

TD: Martingales and Markov Chains

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Exercise 1

We toss twice a coin and record the successive sides that appear. Let $\Omega = \{HH, HT, TH, TT\}$ (where $H \equiv$ HEAD and $T \equiv$ TAIL) be the sample sample space. Let X_k be the random variable which counts the number of Heads that appears after the k first tosses and let \mathcal{G}_1 and \mathcal{G}_2 be the σ -algebras on Ω defined by

$$\mathcal{G}_1 = \{\emptyset, \Omega, \{HT, HH\}, \{TH, TT\}\} \quad \text{and} \quad \mathcal{G}_2 = \{\emptyset, \Omega, \{TH, HH\}, \{HT, TT\}\}$$

1. What is the information contained in \mathcal{G}_1 and \mathcal{G}_2 ?
2. Are the random variable X_1 and X_2 is \mathcal{G}_1 -measurable ?
3. Are the random variable X_1 and X_2 is \mathcal{G}_2 -measurable ?
4. Determine the σ -algebra $\varepsilon = \mathcal{G}_1 \cap \mathcal{G}_2$ and $\mathcal{H} = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$. What is the informations they contain?
 - (a). Are the random variable X_1 and X_2 is \mathcal{H} -measurable?
 - (b). Are the random variable X_1 and X_2 is ε -measurable?
5. Determine $\sigma(X_1), \sigma(X_2)$ and $\sigma(X_1, X_2)$.

Solution

1. Information contained in \mathcal{G}_1 and \mathcal{G}_2 .

We have

$$\mathcal{G}_1 = \{\emptyset, \Omega, \{HT, HH\}, \{TH, TT\}\}, \quad \mathcal{G}_2 = \{\emptyset, \Omega, \{TH, HH\}, \{HT, TT\}\}.$$

- For \mathcal{G}_1 :

$$\{HT, HH\} = \{\omega : \text{first toss is } H\}, \quad \{TH, TT\} = \{\omega : \text{first toss is } T\}.$$

Thus, \mathcal{G}_1 contains *exactly* the information of the **first toss**: we know whether the first toss is Head or Tail, but nothing about the second toss.

- For \mathcal{G}_2 :

$$\{TH, HH\} = \{\omega : \text{second toss is } H\}, \quad \{HT, TT\} = \{\omega : \text{second toss is } T\}.$$

Thus, \mathcal{G}_2 contains *exactly* the information of the **second toss**: we know whether the second toss is Head or Tail, but not the first.

2. Are X_1 and X_2 \mathcal{G}_1 -measurable?

Recall: a random variable X is \mathcal{G}_1 -measurable iff for every Borel set $B \subset \mathbb{R}$,

$$X^{-1}(B) \in \mathcal{G}_1.$$

On a finite space, it is enough to check the preimages of singletons $B = \{x\}$.

- For X_1 :

$$X_1^{-1}(\{1\}) = \{HH, HT\} \in \mathcal{G}_1, \quad X_1^{-1}(\{0\}) = \{TH, TT\} \in \mathcal{G}_1.$$

Thus all level sets of X_1 belong to \mathcal{G}_1 , so X_1 is \mathcal{G}_1 -measurable.

- For X_2 :

$$X_2^{-1}(\{2\}) = \{HH\}.$$

But $\{HH\} \notin \mathcal{G}_1$ (the only nontrivial sets are $\{HH, HT\}$ and $\{TH, TT\}$). Hence X_2 is *not* \mathcal{G}_1 -measurable.

So:

X_1 is \mathcal{G}_1 -measurable, but X_2 is not.

3. Are X_1 and X_2 \mathcal{G}_2 -measurable?

We do the same with \mathcal{G}_2 .

- For X_1 :

$$X_1^{-1}(\{1\}) = \{HH, HT\}.$$

In \mathcal{G}_2 we only have $\emptyset, \Omega, \{TH, HH\}, \{HT, TT\}$. The set $\{HH, HT\}$ is not one of these, so $\{HH, HT\} \notin \mathcal{G}_2$. Hence X_1 is *not* \mathcal{G}_2 -measurable.

- For X_2 :

$$X_2^{-1}(\{2\}) = \{HH\}, \quad X_2^{-1}(\{1\}) = \{HT, TH\}, \quad X_2^{-1}(\{0\}) = \{TT\}.$$

None of the sets $\{HH\}$, $\{HT, TH\}$, $\{TT\}$ belongs to \mathcal{G}_2 . Thus X_2 is also *not* \mathcal{G}_2 -measurable.

