Day 2: Nonlinear Effects.

Tube Screamer Clipping Circuit

New component Diode

New circuit analysis technique

Newton-Raphson Method.

JUCE, C++ implementations.

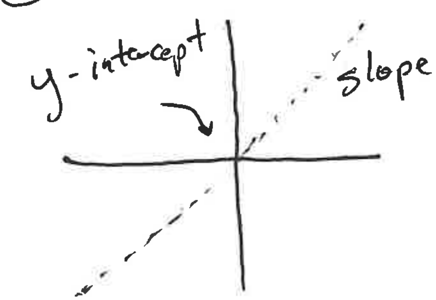
Linear vs. Nonlinear Effects.

Linear	Nonlinear
Gain, DC	Distortion
EQ, Filter	Compression
Echo, Reverb	

Start with digital then analog.

$y = g * x + mu$

Characteristic Curve



Linear operations result in straight line.

Nonlinear operations have a curve.

DC Sweep.   iZotope Trash 2

① ② Common types in audio

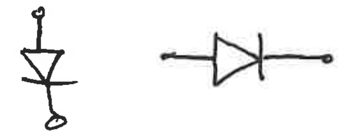
Full-wave rectification : abs(·)

Half-wave rectification : diode

Hard-clipping, soft clipping

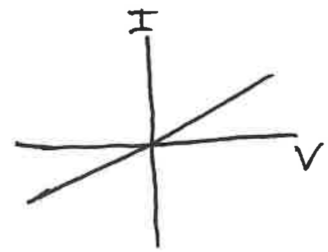
Cubic, arctan, tanh

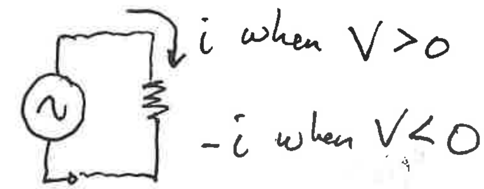
Analog: diodes as a nonlinear component

Symbol 

Basic idea current only flows in one direction.

With a resistor $V = IR$.

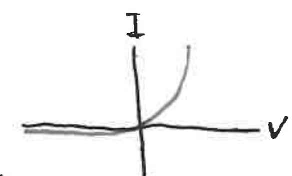




Diode is open circuit ($i = 0$) when $V < 0$

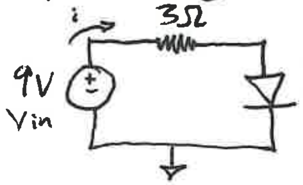
Diode is short circuit when $V > 0$

Ideal



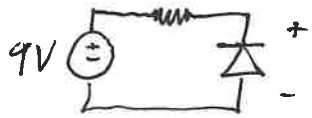
Reality

Ideal Diode Examples

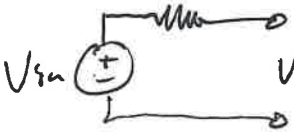


since $V_{in} > 0$ current flows
diode is short

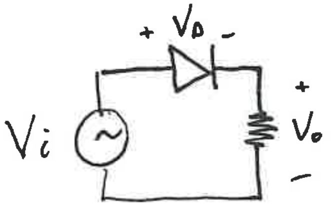
$\downarrow i = \frac{V}{R}$ $i = 3$ $V_0 = 0$



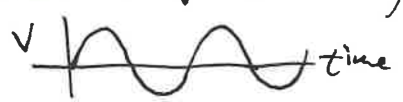
current cannot flow
through diode = open circuit



