chains
- 1923: Wiener measure
- 1933: Kolmogorov axioms

Andrey Kolmogorov (1903-1987)



Wiener Process Path

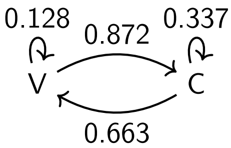
Theorem (Kolmogorov Extension)

If $\{\mu_{t_1,\ldots,t_k}\}$ satisfy consistency conditions, then $\exists!$ probability measure $\mathbb{P}$ on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ with those finite-dim distributions.

Markov's Linguistic Revolution

Markov's 1913 Analysis:

- Studied Pushkin's Eugene Onegin
- Vowel/Consonant sequences
- Discovered memoryless property



" "

(V-C-C-V-C-V-C-C)

Theorem (Markov Property)

$$\mathbb{P}(X_{t+1} = x_{t+1} | X_t = x_t, \ldots, X_0 = x_0) = \mathbb{P}(X_{t+1} = x_{t+1} | X_t = x_t)$$

Transition Matrix for Russian:

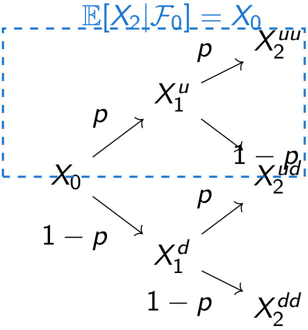
$$P = \begin{pmatrix} 0.128 & 0.872 \\ 0.663 & 0.337 \end{pmatrix}$$

(Vowel $\rightarrow$ Consonant: 87.2%)

The Mathematical Zoo: Classification & Properties

Canonical Processes:

- Wiener process: $dW_t \sim \mathcal{N}(0, dt)$
- Poisson process: $\mathbb{P}(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$
- Markov chains: $P(X_{n+1}|X_n)$
- Martingales: $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$



Theorem (Lévy Characterization)

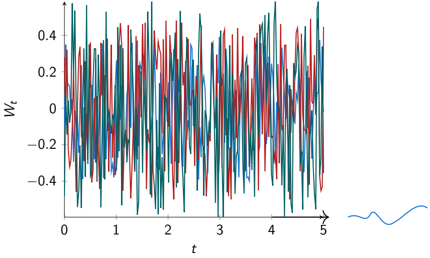
W_t is Brownian motion iff:

1. Continuous paths a.s.
2. $W_0 = 0$
3. $\langle W \rangle_t = t$

Brownian Motion: The Universal Process

Properties:

- $W_0 = 0$
- Independent increments
- $W_t - W_s \sim \mathcal{N}(0, t - s)$
- Continuous paths
- Nowhere differentiable



Theorem (Scaling Property)

$$\{c^{-1/2} W_{ct}\}_{t \geq 0} \stackrel{d}{=} \{W_t\}_{t \geq 0}$$

Fractal Dimension $= 1.5$

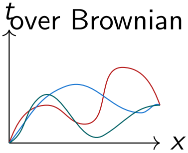
Theorem (Quadratic Variation)

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}(W_{tk/2^n} - W_{t(k-1)/2^n})^2 = t$$

The Unifying Language of Randomness

Field	Key Process	Fundamental Equation
Quantum Physics	Wiener path integrals	$K(x, t; x_0) = \int \mathcal{D}x\, e^{iS[x]/}$
Finance	Geometric Brownian motion	$dS_t = \mu S_t dt + \sigma S_t dW_t$
Biology	Branching processes	$\mathbb{E}[Z_n] = m^n$
Control Theory	Stochastic PDEs	$du = Au\, dt + B\, dW_t$
Machine Learning	Diffusion models	$dx_t = f(x_t, t)dt + g(t)d$

Path integral over Brownian trajecto



Theorem (Feynman-Kac Formula)

Solution to PDE:

Itô's Lemma:

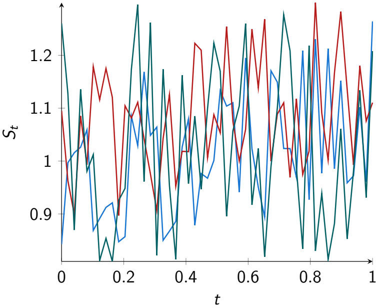
$$df(t, W_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x} dW_t$$

Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

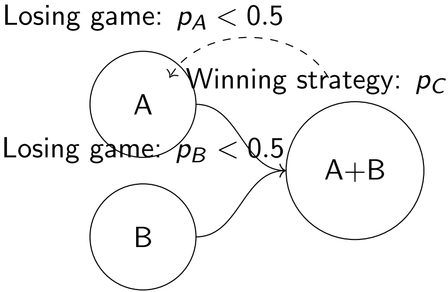
Applications:

