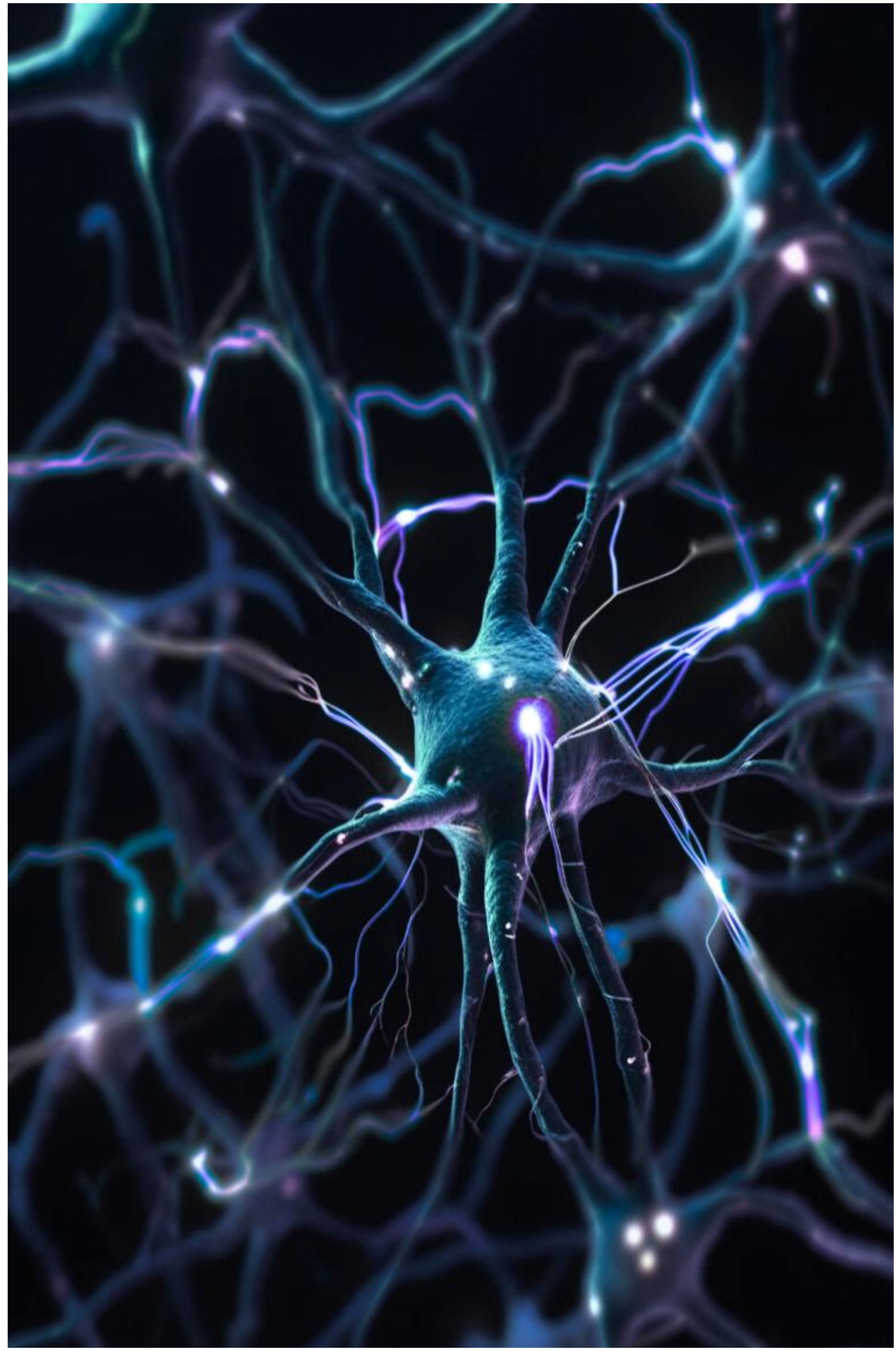
Computational Neuroscience

Session 4: Langevin Equation and Wiener Processes

Instructor: Ashkan Damavandi

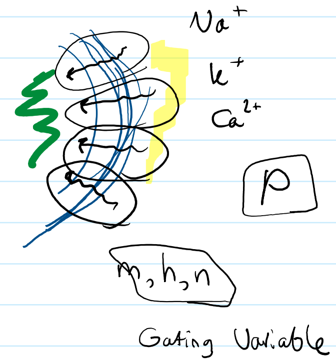
August 16, 2025



August 16, 2025

Stochastic Processes in Neuroscience

Fluctuations in Neural systems → Example: Ion channels in Membrane



Hodgkin Huxley model → Many Ion channels together → Each channel has some stochastic behavior

