Brownian Motion and Stochastic Integration (detailed proofs, split frames)

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Outline

Probability space

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
- Filtration $\{\mathcal{F}_t\}_{t \geq 0}$: increasing family of sub-σ-algebras.

Stochastic process

- A stochastic process is a measurable map $X : [0, \infty) \times \Omega \to \mathbb{R}$.
- Adapted: X_t is $\mathcal{F}_t$-measurable for each t.

Definition (standard Brownian motion)

Definition

A process $(B_t)_{t\geq 0}$ is a standard Brownian motion if:

1. $B_0 = 0$ a.s.
2. Independent Gaussian increments: for $0 \leq s < t$, $B_t - B_s \sim N(0, t-s)$.
3. Continuous paths a.s.

Immediate consequences (1)

- For each fixed t, $B_t \sim N(0, t)$.
- Finite-dimensional distributions are multivariate normal.

Immediate consequences (2)

Covariance identity:

$$\mathbb{E}[B_s B_t] = \min\{s, t\}.$$

Next frames give the short proof and the joint Gaussian argument.

Proof: covariance (step 1)

Let $0 \leq s \leq t$. Write

$$B_t = (B_t - B_s) + B_s.$$

The increment $B_t - B_s$ is independent of B_s and has mean zero.

Proof: covariance (step 2)

Therefore

$$\mathbb{E}[B_s B_t] = \mathbb{E}\big[B_s(B_t - B_s)\big] + \mathbb{E}[B_s^2] = 0 + \mathrm{Var}(B_s) = s.$$

So $\mathbb{E}[B_s B_t] = \min\{s, t\}$ as claimed.

Proof: joint Gaussianity (idea)

Write the vector $(B_{t_1}, \dots, B_{t_n})$ as a linear transform of independent Gaussian increments:

$$B_{t_k} = \sum_{j=1}^{k} (B_{t_j} - B_{t_{j-1}}), \quad t_0 = 0.$$

A linear transform of a Gaussian vector is Gaussian. Hence finite-dimensional distributions are Gaussian.

Kolmogorov continuity — statement (part 1)

Theorem

Let $(X_t)_{t\in[0,T]}$ *be a real-valued process. If there exist constants* $C > 0$, $p \geq 1$, $\delta > 0$ *such that*

$$\mathbb{E}\big[|X_t - X_s|^p\big] \leq C|t-s|^{1+\delta}$$

for all $s, t \in [0, T]$, *then...*

Kolmogorov continuity — statement (part 2)

...there exists a continuous modification $\widetilde{X}$ whose paths are Hölder continuous of any exponent $\gamma \in (0, \delta/p)$.

Proof: overview

Strategy:

1. Use dyadic partitions of $[0, T]$.
2. Control probabilities of large dyadic increments.
3. Apply Borel–Cantelli to get almost sure control.
4. Extend from dyadic points to all times.

Proof: dyadic partition and notation

Fix integer $m \geq 1$ and define dyadic points

$$t_k^{(m)} = k2^{-m}T, \qquad k = 0, \dots, 2^m.$$

Define maximal increment at level m:

$$M_m := \max_{0 \leq k < 2^m} |X_{t_{k+1}^{(m)}} - X_{t_k^{(m)}}|.$$

Proof: tail probability estimate

For a single dyadic interval,

$$\mathbb{P}\{|X_{t_{k+1}} - X_{t_k}| > \varepsilon\} \leq \frac{\mathbb{E}|X_{t_{k+1}} - X_{t_k}|^p}{\varepsilon^p} \leq \frac{C(2^{-m}T)^{1+\delta}}{\varepsilon^p}.$$

(Markov/Chebyshev inequality.)

Proof: union bound and choice of epsilon

Union bound over 2^m intervals:

$$\mathbb{P}\{M_m > \varepsilon\} \le 2^m \cdot \frac{C(2^{-m}T)^{1+\delta}}{\varepsilon^p} = CT^{1+\delta}\varepsilon^{-p}2^{-m\delta}.$$

Choose $\varepsilon_m := 2^{-m\gamma}$ with $0 < \gamma < \delta/p$.

Proof: summability and Borel–Cantelli

With $\varepsilon_m = 2^{-m\gamma}$,

$$\mathbb{P}\{M_m > \varepsilon_m\} \leq CT^{1+\delta} 2^{-m(\delta-\gamma p)}.$$

Since $\delta - \gamma p > 0$, $\sum_m \mathbb{P}\{M_m > \varepsilon_m\} < \infty$. Borel–Cantelli implies $M_m \leq \varepsilon_m$ eventually a.s.

Proof: constructing Hölder bound on dyadics

For dyadic points $s < t$ at level m with m large,

$$|X_t - X_s| \leq (\#\text{intervals})\, \varepsilon_m.$$

As $(\#\text{intervals}) \cdot 2^{-m} T = t - s$, we get

$$|X_t - X_s| \leq \frac{t - s}{2^{-m} T} \varepsilon_m.$$

Selecting m such that $2^{-(m+1)} T < t - s \leq 2^{-m} T$ yields a Hölder estimate.

