ENSIIE. Project simulations: MESIM.

Plan

1. Simulation of a discrete random variable

2. Description of the project
    - Partie introductive
    - Generating the correction of the exercises
        - Correction des exercices
        - Generating exercises

Plan

Simulation

Example. Let X be a random variable valued in $E = \{a_1, a_2, a_3, a_4\}$ with:

$$p_1 = \mathbb{P}(X = a_1),\ p_2 = \mathbb{P}(X = a_2),\ p_3 = \mathbb{P}(X = a_3),\ p_4 = \mathbb{P}(X = a_4).$$

$\diamond$ For example: $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$ and $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{6}$, $p_3 = \frac{1}{3}$ and $p_4 = \frac{1}{4}$.

$\diamond$ *Goal*. Choose randomly $X(\omega)$ among the values taken by X.

$\diamond$ *Principle of simulation*. Let X be a random variable with cdf $F(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$. Let U be a r.v. with uniform distribution on $]0,1[$: $U \sim \mathcal{U}(]0,1[)$ and let F^{-1} be the quantile of X defined as:

$$F^{-1}(u) := \inf\{x \in \mathbb{R} : \quad F(x) \geq u\}, \qquad \text{for all} \quad u \in]0,1[.$$

Then X and $F^{-1}(U)$ have the same distribution: $X \overset{\mathcal{L}}{=} F^{-1}(U)$.

Simulation

We use the previous result to generate a random number from the distribution of X.

◇ The cdf of X reads:

$$F(x) = \begin{cases} 0 & \text{if } x < -2 \\ 1/4 & \text{if } x \in [-2, -1[\\ 5/12 & \text{if } x \in [-1, 0[\\ 9/12 & \text{if } x \in [0, 1[\\ 1 & \text{if } x \geq 1 \end{cases}$$

and its quantile F^{-1} is given by

$$F^{-1}(u) = \begin{cases} a_1 & \text{if } u \in]0, 1/4] \\ a_2 & \text{if } u \in]1/4, 5/12] \\ a_3 & \text{if } u \in]5/12, 9/12] \\ a_4 & \text{if } u \in]9/12, 1[. \end{cases}$$

Simulation

Then, to sample $x = X(\omega)$ from the distribution of X, we use the following steps:

$\diamond$ We sample a random number $u = U(\omega)$ d from de distribution of $U \sim \mathcal{U}(]0,1[)$.

$\diamond$ Then,

- if $u \in]0, 1/4]$ then, we set $x = a_1$,
- if $u \in]1/4, 5/12]$ then, we set $x = a_2$,
- if $u \in]5/12, 9/12]$ then, we set $x = a_3$
- if $u \in]9/12, 1]$ then, we set $x = a_4$.

We repeat n times the previous steps to obtain a sample of size n from the distribution of X.

$\diamond$ *Algorithm*. $c[k] = p_0 + \ldots + p_k$, $X[k] = a_k$, $u = rand()$, $n = 4$

k $\leftarrow$ 0; u $\leftarrow$ rand
while $(u > c[k])$ and $(k < n)$
$k \leftarrow k + 1$
end

$x = X(\omega) \leftarrow X[k]$

Plan

Plan

Aim of the project

The project aims to automate the creation and correction of second-degree equations.

⋄ The project includes an introductory section and a section with corrected exercises

⋄ The introductory section is a short text (and possibly audio) introduction to the purpose of the application. It contains, for example

- We are interested in solving quadratic equations
- A quadratic equation is written in the form $ax^2 + bx + c$, where the coefficients a, b, c, called monomials, are real numbers with $a \neq 0$.
- The number of solutions to this equation, which is either 0, 1, or 2, is determined by the sign of the discriminant Δ of the equation, which is written as
$$\Delta = b^2 - 4ac.$$

Description du projet

Therefore,

1. if $\Delta < 0$ then, the equation has no solution,
2. if $\Delta = 0$, l the equation admits a unique solution $x = -b/(2a)$,
3. if $\Delta > 0$, the equation admits two distinct solutions:

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

Plan

Correction of the exercises

◇ The user is asked to first specify the number of exercises they wish to do.

◇ The exercises are then generated one by one, indicating the time and number of points assigned to them.

◇ Once an exercise has been proposed (we will look at the principle of exercise generation), we go through the following steps to grade the user and propose a correction.

- The number of points associated with the exercise is displayed, along with a timer showing the time remaining to complete it.
- *Question*: What is the discriminant of the equation? (0.5 pt)
- *Question*: What is the number of solutions of the equation (0.5pt)
- depending on the situation, we suggest that they enter the solutions they have found.

Generating exercises

$\diamond$ First, we start by choosing the number of exercises of each type: type 1: $\Delta < 0$, type 2: $\Delta = 0$, type 3: $\Delta > 0$.

$\diamond$ To do this, we consider a random variable X that takes the values $(1, 2, 3)$ with respective probabilities $(1/5, 2/5, 2/5)$, and we generate a random number x according to the distribution X: we then consider case 1, 2, or 3 depending on the value of x.