→ Each channel switches stochastically between closed or open states

Simple Example → A two-state process: a function which can only have two discrete values like σ_- & σ_+ → switches randomly between them with rates like r_- & r_+

Another Example → V_m (Membrane Potential) → 1) Synaptic Noise σ_1 ; 2) channel Noise σ_2

$V_m \to \sigma_1, \sigma_2$; r_1, r_2

Stochasticity in Membrane potential obeys: $\tau_m \frac{\delta V}{\delta t} = -(V - V_L) + \xi(t)$

τ_m → $\frac{C}{g_L}$ → C → capacitance, g_L → leak conductance

$$\tau_m \frac{\delta V}{\delta t} = -(V_m - V_L) + \xi(t)$$

$\xi(t)$: Fluctuations → Gaussian (white) Noise

Random Variable $\rightarrow$ a quantity that under given conditions, can assume different values V_m, $N([Na^+])$

Stochastic process $\rightarrow$ a collection of random variables like $\{X(t)\}_{t \geq 0}$

State at time $= t$

for example: * $V_m = V_m(t)$; $t \in \mathbb{R}^n$

Number of open/closed Ion channels ; $t \in \mathbb{Z}^+$

Stochastic Dynamical systems: How a Stochastic process evolves $\rightarrow$ 1) SDEs or 2) Master Equations

(SDE: Stochastic Differential Equation)

Deterministic Dynamical systems

* A random variable $\boxed{X}$ is completely speified by the range of values $\boxed{x}$ it can assume and the probability $p(x)$ with which each is assumed. $\vec{X} = \{x_1, x_2, \dots, x_n\}$

Statistical Independence: if realization of the outcome $X = x$ does not change the probability $p(y)$ that outcome $Y = y$ obtains & vice-versa, the outcomes $X = x$ and $Y = y$ are statistically independent and we would have:

$p(x \cap y) = p(x) \cdot p(y)$; Dependence $\rightarrow p(x \cap y) \neq p(x) \cdot p(y)$

Expected value: a function that turns the $p(x)$ into a sure variable called "the mean" of X $\longrightarrow$ Mean is the one number that best characterizes the possible values of a random variable ($\text{mean}\{X\}$ or $\langle X \rangle$ or μ) and it is defined like this:

$$\langle X \rangle = \sum_i x_i \, p(x_i)$$

$X \longrightarrow X^2 \longrightarrow X^4 \longrightarrow X^8 \longrightarrow \ldots$

* The square of a random variable is also a random variable. In fact, any algebraic function $f(x)$ of a random variable X is also a random variable. The expected value of the random variable $f(X)$ is defined by:

$$\langle f(X) \rangle = \sum_i f(x_i) \, p(x_i)$$

$\langle X \rangle$ parameterizes the random variable X $\rightarrow$ & so do all Moments $\langle X^n \rangle$ and the moments about the mean $\langle (X - \langle X \rangle)^n \rangle$.

* operation by which a random variable is turned into one of it's moments is one way of asking it to reveal its properties (parameters).

if $n=0 \rightarrow \langle (x - \langle X \rangle)^0 \rangle = \langle 1 \rangle = \sum_i p(x) = 1$

if $n=1 \rightarrow \langle (X - \langle X \rangle)^1 \rangle = \sum_i (x_i - \langle X \rangle) \, p(x_i) = \sum_i x_i \, p(x_i) - \langle X \rangle \sum_i p(x_i) = \underbrace{\langle X \rangle - \langle X \rangle}_{=0}$

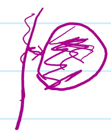
Variance: second moment about the mean: $\text{Var}\{X\}$ or $\sigma^2 = \langle (X - \langle X \rangle)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$

Newton's 2th law: $F_{net} = ma = m\frac{\delta v}{\delta t} = m\frac{\delta^2 x}{\delta t^2}$

* Suppose a particle is suspended in a fluid. The net force F_{net} will be the sum of 3 things:

1) Deterministic force ($F(x)$): Can arise from a external potential field. This force will be represented as $\boxed{F(x) = -\nabla U(x)}$, which $U(x)$ is a potential.

2) Drag force: Opposite of the particle's motion & most of the times is given like: $F_D \propto v \rightarrow F_D \simeq -\gamma v$
$\gamma \rightarrow$ Friction Coefficient.