So:

X_1 and X_2 are not \mathcal{G}_2 -measurable.

4. Determine $\varepsilon = \mathcal{G}_1 \cap \mathcal{G}_2$ and $\mathcal{H} = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$, and their information.

Intersection:

$$\mathcal{G}_1 = \{\emptyset, \Omega, A, A^c\}, \quad A := \{HH, HT\}, \quad A^c = \{TH, TT\},$$

$$\mathcal{G}_2 = \{\emptyset, \Omega, B, B^c\}, \quad B := \{HH, TH\}, \quad B^c = \{HT, TT\}.$$

Clearly \emptyset and Ω are in both. But $A \neq B, B^c$ and $A^c \neq B, B^c$, so there is no other common set.

Thus

$$\varepsilon = \mathcal{G}_1 \cap \mathcal{G}_2 = \{\emptyset, \Omega\}.$$

This is the *trivial* σ -algebra: it contains no information at all (we only know that something happened in Ω).

Generated σ -algebra $\mathcal{H} = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$:

First compute the partition induced by the events of \mathcal{G}_1 and \mathcal{G}_2 . Consider intersections of A, A^c with B, B^c :

$$A \cap B = \{HH\}, \quad A \cap B^c = \{HT\}, \quad A^c \cap B = \{TH\}, \quad A^c \cap B^c = \{TT\}.$$

These four non-empty sets form a partition of Ω into singletons:

$$\{HH\}, \{HT\}, \{TH\}, \{TT\}.$$

The σ -algebra generated by this partition is the whole power set:

$$\mathcal{H} = \mathcal{P}(\Omega) = \{\text{all subsets of } \Omega\}.$$

Information:

- ε has no information (we cannot distinguish any nontrivial event).
- \mathcal{H} has full information: we can distinguish each single outcome, so we know exactly the result of both tosses.

(a). **Are X_1 and X_2 \mathcal{H} -measurable?**

Since $\mathcal{H} = \mathcal{P}(\Omega)$ (the full σ -algebra), every subset of Ω is in \mathcal{H} , so the preimage of any Borel set is in \mathcal{H} . Therefore

$$X_1 \text{ and } X_2 \text{ are } \mathcal{H}\text{-measurable.}$$

(b). **Are X_1 and X_2 ε -measurable?**

For measurability with respect to $\varepsilon = \{\emptyset, \Omega\}$, the preimage of every Borel set must be either \emptyset or Ω . This happens only for constant random variables. But X_1 and X_2 are not constant:

$$X_1^{-1}(\{1\}) = \{HH, HT\} \notin \varepsilon, \quad X_2^{-1}(\{0\}) = \{TT\} \notin \varepsilon.$$

Thus

$$X_1 \text{ and } X_2 \text{ are not } \varepsilon\text{-measurable.}$$

5. Determine $\sigma(X_1)$, $\sigma(X_2)$ and $\sigma(X_1, X_2)$.

By definition

$$\sigma(X_1) = \{X_1^{-1}(B) : B \subset \mathbb{R} \text{ Borel}\}.$$

- For X_1 (values 0 and 1):

$$X_1^{-1}(\{1\}) = \{HH, HT\}, \quad X_1^{-1}(\{0\}) = \{TH, TT\}.$$

Thus

$$\sigma(X_1) = \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\} = \mathcal{G}_1.$$

- For X_2 (values 0, 1, 2):

$$X_2^{-1}(\{2\}) = \{HH\}, \quad X_2^{-1}(\{1\}) = \{HT, TH\}, \quad X_2^{-1}(\{0\}) = \{TT\}.$$

The sets $\{HH\}$, $\{HT, TH\}$, $\{TT\}$ form a partition of Ω . The σ -algebra generated by X_2 is all unions of these three atoms:

$$\sigma(X_2) = \{\emptyset, \Omega, \{HH\}, \{HT, TH\}, \{TT\}, \{HH, HT, TH\}, \{HH, TT\}, \{HT, TH, TT\}\}.$$

This corresponds to knowing only the *number* of heads, not their order.