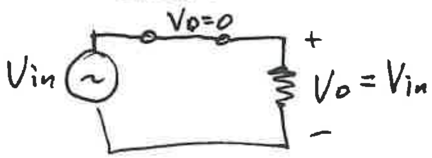
$V_0 = V_{in}$ $i = 0$



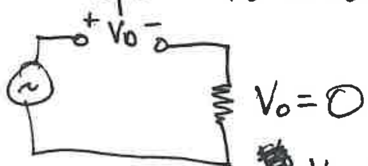
Assume V_i alternating



When $V_i > 0$
short

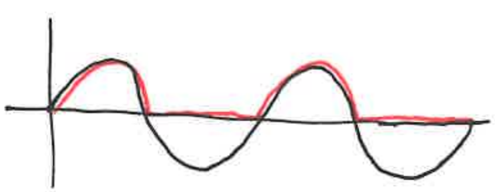


When $V_i < 0$
open no current



$V_{in} = V_D + V_0$

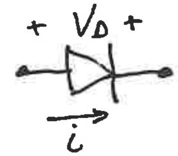
$V_D = V_{in}$



Half-wave Rectifier

③ ④

More accurate model



Shockley Diode Equation
↳ relationship between voltage & current

$$i = I_s\left(e^{\frac{V_D}{nV_T}} - 1\right)$$

I_s - saturation current

V_T - thermal voltage

n - emission coefficient
"quality factor"

Note: different types of diodes can be modeled using different values.

Silicon vs. Germanium.

diodeEquation.m

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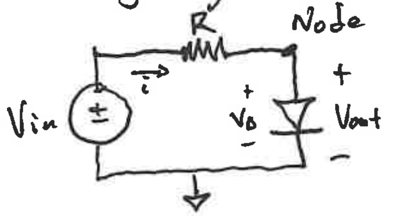
↖ Voltage where diode starts to conduct

$V = 0.7$ silicon

$V = 0.3$ germanium

Silicon	Germanium
I_s: $10^{-12} - 10^{-15}$	10^{-6}
$n = 1$ ideal diode	$n = 1-2$ changes slope

Analyzing circuits using diode equation. ⑤



$V_O = V_{out}$

Currents into node = currents out

$$\frac{V_i - V_{out}}{R} = I_s\left(e^{\frac{V_{out}}{nV_T}} - 1\right)$$

Assume Vin is known: input signal

Need to find Vout : output signal

Can we solve for Vout?

$$\frac{V_i}{R} = I_s\left(e^{\frac{V_{out}}{nV_T}} - 1\right) + \frac{V_{out}}{R}$$

We cannot factor out Vout. due to exponential

This is a special type of equation called an "implicit equation"

Other type called "explicit equation"

↳ we could solve Vout as function of Vin.

Explicit equations can have "analytic" solution

Implicit equations ~~ha~~ do not have analytic solution

instead we use "numerical solution"

↳ iterative process "guess and check"

⑥ Step back

Examples $y = mx + b$ $\quad y = 5x + 2$

Explicit equation the thing we are trying to calculate "y" is isolated and there is only one of it.

More complicated. Solve for x

$-1 = x^2 + 2x$ set equal to zero

$0 = x^2 + 2x + 1$ can we factor?

$0 = (x+1)(x+1)$

$x = -1$

Quadratic Equation $0 = ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 1}}{2\cdot 1} = \frac{-2 \pm \sqrt{0}}{2} = -1$$

Some equations cannot be broken down this way

$0 = x^{\wedge 3} + 2x - 2$ solve for "x"

This is an implicit equation

No analytic solution

Use numerical, guess and check technique.

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Matlab Plot x = [-2 : .001 : 2]

