TD: Martingales and Markov Chains

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Exercise 1

We toss twice a coin and record the successive sides that appear. Let $\Omega = \{HH, HT, TH, TT\}$ (where $H \equiv$ HEAD and $T \equiv$ TAIL) be the sample sample space. Let X_k be the random variable which counts the number of Heads that appears after the k first tosses and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the σ-algebras on Ω defined by

$$\mathcal{G}_1 = \{\emptyset, \Omega, \{HT, HH\}, \{TH, TT\}\} \qquad \text{and} \qquad \mathcal{G}_2 = \{\emptyset, \Omega, \{TH, HH\}, \{HT, TT\}\}$$

1. What is the information contained in $\mathcal{G}_1$ and $\mathcal{G}_2$?
2. Are the random variable X_1 and X_2 is $\mathcal{G}_1$-measurable ?
3. Are the random variable X_1 and X_2 is $\mathcal{G}_2$-measurable ?
4. Determine the σ-algebra $\varepsilon = \mathcal{G}_1 \cap \mathcal{G}_2$ and $\mathcal{H} = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$. What is the informations they contain?
   (a). Are the random variable X_1 and X_2 is $\mathcal{H}$-measurable?
   (b). Are the random variable X_1 and X_2 is ε-measurable?
5. Determine $\sigma(X_1), \sigma(X_2)$ and $\sigma(X_1, X_2)$.

Solution

1. Information contained in $\mathcal{G}_1$ and $\mathcal{G}_2$.
We have

$$\mathcal{G}_1 = \{\emptyset, \Omega, \{HT, HH\}, \{TH, TT\}\}, \qquad \mathcal{G}_2 = \{\emptyset, \Omega, \{TH, HH\}, \{HT, TT\}\}.$$

- For $\mathcal{G}_1$:

$$\{HT, HH\} = \{\omega : \text{first toss is } H\}, \qquad \{TH, TT\} = \{\omega : \text{first toss is } T\}.$$

Thus, $\mathcal{G}_1$ contains *exactly* the information of the **first toss**: we know whether the first toss is Head or Tail, but nothing about the second toss.

- For $\mathcal{G}_2$:

$$\{TH, HH\} = \{\omega : \text{second toss is } H\}, \qquad \{HT, TT\} = \{\omega : \text{second toss is } T\}.$$

Thus, $\mathcal{G}_2$ contains *exactly* the information of the **second toss**: we know whether the second toss is Head or Tail, but not the first.

2. Are X_1 and X_2 $\mathcal{G}_1$-measurable?
Recall: a random variable X is $\mathcal{G}_1$-measurable iff for every Borel set $B \subset \mathbb{R}$,

$$X^{-1}(B) \in \mathcal{G}_1.$$

- For X_1:

$$X_1^{-1}(\{0\}) = \{\omega \in \Omega, X_1(\omega) = 0\} = \{TH, TT\} \in \mathcal{G}_1$$
$$X_1^{-1}(\{1\}) = \{\omega \in \Omega, X_1(\omega) = 1\} = \{HT, HH\} \in \mathcal{G}_1$$

Thus all level sets of X_1 belong to $\mathcal{G}_1$, so X_1 is $\mathcal{G}_1$-measurable.

- For X_2:

$$X_2^{-1}(\{0\}) = \{\omega \in \Omega, X_1(\omega) = 0\} = \{TT\}$$
$$X_2^{-1}(\{1\}) = \{\omega \in \Omega, X_1(\omega) = 1\} = \{HT, TH\}$$
$$X_2^{-1}(\{2\}) = \{\omega \in \Omega, X_1(\omega) = 1\} = \{HH\}$$

But $\{HH\} \notin \mathcal{G}_1$ (the only nontrivial sets are $\{HH, HT\}$ and $\{TH, TT\}$). Hence X_2 is *not* $\mathcal{G}_1$-measurable.

3. Are X_1 and X_2 is $\mathcal{G}_2$-measurable?
We do the same with G_2

- For X_1:

$$X_1^{-1}(\{0\}) = \{TT, TH\}$$
$$X_1^{-1}(\{1\}) = \{HH, HT\}$$

In $\mathcal{G}_2$ we only have $\emptyset, \Omega, \{TH, HH\}, \{HT, TT\}$, from X_1 $\{HH, HT\}$ is not one of these so X_1 is not $\mathcal{G}_2$-measurable.