- For (X_1, X_2) , consider the map

$$\omega \mapsto (X_1(\omega), X_2(\omega)).$$

We have

$$(X_1, X_2)(HH) = (1, 2), \quad (X_1, X_2)(HT) = (1, 1), \quad (X_1, X_2)(TH) = (0, 1), \quad (X_1, X_2)(TT) = (0, 0)$$

and these four pairs are all distinct. Hence the atoms of $\sigma(X_1, X_2)$ are the singletons $\{HH\}, \{HT\}, \{TH\}, \{TT\}$, so

$$\sigma(X_1, X_2) = \mathcal{P}(\Omega) = \mathcal{H}.$$

So finally:

$$\sigma(X_1) = \mathcal{G}_1, \quad \sigma(X_2) = \text{the 8-set algebra generated by } \{HH\}, \{HT, TH\}, \{TT\}, \quad \sigma(X_1, X_2) = \mathcal{H}$$

Exercise 2

We throw a coin three times (the tosses are independent) and record the faces that appear. We denote $\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$ (where $H \equiv \text{HEAD}$ and $T \equiv \text{TAIL}$) all possible outcomes of the random experiment. Let

- X_k denotes the random variables that counts the number of Heads obtained at the k first tosses.
- Y_k denotes the random variables that counts the number of Heads obtained only at the k -th toss and let $\mathcal{G}_1, \mathcal{G}_2$, and \mathcal{G}_3 be the σ -algebras defined by

$$\mathcal{G}_1 = \{\emptyset, \Omega, \{HHT, HHH, HTH, HTT\}, \{TTH, TTT, THT, THH\}\}$$

$$\mathcal{G}_2 = \{\emptyset, \Omega, \{THH, HHT, THT, HHH\}, \{HTT, HHH, HTH, HTT\}\}$$

$$\mathcal{G}_3 = \{\emptyset, \Omega, \{HTH, THH, THH, HHH\}, \{THT, HTT, TTT, HHT\}\}$$

Set $\mathcal{F}_1 = \mathcal{G}_1, \mathcal{F}_2 = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$ and $\mathcal{F}_3 = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3)$

1. What is the informations contained on the sigma-algebra $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$.
2. Are the random variables Y_k, \mathcal{G}_k -measurable
3. Are the random variables X_k, \mathcal{G}_k -measurable
4. Are the random variables Y_k, \mathcal{F}_k -measurable
5. Determine explicitly $\sigma(Y_1), \sigma(Y_2), \sigma(Y_3)$.
6. Are the random variables X_k, \mathcal{F}_k -measurable
7. Let p be the probability of having Heads at every toss of the coin.
 - (a) Compute $\mathbb{E}(X_3|X_1)$
 - (b) Deduce the value of $\mathbb{E}(X_3)$
 - (c) Compute the $\mathbb{E}(X_1|X_2)$

Exercise 3

(Application of the course)

1. Let X be a random variable defined on Ω and taking value in (E, \mathcal{E}) and let f be a measurable function from (E, \mathcal{E}) to \mathbb{R} . Do you have $\sigma(f(X)) \subset \sigma(X)$ or $\sigma(X) \subset \sigma(f(X))$?
2. Let \mathcal{F} and \mathcal{G} be two σ -algebra, Is $\mathcal{F} \cap \mathcal{G}$ always σ -algebra? Is $\mathcal{F} \cup \mathcal{G}$ always σ -algebra ?
3. Let X be a integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
 - (a) What is $\mathbb{E}(X|\mathcal{F})$ if $\mathcal{F} = \{\emptyset, \Omega\}$? if $\mathcal{F} = \mathcal{P}(\Omega)$?
 - (b) What is $\mathbb{E}(X|\mathcal{F})$? if X is independent from \mathcal{F} .

Exercise 4

We flip three times a coin which probability of having Heads is $p \in]0, 1[$ and we record the successive sides that appear.

