$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \qquad \hat{U}(t,t_0) : |\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = i\hbar \overbrace{\frac{d\hat{U}(t,t_0)}{dt}}^{\dot{U}}|\psi(t_0)\rangle \quad : \langle\psi(t)|\psi(t)\rangle = 1$$

$$\hookrightarrow \langle\psi(t_0)|U^{\dagger}U|\psi(t_0)\rangle = 1$$

$$U^{\dagger} = U^{-1} \Rightarrow |\psi(t_0)\rangle = U^{-1}|\psi(t)\rangle \qquad \hookrightarrow \underline{U^{\dagger}U = \mathbb{1}}$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \underbrace{i\hbar \dot{U}U^{\dagger}}_{H}|\psi(t)\rangle \qquad U(t,t_0) = \hat{1}$$

$$\Rightarrow \dot{U} = -\frac{i}{\hbar}HU$$

$$\hookrightarrow U = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} HU\,dt_1$$

$$U = T\left[e^{-\frac{i}{\hbar}\int H\,dt}\right] \quad ; [H(t_1), H(t_2)] = 0$$

$$\hookrightarrow U = \hat{\mathbb{1}} - \frac{i}{\hbar}\int_{t_0}^{t} \underset{H(t_1)}{H}\left(\hat{\mathbb{1}} - \frac{i}{\hbar}\int_{t_0}^{t_1} \underset{t_2}{H}U\,dt_2\right)dt_1$$

$$\hat{U}(t,t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}} \qquad \vdots$$

$$1 \equiv \int dx'|x'\rangle\langle x'|$$

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|\hat{U}(t,t_0)|\psi(t_0)\rangle$$

$$= \int dx' \langle x|U(t,t_0)|x'\rangle \underbrace{\langle x'|\psi(t_0)\rangle}_{\psi(x',t_0)}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \underbrace{\langle x|U(t,t_0)|x'\rangle}_{K(x,t;x',t_0)} \psi(x',t_0)$$

Free Particle. $H = \frac{P^2}{2m} \Rightarrow U(t,0) = U(t) = e^{-\frac{it}{\hbar}\frac{P^2}{2m}}$

$$K(x,t,x') = \langle x|e^{-\frac{it}{2m\hbar}P^2}|x'\rangle$$

$$\rightarrow \langle x|k\rangle = e^{ikx}$$

$$K(x,t,x') = \langle x| e^{-\frac{it}{2m\hbar}p^2} |x'\rangle \quad ; \quad \hat{P}|k\rangle = \hbar k|k\rangle$$

$$\mathbb{1} = \int dk\, |k\rangle\langle k|$$

$$= \int dk\, e^{-\frac{i\hbar t}{2m}k^2} \underbrace{\langle x|k\rangle}_{e^{ikx}}\underbrace{\langle k|x'\rangle}_{e^{-ikx'}} = \int_{-\infty}^{\infty} dk\, e^{-\frac{i\hbar t}{2m}k^2}\, e^{ik(x-x')}$$

$$e^{\hat{A}}|a\rangle = \sum_{n=0}^{\infty}\frac{\hat{A}^n}{n!}|a\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|a\rangle = e^{\alpha}|a\rangle$$

$$\hat{A}|a\rangle = \alpha|a\rangle$$

$$K(x,t,x') = \int_{-\infty}^{\infty} dk\, e^{-i\left(\frac{\hbar t}{2m}k^2 - (x-x')k\right)}$$

$$= \boxed{\frac{1}{\sqrt{2\pi i t}}\, e^{\frac{i(x-x')^2}{2t}}} \Rightarrow \text{Propagator of free Particle}$$

$$\boxed{\psi(x,t) = \int_{-\infty}^{\infty} K(x,t,x')\,\psi(x',0)\,dx'}$$

$$\psi(x',0) = \frac{e^{-\frac{ix'^2}{2}}}{(1+x'^2)^{1/3}}$$

$$\psi(x,t) = \frac{e^{\frac{ix^2}{2t}}}{\sqrt{2\pi i t}}\int_{-\infty}^{\infty} e^{\frac{ix'^2}{2t}(1-t)}\,\frac{e^{-\frac{ixx'}{t}}}{(1+x'^2)^{1/3}}\,dx'$$

$\psi(0,1) \longrightarrow$ divergent !

$$\psi(x,1) = \frac{2\sqrt{\pi}}{\Gamma(1/3)} \left(\frac{|x|}{2}\right)^{1/6} \overbrace{K_{1/6}(x)}^{\text{Mod. Bessel of third kind}}$$

↳ diverges for $x=0$!