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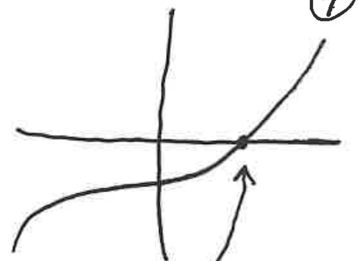
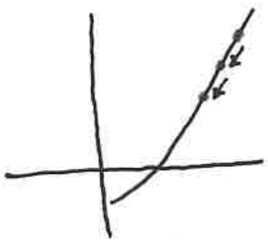
plot(x, x.^3 + 2*x - 2);
```



This is the point where $0 = x^3 + 2x - 2$

In other words, this is the value of "x" that makes this equation true.

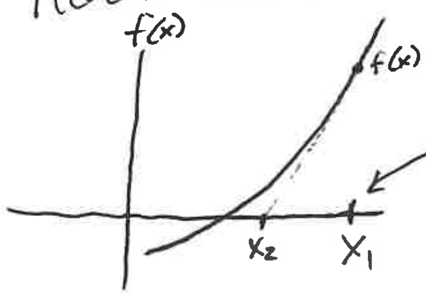
How do we find this? Gradual decent.



Go down (up) by small changes, check if we found zero, continue on.

Too slow. Faster way: Newton-Raphson Method.

Here's how it works for function $f(x) = x^3 + 2x - 2$



Find $f(x)$

If not ≈ 0 use slope to find next guess X_2.

Repeat until ≈ 0

slope $f'(x)$
↳ derivative of $f(x)$

$$X_2 = X_1 - \frac{f(x_1)}{f'(x_1)}$$

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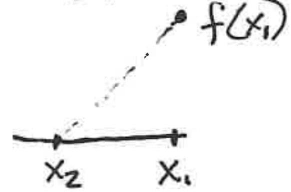
Why does this work?

Start with equation to find slope.



Note $f(x_2) = 0$

$$\frac{\text{Change "y"}}{\text{Change "x"}} = \text{slope}$$

$$\frac{f(X_1) - 0}{X_1 - X_2} = \text{slope} = f'(X_1)$$

$$f(x_1) = f'(x_1) \cdot (X_1 - X_2)$$

$$\frac{f(x_1)}{f'(x_1)} = X_1 - X_2 \qquad X_2 = X_1 - \frac{f(x_1)}{f'(x_1)}$$

If we have $0 = X^3 + 2x - 2$

Guess X_1

Calculate $f(x_1) = (x_1)^3 + 2(x_1) - 2$

Need to know slope $f'(x)$.

Many functions have known derivatives.

Look-up Wolfram Alpha Calculus Class

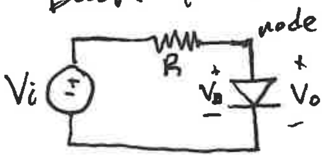
$f(x) = x^3 + 2x - 2 \qquad f'(x) = 3x^2 + 2$

Plug in X_1 & calculate

$$X_2 = X_1 - \frac{(x_1)^3 + 2 \cdot (x_1) - 2}{3(x_1)^2 + 2}$$

$\boxed{8}$ NewtonRaphson.m

Back to our circuit.



$$\frac{V_i - V_o}{R} = I_s * \left(e^{\frac{V_o}{nV_T}} - 1\right)$$

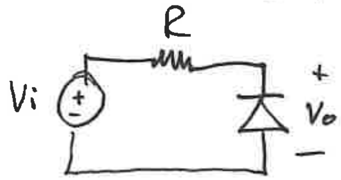
$$\frac{V_i}{R} = I_s\left(e^{\frac{V_o}{nV_T}} - 1\right) + \frac{V_o}{R}$$

$$0 = I_s\left(e^{\frac{V_o}{nV_T}} - 1\right) + \frac{V_o}{R} - \frac{V_i}{R}$$

9 diode Resistor

$$f(V_o) = I_s\left(e^{\frac{V_o}{nV_T}} - 1\right) + \frac{V_o}{R} - \frac{V_i}{R}$$

$$f'(V_o) = \frac{I_s}{nV_T}\left(e^{\frac{V_o}{nV_T}}\right) + \frac{1}{R}$$