- For X_2:

$$X_2^{-1}(\{0\}) = \{TT\}$$
$$X_2^{-1}(\{1\}) = \{HT, TH\}$$
$$X_2^{-1}(\{2\}) = \{HH\}$$

None of these set are in $\{HH\}, \{HT, TH\}, \{TT\}$ belong to $\mathcal{G}_2$, therefore X_2 is not $\mathcal{G}_2$-measurable.

4. Determine the σ-algebra $\varepsilon = \mathcal{G}_1 \cap \mathcal{G}_2$ and $\mathcal{H} = \sigma\left(\mathcal{G}_1 \cup \mathcal{G}_2\right)$?

<u>Intersection</u>:

$$\mathcal{G}_1 = \{\emptyset, \Omega, A, A^c\}, \quad A := \{HH, HT\} \quad A^c = \{TT, TH\}$$
$$\mathcal{G}_2 = \{\emptyset, \Omega, B, B^c\}, \quad B := \{HH, TH\} \quad B^c = \{TT, HT\}$$

From this, one can easy get

$$\varepsilon = \mathcal{G}_1 \cap \mathcal{G}_2 = \{\emptyset, \Omega\}$$

Therefore ε is a trivial σ-algebra, it does not contains any information. We only know what happen inside Ω.

<u>Generated σ-algebra $\mathcal{H} = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$</u>:

Consider intersection of A, A^c, B, B^c:

$$A \cap B = \{HH\}, \quad A \cap B^c = \{HT\}, \quad A^c \cap B = \{TH\}, \quad A^c \cap B^c = \{TT\}.$$

These set is $\Omega!$, hence the σ-algebra generated by $\mathcal{H}$, then

$$\mathcal{H} = \mathcal{P}(\Omega)$$

a. Are X_1 and X_2 is $\mathcal{H}$-measurable ?
Since $\mathcal{H} = \mathcal{P}(\Omega)$, full span set of the σ-algebra, so the preimage of any Borel set in $\mathcal{H}$,
Therefore

$$\boxed{X_1 \text{ and } X_2 \text{ are } \mathcal{H}\text{-measurable.}}$$

b. Are X_1 and X_2 is ε-measurable ?
Within $\varepsilon = \{\emptyset, \Omega\}$, one can say the preimage of every Borel set must either $\emptyset$ or Ω, but as the X_1 and X_2 are not constant.

$$X_1^{-1}(1) = \{HH, HT\} \notin \varepsilon$$
$$X_2^{-1}(0) = \{TT\} \notin \varepsilon$$

Thus

$$\boxed{X_1 \text{ and } X_2 \text{ are not } \varepsilon\text{-measurable.}}$$

5. Determine $\sigma(X_1), \sigma(X_2)$ and $\sigma(X_1, X_2)$
By definition

$$\sigma(X_1) = \{X_1^{-1}(B) : B \in \mathbb{R} \quad \text{Borel Law}\}$$

- For $\sigma(X_1)$

$$X_1^{-1}(0) = \{TT, TH\}, \quad X_1^{-1}(1) = \{HH, HT\}$$

Thus

$$\sigma(X_1) = \{\emptyset, \Omega, \{HH, HT\}, \{TT, TH\}\} = \mathcal{G}_1$$

- For $\sigma(X_2)$

$$X_2^{-1}(\{0\}) = \{TT\}$$
$$X_2^{-1}(\{1\}) = \{HT, TH\}$$
$$X_2^{-1}(\{2\}) = \{HH\}$$

Thus

$$\sigma(X_2) = \{\emptyset, \Omega, \{TT\}, \{HT, TH\}, \{HH\}, \{TT, HT, TH\}, \{TT, HH\}, \{HT, TH, HH\}\}$$

- For $\sigma(X_1, X_2)$,

$$\omega \to (X_1(\omega), X_2(\omega))$$

We have

$$
\begin{aligned}
(X_1, X_2)(HH) &= (1, 2)\\
(X_1, X_2)(HT) &= (1, 1)\\
(X_1, X_2)(TH) &= (0, 1)\\
(X_1, X_2)(TT) &= (0, 0)
\end{aligned}
$$