$$\psi(x,0) = e^{-\frac{ix^2}{2}} \frac{\sin x}{x}$$

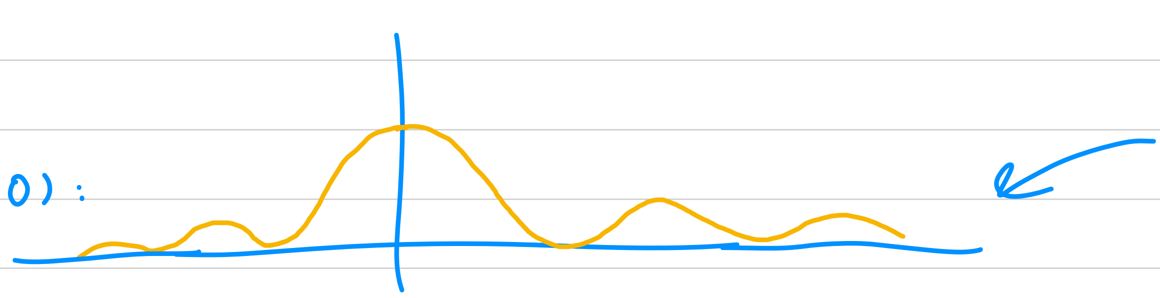
$$\psi(x,t) = \frac{e^{i\frac{x^2}{2t}}}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} e^{\frac{ix'^2}{2t}(1-t)} \, e^{-\frac{ixx'}{t}} \, \frac{\sin x'}{x'} \, dx'$$

$$\psi(x,1) = \frac{e^{i\frac{x^2}{2}}}{\sqrt{2\pi i}} \underbrace{\int_{-\infty}^{\infty} e^{-ixx'} \frac{\sin x'}{x'} \, dx'}_{\pi \operatorname{Rect}(x)}$$

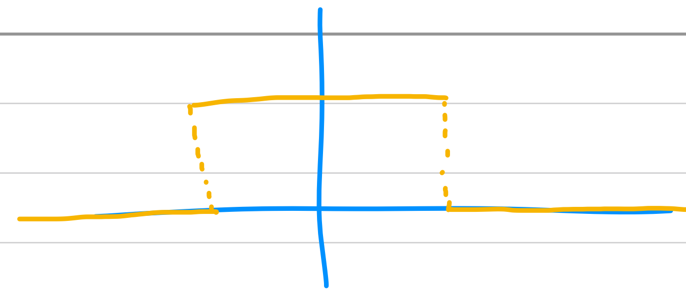
$$\operatorname{Rect}(x) := \begin{cases} 1 & |x|<1 \\ \frac{1}{2} & |x|=1 \\ 0 & |x|>1 \end{cases} \Longrightarrow \psi(x,1) = \pi \frac{e^{\frac{ix^2}{2}}}{\sqrt{2\pi i}} \operatorname{Rect}(x)$$

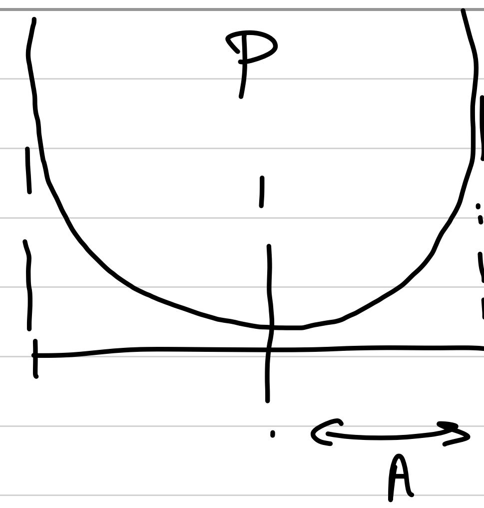
$$\psi(x,1) = \begin{cases} e^{ix^2/2} \sqrt{\frac{\pi}{2i}} & |x|<1 \\ 0 & |x|>1 \end{cases}$$

$|\psi|^2 \; (t=0)$:



$|\psi|^2 (t=1)$:



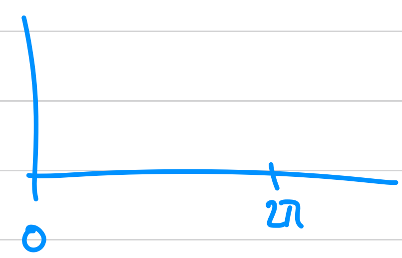


Separability :

$\mathcal{H}$. separable $\Longleftrightarrow$ $\mathcal{H} = \text{Span}\{|e_n\rangle\}$

n: Countable

$\mathcal{H} = \mathcal{L}^2([0,2\pi])$, $\langle v|u\rangle = \int_0^{2\pi} v^*(x)\, u(x)\, dx$



if $f(x) \in \mathcal{H}$: has finite max , min , discontinuity

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_n v_n e^{inx} \quad , \quad v_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x)\, dx$$

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$A^{+} = A$$

$$P = -i\hbar\, \partial/\partial x \longrightarrow -i\hbar\, \partial/\partial x$$