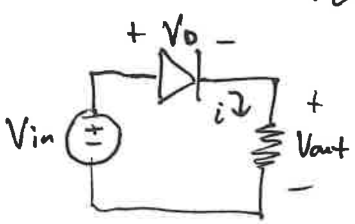


$V_o = -V_D \quad V_D = -V_o$

$$\frac{V_i - V_o}{R} = -I_s\left(e^{\frac{V_o}{nV_T}} - 1\right)$$

$$f(x) = -I_s\left(e^{-V_o/nV_T} - 1\right) + \frac{V_o}{R} - \frac{V_i}{R}$$

$$f'(x) = +\frac{I_s}{nV_T}\left(e^{\frac{-V_o}{nV_T}}\right) + \frac{1}{R}$$



$V_{in} = V_D + V_{out}$

$V_{out} = V_{in} - V_D$

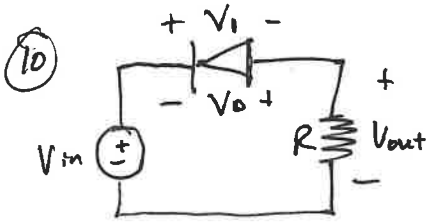
Current $i = \frac{V_{out}}{R} \quad i = I_s\left(e^{\frac{V_D}{nV_T}} - 1\right)$

Set equal $\frac{V_{out}}{R} = I_s\left(e^{\frac{V_D}{nV_T}} - 1\right)$

$$\frac{V_{in} - V_D}{R} = I_s\left(e^{\frac{V_D}{nV_T}} - 1\right)$$

$$0 = I_s\left(e^{\frac{V_D}{nV_T}} - 1\right) + \frac{V_D}{R} - \frac{V_{in}}{R} \qquad f'(x) = \frac{I_s}{nV_T}\left(e^{\frac{V_D}{nV_T}}\right) + \frac{1}{R}$$

at end $V_{out} = V_{in} - V_D$



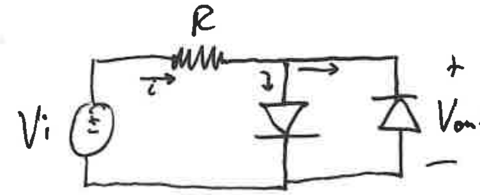
$V_1 = -V_D \qquad V_{out} = V_{in} - V_1$

$$i = \frac{V_{out}}{R} \qquad i = -I_s\left(e^{\frac{-V_1}{nV_T}} - 1\right)$$

$$\frac{V_{out}}{R} = -I_s\left(e^{\frac{-V_1}{nV_T}} - 1\right) \qquad \frac{V_{in} - V_1}{R} = -I_s\left(e^{-\frac{V_1}{nV_T}} - 1\right)$$

$$0 = -I_s\left(e^{\frac{-V_1}{nV_T}} - 1\right) + \frac{V_1}{R} - \frac{V_{in}}{R}$$

$$f'(V_1) = \frac{I_s}{nV_T}\left(e^{\frac{-V_1}{nV_T}}\right) + \frac{1}{R}$$



Current i split across diodes.

$$\frac{V_i - V_o}{R} = I_s\left(e^{\frac{V_o}{nV_T}} - 1\right) + -I_s\left(e^{\frac{-V_o}{nV_T}} - 1\right)$$

$$f(x) = I_s\left(e^{\frac{V_o}{nV_T}} - 1\right) + -I_s\left(e^{\frac{-V_o}{nV_T}} - 1\right) + \frac{V_o}{R} - \frac{V_i}{R}$$

$$f'(x) = \frac{I_s}{nV_T}\left(e^{\frac{V_o}{nV_T}}\right) + \frac{I_s}{nV_T}\left(e^{\frac{-V_o}{nV_T}}\right) + \frac{1}{R}$$

Note instead of $I_s\left(e^{\frac{V_o}{nV_T}} - 1\right) + -I_s\left(e^{\frac{-V_o}{nV_T}} - 1\right)$

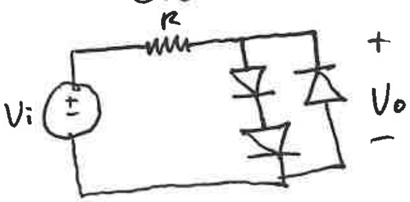
we can use $2 I_s \cdot \sinh\left(\frac{V_o}{nV_T}\right)$