- Option pricing
- Interest rate modeling
- Risk management



$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{t}{2}$$

Parrondo's Paradox



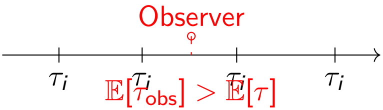
Inspection Paradox

Theorem (Renewal Theory)

For renewal process with i.i.d. τ_i:

$$\mathbb{E}[\tau_{\text{observed}}] = \frac{\mathbb{E}[\tau^2]}{\mathbb{E}[\tau]} \geq \mathbb{E}[\tau]$$

Equality iff τ constant.



Stochastic Resonance: Noise-Enhanced Detection

Phenomenon:

- Weak signal undetectable
- Add optimal noise
- Signal becomes detectable

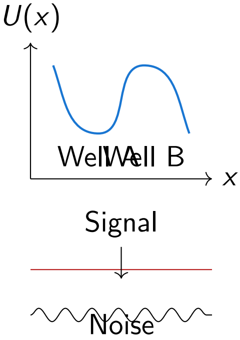
Mathematical Model:

$$dx = [-U'(x) + A\cos(\omega t)]dt + \sigma dW_t$$

where $U(x)$ is bistable potential

Applications:

- Neuroscience: weak signal detection
- Medical: balance therapy
- Climate: ice age cycles



Beyond Classical Theory: Current Research Horizons

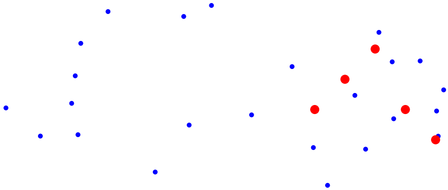
Rough Path Theory

$$dY_t = f(Y_t)d\mathbf{X}_t$$

where $\mathbf{X}_t = (X_t, \mathbb{X}_{s,t})$ with:

$$\mathbb{X}_{s,t} = \int_s^t (X_u - X_s) \otimes dX_u$$

Stochastic Geometry



Active Research Areas:

- Regularity structures (Hairer)
- Stochastic homogenization
- Kardar-Parisi-Zhang equation
- Neuro-stochastic processes
- Algorithmic randomness

Theorem (Universality Principle)

Limiting behavior of complex systems depends only on aggregate statistics:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} X_i^{(N)} \xrightarrow{d} \mu$$

independent of microscopic details

Stochastic PDEs: Modeling Complexity

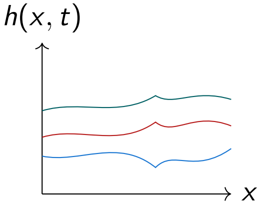
General Form:

$$\partial_t u = \mathcal{A}u + \sigma(u)\dot{W}$$

where $\dot{W}$ is space-time white noise

KPZ Equation:

$$\partial_t h = \nu\partial_x^2 h + \lambda(\partial_x h)^2 + \eta(x, t)$$



Interface growth with noise

Regularity Structures:

- Hairer's reconstruction theorem
- Renormalization techniques
- Models: Φ^4_3, KPZ, etc.

The Unreasonable Effectiveness of Randomness

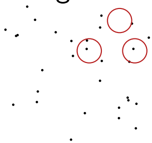
Core Principles:

1. Measure-theoretic foundation
2. Universality across scales
3. Emergent order from chaos
4. Noise as information carrier

Future Directions:

- Quantum stochastic calculus
- Stochastic machine learning
- Randomness in number theory
- Biological computation

Emergent Structure



Final Theorem:

> **Theorem (Benjamini-Schramm)**
>
> Local weak convergence of random graphs preserves global properties.

“Stochastic processes reveal the hidden deterministic soul of

References & Further Reading

Foundational Texts:

- Doob: *Stochastic Processes*
- Karatzas & Shreve: *Brownian Motion and Stochastic Calculus*
- Revuz & Yor: *Continuous Martingales and Brownian Motion*

Modern Research:

- Hairer: *Theory of Regularity Structures*
- Stroock: *Mathematics of Statistical Mechanics*
- Van Handel: *Stochastic Calculus and Applications*

Interdisciplinary:

- Oksendal: *Stochastic Differential Equations*
- Kallenberg: *Foundations of Modern Probability*
- Ebeling & Sokolov: *Statistical Thermodynamics and Stochastic Theory*