Proof: extension to all times and continuity

The Hölder estimate holds for dyadic rationals. Using density of dyadics and completeness of reals, extend the bound to all $s, t \in [0, T]$, producing a continuous modification with the claimed exponent.

Motivation

Brownian sample paths almost surely have infinite variation, so pathwise Riemann–Stieltjes integrals do not generally make sense. The Wiener integral constructs $\int f(t)\,dB_t$ for deterministic $f \in L^2$ as an $L^2(\Omega)$ limit.

Step functions: definition

Let f be a step function on $[a, b]$:

$$f(t) = \sum_{j=1}^{n} c_j \mathbf{1}_{[t_{j-1}, t_j)}(t).$$

Define the stochastic Riemann sum

$$I(f) := \sum_{j=1}^{n} c_j \big(B_{t_j} - B_{t_{j-1}}\big).$$

Step functions: Gaussianity and variance

Each increment $B_{t_j} - B_{t_{j-1}}$ is Gaussian and independent across j. Thus $I(f)$ is Gaussian, mean zero, and

$$\mathrm{Var}(I(f)) = \sum_{j=1}^{n} c_j^2 (t_j - t_{j-1}).$$

(Cross-terms vanish by independence.)

Isometry for step functions

For step functions f and g (use a common refinement),

$$\mathbb{E}[I(f)I(g)] = \int_a^b f(t)g(t)\,\mathrm{d}t.$$

Hence I preserves the L^2 inner product on step functions.

Extension to $L^2[a, b]$

Step functions are dense in $L^2[a, b]$. If $f_n \to f$ in L^2, then $I(f_n)$ is Cauchy in $L^2(\Omega)$:

$$\mathbb{E}|I(f_n) - I(f_m)|^2 = \|f_n - f_m\|_{L^2}^2.$$

Define $\int_a^b f \, dB$ as the $L^2(\Omega)$ limit of $I(f_n)$.

Properties of the Wiener integral

- $\int_a^b f\, dB$ is Gaussian with mean 0 and variance $\|f\|_{L^2}^2$.
- Isometry: $\mathbb{E}\big[\big(\int_a^b f\, dB\big)^2\big] = \|f\|_{L^2}^2$.
- Inner product preservation holds for all $f, g \in L^2$.

Compatibility with pathwise integral

If f is continuous and of bounded variation, then for almost every Brownian sample path the pathwise Riemann–Stieltjes integral $\int f \, dB$ exists and coincides a.s. with the Wiener integral (approximate f by step functions).

Elementary adapted processes

Let $\{\mathcal{F}_t\}$ be a filtration. An elementary adapted (predictable simple) process is

$$\xi_t(\omega) = \sum_{k=1}^{n} H_{k-1}(\omega)\,\mathbf{1}_{(t_{k-1},t_k]}(t),$$

where H_{k-1} is $\mathcal{F}_{t_{k-1}}$-measurable and square-integrable.

Definition of the Itô integral (elementary)

For such ξ, define

$$\int_0^T \xi_t \, \mathrm{d}B_t := \sum_{k=1}^{n} H_{k-1}\big(B_{t_k} - B_{t_{k-1}}\big).$$

This is a random variable in $L^2(\Omega)$.

Itô isometry (elementary) — statement

For elementary ξ as above,

$$\mathbb{E}\Big[\Big(\int_0^T \xi_t \,\mathrm{d}B_t\Big)^2\Big] = \mathbb{E}\Big[\int_0^T \xi_t^2 \,\mathrm{d}t\Big].$$

Proof split over the next few frames.

Itô isometry — proof (part 1)

Expand the square:

$$\left(\sum_{k=1}^{n} H_{k-1} \Delta B_k \right)^2 = \sum_{k=1}^{n} H_{k-1}^2 (\Delta B_k)^2 + 2 \sum_{i<j} H_{i-1} H_{j-1} \Delta B_i \Delta B_j,$$

where $\Delta B_k := B_{t_k} - B_{t_{k-1}}$.

Itô isometry — proof (part 2)

Take expectation. For $i < j$, ΔB_j is independent of $\mathcal{F}_{t_{j-1}}$, hence independent of H_{i-1} and of ΔB_i. Moreover $\mathbb{E}[\Delta B_j] = 0$. Thus each cross-term has expectation zero.

Itô isometry — proof (part 3)

For diagonal terms,

$$\mathbb{E}[H_{k-1}^2(\Delta B_k)^2] = \mathbb{E}[H_{k-1}^2]\,\mathbb{E}[(\Delta B_k)^2] = \mathbb{E}[H_{k-1}^2](t_k - t_{k-1}).$$

Summing yields the desired equality.