Derivative $2\frac{I_s}{nV_T}\cosh\left(\frac{V_o}{nV_T}\right)$

If diode are assumed to be identical

Asymmetrical Distortion

even & odd harmonics



2 diodes in series same direction
1 diode in parallel opposite direction.

$$i = I_s\left(e^{\frac{V_D}{2\cdot \eta V_T}} - 1\right)$$

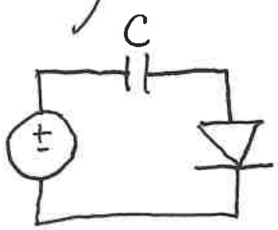
↑ when there are 2 diodes.

$$f(x) = I_s\left(e^{\frac{V_D}{2\cdot \eta V_T}} - 1\right) - I_s\left(e^{\frac{-V_D}{\eta V_T}} - 1\right) + \frac{V_o}{R} - \frac{V_i}{R}$$

$$f'(x) = \frac{I_s}{2\cdot \eta V_T} e^{\frac{V_o}{2\eta V_T}} + \frac{I_s}{\eta V_T}\left(e^{\frac{-V_o}{\eta V_T}}\right) + \frac{1}{R}$$

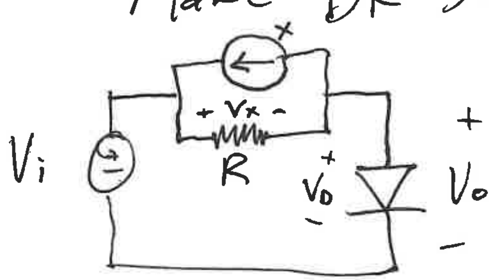
Other options: use germanium & silicon together.

Using diodes and capacitors together.



Weird circuit. Introduces DC offset (negative)

Make DK substitution



$V_o = V_D$

$R = \frac{T_s}{2\cdot C}$

$x \rightarrow$ state variable

Current through diode

$$i = I_s\left(e^{\frac{V_d}{\eta V_T}} - 1\right)$$

Current through capacitor

$$i = \frac{V_i - V_d}{R} - x[n-1]$$

$$\frac{V_i - V_d}{R} - x = I_s\left(e^{\frac{V_d}{\eta V_T}} - 1\right)$$

$$0 = I_s\left(e^{\frac{V_d}{\eta V_T}} - 1\right) + \frac{V_d}{R} + x - \frac{V_i}{R}$$

$$f'(x) = \frac{I_s}{\eta V_T}\left(e^{\frac{V_d}{\eta V_T}}\right) + \frac{1}{R}$$

$$V_d[n+1] = V_d[n] - \frac{f(V_d)}{f'(V_d)}$$

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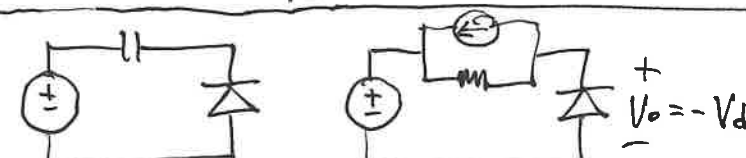
diodeCapacitor

State update equation

$$x[n] = \frac{2}{R}(V_i - V_d) - x[n-1]$$



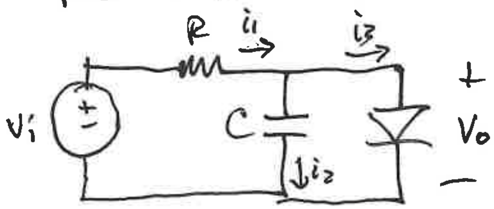
$$\frac{V_i - V_o}{R} - x = -I_s\left(e^{\frac{-V_o}{\eta V_T}} - 1\right)$$

$$0 = -I_s\left(e^{\frac{-V_o}{\eta V_T}} - \right) + \frac{V_o}{R} + x - \frac{V_i}{R}$$

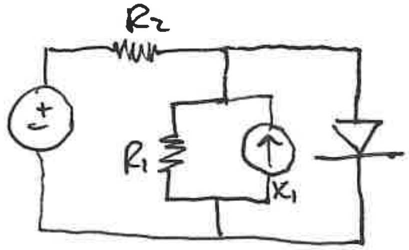
$$f'(x) = \frac{I_s}{\eta V_T}\left(e^{\frac{-V_o}{\eta V_T}}\right) + \frac{1}{R}$$

$$x[n] = \frac{2}{R}(V_i - V_o) - x[n-1]$$

RC Filter with Diode.



$i_1 = i_2 + i_3$



$$i_1 = \frac{V_i - V_o}{R_2}$$