All set are distinct pair and actually is Ω, therefore

$$\sigma(X_1, X_2) = \mathcal{H} = \mathcal{P}(\Omega)$$

Exercise 2

We throw a coin three times (the tosses are independent) and record the faces that appear. We denote $\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$ (where $H \equiv$ HEAD and $T \equiv$ TAIL) all possible outcomes of the random experiment. Let

- X_k denotes the random variables that counts the number of Heads obtained at the k first tosses.
- Y_k denotes the random variables that counts the number of Heads obtained only at the k-th toss and let $\mathcal{G}_1, \mathcal{G}_2,$ and $\mathcal{G}_3$ be the σ-algebras defined by

$$
\begin{aligned}
\mathcal{G}_1 &= \{\emptyset, \Omega, \{HHT, HHH, HTH, HTT\}, \{TTH, TTT, THT, THH\}\}\\
\mathcal{G}_2 &= \{\emptyset, \Omega, \{THH, HHT, THT, HHH\}, \{HTT, HHH, HTH, HTT\}\}\\
\mathcal{G}_3 &= \{\emptyset, \Omega, \{HTH, THH, THH, HHH\}, \{THT, HTT, TTT, HHT\}\}
\end{aligned}
$$

Set $\mathcal{F}_1 = \mathcal{G}_1, \mathcal{F}_2 = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$ and $\mathcal{F}_3 = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3)$

1. What is the informations contained on the sigma-algebra $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$.
2. Are the random variables $Y_k, \mathcal{G}_k$-measurable
3. Are the random variables $X_k, \mathcal{G}_k$-measurable
4. Are the random variables $Y_k, \mathcal{F}_k$-measurable
5. Determine explicitly $\sigma(Y_1), \sigma(Y_2), \sigma(Y_3)$.
6. Are the random variables $X_k, \mathcal{F}_k$-measurable
7. Let p be the probability of having Heads at every toss of the coin.
   (a) Compute $\mathbb{E}(X_3|X_1)$
   (b) Deduce the value of $\mathbb{E}(X_3)$
   (c) Compute the $\mathbb{E}(X_1|X_2)$

Solution

1. Information contained in $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$.
Each $\mathcal{G}_k$ corresponds to knowing only the k-th toss:

- $\mathcal{G}_1$ contains the information "is the first toss H or T?"

- $\mathcal{G}_2$ contains the information "is the second toss H or T?"
- $\mathcal{G}_3$ contains the information "is the third toss H or T?"

2. Are Y_k is $\mathcal{G}_k$-measurable?
Since

$$Y_k^{-1}(\{1\}) = \{\omega : k\text{-th toss} = \text{H}\}, \qquad Y_k^{-1}(\{0\}) = \{\omega : k\text{-th toss} = \text{T}\}$$

and these two sets belong to $\mathcal{G}_k$, we have

$$\boxed{Y_k \text{ is } \mathcal{G}_k\text{-measurable.}}$$

3. Are X_k is $\mathcal{G}_k$-measurable?

- For X_1:

  $$X_1^{-1}(\{0\}) = \{THH, THT, TTH, TTT\}, \quad X_1^{-1}(\{1\}) = \{HHH, HHT, HTH, HTT\}.$$

  These sets correspond exactly to "first toss T" and "first toss H". Thus X_1 is $\mathcal{G}_1$-measurable.
- For X_2:

  $$\begin{aligned} X_2^{-1}(\{0\}) &= \{TTT, TTH\}, \\ X_2^{-1}(\{1\}) &= \{HTT, HTH, THH, THT\}, \\ X_2^{-1}(\{2\}) &= \{HHH, HHT\}. \end{aligned}$$

  None of these sets correspond to the two blocks of $\mathcal{G}_2$, so X_2 is not $\mathcal{G}_2$-measurable.
- For X_3: X_3 depends on all three tosses, but $\mathcal{G}_3$ knows only the third toss. Thus X_3 is not $\mathcal{G}_3$-measurable.