1. Determine the sample space Ω .
2. We denote $X_k(\omega)$ for $k = 1, 2, 3, \dots$ the random variable that counts the number of Tails appeared after the k first toss when we observe $\omega \in \Omega$.
3. Determine the σ -algebra $\mathcal{P}_1 = \sigma(X_1)$ and $\mathcal{P}_2 = \sigma(X_2)$ and then $\mathcal{P}_{1,2} = \sigma(X_1, X_2)$.
4. Are the following inclusions true ? $\mathcal{P}_1 \subset \mathcal{P}_2$; $\mathcal{P}_1 \subset \mathcal{P}_{1,2}$.
5. Determine $\mathbb{E}[X_3|\mathcal{P}_1]$ and $\mathbb{E}[\mathbb{E}[X_3|\mathcal{P}_2]|\mathcal{P}_1]$?
6. Does one have the equality

$$\mathbb{E}[X_3|\mathcal{P}_1] = \mathbb{E}[\mathbb{E}[X_3|\mathcal{P}_2]|\mathcal{P}_1]$$

7. Does one have the equality

$$\mathbb{E}[X_3|\mathcal{P}_1] = \mathbb{E}[\mathbb{E}[X_3|\mathcal{P}_{1,2}]|\mathcal{P}_1]$$

Exercise 5

Let X be a random variable defined on (Ω, \mathcal{F}) and let \mathbb{P} be a probability on (Ω, \mathcal{F}) such that $\mathbb{E}(X^2) < +\infty$. Let \mathcal{P} be a sub σ -algebra of \mathcal{F} . Set

$$Var(X|\mathcal{P}) = \mathbb{E}[(X - \mathbb{E}(X|\mathcal{P}))^2|\mathcal{P}]$$

Starting from

$$X - \mathbb{E} = X - \mathbb{E}(X|\mathcal{P}) + \mathbb{E}(X|\mathcal{P}) - \mathbb{E}(X)$$

Show that

$$Var(X) = \mathbb{E}(Var(X|\mathcal{P})) + Var(X\mathbb{E}(X|\mathcal{P})).$$

Exercise 6

We toss n times a coin and consider a gambling which consists on scoring 1 point when Heads appears or losing 2 points otherwise. The probability of having Heads with the coin is p . Let Y_i be the random variable representing the scored point on the i -th toss only and let X_n be the cumulated scored point after the n first tosses with X_0 . Let $(\mathcal{F}_n)_{n \geq 0}$ with $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ be the natural filtration of the process $(Y_n)_{n \geq 1}$.

1. Write X_n as a function of Y_1, \dots, Y_n .
2. Determine the value of p which makes the process $(X_n)_{n \geq 0}$ a \mathcal{F}_n martingale.
3. Determine the value of p which makes the process $(X_n)_{n \geq 0}$ a \mathcal{F}_n sur-martingale.
4. Determine the value of p which makes the process $(X_n)_{n \geq 0}$ a \mathcal{F}_n sub-martingale.

Exercise 7

Let $(Y_n)_{n \geq 1}$ be a sequence of independent random variables with the same distribution $\mathcal{N}(0, \sigma^2)$ where $\sigma > 0$. Set $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ and $X_n = Y_1 + \dots + Y_n$.

1. Show that

$$\mathbb{E}[e^{uY_1}] = e^{\frac{u^2\sigma^2}{2}}$$

2. Let $Z_n^u = \exp\left(uX_n - n\frac{u^2\sigma^2}{2}\right)$. Show that, for any $u \in \mathbb{R}$, $(Z_n^u)_{n \geq 1}$ is an \mathcal{F} -martingale.
3. Show that for any $u \in \mathbb{R}$, $(Z_n^u)_{n \geq 1}$ converges almost surely toward a random variable Z_∞^u which is finite.

Exercise 8

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(X_k)_{k=0, \dots, n}$ be a sequence of random variables. Let $(\mathcal{G}_k)_{k=0, \dots, n}$ be the filtration generated by the process $(Y_k)_{k=1, \dots, n}$ with $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and let $\bar{\mathbb{P}}$ be the new probability defined as

$$d\bar{\mathbb{P}} = Z_n d\mathbb{P}, \quad \text{with} \quad Z_n = \exp\left(\sigma(Y_1 + \dots + Y_n) - \frac{n\sigma^2}{2}\right)$$

1. Show that the process $(Z_k)_{k=0, \dots, n}$ defined for every $k \in 0, \dots, n$ by $Z_k = \mathbb{E}_{\mathbb{P}}(Z_n | \mathcal{G}_k)$ is a martingale.
2. Determine $\mathbb{E}_{\mathbb{P}}(X_{k+1} | \mathcal{G}_k)$ for all $k \in \{0, \dots, n-1\}$. Deduce the value of μ such that $(X_k)_{k=0, \dots, n}$ is a martingale.
3. Show that

$$\mathbb{E}_{\mathbb{P}} = x_0 e^\mu$$

4. We consider the process $(\bar{X}_k)_{k=0, \dots, n}$ defined for every $k \in \{0, \dots, n\}$ by $\bar{X}_k = e^{-\mu k/n} X_k$. Show that $(\bar{X}_k)_{k=0, \dots, n}$ is \mathbb{P} -martingale.
5. Determine the value of σ such that $(\bar{X}_k)_{k=0, \dots, n}$ is $\bar{\mathbb{P}}$ -martingale.

