$$\oint p\, dq = nh$$ $\longrightarrow$ Hydrogen atom ; Cross-field

$\longrightarrow$ Fermi Gas ; many-electron atom

Paragraph 1:

$$\vec{E} = \frac{e}{r^3 c}\left(\vec{r} \times (\vec{r} \times \vec{v})\right) \; ; \; \vec{B} = \frac{e}{r^2 c^2}(\vec{v} \times \vec{r})$$

$$\frac{e}{rc^3}(\dot{\vec{v}} \times \vec{v}) \; , \; \frac{e}{rc^4}(\dot{\vec{v}} \times \vec{v}^2)$$

$\hookrightarrow$ Fourier expand.

$\nu(n, \alpha)$; A

Classical: $x(t) = \sum_\alpha X_\alpha(n)\, e^{2\pi i \alpha \nu(n) t}$; $\nu(n,\alpha) = \alpha \nu(n)$

$$J := \oint p\, dq \longrightarrow T = \int \frac{dq}{\dot{q}} = \int \frac{dq}{(\partial H/\partial p)_q}$$

$$\frac{dJ}{dE} = \frac{d}{dE}\oint p_{(q,E)}\, dq = \oint \left(\frac{\partial p}{\partial E}\right)_q dq = \oint \frac{dq}{(\partial E/\partial p)_q}$$

$$\frac{dJ}{dE} = T \longrightarrow \underline{\underline{\underset{\uparrow \atop \nu(n)}{\nu} = \frac{1}{T} = \frac{dE}{dJ}}}$$; $J = nh$

$$n := \frac{J}{h}$$

$$\nu(n) = \frac{\nu(n,\alpha)}{\alpha} = \frac{dE}{d(nh)} = \frac{1}{h}\frac{dE}{dn}$$

$$\nu(n,\alpha) = \frac{\alpha}{h}\frac{dE}{dn}$$

$$\alpha \frac{d}{dn} Q \longleftrightarrow Q_n - Q_{n-\alpha}$$

$$h\,\nu(n,\alpha) = E_n - E_{n-\alpha}$$

c ← Classical : $\nu(n,\alpha) + \nu(n,\beta) = \nu(n, \alpha+\beta)$

$E_n - E_\alpha \quad + E_\alpha - E_{n-\alpha-\beta}$

Q ← Quantum : $\nu(n, n-\alpha) + \nu(n-\alpha, n-\alpha-\beta) = \nu(n, n-\alpha-\beta)$

C : $X(t) \rightarrow Re\left\{ \underline{A_\alpha(n)}\, e^{i\omega(n)\alpha t} \right\}$ $\quad X(t) = \sum_\alpha ---$

$\omega(n) \longrightarrow \omega(n, n-\alpha)$

Q : $X(t) \rightarrow Re\left\{ \underline{A(n,n-\alpha)}\, e^{i\omega(n,n-\alpha)t} \right\}$ $\quad A_\alpha(n) \longrightarrow A(n, n-\alpha)$

$\underline{X(t)^2}$: $\quad$ C : $X(t) = \sum_\alpha A_\alpha(n)\, e^{i\omega(n)\alpha t}$

$$X^2 = \sum_{\alpha,\alpha'} A_\alpha(n) A_{\alpha'}(n)\, e^{i\omega(n)(\underline{\alpha+\alpha'})t}$$

$$\downarrow \quad X^2 = \sum_\beta B_\beta(n)\, e^{i\omega(n)\beta t}$$

$\alpha + \alpha' = \beta$

$\hookrightarrow \alpha' = \beta - \alpha$

$$B_\beta(n) = \sum_\alpha A_\alpha(n)\, A_{\beta-\alpha}(n)$$

$$Q: \quad \underline{B(n, n-\beta) = \sum_\alpha A(n, n-\alpha)\, A(n-\alpha, n-\beta)}$$