Thus:

$$\boxed{X_1 \in \mathcal{G}_1, \qquad X_2 \notin \mathcal{G}_2, \qquad X_3 \notin \mathcal{G}_3.}$$

4. Are Y_k is $\mathcal{F}_k$-measurable?

- $\mathcal{F}_1 = \mathcal{G}_1$ contains Y_1.
- $\mathcal{F}_2$ contains the information of both tosses 1 and 2, so it contains Y_1, Y_2.
- $\mathcal{F}_3$ contains full information of all three tosses, so it contains Y_1, Y_2, Y_3.

Thus

$$\boxed{Y_k \text{ is } \mathcal{F}_k\text{-measurable for all } k.}$$

5. Determine $\sigma(Y_1), \sigma(Y_2), \sigma(Y_3)$. Each Y_k takes values in $\{0, 1\}$, so

$$\sigma(Y_k) = \{\emptyset, \Omega, \{Y_k = 1\}, \{Y_k = 0\}\} = \mathcal{G}_k.$$

Thus

$$\boxed{\sigma(Y_k) = \mathcal{G}_k.}$$

6. Are X_k $\mathcal{F}_k$-measurable?

- $\mathcal{F}_1 = \mathcal{G}_1$ contains X_1.
- $\mathcal{F}_2$ contains full information of tosses 1 and 2, hence contains X_2.
- $\mathcal{F}_3$ contains full information of all three tosses, hence contains X_3.

Thus

$$\boxed{X_k \text{ is } \mathcal{F}_k\text{-measurable.}}$$

7. Let $p = \mathbb{P}(H)$.

(a) Compute $\mathbb{E}(X_3 \mid X_1)$.
Recall that

$$X_3 = Y_1 + Y_2 + Y_3,$$

where Y_k are independent Bernoulli(p) random variables. We treat the cases according to the value of $X_1 = Y_1$.

- If $X_1 = 1$ (the first toss is H), then

$$X_3 = 1 + Y_2 + Y_3,$$

and using independence of Y_2, Y_3 from Y_1,

$$\mathbb{E}(X_3 \mid X_1 = 1) = 1 + \mathbb{E}(Y_2) + \mathbb{E}(Y_3) = 1 + 2p.$$

- If $X_1 = 0$ (the first toss is T), then

$$X_3 = Y_2 + Y_3,$$

and again by independence,

$$\mathbb{E}(X_3 \mid X_1 = 0) = \mathbb{E}(Y_2) + \mathbb{E}(Y_3) = 2p.$$

Hence,

$$\boxed{\mathbb{E}(X_3 \mid X_1) = \begin{cases} 1 + 2p, & X_1 = 1, \\ 2p, & X_1 = 0. \end{cases}}$$

(b) Deduce $\mathbb{E}(X_3)$.
Using the law of total expectation:

$$\mathbb{E}(X_3) = \mathbb{E}(\mathbb{E}(X_3 \mid X_1)) = (1 + 2p)\mathbb{P}(X_1 = 1) + (2p)\mathbb{P}(X_1 = 0).$$

Since X_1 is Bernoulli(p), this gives

$$\mathbb{E}(X_3) = (1 + 2p)p + (2p)(1 - p) = 3p.$$

Thus

$$\boxed{\mathbb{E}(X_3) = 3p.}$$

(c) Compute $\mathbb{E}(X_1 \mid X_2)$.
Since $X_1 = Y_1$ and $X_2 = Y_1 + Y_2$, we distinguish the three possible values of X_2.

- If $X_2 = 0$, then $(Y_1, Y_2) = (0, 0)$, hence

$$\mathbb{E}(X_1 \mid X_2 = 0) = 0.$$

- If $X_2 = 1$, the pairs $(1, 0)$ and $(0, 1)$ occur with equal probability (by independence):

$$\mathbb{P}(Y_1 = 1 \mid X_2 = 1) = \frac{1}{2}$$

- If $X_2 = 2$, then $(Y_1, Y_2) = (1, 1)$, hence
$$\mathbb{E}(X_1 \mid X_2 = 2) = 1.$$

Hence,

$$\boxed{\mathbb{E}(X_1 \mid X_2) = \begin{cases} 1, & X_2 = 2, \\ \frac{1}{2}, & X_2 = 1, \\ 0, & X_2 = 0. \end{cases}}$$

Also we can do By Definition 1.5.1 for discrete random variables, for any value x of X_2 we have

$$\mathbb{E}(X_1 \mid X_2 = x) = \sum_a a\,\mathbb{P}(X_1 = a \mid X_2 = x) = \frac{\sum_a a\,\mathbb{P}(X_1 = a, X_2 = x)}{\mathbb{P}(X_2 = x)}.$$

Here $X_1 \in \{0, 1\}$ and $X_2 \in \{0, 1, 2\}$. Let $p = \mathbb{P}(H)$ and $q = 1 - p$. We use that the first two tosses are independent Bernoulli(p).

