$$\left.\begin{array}{l} |V\rangle \in \mathcal{H}_1 \\ |U\rangle \in \mathcal{H}_2 \end{array}\right\} \quad |W\rangle = |U\rangle \otimes |V\rangle \longrightarrow W_{m\nu} = u_m v_\nu$$

$$\mathcal{H}_1 = \text{span}\{|m\rangle\} \qquad |m\rangle\langle n| : \mathcal{H}_1 \to \mathcal{H}_1$$

$$\mathcal{H}_2 = \text{span}\{|\mu\rangle\} \qquad |\mu\rangle\langle \nu| : \mathcal{H}_2 \to \mathcal{H}_2$$

$$\left(|m\rangle\langle n|\right) \otimes \left(|\mu\rangle\langle\nu|\right) := \left(|m\rangle \otimes |\mu\rangle\right)\left(\langle n| \otimes \langle \nu|\right)$$

$$\left.\begin{array}{l} \dim\{\mathcal{H}_1\} = N \\ \dim\{\mathcal{H}_2\} = M \end{array}\right\} \quad \dim\{\mathcal{H}_1 \otimes \mathcal{H}_2\} = MN \quad \gg \quad M+N$$

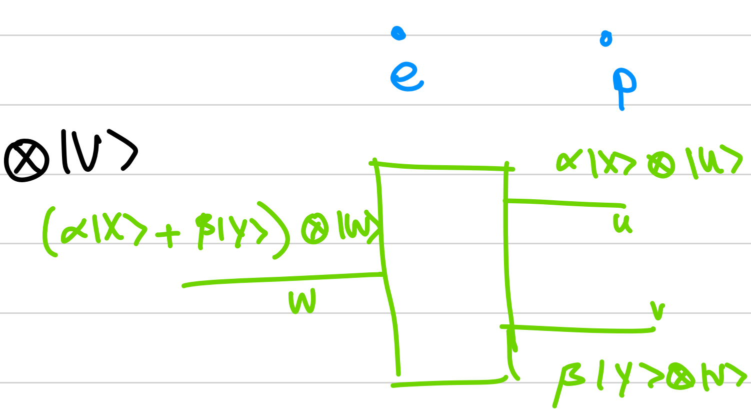
$$|\psi\rangle = \sum_{i,j} C_{ij}\, |i\rangle \otimes |j\rangle \neq (\quad) \otimes (\quad)$$

$$(A \otimes \mathbb{1} + \mathbb{1} \otimes B)_{m\mu,\, n\nu} = A_{mn}\, \delta_{\mu\nu} + \delta_{mn}\, B_{\mu\nu}$$

$$(A \otimes \mathbb{1} + \mathbb{1} \otimes B)(|u\rangle \otimes |v\rangle) = (A|u\rangle) \otimes |v\rangle + |u\rangle \otimes (B|v\rangle)$$

## InComplete Quantum Tests:

$$(\alpha|X\rangle + \beta|Y\rangle) \otimes |W\rangle \longrightarrow \alpha|X\rangle \otimes |U\rangle + \beta|Y\rangle \otimes |V\rangle$$




$$\mathcal{H} = (\quad_{\text{tested}}\quad) \otimes (\quad_{\text{not tested}}\quad)$$

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 0 \end{pmatrix} \;,\quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \end{pmatrix}$$

$$\hat{A} = \sigma_y \otimes \mathbb{1} \qquad ; \quad |\psi\rangle = \alpha|X\rangle\otimes|u\rangle + \beta|Y\rangle\otimes|V\rangle$$

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$$

$$= \left(\alpha^*\langle X|\otimes\langle u| + \beta^*\langle Y|\otimes\langle V|\right)\left(\sigma_y\otimes\mathbb{1}\right)\left(\alpha|X\rangle\otimes|u\rangle + \beta|Y\rangle\otimes|V\rangle\right)$$

$$* \left(\langle a|\otimes\langle b|\right)\left(|c\rangle\otimes|d\rangle\right) = \langle a|c\rangle\langle b|d\rangle \quad *$$

$$|W\rangle = |u\rangle\otimes|V\rangle \longrightarrow W_{m\nu} = u_m V_\nu$$

$$|1\rangle = |a\rangle\otimes|b\rangle \longrightarrow |1\rangle_{m\nu} = a_m b_\nu$$

$$|2\rangle = |c\rangle\otimes|d\rangle \longrightarrow |2\rangle_{m\nu} = c_m d_\nu$$