3) Stochastic (Random) force $\eta(t)$: Representing the random collisions of the particles with the surrounding fluid's molecules $\rightarrow$ Gives the motion a random character

$F_{net} = \sum_i f_i = -\nabla U(x) - \gamma v + \eta(t)$

* Langevin Equation: $F_{net} = -\nabla U(x) + F_\delta + \eta(t)$

$$\rightarrow m\frac{\delta v}{\delta t} = -\nabla U(x) - \gamma v + \eta(t)$$

Overdamped limit: Systems that momentum relaxes in much faster timescale than position $\rightarrow m\frac{\delta v}{\delta t} \rightarrow 0 \longrightarrow 0 \simeq -\nabla U(x) - \gamma v + \eta(t)$

$m\frac{\delta v}{\delta t} = 0$

$$\xrightarrow{v = \frac{\delta x}{\delta t}} 0 = -\nabla U(x) - \gamma\frac{\delta x}{\delta t} + \eta(t) \Longrightarrow \boxed{\frac{\delta x}{\delta t} = \frac{1}{\gamma}\left(-\nabla U(x) + \eta(t)\right)} \rightarrow dx$$

A first order SDE, which describes the evolution of the particle's position (x), where v is directly determined by the instantaneous balance of the deterministic & random forces $\rightarrow$ Forgets its past momentum

Gaussian White Noise: $\eta(t) \to$ Typically & has these properties:

1) $\langle \eta(t) \rangle = 0$ 2) Uncorrelated time: $\langle \eta(t)\eta(t') \rangle = 2D\,\boxed{\delta(t-t')}$ → Diffusion Coefficient

$\delta(x)=0 \leftarrow x \neq 0$, $\int \delta(x)\,dx = 1$ — Dirac Delta function

3) Strength of the noise is related to the friction coefficient & temperature

→ Einstein relation: $D = \frac{k_B T}{\gamma}$ (k_B → Boltzman Constant)

* $\langle \eta(t)\eta(t') \rangle$ → Autocorrelation function for $\eta(t)$ → Calculates the statistical relationship between a random force in t' & t

$\eta(t)$ → 1) $\langle \eta(t) \rangle = 0$; 2) $\langle \eta(t)\eta(t') \rangle =$

$t \to \eta(t)$ $t = t'$

$t' \to \eta(t')$ $t \neq t'$

$x \neq 0 \to \delta(x) = 0$

$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$

T

$\eta(t)$

γ

$D = \frac{k_B T}{\gamma}$

Brownian Motion: Random movement of microscopic particles, suspended in a fluid.

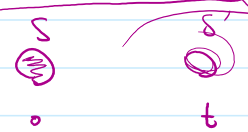
Wiener process: One-Dimensional Brownian motion. A Wiener process ($W(t)$) has these properties:

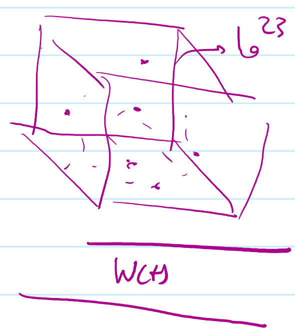
1) in $t=0 \to W(0)=0$ (starts at the origin of time points)

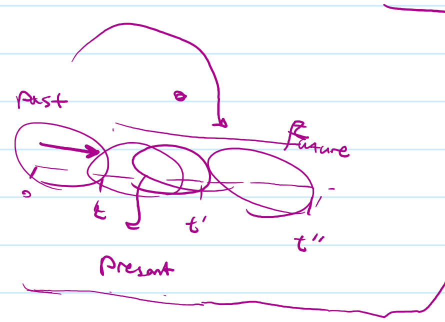
2) $W(t)$ has a continuous path over time.

*3) knowing how a particle moved in a pasted time interval, will not give any information about how it will move in the future; only gives information about the present *

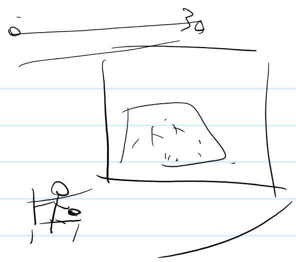
4) $\boxed{W(t) - W(s) \sim \mathcal{N}(0, t-s)}$ → $\mu = 0$ $t > s$; → $\sigma = t - s$

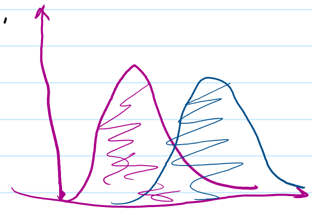






$$dX(t) = \underbrace{\mu}_{\text{Drift}}\, dt + \underbrace{\sigma\, dW(t)}_{\text{Diffusion}}$$





$dX(t) =$

G_d

C_s

$[Ka^+]$

$[Na^+]$

$[Ca^{2+}]$

$\bar{g}$

$[Ca^{2+}]$

V_L

Pinsky_Rinzel $\longrightarrow$ Nernst_planck

$$J = -D \frac{dx}{dz}$$