Computational Neuroscience

Session 2: Mathematical Foundations (1)

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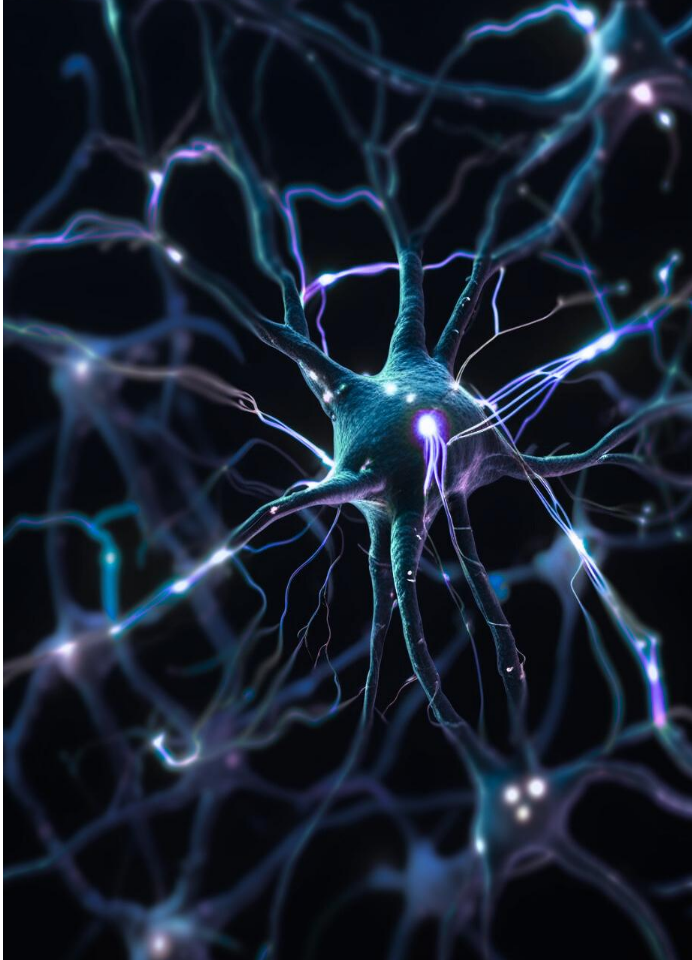

Neuron's Electrophysiology (Review)

- Membrane Capacitance: $Q = C_m \cdot V$
- Membrane Time Constant: $\tau_m = R_m C_m$
- Membrane Resistance:

  $V = I \cdot R_m$ or $I = G_m \cdot V$

  $G_m = 1/R_m$

  $I_{\text{ion}} = G_{\text{ion}} \left(V - E_{\text{ion}}\right)$
- Leaky Integrate-and-Fire (LIF) Model:

  $$C_m \frac{dV}{dt} = -\frac{1}{R_m}\left(V - E_{\text{rest}}\right) + I_{\text{inj}}(t)$$

  $$\tau_m \frac{dV_m}{dt} = -\left(V_m - V_{\text{rest}}\right) + R_m I_{\text{inj}}(t)$$

- Nernst Potential: $E_{\text{ion}} = \frac{RT}{zF} \ln\left(\frac{[\text{Ion}]_{\text{out}}}{[\text{Ion}]_{\text{in}}}\right)$
- Goldman-Hodgkins-Katz (GHK) Equation:

  $$V_m = \frac{RT}{F} \ln\left(\frac{P_K[\text{K}^+]_{\text{out}} + P_{Na}[\text{Na}^+]_{\text{out}} + P_{Cl}[\text{Cl}^-]_{\text{in}}}{P_K[\text{K}^+]_{\text{in}} + P_{Na}[\text{Na}^+]_{\text{in}} + P_{Cl}[\text{Cl}^-]_{\text{out}}}\right)$$
- Synapses as Modulators of Permeability:

  $$I_{\text{syn}} = G_{\text{syn}}(t)\left(V_m - E_{\text{syn}}\right)$$

  $$G_{\text{syn}}(t) = G_{\text{max}} \frac{t}{\tau_{\text{syn}}} e^{\left(1 - t/\tau_{\text{syn}}\right)}$$
- Cable Equation:

  $$\tau_m \frac{\partial V}{\partial t} = \lambda^2 \frac{\partial^2 V}{\partial x^2} - \left(V - V_{\text{rest}}\right), \lambda = \sqrt{\frac{r \cdot R_m}{2 \cdot R_a}}$$

One-Dimensional Dynamical Systems

Definition: A dynamical system consists of two primary components:

1. State space ($X(t)$), which is the set of all possible states a system can be in (often represented by a vector in a geometric space).
2. An evolution rule ($\Phi(X, t)$), which is a fixed function that describes how the system transitions from its current state to a future state (evolution rule is deterministic, which means that for a given time interval, only one future state follows from the current one).

