Linear Algebra

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Contents

1 What Vectors Actually are?

Group: We call a set like G and a binary operator like "+" (say addition), togeather, *the group* $(G, +)$ if and only if the following axioms are satisfied:

1. $\forall g_1, g_2 \in G : g_1 + g_2 \in G$ Closeness
2. $\exists 0, \forall g \in G : g + 0 = 0 + g = g$ Addition Identity
3. $\forall g \in G, \exists(-g) : g + (-g) = 0$ Addition Inverse
4. $\forall g_1, g_2, g_3 \in G : (g_1 + g_2) + g_3 = g_1 + (g_2 + g_3)$ Addition Associativity

Abelian Group: We call a group like $(G, +)$ *abelian* or *commutative* if and only if satisfies the following axiom:

- $\forall g_1, g_2 \in G : g_1 + g_2 = g_2 + g_1$ Addition Commutativity

Ring: We call a set like R and two binary operators like "+" and "$\cdot$" (say addition and multiplication), togeather, *the ring* $(R, +, \cdot)$ if and only if the following axioms are satisfied:

1. $(R, +) \rightarrow$ Abelian Group
2. $\forall r_1, r_2, r_3 \in R : r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$ Distributivity
3. $\forall r_1, r_2, r_3 \in R : r_1 \cdot (r_2 \cdot r_3) = (r_1 \cdot r_2) \cdot r_3$ Multiplication Associativity

Unital Ring: We call a ring like $(R, +, \cdot)$ *unital* if and only if the following axiom is satisfied:

- $\exists 1, \forall r \in R : 1 \cdot r = r \cdot 1 = r$ Multiplication Identity

Field: We call a unital ring like $(F, +, \cdot)$ *a field* if and only if the following axiom is satisfied:

- $\forall f \in F, \exists f^{-1} : f^{-1} \cdot f = f \cdot f^{-1} = 1$ Multiplication Inverse

Vector Space: We call a set like V and a binary operator like "+" (say addition), togeather, the *vector space* $(V, +)$ *over the field* $(F, +', \cdot')$ if and only if the following axioms are satisfied:

- $\cdot : F \times V \rightarrow V$

1. $(V, +) \rightarrow$ Abelian Group
2. $\forall f \in F, \forall \left|v_1\right\rangle, \left|v_2\right\rangle \in V : f \cdot (\left|v_1\right\rangle + \left|v_2\right\rangle) = f \cdot \left|v_1\right\rangle + f \cdot \left|v_2\right\rangle \in V$
3. $\forall f_1, f_2 \in F, \forall \left|v\right\rangle \in V : (f_1 +' f_2) \left|v\right\rangle = f_1 \cdot \left|v\right\rangle + f_2 \cdot \left|v\right\rangle \in V$
4. $\forall f_1, f_2 \in F, \forall \left|v\right\rangle \in V : (f_1 \cdot' f_2) \cdot \left|v\right\rangle = f_1 \cdot (f_2 \cdot \left|v\right\rangle) \in V$
5. $\forall \left|v\right\rangle \in V : 1 \cdot \left|v\right\rangle = \left|v\right\rangle$

-The field elements are called *scalers* of the vector space.
-Vector space elements are shown as *kets* such as $|v\rangle$.
-Multiplication of a scaler like f and a vector like $|v\rangle$ is denoted as $f \cdot |v\rangle \equiv f\,|v\rangle$.
-Multiplication of scalers like f_1 and f_2 is denoted as $f_1 \cdot' f_2 \equiv f_1 f_2$.

Hereafter, the vector space elements are shown as kets such as $|v\rangle$ and the vector space scalers as greek letters such as α, β, etc.