$\underline{X^3:}$

$$C: \quad C_\gamma(n) = \sum_{\alpha,\beta} A_\alpha(n)\, A_\beta(n)\, A_{\gamma-\alpha-\beta}(n)$$

$$Q: \quad C(n, n-\gamma) = \sum_{\alpha,\beta} A(n, n-\alpha)\, A(n-\alpha, n-\alpha-\beta)\, A(n-\alpha-\beta, n-\gamma)$$

$X(t)\, Y(t):$

$$X(t) = \sum_\alpha A_\alpha(n)\, e^{i\omega(n)\alpha t} \quad ; \quad Y(t) = \sum_\alpha B_\alpha(n)\, e^{i\omega(n)\alpha t}$$

$$XY = \sum C \quad \ldots$$

$$C: \quad C_\beta(n) = \sum_\alpha A_\alpha(n)\, B_{\beta-\alpha}(n)$$

$$Q: \quad C(n, n-\beta) = \sum_\alpha A(n, n-\alpha)\, B(n-\alpha, n-\beta)$$

$$XY \neq YX \qquad [X, Y] \neq 0$$

$$V\dot{V} = \frac{d}{dt}(V^2) \longrightarrow \frac{V\dot{V} + \dot{V}V}{2}$$

Paragraph 2 :

$$\ddot{x} + f(x) = 0$$

Dynamics :

$$\oint p\, dq = J = nh \longrightarrow (n+a)h$$

$$\hookrightarrow m \oint \dot{x}\, dx = m \oint \dot{x}^2 dt = J = nh$$

Classical:
$$x(t) = \sum_\alpha A_\alpha(n)\, e^{i\omega(n)\alpha t}$$

$$\dot{x}(t) = \sum_\alpha A_\alpha(n)\, i\omega(n)\alpha\, e^{i\omega(n)\alpha t}$$

$$|\dot{x}(t)|^2 = \sum_{\alpha,\beta} A_\alpha(n) A^*_\beta(n)\, \omega(n)^2 \alpha^2\, e^{i\omega(n)(\alpha+\beta)t}$$

$$m\int |\dot{x}|^2 dt = 2\pi m \sum_\alpha A_\alpha(n) A_{-\alpha}(n)\, \alpha^2 \omega(n)$$

$$x(t) \in \mathbb{R} \Longrightarrow A_\alpha = A^*_{-\alpha}$$

$$\Longrightarrow 2\pi m \sum_\alpha |A_\alpha(n)|^2 \alpha^2 \omega(n) = J \overset{?}{=} nh$$

$$\frac{d}{dn} \overbrace{\oint P\, dq}^{J} = \frac{d}{dn}(nh) = h$$

$$h = 2\pi m \sum_\alpha \alpha \frac{d}{dn}\left(|A_\alpha(n)|^2 \omega(n)\, \alpha \right)$$

$$\alpha \frac{d}{dn} Q \longleftrightarrow Q_n - Q_{n\alpha}$$

$$h = 4\pi m \sum_{\alpha>0} \left[|A(n+\alpha, n)|^2 \omega(n+\alpha, n) - |A(n, n-\alpha)|^2 \omega(n, n-\alpha) \right]$$

$$\ddot{x} + f(x) = 0$$

$$+ \text{B.C.}$$

Paragraph 3:

$$\ddot{x} + \omega^2 x = 0$$

$A_\alpha(n) \longrightarrow A(n, n-\alpha)$

$\omega(n)\alpha \longrightarrow \omega(n, n-\alpha)$

$$x(t) = \sum_\alpha A_\alpha(n)\, e^{i\omega(n)\alpha t}$$

$$\dot{x}(t) = \sum_\alpha A_\alpha(n)\, i\omega\alpha\, e^{i\omega\alpha t} \longrightarrow \ddot{x} = -\sum_\alpha A_\alpha(n)\, \omega_\alpha^2 \alpha^2\, e^{i\omega\alpha t}$$

$$\ddot{x} + \omega^2 x = 0$$

$$\sum_\alpha A_\alpha(n) \left(\omega^2 - \omega_{(n)}^2 \alpha^2 \right) e^{i\omega\alpha t} = 0$$