- **Case $X_2 = 0$.**
Then the only possible pair is $(Y_1, Y_2) = (0, 0)$ (outcome TT), so
$$\mathbb{P}(X_2 = 0) = \mathbb{P}(TT) = q^2,$$
and necessarily $X_1 = 0$. Hence
$$\mathbb{E}(X_1 \mid X_2 = 0) = \frac{0 \cdot \mathbb{P}(X_1 = 0, X_2 = 0)}{\mathbb{P}(X_2 = 0)} = 0.$$

- **Case $X_2 = 1$.**
Now $X_2 = 1$ corresponds to exactly one head in the first two tosses: outcomes HT or TH.
$$\mathbb{P}(HT) = pq,$$
$$\mathbb{P}(TH) = qp.$$
Therefore
$$\mathbb{P}(X_2 = 1) = \mathbb{P}(HT) + \mathbb{P}(TH) = 2pq.$$
On HT we have $(X_1, X_2) = (1, 1)$; on TH we have $(X_1, X_2) = (0, 1)$. Thus
$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(HT) = pq, \qquad \mathbb{P}(X_1 = 0, X_2 = 1) = \mathbb{P}(TH) = pq.$$
Hence, by the discrete conditional expectation formula,
$$\mathbb{E}(X_1 \mid X_2 = 1) = \frac{1 \cdot \mathbb{P}(X_1 = 1, X_2 = 1) + 0 \cdot \mathbb{P}(X_1 = 0, X_2 = 1)}{\mathbb{P}(X_2 = 1)} = \frac{pq}{2pq} = \frac{1}{2}.$$

- **Case $X_2 = 2$.**
Then the only possible pair is $(Y_1, Y_2) = (1, 1)$ (i.e. outcome HH), so
$$\mathbb{P}(X_2 = 2) = \mathbb{P}(HH) = p^2,$$
and necessarily $X_1 = 1$. Thus
$$\mathbb{E}(X_1 \mid X_2 = 2) = \frac{1 \cdot \mathbb{P}(X_1 = 1, X_2 = 2)}{\mathbb{P}(X_2 = 2)} = \frac{1 \cdot p^2}{p^2} = 1.$$

So again we obtain

$$\boxed{\mathbb{E}(X_1 \mid X_2) = \begin{cases} 1, & X_2 = 2, \\ \frac{1}{2}, & X_2 = 1, \\ 0, & X_2 = 0. \end{cases}}$$

Exercise 3

(Application of the course)

1. Let X be a random variable defined on Ω and taking value in $(E, \mathcal{E})$ and let f be a measurable function from $(E, \mathcal{E})$ to $\mathbb{R}$. Do you have $\sigma(f(X)) \subset \sigma(X)$ or $\sigma(X) \subset \sigma(f(X))$?

2. Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebra, Is $\mathcal{F} \cap \mathcal{G}$ always σ-algebra? Is $\mathcal{F} \cup \mathcal{G}$ always σ-algebra ?

3. Let X be a integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
   (a) What is $\mathbb{E}(X|\mathcal{F})$ if $\mathcal{F} = \{\emptyset, \Omega\}$? if $\mathcal{F} = \mathcal{P}(\Omega)$?
   (b) What is $\mathbb{E}(X|\mathcal{F})$? if X is independent from $\mathcal{F}$.

Solution

1. We show that one always has $\sigma(f(X)) \subset \sigma(X)$. We know that

$$X : (\Omega, \mathcal{F}) \to (E, \mathcal{E}), \qquad f : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

and f is $\mathcal{E}/\mathcal{B}(\mathbb{R})$measurable.