$$i_2 = \frac{V_o}{R_1} - X_1$$

$$i_3 = I_s\left(e^{\frac{V_o}{nV_T}} - 1\right)$$

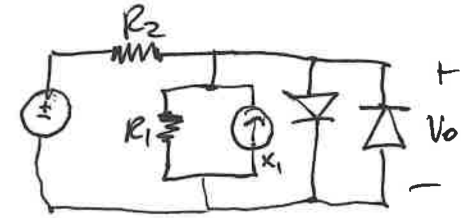
$$\frac{V_i - V_o}{R_2} = \frac{V_o}{R_1} - X_1 + I_s\left(e^{\frac{V_o}{nV_T}} - 1\right)$$

$$0 = V_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - X_1 + I_s\left(e^{\frac{V_o}{nV_T}} - 1\right) - \frac{V_i}{R_2}$$

$$f'(V_o) = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{I_s}{nV_T}\left(e^{\frac{V_o}{nV_T}}\right)$$

$$X_1[n] = \frac{2}{R_1} V_o - X_1[n-1]$$

13 14 RC Filter with parallel diodes



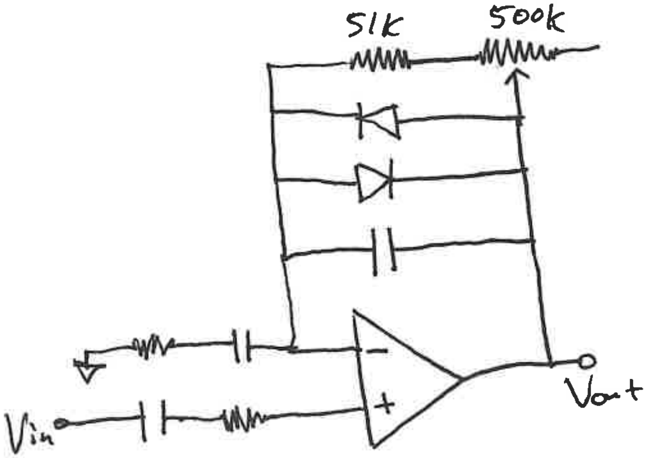
Recall $2 \cdot I_s \cdot \sinh\left(\frac{V_d}{nV_T}\right)$

$$\frac{V_i - V_o}{R_2} = \frac{V_o}{R_1} - X_1 + 2 \cdot I_s \cdot \sinh\left(\frac{V_d}{nV_T}\right)$$

$$0 = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_o - X_1 + 2 I_s \cdot \sinh\left(\frac{V_d}{nV_T}\right) - \frac{V_i}{R_2}$$

$$f'(x) = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_o + \frac{2I_s}{nV_T} \cdot \cosh\left(\frac{V_d}{nV_T}\right)$$

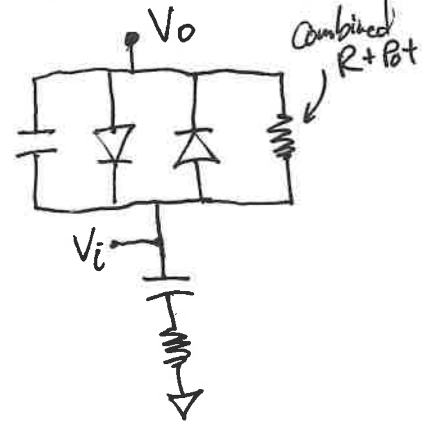
$$X_1[n] = \frac{2}{R_1} \cdot V_o - X_1[n-1]$$



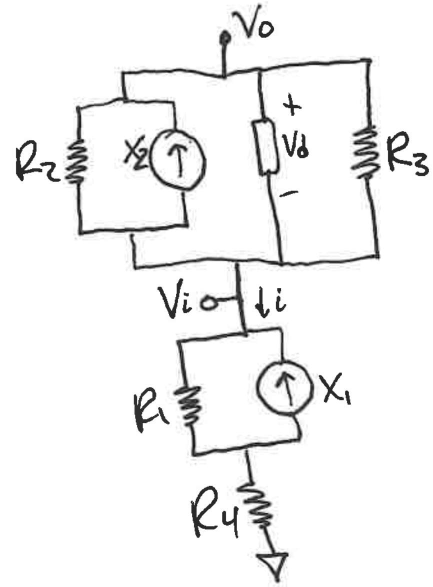
Notes: no current can flow into "+" terminal of op-amp. As a result, there is no voltage drop across input R & C. Node at input to op-amp = Vin.

* The op-amp is an important buffer for this sub-circuit in the context of the entire circuit. However, for our model we can actually work without it.
* We can replace the two diodes with a single block. $i = 2 I_s \cdot \sinh\left(e^{\frac{V_d}{n V_T}}\right)$
* The potentiometer is in series with 51k resistor. Combined they have a resistance between 51k – 551k

15 16 Practical Circuit



DK



Voltage Divider

$V_o = V_d + V_i$

Voltage across 3 parallel components.

Use voltage across diode for all

$V_{R2} = V_d = V_{R3}$

Total current

$i = i_2 + i_d + i_3$

$i_2 = \frac{V_{R2}}{R_2} - X_2 = \frac{V_d}{R_2} - X_2$