Extension to predictable L^2 integrands

Let $\mathcal{H}$ be the closure (under $\mathbb{E} \int_0^T \xi_t^2 dt$) of elementary adapted processes. By the isometry, the map $\xi \mapsto \int_0^T \xi_t \, dB_t$ extends uniquely to an isometry on $\mathcal{H}$.

Martingale property (elementary case)

For elementary ξ, for $s < t$ the increment $M_t - M_s$ is a sum of terms each independent of $\mathcal{F}_s$ and mean zero. So $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$. Thus M is a martingale.

Martingale property (general case)

For $\xi \in \mathcal{H}$ pick elementary $\xi^{(n)} \to \xi$. The integrals converge in L^2 uniformly in t; conditional expectation is continuous in L^2, so the martingale property passes to the limit. Hence $M_t = \int_0^t \xi_s \, dB_s$ is a square-integrable martingale.

Quadratic variation — definition

For partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ define

$$Q(\Pi) := \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2.$$

We study the limit as $\|\Pi\| \to 0$.

Q.V. for Brownian motion — statement

For Brownian motion,

$$\lim_{\|\Pi\|\to 0} Q(\Pi) = t \quad \text{in } L^2(\Omega),$$

hence in probability. So $[B]_t = t$.

Proof — expectation

Compute expectation:

$$\mathbb{E}[Q(\Pi)] = \sum_k \mathbb{E}[(\Delta B_k)^2] = \sum_k (t_{k+1} - t_k) = t.$$

Thus $\mathbb{E}[Q(\Pi) - t] = 0$.

Proof — second moment expansion

Expand

$$Q(\Pi)^2 = \sum_k (\Delta B_k)^4 + 2 \sum_{i<j} (\Delta B_i)^2 (\Delta B_j)^2.$$

Take expectation and use independence for cross terms.

Proof — use Gaussian fourth moment

For $Y \sim N(0, \sigma^2)$ we have $\mathbb{E}[Y^4] = 3\sigma^4$. Thus $\mathbb{E}[(\Delta B_k)^4] = 3(t_{k+1} - t_k)^2$. Combine to get

$$\mathbb{E}[Q(\Pi)^2] = t^2 + 2\sum_k (t_{k+1} - t_k)^2.$$

Proof — variance tends to zero

Therefore

$$\mathrm{Var}(Q(\Pi)) = \mathbb{E}[Q(\Pi)^2] - t^2 = 2\sum_k (t_{k+1} - t_k)^2 \leq 2t\|\Pi\| \to 0.$$

Hence $Q(\Pi) \to t$ in L^2 as desired.

Itô formula — setup

Suppose

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s,$$

with adapted μ, σ satisfying integrability assumptions. Let $f \in C^2(\mathbb{R})$.

Itô formula — discrete Taylor

On a partition $0 = t_0 < \cdots < t_n = t$ apply Taylor:

$$f(X_{t_{k+1}}) - f(X_{t_k}) = f'(X_{t_k})\Delta X_k + \frac{1}{2}f''(X_{t_k})(\Delta X_k)^2 + r_k,$$

where $\Delta X_k = X_{t_{k+1}} - X_{t_k}$ and r_k is the remainder.

Itô formula — analyze terms: first term

Sum of first terms approximates

$$\sum_k f'(X_{t_k})\Delta X_k \to \int_0^t f'(X_s)\, dX_s$$

which splits into Lebesgue and Itô parts corresponding to μ and $\sigma\, dB$.

Itô formula — analyze terms: quadratic term

Note $(\Delta X_k)^2$ contains $(\Delta M_k)^2$ where M is the martingale part. Using quadratic variation, $\sum(\Delta M_k)^2 \to [M]_t = \int_0^t \sigma_s^2 ds$. Hence quadratic term yields $\frac{1}{2}\int_0^t f''(X_s)\sigma_s^2 ds$.

Itô formula — remainder control

Remainder r_k is of higher order: typically $O((\Delta t)^{3/2})$ because $\Delta M_k = O(\sqrt{\Delta t})$. Sum of r_k tends to zero in probability (or in L^1) under standard integrability assumptions, by direct L^2 estimates.

Itô formula — final form

Putting pieces together:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mu_s \, ds + \int_0^t f'(X_s)\sigma_s \, dB_s + \frac{1}{2}\int_0^t f''(X_s)\sigma_s^2 \, ds.$$

This is the Itô formula in one dimension.

Exercise A

Show $\mathbb{E}|B_t - B_s|^4 = 3|t-s|^2$.

Exercise A — hint

Use: if $Z \sim N(0, \sigma^2)$ then $\mathbb{E}Z^4 = 3\sigma^4$. Apply with $\sigma^2 = |t - s|$.

Exercise B

Compute the distribution of $Y = \int_0^1 B_t \, dt$.

Exercise B — hint

Integrate by parts (or use kernel representation):

$$\int_0^1 B_t \, dt = \int_0^1 (1-t) \, dB_t,$$

so variance is $\int_0^1 (1-t)^2 dt = 1/3$.