$\diamond$ *Choice of coefficients in case 3: when $\Delta > 0$.*

- We set $\Delta = e^2$. We want to choose a, b, c such that

$$\begin{cases} -b - \sqrt{\Delta} = 2ax_1 \\ -b + \sqrt{\Delta} = 2ax_2 \\ b^2 - 4ac = \Delta \end{cases}$$

- This leads to

$$a = 1, \quad b = -(x_1 + x_2), \quad c = x_1 x_2.$$

Examples

◇ *Example 1*. Choose a, b, c such that $(x_1, x_2) = (-2, 3)$. We have

$$a = 1,\ b = -1,\ c = -6: \quad \text{the associated equation is}\ \ x^2 - x - 6 = 0.$$

◇ *Example 2*. Choose a, b, c such that $(x_1, x_2) = \big(-\frac{1}{2}, \frac{3}{2}\big)$. We get

$$a = 1,\ b = -1,\ c = -\frac{3}{4}: \quad \text{the associated equation is}\ \ 4x^2 - 4x - 3 = 0.$$

◇ *Example 3*. Choose a, b, c such that $(x_1, x_2) = \big(1, \frac{2}{3}\big)$. We have

$$a = 1,\ b = -\frac{5}{3},\ c = \frac{2}{3}: \quad \text{the associated equation is}\ 3x^2 - 5x + 2 = 0.$$

◇ *Example 4*. Choose a, b, c such that $(x_1, x_2) = \big(\frac{-1-2\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\big)$. We have

$$a = 1,\ b = 1,\ c = -\frac{11}{4}: \quad \text{the associated equation is}\ \ 4x^2 + 4x - 11 = 0.$$

Summary for case 3.

⋄ Let $E = \{x_1, \ldots, x_n\}$ of cardinal n. We denote $\mathcal{U}(E)$ the uniform distribution on E which assigns to the r.v. X valued in E the distribution: $\mathbb{P}(X = x_i) = 1/n$, $\forall i = 1, \ldots, n$.

⋄ $x \sim X$ or $x \sim \mathcal{U}(E)$ means: x is a sample from the distribution of X or from the distribution $\mathcal{U}(E)$.

⋄ We will consider the sets:

$$\mathbb{E} = \{-9, -8, \ldots, -1, 1, \ldots, 8, 9\} \quad \text{and} \quad \mathbb{E}^+ = \{1, \ldots, 8, 9\}$$

⋄ To automate the generation of the corrections in case 3 we consider the 2 following situations, each being chosen once in every two :

- Case 3.1. Choose randomly x_1 and x_2 as

$$x_1 = \frac{h}{\ell}, \quad x_2 = \frac{k}{\ell} \quad \text{with } k, h \sim \mathcal{U}(\mathbb{E}) \text{ and } \ell \sim Z$$

where Z is a r.v. valued in $\mathbb{E}$ (of cardinality 18) with $\mathbb{P}(Z = 1) = \frac{1}{2}$ and $\mathbb{P}(Z \neq 1) = \frac{1}{2 \times 17}$.

Résumé pour le cas 3.

- Case 3.2. Choose randomly x_1 and x_2 as

$$x_1 = \frac{-h - e\sqrt{p}}{\ell}, \; x_2 = \frac{-h + e\sqrt{p}}{\ell} \quad \text{where } h \sim \mathcal{U}(\mathbb{E}), e, p \sim \mathcal{U}(\mathbb{E}^+)$$

and $\ell \sim \mathcal{U}(E)$, with $E = \{-3, -2, -1, 1, 2, 3\}$. In this case

$$a = 1, \; b = \frac{2h}{\ell}, \; c = \frac{h^2 - pe^2}{\ell^2} \implies \ell^2 x^2 + 2h\ell x + (h^2 - pe^2) = 0.$$

◇ In summary (of the page 13) if a user chooses to do 5 exercises,

- 1 out of 5 exercises will be of type 1 ($\Delta < 0$)
- 2 out of 5 exercises will be of type 2 ($\Delta = 0$)
- 2 out of 5 exercises will be of type 3 ($\Delta > 0$)

◇ In case 3,

- 1 out of every two exercises will be of type 3.1
- 1 out of every two exercises will be of type 3.2

Summary of case 2.

$\diamond$ *Choice of the coefficients in case 2: when* $\Delta = 0$.

$\diamond$ *Example*. Choose a, b, c such that the solution x_0 of the equation is $x_0 = -2$. Using the fact that $b = -2ax_0$ and $c = ax_0^2$, this leads to the equation

$$x^2 - 2x_0x + x_0^2 = 0.$$

- The solution $x_0 = -2$ is associated with the equation $x^2 + 4x + 4 = 0$.
- The solution $x_0 = \frac{2}{\sqrt{3}}$ is associated with the equation $3x^2 - 4\sqrt{3}x + 4 = 0$.

$\diamond$ In summary, one can generally choose x_0 in the form

$$x_0 = \frac{e}{\sqrt{\ell}}, \qquad e \sim \mathcal{U}(\mathbb{E}),\ \ell \sim X$$

where X is a r.v. valued in $\mathbb{E}^+$ with $\mathbb{P}(X = 1) = \frac{1}{2}$, $\mathbb{P}(X = i) = \frac{1}{6}$ for $i \in \{4, 9\}$, $\mathbb{P}(X = i) = \frac{1}{6\times 6}$ for $i \in \{2, 3, 5, 6, 7, 8\}$.

Summary for case 1.

$\diamond$ *Choice of the coefficients in case 1: when* $\Delta < 0$.

$\diamond$ We want to choose a, b, c such that $\Delta < 0$. We fix a and b and choose c such that $c > \frac{b^2}{4a}$.

$\diamond$ Therefore, to automate the generation and the correction of exercise we may choose $a, b \sim \mathcal{U}(\mathbb{E})$, and then $e \sim \mathcal{U}(E)$, with $E = \{1, 2, 3\}$ and set

$$c = \frac{b^2 + e}{4a}$$

$\diamond$ If, for example, $a = -2$, $b = 3$ et $e = 1$,then, $c = -\frac{5}{4}$ and the associated equation is

$$-8x^2 + 12x - 5 = 0.$$

$\diamond$ We will seek (as was the case previously where $c = -\frac{10}{8} = -\frac{5}{4}$) to make all fractions irreducible.