$i_3 = \frac{V_{R3}}{R_3} = \frac{V_d}{R_3}$ $\quad i_d = 2 \cdot I_s \cdot \sinh\left(\frac{V_d}{n V_T}\right)$

$i = \frac{V_d}{R_2} - X_2 + 2 \cdot I_s \cdot \sinh\left(\frac{V_d}{n V_T}\right) + \frac{V_d}{R_3}$

save for later.

$$V_i = V_{R1} + V_{R4}$$

Currents must be equal through R & C

$$\frac{V_{R4}}{R_4} = \frac{V_{R1}}{R_1} - X_1$$ Solve for & replace V_{R1}

$$\frac{V_{R4}}{R_4} + X_1 = \frac{V_{R1}}{R_1} \Rightarrow \frac{R_1}{R_4} V_{R4} + R_1 X_1 = V_{R1}$$

$$V_i = \frac{R_1}{R_4} V_{R4} + R_1 X_1 + V_{R4}$$ Solve for & replace V_{R4}

$$V_i - R_1 X_1 = V_{R4} \underbrace{\left(\frac{R_1}{R_4} + 1\right)}_{G_4}$$

$$V_{R4} = \frac{V_i}{G_4} - \frac{R_1}{G_4} X_1$$ Plug back in

$$i = \frac{V_{R4}}{R_4} \qquad i = \frac{V_i}{G_4 R_4} - \frac{R_1}{G_4 R_4} X_1$$

Current in bottom of circuit must equal current in top.

$$\frac{V_i}{G_4 R_4} - \frac{R_1}{G_4 R_4} X_1 = \frac{V_d}{R_2} - X_2 + \frac{V_d}{R_3} + 2 \cdot I_s \cdot \sinh\left(\frac{V_d}{\eta V_t}\right)$$

$$0 = \frac{-V_i}{G_4 R_4} + \frac{R_1}{G_4 R_4} X_1 + \frac{V_d}{R_2} - X_2 + \frac{V_d}{R_3} + 2 \cdot I_s \cdot \sinh\left(\frac{V_d}{\eta V_t}\right)$$

$$f'(V_d) = \frac{1}{R_2} + \frac{1}{R_3} + \frac{2 \cdot I_s}{\eta V_t} \sinh\left(\frac{V_d}{\eta V_t}\right)$$

Use in Newton solver for V_d. Then $V_{out} = V_d + V_{in}$

(17) (18) State Update Equations.

$$X_2[n] = \frac{2}{R_2} \cdot V_{R2} - X_2[n-1] = \frac{2}{R_2} V_d - X_2[n-1]$$

$$X_1[n] = \frac{2}{R_1} V_{R1} - X_1[n-1]$$

$$V_i = V_{R1} + V_{R4}$$ ← replace V_{R4} so everything is a function of V_{R1}

$$\frac{V_{R4}}{R_4} = \frac{V_{R1}}{R_1} - X_1$$

$$V_{R4} = \frac{R_4}{R_1} V_{R1} - X_1 \cdot R_4$$

$$V_i = V_{R1} + \frac{R_4}{R_1} V_{R1} - R_4 X_1$$ isolate V_{R1}

$$V_i + R_4 X_1 = V_{R1} \underbrace{\left(1 + \frac{R_4}{R_1}\right)}_{G_1}$$

$$V_{R1} = \frac{V_i}{G_1} + \frac{R_4}{G_1} X_1$$ Plug in state equation

$$X_1[n] = \frac{2}{R_1}\left[\frac{V_i}{G_1} + \frac{R_4}{G_1} X_1\right] - X_1$$