Linearly Independence: we call a set of vectors like $S = \{|v_1\rangle, |v_2\rangle, ...\}$ the *linearly independent* if and only if the following axiom is satisfied:

$$\sum_i \alpha_i |v_i\rangle = 0 \iff \forall |v_i\rangle \in S : \alpha_i = 0$$

Dimensions: We call a vector space like $(V, +)$ N *dimensional* if and only if the maximum number of linearly independent vectors can be chosen in V is equal to N

Basis: We call a set of vectors like $B = \{|b_1\rangle, |b_2\rangle, ...\}$ the *basis* of the vector space V if and only if the following axiom is satisfied:

1. B is linearly independent.
2. $\forall |v\rangle \in V$, $\forall |b_i\rangle \in B$, $\exists \alpha_i : |v\rangle = \sum_i \alpha_i |b_i\rangle$
   where α_i is the i'th *component* of the vector $|v\rangle$ in basis B.

Subspace: We call a set W a *subspace* of the vector space $(V, +)$ if and only if the $(W, +)$ itself is a vector space over the same field.

Inner Product: We call a binary operator like $\langle , \rangle$ an *inner product in the vector space (V,+)* if and only if the following axioms are satisfied:

1. $\langle , \rangle : V \times V \to F$
2. $\forall |v\rangle \in V : \langle v, v\rangle \geq 0$
3. $\forall |v\rangle \in V : \langle v, v\rangle = 0 \iff |v\rangle = |0\rangle$
4. $\forall |v_1\rangle, |v_2\rangle, |v_3\rangle \in V$, $\forall \alpha, \beta \in F : \langle v_1, \alpha v_2 + \beta v_3\rangle = \alpha\langle v_1, v_2\rangle + \beta\langle v_1, v_3\rangle$
5. $\forall |v_1\rangle, |v_2\rangle \in V : \langle v_1, v_2\rangle = \langle v_2, v_1\rangle^*$

- The inner product of $|a\rangle$ and $|b\rangle$ is denoted by $\langle a, b\rangle$.
- F is the field of the vector space.
- The symbol a^* represents the complex conjugation of a.

Inner Product Space: We call a vector space like $(V, +)$ an *inner product space* if and only if we define at least one type of inner product on the space.

Orthogonality: We call two vectors such as $|a\rangle, |b\rangle$ *orthogonal* if and only if the inner product of them is zero: $\langle a, b\rangle$=0

Norm: We call $|v|$ the *norm* of the vector $|v\rangle$ which is defined by: $|v| \equiv \sqrt{\langle v, v\rangle}$

Orthonormal Basis: We call a basis $B = \{|b_1\rangle, |b_2\rangle, ...\}$ *orthonormal* if and only if:

$$\forall\, |b_i\rangle, |b_j\rangle \in B : \langle b_i, b_j\rangle = \delta_{ij}$$

As we can see, using basis, we represent the inner product as:

$$|v\rangle = \sum_i v_i |b_i\rangle \quad , \quad |w\rangle = \sum_j w_j |b_j\rangle$$

$$\langle v, w\rangle = \sum_i \sum_j v_i^* w_j \langle b_i, b_j\rangle$$

Using an orthonormal basis, we obtain:

$$\langle v, w\rangle = \sum_i v_i^* w_i$$

As we see, $\langle v, v\rangle = \sum_i |v_i|^2 \geq 0$ and $\langle v, v\rangle = 0 \iff |v\rangle = |0\rangle$

2 The Dual Space

We can represent each vector as a column matrix containing its components:

$$|v\rangle = \sum_i v_i |b_i\rangle = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}$$

$$|w\rangle = \sum_i w_i |b_i\rangle = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{bmatrix}$$

Using this notation we can represent the inner product in an orthonormal basis as a matrix multiplications:

$$\langle v, w\rangle = \begin{bmatrix} v_1^* & v_2^* & v_3^* & ... \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{bmatrix} = \sum_i v_i^* w_i$$

Hence for any vector in vector space there exists a column matrix and the multiplication of them is defined as:

$$\langle v, w\rangle = v^{*T} w$$

Dual Space: the vector space containing all transpose conjugated elements of v is called *the dual space of V*.