Exercise 9

A stochastic process $(M_n)_{n \geq 0}$ is said to be with independent increments if for any n , the random variable $M_{n+1} - M_n$ is independent from $\mathcal{F} = \sigma(M_0, \dots, M_n)$. A real random variable M is square integrable if $\mathbb{E}(M^2)$ is finite. $\mathbb{E}(M^2) < +\infty$.

1. Let $(M_n)_{n \geq 0}$ be a square integrable with independent increments.
 - (a) Show that $\mathbb{E}[M_n] = \mathbb{E}[M_k]$ for every $n \leq k$ and verify that $\text{cov}(M_n, M_k) = \text{Var}(M_n)$.
We recall that for a given random variables X and Y $\text{cov}(X, Y) = E[XY] - E[X]E[Y]$.
 - (b) We set $\sigma_0^2 = \text{Var}(M_0)$ and for $n \geq 0$, $\sigma_k^2 = \text{Var}(M_k - M_{k-1})$. Show that for every $n \geq 0$ $\text{Var}(M_n) = \sum_{k=0}^n \sigma_k^2$.
2. Let $(\langle M_n \rangle)_{n \geq 0}$ be the compensator of the sub-martingale $(M_n^2)_{n \geq 0}$, defined by $\langle M \rangle_0 = 0$ and for any $n \geq 1$ by $\langle M \rangle_n = \langle M_n \rangle = \langle M_{n-1} \rangle + \mathbb{E}[M_n^2 - M_{n-1}^2 | \mathcal{F}_{n-1}]$
 - a Show that $\mathbb{E}[M_n^2 - M_{n-1}^2 | \mathcal{F}_{n-1}] = \mathbb{E}[(M_n - M_{n-1})^2 | \mathcal{F}_{n-1}]$
 - b Determine $\langle M \rangle_n$.
 - c Deduce that the process defined for every $n \geq 1$ by $M_n^2 - \sum_{k=1}^n \sigma_k^2$ is an \mathcal{F}_n -martingale.
3. Now, let $(M_n)_{n \geq 0}$ be a gaussian process, means: for any $n \geq 0$ the vector (M_0, \dots, M_n) is a gaussian vector. Suppose that $(M_n)_{n \geq 0}$ is a martingale.
 - (a) Show that for any $k = 0, \dots, n$, we have $\mathbb{E}(M_k(M_{n+1} - M_n)) = 0$.
 - (b) Deduce that $(M_n)_{n \geq 0}$ is with independent increments.
 - (c) Show that for any fixed $\theta \in \mathbb{R}$ the process

$$Z_n^\theta = e^{\theta M_n - \theta^2 \langle M_n \rangle / 2}$$

is martingale. Is this process converge almost surely ?

Exercise 10

Let $(S_n)_{n \geq 0}$ be a random walk on \mathbb{Z} : $S_0 = 0, S_n = U_1 + \dots + U_n$ where the $U_i \in \{x, y\}$ are independent with the same distribution means, $0 < \mathbb{P}(U_i = x) = p < 1, \mathbb{P}(U_i = -y) = 1 - p$, for any $i \in \{1, \dots, n\}$. We define $(\mathcal{F}_n)_{n \geq 0}$ as the filtration generated by $(S_n)_{n \geq 0}$: $\mathcal{F}_n = \sigma(S_0, \dots, S_n)$ for every $n \geq 1$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

I We set $x = 1, y = 2$. Determine the value of p such that $(S_n)_{n \geq 1}$ is sub-martingale.