Recall the definition (see Definition 1.4.4 of the course):

$$\sigma(X) = \{X^{-1}(A) : A \in \mathcal{E}\} \subset \mathcal{F},$$

and similarly

$$\sigma(f(X)) = \{(f(X))^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\} \subset \mathcal{F}.$$

Now let $B \in \mathcal{B}(\mathbb{R})$. Since f is measurable, we have

$$f^{-1}(B) \in \mathcal{E}.$$

Then

$$(f(X))^{-1}(B) = X^{-1}(f^{-1}(B)).$$

Because $f^{-1}(B) \in \mathcal{E}$, we get

$$X^{-1}(f^{-1}(B)) \in \sigma(X).$$

Thus every generator of $\sigma(f(X))$ belongs to $\sigma(X)$, and therefore

$$\boxed{\sigma(f(X)) \subset \sigma(X)}.$$

This means that $f(X)$ contains less information than X: to know $f(X)$ it is enough to know X, but not necessarily the opposite.

2. Is $\mathcal{F} \cap \mathcal{G}$ and $\mathcal{F} \cup \mathcal{G}$ always a σ-algebra?

Intersection.
Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras on Ω. We show that $\mathcal{F} \cap \mathcal{G}$ is again a σ-algebra.

- Since $\emptyset, \Omega \in \mathcal{F}$ and $\emptyset, \Omega \in \mathcal{G}$, we have

$$\emptyset, \Omega \in \mathcal{F} \cap \mathcal{G}.$$

- If $A \in \mathcal{F} \cap \mathcal{G}$, then $A \in \mathcal{F}$ and $A \in \mathcal{G}$. As $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras, $A^c \in \mathcal{F}$ and $A^c \in \mathcal{G}$, hence $A^c \in \mathcal{F} \cap \mathcal{G}$.

- If $(A_n)_{n\geq 1} \subset \mathcal{F} \cap \mathcal{G}$, then each $A_n \in \mathcal{F}$ and $A_n \in \mathcal{G}$. Since both $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras,
$$\bigcup_{n\geq 1} A_n \in \mathcal{F} \quad \text{and} \quad \bigcup_{n\geq 1} A_n \in \mathcal{G},$$
so $\bigcup_{n\geq 1} A_n \in \mathcal{F} \cap \mathcal{G}$.

Thus $\mathcal{F} \cap \mathcal{G}$ satisfies the three axioms of a σ-algebra, and we conclude

$$\boxed{\mathcal{F} \cap \mathcal{G} \text{ is always a } \sigma\text{-algebra.}}$$

Union.
In general, $\mathcal{F} \cup \mathcal{G}$ is *not* a σ-algebra.

Consider the example of a die roll with sample space

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Define
$$\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4, 5, 6\}\}, \qquad \mathcal{G} = \{\emptyset, \Omega, \{6\}, \{1, 2, 3, 4, 5\}\}.$$

Then
$$\mathcal{F} \cup \mathcal{G} = \{\emptyset, \Omega, \{1\}, \{6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5\}\}.$$

Note that $\{1\} \in \mathcal{F} \cup \mathcal{G}$ and $\{6\} \in \mathcal{F} \cup \mathcal{G}$, but

$$\{1\} \cup \{6\} = \{1, 6\} \notin \mathcal{F} \cup \mathcal{G}.$$

So $\mathcal{F} \cup \mathcal{G}$ is not closed under finite (hence not under countable) unions, and therefore it is not a σ-algebra.

Thus, in general,
$$\boxed{\mathcal{F} \cup \mathcal{G} \text{ is not necessarily a } \sigma\text{-algebra.}}$$

3. $E[X \mid \mathcal{F}]$ if $\mathcal{F} = \{\emptyset, \Omega\}$ and if $\mathcal{F} = \mathcal{P}(\Omega)$.

By definition, a random variable Y is a version of $E[X \mid \mathcal{F}]$ if

- Y is $\mathcal{F}$-measurable,
- for all $A \in \mathcal{F}$,
$$\int_A Y \, d\mathbb{P} = \int_A X \, d\mathbb{P}.$$

For $\mathcal{F} = \{\emptyset, \Omega\}$ (trivial σ-algebra)

An $\mathcal{F}$-measurable random variable must be *constant* a.s., say $Y(\omega) \equiv c$. Then, for $A = \Omega$,

$$\int_\Omega Y \, d\mathbb{P} = \int_\Omega c \, d\mathbb{P} = c$$

and by the defining property of the conditional expectation,

$$\int_\Omega Y \, d\mathbb{P} = \int_\Omega X \, d\mathbb{P} = \mathbb{E}[X].$$

Hence $c = \mathbb{E}[X]$, so
$$\boxed{E[X \mid \{\emptyset, \Omega\}] = \mathbb{E}[X].}$$

For $\mathcal{F} = \mathcal{P}(\Omega)$ (full σ-algebra)

Here every random variable X is $\mathcal{F}$-measurable, since $\mathcal{F}$ contains *all* subsets of Ω.