3 Dirac Notation

Notation
The vectors of V is represented by $|v\rangle$ and the corresponding vectors of dual space of V is represented by $\langle v|$. where:

$$|v\rangle = \sum_i v_i |b_i\rangle$$

$$\langle v| = \sum_i v_i^* \langle b_i|$$

Therefore, the inner product of vectors in V is obtained by the following matrix product:

$$\langle v, w\rangle = \langle v| |w\rangle = \langle v|w\rangle$$

Hereover, we represent the inner product of the vectors $|v\rangle$ and $|w\rangle$ as bra-ket notation: $\langle v|w\rangle$. It's literally a matrix product. Bra and Ket notations can therefore be converted to each other by a transpose conjugation.
We also obtain the bra of $a|v\rangle$:

$$w = a|v\rangle = |av\rangle \iff \langle w| = \langle av| = a^* \langle v|$$

Expansion of a vector in an orthonormal basis

$$|v\rangle = \sum_i v_i |b_i\rangle$$

Each component of $|v\rangle$ is given by:

$$\langle b_j|v\rangle = \sum_i v_i \langle b_j|b_i\rangle = \sum_i v_i \delta_{ij} = v_j$$

Using this expression, the inner product of vectors can be represented for any basis:

$$|v\rangle = \sum_i (\langle b_i|v\rangle) |b_i\rangle = \sum_i |b_i\rangle \langle b_i|v\rangle = \sum_i (|b_i\rangle \langle b_i|) |v\rangle$$

$$\langle v| = \sum_i \langle v| (|b_i\rangle \langle b_i|)$$

4 Operators

An operator such as Ω is a function from a vector to a vector:

$$\Omega |v\rangle = |v'\rangle = |\Omega v\rangle \quad , \quad \langle v'| \Omega = \langle v''|$$

Linear Operators
Linear operators are those which

$$\Omega (\alpha |v\rangle + \beta |w\rangle) = \alpha\Omega |v\rangle + \beta\Omega |w\rangle$$

$$\Omega\left(\langle v|\,\alpha + \langle w|\,\beta\right) = \langle v|\,\Omega\alpha + \langle w|\,\Omega\beta$$

We tend to work with linear operators because their operation on vectors can be obtained if their operation is determined on the basis only:

$$\Omega\,|b_i\rangle = |b'_i\rangle$$

$$\Omega\,|v\rangle = \Omega\left(\sum_i v_i\,|b_i\rangle\right) = \sum_i v_i\Omega\,|b_i\rangle = \sum_i v_i\,|b'_i\rangle$$

Commutator: The commutator of two operator Ω and Λ is defined as:

$$[\Omega, \Lambda] \equiv \Omega\Lambda - \Lambda\Omega$$

Inverse: The inverse of an operator Ω is represented as Ω^{-1} where its defined as

$$\Omega\Omega^{-1} = \Omega^{-1}\Omega = \mathbb{1}$$

Where $\mathbb{1}$ is the unit matrix. It's clear that

$$(\Omega\Lambda)^{-1} = \Lambda^{-1}\Omega^{-1}$$

Because:

$$(\Omega\Lambda)(\Omega\Lambda)^{-1} = (\Omega\Lambda)(\Lambda^{-1}\Omega^{-1}) = \Omega\Lambda\Lambda^{-1}\Omega^{-1} = \Omega\Omega^{-1} = \mathbb{1}$$

Matrix Representation of Linear Operators

$$\Omega\,|b_i\rangle = |b'_i\rangle$$

$$\Omega\,|v\rangle = \Omega\sum_i v_i\,|b_i\rangle = \sum_i v_i\,|b'_i\rangle$$

to find a vector $|v\rangle = \sum_i v'_i\,|b'_i\rangle$ in $|b_i\rangle$ basis, inner product can be used:

$$|v\rangle = \sum_i v_i\,|b_i\rangle = \sum_i v'_i\,|b'_i\rangle$$

$$v_i = \langle b_i|v\rangle = \sum_j v'_j\,\langle b_i|b'_j\rangle$$

Here to find $\Omega\,|v\rangle$ in $|b_i\rangle$ basis, we use the given method:

$$\Omega\,|v\rangle = \sum_i v'_i\,|b_i\rangle$$

$$v'_i = \langle b_i|\Omega|v\rangle$$

$$\Omega\,|v\rangle = \sum_i |b_i\rangle\,\langle b_i|\Omega|v\rangle = \sum_i |b_i\rangle\,\langle b_i|\sum_j v_j\Omega\,|b_j\rangle$$

$$\Omega |v\rangle = \sum_{i,j} v_j |b_i\rangle \langle b_i|\Omega|b_j\rangle$$

Let's represent the Ω operator as a matrix Ω_{ij} with indices defined as follow:

$$\Omega_{ij} \equiv \langle b_i|\Omega|b_j\rangle$$

The operation of $|v\rangle$ is then obtain by:

$$\Omega |v\rangle = \sum_{i,j} \Omega_{ij} v_j |b_i\rangle$$

$$[\Omega |v\rangle]_i = \langle b_i|\Omega|v\rangle = \sum_j \Omega_{ij} v_j$$

which can be represented as a matrix product:

$$\begin{bmatrix} \Omega |v\rangle_1 \\ \Omega |v\rangle_2 \\ \Omega |v\rangle_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \dots \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \dots \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}$$

Identity Operator

$$\mathbb{1} |v\rangle = \mathbb{1} \sum_i v_i |b_i\rangle = \sum_i v_i |b_i\rangle = \sum_i \left(\sum_j v_j \langle b_i|\mathbb{1}|b_j\rangle \right) |b_i\rangle$$

$$v_i = \sum_j v_j \langle b_i|\mathbb{1}|b_j\rangle$$

$$\langle b_i|\mathbb{1}|b_j\rangle = \mathbb{1}_{ij} = \delta_{ij}$$

$$\mathbb{1} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Projection Operator

$$|v\rangle = \sum_i v_i |b_i\rangle$$

$$v_i = \langle b_i|v\rangle$$

$$|v\rangle = \sum_i |b_i\rangle \langle b_i|v\rangle = \left(\sum_i |b_i\rangle \langle b_i| \right) |v\rangle$$

$$\sum_i |b_i\rangle\langle b_i| = \mathbb{1}$$

$$[|b_k\rangle\langle b_k|]_{ij} = \delta_{ik}\delta_{jk}$$

Let's define the projection operator as:

$$\mathbb{P}_i \equiv |b_i\rangle\langle b_i|$$

where it clearly projects the vector on the given basis

$$\mathbb{P}_i |v\rangle = v_i |b_i\rangle$$

$$\langle v| \mathbb{P}_i = \langle b_i| v_i^*$$

the multiplication of two projector is obtained as:

$$\mathbb{P}_i\mathbb{P}_j = |b_i\rangle\langle b_i|b_j\rangle\langle b_j| = \delta_{ij}|b_i\rangle\langle b_j| = \delta_{ij}\mathbb{P}_i = \delta_{ij}\mathbb{P}_j$$

Multiplication of Operators

$$\Omega(\Lambda|v\rangle) = (\Omega\Lambda)|v\rangle$$

$$[\Omega\Lambda]_{ij} = \langle b_i|\Omega\Lambda|b_j\rangle = \langle b_i|\Omega\mathbb{1}\Lambda|b_j\rangle = \langle b_i|\Omega\left(\sum_k |b_k\rangle\langle b_k|\right)\Lambda|b_j\rangle$$

$$[\Omega\Lambda]_{ij} = \sum_k \langle b_i|\Omega|b_k\rangle\langle b_k|\Lambda|b_j\rangle$$

$$[\Omega\Lambda]_{ij} = \sum_k \Omega_{ik}\Lambda_{kj}$$

Adjoint Operator

$$|v'\rangle = \Omega|v\rangle \equiv |\Omega v\rangle$$

The adjoint operator of Ω called "Ω dagger" represented by $\Omega^\dagger$ is defined as:

$$\langle v'| = \langle\Omega v| = \langle v|\Omega^\dagger$$

Let's find the matrix representation of $\Omega^\dagger$:

$$\langle b_i|\Omega^\dagger|b_j\rangle = \langle\Omega b_i|b_j\rangle = \langle b_j|\Omega b_i\rangle^* = \langle b_j|\Omega|b_i\rangle^*$$

$$\Omega^\dagger_{ij} = \Omega^*_{ji}$$

$$\Omega^\dagger = \left(\Omega^T\right)^* = (\Omega^*)^T$$

So that the adjoint operator is equivalent to transpose conjugate of the operator matrix form. We also can find the adjoint of two operators multiplication $(\Omega\Lambda)^\dagger$:

$$\langle \Omega\Lambda v| = \langle v| (\Omega\Lambda)^\dagger = \langle \Lambda v| \Omega^\dagger = \langle v| \Lambda^\dagger \Omega^\dagger$$

$$(\Omega\Lambda)^\dagger = \Lambda^\dagger \Omega^\dagger$$

Adjoint operator of sum of two operators is also given by:

$$\langle (\Omega + \Lambda) v| = \langle \Omega v| + \langle \Lambda v| = \langle v| \Omega^\dagger + \langle v| \Lambda^\dagger = \langle v| \left(\Omega^\dagger + \Lambda^\dagger\right)$$

$$(\Omega + \Lambda)^\dagger = \Omega^\dagger + \Lambda^\dagger$$

Hermitian Operators
We call an operator hermitian if and only if:

$$\Omega^\dagger = \Omega$$

and we call an operator anti-hermitian if and only if:

$$\Omega^\dagger = -\Omega$$

Any linear operator can be divided into a hermitan and an anti-hermitian parts:

$$\Omega = \frac{\Omega + \Omega^\dagger}{2} + \frac{\Omega - \Omega^\dagger}{2}$$

If Ω and Λ are hermitian operators, then $\Omega\Lambda$, $\Omega\Lambda + \Lambda\Omega$, $[\Omega, \Lambda]$ are hermitian too. Also $i[\Omega, \Lambda]$ is anti-hermitian.

A Vectors Theorems

Theorem 1. *For an N dimensional vector space like $(V, +)$, any set with N linearly independent vector like $S = \{|v_1\rangle, |v_2\rangle, ..., |v_N\rangle\}$ is a basis.*

Proof. N is the number of maximum possible linearly independent vectors in V. It means that if we add any vector like $|v\rangle$ to S, it's no longer linearly independent:

$$\sum_{i=1}^{N} \alpha_i |v_i\rangle + \alpha |v\rangle = 0, \qquad \alpha \neq 0$$

$$\forall |v\rangle \in V : |v\rangle = \sum_{i=1}^{N} \left(-\frac{\alpha_i}{\alpha}\right) |v_i\rangle$$

We have written every elements of the vector space in terms of linear combination of $|v_i\rangle$'s. so S is a basis. $\square$

Theorem 2. *The number of vectors of any basis in an N dimensional vector space like V is equal to N*

Proof. Clearly, by definition, the number of basis cannot exceed the number of dimensions. Assume the number of basis of the N dimensional vector space $(V,+)$ is $m < N$. The basis set is $B = \{|b_1\rangle, |b_2\rangle, ..., |b_m\rangle\}$ and the set with maximum number of linearly independent vectors is $S_0 = \{|v_1\rangle, |v_2\rangle, ..., |v_N\rangle\}$. according to theorem **??**, S_0 itself is a basis:

$$|b_i\rangle = \sum_{j=1}^{N} \alpha_{ij} |v_j\rangle = \sum_{j=1}^{i-1} \alpha_{ij} |v_j\rangle + \alpha_{ii} |v_i\rangle + \sum_{j=i+1}^{N} \alpha_{ij} |v_j\rangle$$

$$|v_i\rangle = \left(\frac{1}{\alpha_{ii}}\right) |b_i\rangle + \sum_{j=1}^{i-1} \left(-\frac{\alpha_{ij}}{\alpha_{ii}}\right) |v_j\rangle + \sum_{j=i+1}^{N} \left(-\frac{\alpha_{ij}}{\alpha_{ii}}\right) |v_j\rangle$$