- 1 Let $\mu = \mathbb{E}(U_1)$ and let $X_n = S_n - n\mu$ for every $n \geq 0$.
 - (a) Show that $(X_n)_{n \geq 0}$ is a martingale.
 - (b) Let $a > 0$ and let τ be a stopping time defined by

$$\tau = \inf \{n \geq 0, |X_n| > a\}$$

Show that for all $n \geq 0, \mathbb{E}(S_{n \wedge \tau})$

- (c) Deduce that $\mathbb{E}(S_\tau) = \mu \mathbb{E}(\tau)$
- 2 We set $\sigma^2 = \text{Var}(U_1)$ and $Y_n = (S_n - n\mu)^2 - n\sigma^2$ for all $n \geq 0$. Show that $(Y_n)_{n \geq 0}$ is an $(\mathcal{F}_n)_{n \geq 0}$ -martingale.

II Suppose now $x = y = 1$.

- (a) Let $Z_n = (q/p)^{S_n}$. Show that $(Z_n)_{n \geq 0}$ is a positive martingale.
- (b) Show that

$$\mathbb{P} \left(\sup_{n \geq 0} Z_n \geq (q/p)^k \right) \leq \left(\frac{p}{q} \right)^k.$$

- (c) Deduce that

$$\mathbb{P} \left(\sup_{n \geq 0} S_n \geq k \right) \leq \left(\frac{p}{q} \right)^k.$$

Exercise 11

We consider a binomial model with N period on an underlying asset S with initial value $S_0 = s_0 > 0$ a constant value. We suppose that any time $n \in \{1, \dots, N\}$, the value S_n of the asset may go up or go down and the asset take upward value uS_{n-1} and the downward value dS_{n-1} , with $0 < d < r + 1 < u$, where r is the interest and where S_{n-1} is the value asset at time $n - 1$. We suppose we may write down for every $n \geq 1$.

$$S_n = S_{n-1} Y_n$$

where $(Y_n)_{n \geq 1}$ is independent and identical distributed sequence of random variable with

$$Y_n = \begin{cases} u, & \text{with probability } \tilde{p}, \\ d, & \text{with probability } \tilde{q} = 1 - \tilde{p}, \end{cases}$$

with $\tilde{p} = \frac{1+r-d}{u-d}$. We define the updated value $(\tilde{S}_n)_{n \geq 0}$ of the asset process $(S_n)_{n \geq 0}$ by

$$\tilde{S}_n = \frac{1}{(1+r)^n} S_n, \quad \forall n \geq 0$$

and denote by $(\mathcal{F}_n)_{n \geq 0}$ the filtration generated by $(Y_n)_{n \geq 1}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$

1. Show that $(\tilde{S}_n)_{n \geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n \geq 0}$.
2. We define the process $(V_n)_{n \geq 0}$ by

$$V_n = \Delta_n S_n + (1+r)(V_{n+1} - \Delta_n S_{n-1})$$

where $(\Delta_n)_{n \geq 0}$ is predictable with respect to $(\mathcal{F}_n)_{n \geq 0}$. Let

$$\bar{V}_n = \frac{1}{(1+r)^n} V_n, \quad \forall n \geq 0$$

- a Show that $(\bar{V}_n)_{n \geq 0}$ is an \mathcal{F}_n -martingale.

- b We suppose that $V_N = \max(K - S_N, 0)$ for $K > 0$ and $N \geq 1$. Show that

$$V_0 = \frac{1}{(1+r)^N} \mathbb{E}[\max(K - S_N, 0)]$$

- c Compute explicitly V_0 when $N = 1$ and $dS_0 < K < uS_0$.

Exercise 12

We consider a gambling where we toss a coin and gain 2 points if Heads appears and loss 2 points when Tails appears. We denote by p the probability that Head appears. Let's X_n denotes our score at the n -th toss of the coin and let $(Y_i)_{i \geq 1}$ be independent and identical distributed sequence of random variables such that

$$Y_i = \begin{cases} 2, & \text{with probability } p, \\ -2, & \text{with probability } 1 - p \end{cases}$$

1. Show that $(X_n)_{n \geq 0}$ is a Markov chain and determine its transition matrix.
2. Compute the following quantities: $\mathbb{P}(X_0 = 0, X_1 = 2, X_2 = 4, X_3 = 6)$ and $\mathbb{P}(X_0 = 0, X_1 = -2, X_2 = 0)$.