$$\langle 1|2\rangle = \sum_{m,\nu}\langle 1|_{m\nu}\,|2\rangle_{m\nu} = \sum_{m,\nu} a_m^* b_\nu^* c_m d_\nu = \underbrace{\left(\sum_m a_m^* c_m\right)}_{\langle a|c\rangle}\underbrace{\left(\sum_\nu b_\nu^* d_\nu\right)}_{\langle b|d\rangle}$$

$$\langle A\rangle = \left(\alpha^*\langle X|\otimes\langle u| + \beta^*\langle Y|\otimes\langle V|\right)\underbrace{\left(\sigma_y\otimes\mathbb{1}\right)\left(\alpha|X\rangle\otimes|u\rangle + \beta|Y\rangle\otimes|V\rangle\right)}_{\alpha(\sigma_y|X\rangle)\otimes|u\rangle + \beta(\sigma_y|Y\rangle)\otimes|V\rangle}$$

$$= |\alpha|^2\langle X|\sigma_y|X\rangle + |\beta|^2\langle Y|\sigma_y|Y\rangle$$

$|\alpha|^2\,|X\rangle$ $\quad$ $|\beta|^2\,|Y\rangle$

$$|\psi\rangle = \left(\alpha|X\rangle + \beta|Y\rangle\right)\otimes|W\rangle \;,\quad \hat{A} = \hat{\sigma}_y\otimes\hat{\mathbb{1}}$$

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \left(\alpha^*\langle X| + \beta^*\langle Y|\right)\otimes\langle W|\;\left(\hat{\sigma}_y\otimes\mathbb{1}\right)\left(\alpha|X\rangle + \beta|Y\rangle\right)\otimes|W\rangle$$

$$\langle A\rangle = |\alpha|^2 \langle X|\sigma_y|X\rangle + |\beta|^2 \langle Y|\sigma_y|Y\rangle + \alpha^*\beta \langle X|\sigma_y|Y\rangle + \alpha\beta^* \langle Y|\sigma_y|X\rangle$$

## Partial Trace:

$$\langle A\rangle = |\alpha|^2 \langle X|\underbrace{\sigma_y}_{\sum_n |n\rangle\langle n|}|X\rangle + |\beta|^2 \langle Y|\underbrace{\sigma_y}_{\sum_n |n\rangle\langle n|}|Y\rangle$$

$$= \sum_n |\alpha|^2 \langle X|\sigma_y|n\rangle\langle n|X\rangle + |\beta|^2 \langle Y|\sigma_y|n\rangle\langle n|Y\rangle$$

$$= \sum_n |\alpha|^2 \langle n|X\rangle\langle X|\sigma_y|n\rangle + |\beta|^2 \langle n|Y\rangle\langle Y|\sigma_y|n\rangle$$

$$= \sum_n \langle n| \left( |\alpha|^2 |X\rangle\langle X| + |\beta|^2 |Y\rangle\langle Y| \right) \sigma_y |n\rangle$$

$$= \mathrm{Tr}[\rho\, \sigma_y] \longrightarrow \rho = |\alpha|^2 |X\rangle\langle X| + |\beta|^2 |Y\rangle\langle Y|$$

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## General:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \qquad \hat{A} = \sigma_y \otimes \mathbb{1}$$

$$\langle A\rangle = \mathrm{Tr}[\rho A] = \sum_{m,n,\mu,\nu} \rho_{n\nu,m\mu}\, \sigma_{y\,mn}\, \delta_{\mu\nu}$$

$$= \sum_{m,n} \underbrace{\left( \sum_\mu \rho_{n\mu,m\mu} \right)}\sigma_{y\,mn} = \sum_{n,m} \tilde{\rho}_{nm} A_{mn} = \mathrm{Tr}[\tilde{\rho}\,\sigma_y]$$

$$\tilde{\rho}_{nm} = \sum_\mu \rho_{n\mu,m\mu} \longrightarrow \text{Reduced density matrix}$$

$$(A) + (B) \qquad \rho_{AB} : \qquad \rho_A = \mathrm{Tr}_B[\rho_{AB}]$$

$$\rho^\dagger = \rho \longrightarrow \rho = \sum_r \lambda_r |r\rangle\langle r|$$

$$\{p_i \, , \, |\psi_i\rangle\} \longrightarrow \rho$$

$$\{\lambda_r \, , \, |r\rangle\}$$

$$\left.\begin{array}{l} \{\tfrac{1}{2} \to |+\rangle \, , \, \tfrac{1}{2} \to |-\rangle\} \\ \{\tfrac{1}{2} \to |x\rangle \, , \, \tfrac{1}{2} \to |y\rangle\} \end{array}\right\} \underline{\underline{\rho = \tfrac{1}{2}\hat{\mathbb{1}}}}$$