$|v_i\rangle$ itself can be written as a linear combination of other $|v_j\rangle$'s and $|b_i\rangle$. So the set $S_1 = \{|b_1\rangle, |v_2\rangle, |v_3\rangle, ..., |v_N\rangle\}$ is a basis too. Assume $m < N$. Iterating this method, we finally get:

$$S_m = \{|b_1\rangle, |b_2\rangle, ..., |b_m\rangle, |v_{m+1}\rangle, ..., |v_N\rangle\}$$

As $|b_i\rangle$'s are basis, $|v_i\rangle$'s can be written in terms of them. so S_m is not linearly independent anymore unless our assumption is revoked and there's no $|v_i\rangle$ remained after iterating this proccess. So $m = N$.

Precise Rest of The Proof

We've found this set linearly independent meaning that no linear combination with non-zero coefficients could lead to zero but here we see that if B is a basis meaning that $|v_i\rangle$'s can be written as a linear combination of them:

$$|v_i\rangle = \sum_{j=1}^{m} \beta_{ij} |b_j\rangle$$

its clear that S_m woudln't be linearly independent. Here's an instance linear combination of S_m elements equal to zero to verify its linearly independence:

$$\sum_{i=1}^{m} \gamma_i |b_i\rangle + \sum_{i=m+1}^{N} \gamma_i |v_i\rangle = 0$$

Substituting $|v_i\rangle$'s, we get:

$$\sum_{i=1}^{m} \gamma_i |b_i\rangle + \sum_{i=m+1}^{N} \gamma_i \sum_{j=1}^{m} \beta_{ij} |b_j\rangle = \sum_{i=1}^{m} \gamma_i |b_i\rangle + \sum_{j=m+1}^{N} \gamma_j \sum_{i=1}^{m} \beta_{ji} |b_i\rangle = 0$$

$$\sum_{i=1}^{m} \gamma_i |b_i\rangle + \sum_{i=1}^{m} \sum_{j=m+1}^{N} \gamma_j \beta_{ji} |b_i\rangle = \sum_{i=1}^{m} \left[\gamma_i + \sum_{j=m+1}^{N} \gamma_j \beta_{ji} \right] |b_i\rangle = 0$$

As B is linearly independent, the last equatiton leads to:

$$\gamma_i + \sum_{j=m+1}^{N} \gamma_j \beta_{ji} = 0$$

Which doesn't necessarily need all γ_i's to be zero. So there exists linear combinations of S_m elements to be zero with non-zero coefficients which denies its linear independence.

□

Theorem 3. *Addition inverse of a vector like $|v\rangle$ in basis $B = \{|b_1\rangle, |b_2\rangle, ..., |b_N\rangle$ is equivalent to inverse of all components:*

$$|v\rangle = \sum_{i=1}^{N} \alpha_i |b_i\rangle \iff |-v\rangle = \sum_{i=1}^{N} -\alpha_i |b_i\rangle$$

Proof.

$$|v\rangle = \sum_{i=1}^{N} \alpha_i |b_i\rangle \quad , \quad |-v\rangle = \sum_{i=1}^{N} \beta_i |b_i\rangle$$

$$|v\rangle + |-v\rangle = |0\rangle = \sum_{i=1}^{N} (\alpha_i + \beta_i) |b_i\rangle$$

$$\beta_i = -\alpha_i$$

□

Theorem 4. *The representation of a vector like $|v\rangle$ in $B = \{|b_1\rangle, |b_2\rangle, ..., |b_N\rangle\}$ basis is unique.*

Proof. Assume there is two representation of $|v\rangle$ in this basis:

$$|v\rangle = \sum_{i=1}^{N} \alpha_i \quad , \quad |b_i\rangle = \sum_{i=1}^{N} \beta_i |b_i\rangle$$

Substracting them we obtain:

$$|v\rangle + |-v\rangle = |0\rangle = \sum_{i=1}^{N} (\alpha_i - \beta_i) |b_i\rangle$$

$$\alpha_i = \beta_i$$

□