Note: Convergence in the Continuous Spectrum

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Abstract

In this note, I will show that although continuous operators do not have eigenvectors in the Hilbert space, it is possible to define converging sequences for them in the Hilbert space.

1 Initial Definitions

I will work with the operator under consideration. For simplicity, and without reducing the generality of the problem, I consider the position operator. I define the operator $\hat{\mathbb{E}}(\xi)$ as follows:

$$\hat{\mathbb{E}}(\xi) := \begin{cases} 1 & x < \xi \\ 0 & x > \xi \end{cases}$$

Thus,

$$\mathrm{d}\hat{\mathbb{E}}(\xi) = \hat{\mathbb{E}}(\xi + \mathrm{d}\xi) - \hat{\mathbb{E}}(\xi)$$

This limit is well-defined and bounded. Additionally, this operator satisfies the following relations:

$$\hat{\mathbb{E}}(x_{\min}) = 0 \quad , \quad \hat{\mathbb{E}}(x_{\max}) = \mathbb{1}$$

$$\int_0^{\mathbb{1}} \mathrm{d}\hat{\mathbb{E}}(\xi) = \hat{\mathbb{1}}$$

$$\hat{\mathbb{E}}(\xi)\hat{\mathbb{E}}(\xi') = \hat{\mathbb{E}}(\xi')\hat{\mathbb{E}}(\xi) = \begin{cases} \hat{\mathbb{E}}(\xi) & \xi \leq \xi' \\ \hat{\mathbb{E}}(\xi') & \xi \geq \xi' \end{cases} \tag{1}$$

The position operator (which is x in position representation) can be written as:

$$x = \int_0^{\mathbb{1}} \xi \, \mathrm{d}\hat{\mathbb{E}}(\xi)$$

And similarly:

$$f(x) = \int_0^{\mathbb{1}} f(\xi) \, \mathrm{d}\hat{\mathbb{E}}(\xi)$$

From equation (1), we can deduce:

$$\left(\int f(\xi) \, \mathrm{d}\hat{\mathbb{E}}(\xi) \right) \left(\int g(\eta) \, \mathrm{d}\hat{\mathbb{E}}(\eta) \right) = \int f(\sigma)g(\sigma) \, \mathrm{d}\hat{\mathbb{E}}(\sigma) \tag{2}$$

2   Strong Convergence to Eigenvectors

Consider any desired ε. I define the following operator:

$$\hat{P}_\lambda := \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \mathrm{d}\hat{\mathbb{E}}\left(\xi\right)$$

It can be easily shown that this is a projection operator. Also, assume that ψ_λ is the eigenvector of this operator with eigenvalue 1. Constructing such a vector is very simple. It is enough to find a vector such as ϕ such that $\hat{P}_\lambda \phi \neq 0$. Then,

$$\psi_\lambda = \frac{\hat{P}_\lambda \phi}{\left\|\hat{P}_\lambda \phi\right\|}$$

Thus,

$$\hat{P}_\lambda \psi_\lambda = \psi_\lambda$$

Now, to prove convergence, I write:

$$\hat{X}\psi_\lambda = \hat{X}\hat{P}_\lambda\psi_\lambda = \left(\int_0^{\mathbb{1}} \xi\,\mathrm{d}\hat{\mathbb{E}}\left(\xi\right)\right)\left(\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \mathrm{d}\hat{\mathbb{E}}\left(\eta\right)\right)\psi_\lambda$$

From equation (2), we get:

$$\hat{X}\psi_\lambda = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \xi\,\mathrm{d}\hat{\mathbb{E}}\left(\xi\right)\psi_\lambda$$

Thus,

$$\left\|\left(\hat{X}-\lambda\hat{\mathbb{1}}\right)\psi_\lambda\right\|^2 = \langle\psi_\lambda|\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\xi-\lambda)^2\,\mathrm{d}\hat{\mathbb{E}}\left(\xi\right)|\psi_\lambda\rangle$$

In this, I have once again used equation (2). On the other hand,

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\xi-\lambda)^2\,\langle\psi_\lambda|\,\mathrm{d}\hat{\mathbb{E}}\left(\xi\right)|\psi_\lambda\rangle \leq \varepsilon^2 \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \langle\psi_\lambda|\,\mathrm{d}\hat{\mathbb{E}}\left(\xi\right)|\psi_\lambda\rangle$$

The integral on the right side equals 1:

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \langle\psi_\lambda|\,\mathrm{d}\hat{\mathbb{E}}\left(\xi\right)|\psi_\lambda\rangle = \langle\psi_\lambda|\left(\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \mathrm{d}\hat{\mathbb{E}}\left(\xi\right)\right)|\psi_\lambda\rangle = \langle\psi_\lambda|\hat{P}_\lambda|\psi_\lambda\rangle = 1$$

Finally, we obtain:

$$\boxed{\left\|\left(\hat{X}-\lambda\hat{\mathbb{1}}\right)\psi_\lambda\right\| \leq \varepsilon}$$

Similarly:

$$\left\|\left(f(\hat{X})-f(\lambda)\hat{\mathbb{1}}\right)\psi_\lambda\right\| \leq \varepsilon|f'(\lambda)| + \mathcal{O}(\varepsilon^2)$$

3   My Question

We know that operators with continuous spectra do not have exact eigenvectors. The proof is simple:

$$x\psi(x) = \lambda\psi(x) \Rightarrow (x-\lambda)\psi(x) = 0$$

Therefore, $\psi(x) = 0$ for all x except $x = \lambda$. Since the Dirac delta distribution is not square-integrable, it's not a member of the Hilbert space. Hence, there doesn't exist such $\psi(x)$ satisfying the eigenvalue equation. On the other hand, the completeness axiom of Hilbert space asserts that

if the norm of a sequence of vectors like ψ_n converges to a vector like ψ, then ψ itself is a member of Hilbert space.

Here, I proved that there exists a sequence of ψ_n's such that $\left\| \left(\hat{X} - \lambda \hat{\mathbb{1}} \right) \psi_\lambda \right\|$ converges to zero but there's not any vector in Hilbert space that satisfies $\left(\hat{X} - \lambda \right) \psi_\lambda = 0$. This seems to contradict the completeness of Hilbert space.