Exercise 13

Let $(X_n)_{n \geq 0}$ be a Markov chain defined on $E = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1. Show that P^2 is a transition matrix
2. Let $j \in E$ and define the number of passage of the Markov Chain at the state j by

$$N_j = \sum_{k=0}^{+\infty} \mathbf{1}_{\{X_k=j\}}$$

Compute $E_i(N_1)$ for all $i \in E$

3. Let τ_3 be the shopping time (the first passage time at state 3) defined by $\tau_3 = \inf \{n \geq 0 : X_n = 3\}$. Compute $\mathbb{P}_i(\tau_3 < +\infty)$ for any $i \in E$
4. Show that the Markov Chain is irreducible and recurrent.
5. Determine the invariante probability of the Markov Chain.
6. Deduce for any $i \in E$, the almost sure limit of $\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k=i\}}$.
7. Compute the almost sure limit of $\frac{1}{n} \sum_{k=0}^{n-1} X_k^2$.

Exercise 14

A mobile moves randomly on \mathbb{Z} following Markov Chain with transition matrix P with components

$$P(i, j) = \begin{cases} p, & \text{if } j = i + 1 \\ 1 - p, & \text{if } j = i - 1 \\ 0, & \text{otherwise} \end{cases}$$

with $0 < p < 1$.

1. Let $(Z_n)_{\geq 1}$ be an independent and identical distributed sequence of random variables such that $\mathbb{P}(Z_n = 1) = p, \mathbb{P}(Z_n = -1) = 1 - p$. We set $X_0 = 0$ and $X_n = Z_1 + \dots + Z_n$. Show that $(X_n)_{n \geq 0}$ is a Markov Chain starting at 0 with transition matrix P .
2. What is the limit of $\lim_{n \rightarrow +\infty} \frac{1}{n} X_n$? If $p \neq 1/2$, the chain is recurrent or transient?
3. Set $Y_i = \frac{1}{2}(Z_i + 1)$. What are the distributions of Y_i and $T_n = \frac{1}{2}(X_n + n)$? Compute $P^{(n)}(0, 0) = \mathbb{P}(X_n = 0)$ when n is even and when it is odd.
4. If $p = \frac{1}{2}$, show that the Markov Chain is recurrent. Is it positive recurrent?
5. We suppose that $p = \frac{1}{2}$, and for $a, b > 0$ we define the stopping times τ_{-a} and τ_b by

$$\tau_{-a} = \inf \{n \geq 0, X_n = -a\} \quad \text{and} \quad \tau_b = \inf \{\tau_b \geq 0, X_n = b\}$$

and set $\tau = \tau_{-a} \wedge \tau_b$.

- a Show that the process defined by $S_n = X_n^2 - n$, for all $n \geq 0$ is martingale.
- b Show that $\mathbb{E}(S_\tau) = 0$ and deduce the values of $\mathbb{P}(\tau_{-a} < \tau_b)$ and $\mathbb{P}(\tau_{-a} > \tau_b)$.

Exercise 15

Let $(X_n)_{n \geq 0}$ be a Markov Chain with state space $E = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix}$$

1. Calculate the probability that, starting from $i \in E$, the hitting time of $A = \{2, 3\}$ is finite: $\mathbb{P}_i(\tau_A < +\infty)$ for all $i \in E$ where

$$\tau_A = \inf \{n \geq 0, X_n \in A\}$$

2. Calculate the expectation of the time until reaching A starting from $i \in E$: $\mathbb{E}_i(\tau_A)$
3. Compute $\mathbb{P}_i(\tau_1 < \tau_2)$ for $i \in E$.

Exercise 16

Consider the total growth process $(X_n)_{n \geq 0}$ defined by $X_0 = 0$ and for every $n \geq 1$ by

$$X_n = Y_1 + \dots + X_n$$

where $(Y_n)_{n \geq 1}$ is independent and identical distributed sequence of the random variable valued in $\{-d, e\}$, $d, e > 0$ satisfying for any $n \geq 1$, $\mathbb{P}(Y_n = -d) = q$ and $\mathbb{P}(Y_n = e) = p$ with $q = 1 - p$. Let $(\bar{X}_n)_{n \geq 0}$ be a stochastic process defined by $\bar{X}_0 = 0$ and for any $n \geq 1$, by

$$\bar{X}_n = \bar{Y}_1 + \dots + \bar{Y}_n \quad \text{with} \quad \bar{Y}_k = \frac{1}{e+d}(2Y_k + d - e), \quad k \geq n$$