The path a system traces through its state space over time is referred to as its trajectory. The evolution rule of a dynamical system can be formulated within continuous or discrete time.

$t \in \mathbb{R}$

Continuous Time: The system's state evolves in a way that every instant has a defined value. These systems are modeled by differential equations, which describe the instantaneous rate of change of the state variables. For example, a simple one-dimensional system with state variable x with an evolution rule $f(x)$ is given by an Ordinary Differential Equation (ODE) like:

$$\frac{dx}{dt} = f(x)$$

$f(x)$ can be a vector field on the state space (assigning a velocity to each point x). The solution $x(t)$ is a continuous function of time.

$n \in \mathbb{Z}$

Discrete Time: The system's state is evolving during separate “points” in time. The state variable jumps from one value to the next in discrete steps. These systems are modeled by **Numerical Differential Equations**. For example, a one-dimensional system in discrete time has a form like:

$$x_{n+1} = f(x_n)$$

In this notation, the function f takes the current state x_n at time step n and maps it to the state x_{n+1} at the next time step $n + 1$.

In a one-dimensional state space, a point x_i can only move in one of two directions: right (increasing) or left (decreasing). The sign of the function $f(x)$ in the equation $\frac{dx}{dt} = f(x)$ determines this direction. If $f(x) > 0$, the velocity is positive and x increases; if $f(x) < 0$, the velocity is negative and x decreases.

Stability

Fixed Points: A fixed point, denoted $x*$, is a state where the system ceases to evolve over time. For a continuous one-dimensional system described by $\frac{dx}{dt} = f(x)$, a fixed point is a value $x*$ such that the rate of change is zero: $f(x*) = 0$
If the system starts exactly at a fixed point, $x(0) = x*$ it will remain there for all future time $x(t) = x*$. Fixed points are simply the roots of the function $f(x)$.
Stability: A fixed point can be either stable or unstable.
A stable fixed point is one to which nearby trajectories converge (if the system is slightly perturbed from this point, it will return). An unstable fixed point is one from which nearby trajectories diverge (a small perturbation will cause the system to move away).
The stability of a fixed point in a one-dimensional flow can be determined directly by inspecting the graph of $f(x)$ versus x. The sign of $f(x)$ on either side of the fixed point determines the direction of flow:

- **Stable Fixed Point**: If the function $f(x)$ crosses the x-axis with a negative slope at $x*$, the fixed point is stable. To the left of $x*$, $f(x) > 0$, so trajectories move to the right (towards $x*$). To the right of $x*$, $f(x) < 0$, so trajectories move to the left (also towards $x*$).

- **Unstable Fixed Point**: If $f(x)$ crosses the x-axis with a positive slope at $x*$, the fixed point is unstable. To the left of $x*$, $f(x) < 0$, so trajectories move to the left (away from $x*$). To the right of $x*$, $f(x) > 0$, so trajectories move to the right (away from $x*$).
- **Semi-stable Fixed Point**: If $f(x)$ is tangent to the x-axis at $x*$ (it touches it but doesn't cross) the fixed point is semi-stable. Trajectories on one side are attracted to $x*$, while trajectories on the other side are repelled.

Now, suppose we approximate the nonlinear function $f(x)$ with a linear one. Consider a small perturbation $\epsilon(t)$ from a fixed point $x*$ such that $x(t) = x* + \epsilon(t)$. The rate of change of the perturbation is $d\epsilon/dt = dx/dt = f\big(x* + \epsilon(t)\big)$. We can approximate $f(x* + \epsilon)$ using a Taylor series expansion around $x*$:

$$d\epsilon/dt = f(x*) + \epsilon f'(x*) + O(\epsilon 2)$$

Since $f(x*) = 0$ by definition and for very small ϵ we can neglect higher-order terms, the dynamics of the perturbation are governed by the linear differential equation:

$$d\epsilon/dt \approx f'(x*)\epsilon$$

The solution to this equation is $\epsilon(t) = \epsilon(0)\mathrm{e}^{\mathrm{f}'(\mathrm{x}^*)\mathrm{t}}$.

The stability of the fixed point is thus determined by the sign of the derivative of $f(x)$ evaluated at the fixed point, $f'(x*)$:

- If $f'(x*) < 0$, the exponent is negative, and the perturbation $\epsilon(t)$ decays exponentially to zero. The fixed point $x*$ is stable.
- If $f'(x*) > 0$, the exponent is positive, and the perturbation $\epsilon(t)$ grows exponentially. The fixed point $x*$ is unstable.
- If $f'(x*) = 0$, the fixed point is marginally stable. The stability depends on higher-order terms in the Taylor expansion. This marginal case is very important, because it is where bifurcations occur.

For the Next Session…

Bistability
Bifurcations
Hartman-Grobman Theorem
Two-Dimensional Dynamical Systems