$$A_\alpha = 0 \qquad \omega_{(n)}^2 \alpha^2 = \omega^2 \longrightarrow x(t) = A e^{i\omega t} + A^* e^{-i\omega t}$$

$$\omega(n, n-\alpha) \Longleftrightarrow \alpha\, \omega(n)$$

$$A(n, n\pm 1) \neq 0$$

$$;\ \ddot{x} + \omega^2 x = 0 \longrightarrow \left(\omega(n, n-\alpha)^2 - \omega^2 \right) A(n, n\pm 1) = 0$$

$$\omega(n, n\pm 1) = \pm\, \omega$$

$$h = 4\pi m \sum_{\alpha>0}\left[|A(n+\alpha,n)|^2 \omega(n+\alpha,n) - |A(n,n-\alpha)|^2 \omega(n,n-\alpha)\right]$$

$$= 4\pi m\,\omega\left(|A(n+1,n)|^2 - |A(n,n-1)|^2 \right) = h = 2\pi\hbar$$

$$\longrightarrow \frac{\hbar}{2m\omega} = |A(n+1,n)|^2 - |A(n,n-1)|^2$$

$$|A(n+1,n)|^2 = \frac{\hbar}{2m\omega} + \overbrace{|A(n,n-1)|^2}^{a_n}$$

$$a_{n+1} = \frac{\hbar}{2m\omega} + a_n \longrightarrow |A(n+1,n)|^2 = \frac{n\hbar}{2m\omega} + \text{const}$$

$$a_{n_0} = 0 \longrightarrow |A(n+1,n)|^2 = \frac{n\hbar}{2m\omega}$$

$$E = \frac{m}{2}\dot{x}^2 + \frac{1}{2}m\omega^2 x^2$$

$$E(n,n-\beta) = \frac{m}{2}\sum_{\alpha}\left(\omega^2 - \omega(n,n-\alpha)\,\omega(\underset{\substack{n-1\\n+1}}{n-\alpha},\underset{\substack{n-2\\n+2}}{n-\beta})\right)\underbrace{A(n,n-\alpha)}_{A(n-\alpha,n-\beta)}$$

$$A(n,n\pm1) \neq 0 \qquad \cdots A(n,n-\alpha)\,A(\underset{n+1}{n-\alpha},\underset{n+2}{n-\beta})\ \ 0$$

$$\alpha = \begin{cases} +1 \longrightarrow \beta = \begin{cases} 0 \\ 2 \ \times \end{cases} \\ -1 \longrightarrow \beta = \begin{cases} 0 \\ -2 \ \times \end{cases} \end{cases}$$

$$\beta = 2\alpha$$

$$A(n,m)^* = A(m,n)$$

$$\cancel{E(n,n) =}$$

$$E(n,n) = m\omega^2 \Big(A(n,n-1)\, A(n-1,n) + A(n,n+1)\, A(n+1,n) \Big)$$

$$= m\omega^2 \Big(|A(n,n-1)|^2 + |A(n,n+1)|^2 \Big)$$

$$|A(n+1,n)|^2 = \frac{n\hbar}{2m\omega} \longrightarrow \underline{\underline{E(n,n) = \hbar\omega \left(n + \tfrac{1}{2}\right)}}$$: Q

$$\oint p\,dq = nh \longrightarrow E_n = \hbar\omega n$$

$$a = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{X} + i\hat{p} \right)$$

$$X = \sqrt{\frac{2m\omega}{\hbar}} \left(a + a^\dagger \right)$$

$$\langle n | X | m \rangle = \sqrt{\frac{2m\omega}{\hbar}} \Big(\underbrace{\langle n | \overbrace{a | m\rangle}^{\propto |m-1\rangle}}_{\delta_{n,m-1}} + \underbrace{\langle n | \overbrace{a^\dagger | m\rangle}^{\propto |m+1\rangle}}_{\delta_{n,m+1}} \Big)$$

$$m = n \pm 1$$