Let E be the set of values taken by $(\bar{X}_n)_{n \geq 0}$ and let $-b, a \in E$ with $a, b > 0$. Define the stopping time τ by

$$\tau = \tau_{-b} \wedge \tau_a \quad \text{with} \quad \tau_l = \inf \{k \geq 0, \bar{X}_k = l\}, \quad \text{for } l \in \{-b, a\}$$

1. Is the process $(\bar{X})_{n \geq 0}$ a martingale with respect to its natural filtration ? Justify your answer.

2. We set $A_n = \{-b < \bar{X}_n < a\}$ for any $n \geq 0$. Prove that

$$\mathbb{P}(\tau = +\infty) = \mathbb{P}\left(\bigcup_{n \geq 0} A_n\right) \leq \lim_{n \rightarrow +\infty} \mathbb{P}(A_n).$$

3. Use the Central Limit Theorem 1 and the result 2 to show that τ is finite almost surely.
4. We define the process $(Z_n)_{n \geq 0}$ by $Z_n = (q/p)^{\bar{X}_n}$, for every $n \geq 0$. Show that there exists a constant $C > 0$ such that $\sup_{n \geq 0} \mathbb{E}(|Z_{n \wedge \tau}|) < C$
5. Show that the process $(Z_n)_{n \geq 0}$ is a martingale with respect to the natural filtration of the process $(\bar{X}_n)_{n \geq 0}$ and that $\mathbb{E}(Z_\tau) = 1$
6. Show that

$$\mathbb{P}(\tau_{-b} < \tau_a) = \frac{1 - (q/p)^a}{(q/p)^{-b} - (q/p)^a} \quad \text{and} \quad \mathbb{P}(\tau_a < \tau_{-b}) = \frac{(q/p)^{-b} - 1}{(q/p)^{-b} - (q/p)^a}$$

Exercise 17

Let $(Z_n)_{n \geq 1}$ be a sequence of independent random variables such that $\mathbb{P}(Z_n = 1) = p, \mathbb{P}(Z_n = -1) = 1 - p$. Define the random walk $(X_n)_{n \geq 0}$ by $X_0 = x \in \mathbb{Z}$ and $X_n = Z_1 + \dots + Z_n$. Let $a, b \in \mathbb{Z}$ be such that $a < x < b$ and let τ_y be the stopping time defined by $\tau_y = \inf\{n \geq 0, X_n \geq y\}$, for $y \in \mathbb{Z}$.

$$\mathbb{P}_x(\tau_b < \tau_a) = \begin{cases} \frac{\left(\frac{q}{p}\right)^{x-a} - 1}{\left(\frac{q}{p}\right)^{b-a} - 1}, & \text{if } p \neq q, \\ \frac{x-a}{b-a}, & \text{if } p = q. \end{cases} \quad (1)$$

1. We define $\phi(x, l) = \mathbb{P}_x(\tau_l < \tau_0)$, for $0 < x < l \in \mathbb{Z}$. Show that $\mathbb{P}_x(\tau_b < \tau_a) = \phi(x-a, b-a)$, for $a < x < b$. What are the values of $\phi(0, l)$ and $\phi(l, l)$?
2. Suppose that $a = 0$ and $b = l$ and define $\delta(x) = \phi(x+1, l) - \phi(x, l)$ for $x \in \mathbb{N}^*$.
- Show that for all $x \in \mathbb{N}^*$, $p\delta(x) = q\delta(x-1)$.
 - Deduce that $\delta(x) = \left(\frac{q}{p}\right)^x \delta(0)$.
 - We suppose that $q \neq p$
 - Show that $\delta(0) = \frac{\frac{q}{p}-1}{\left(\frac{q}{p}\right)^l-1}$
 - Using the previous parts show that $\phi(x, l) = \frac{\left(\frac{q}{p}\right)^x-1}{\left(\frac{q}{p}\right)^l-1}$
 - We suppose that $p = q$. Show that in this case $\delta(0) = 1/l$ and that $\phi(x, l) = x/l$.
3. Deduce the equation (1).
4. Let $x > 0$. Observing that $\mathbb{P}_x(\tau_0 < +\infty) = \lim_{l \rightarrow +\infty} \mathbb{P}_x(\tau_0 < \tau_l)$, show that

$$\mathbb{P}_x(\tau_0 < +\infty) = \begin{cases} (q/p)x, & \text{if } p > 1/2, \\ 1, & \text{if } p \leq 1/2. \end{cases} \quad (2)$$