Consider $Y := X$. Then Y is $\mathcal{F}$-measurable, and for every $A \in \mathcal{F} = \mathcal{P}(\Omega)$,

$$\int_A Y \, d\mathbb{P} = \int_A X \, d\mathbb{P}.$$

So Y satisfies the defining property of $E[X \mid \mathcal{F}]$.

By uniqueness (up to a.s. equality) of conditional expectation, we obtain

$$\boxed{E[X \mid \mathcal{P}(\Omega)] = X.}$$

Exercise 4

We flip three times a coin which probability of having Heads is $p \in]0, 1[$ and we record the successive sides that appear.

1. Determine the sample space Ω.
2. We denote $X_k(\omega)$ for $k = 1, 2, 3, ...$ the random variable that counts the number of Tails appeared after the k first toss when we observe $\omega \in \Omega$.
3. Determine the σ-algebra $\mathscr{P}_1 = \sigma(X_1)$ and $\mathscr{P}_2 = \sigma(X_2)$ and then $\mathscr{P}_{1,2} = \sigma(X_1, X_2)$.
4. Are the following inclusions true ? $\mathscr{P}_1 \subset \mathscr{P}_2$; $\mathscr{P}_1 \subset \mathscr{P}_{1,2}$.
5. Determine $\mathbb{E}[X_3|\mathscr{F}_1]$ and $\mathbb{E}[\mathbb{E}[X_3|\mathscr{P}_2]|\mathscr{P}_1]$?
6. Does one have the equality
$$\mathbb{E}[X_3|\mathscr{P}_1] = \mathbb{E}[\mathbb{E}[X_3|\mathscr{P}_2]|\mathscr{P}_1]$$
7. Does one have the equality
$$\mathbb{E}[X_3|\mathscr{P}_1] = \mathbb{E}[\mathbb{E}[X_3|\mathscr{P}_{1,2}]|\mathscr{P}_1]$$

Exercise 5

Let X be a random variable defined on $(\Omega, \mathcal{F})$ and let $\mathbb{P}$ be a probability on $(\Omega, \mathcal{F})$ such that $\mathbb{E}(X^2) < +\infty$. Let $\mathscr{P}$ be a sub σ-algebra of $\mathscr{P}$. Set

$$Var(X|\mathscr{P}) = \mathbb{E}\left[(X - \mathbb{E}(X|\mathscr{P}))^2|\mathscr{P}\right]$$

Starting from

$$X - \mathbb{E} = X - \mathbb{E}(X|\mathscr{P}) + \mathbb{E}(X|\mathscr{P}) - \mathbb{E}(X)$$

Show that

$$Var(X) = \mathbb{E}\left(Var(X|\mathscr{P})\right) + Var(X\mathbb{E}(X|\mathscr{P})).$$

Solution

Show that

$$\operatorname{Var}(X) = \mathbb{E}\Big(\operatorname{Var}(X \mid \mathscr{P})\Big) + \operatorname{Var}\left(\mathbb{E}(X \mid \mathscr{P})\right),$$

where $\mathscr{P}$ is a sub-σ-algebra of $\mathcal{F}$.

Starting from
$$X - \mathbb{E}[X] = X - \mathbb{E}[X \mid \mathscr{P}] + \mathbb{E}[X \mid \mathscr{P}] - \mathbb{E}[X].$$
Let
$$A := X - \mathbb{E}[X \mid \mathscr{P}], \qquad B := \mathbb{E}[X \mid \mathscr{P}] - \mathbb{E}[X].$$
Then $X - \mathbb{E}[X] = A + B$ and
$$\mathrm{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = \mathbb{E}\big[(A+B)^2\big] = \mathbb{E}[A^2] + 2\mathbb{E}[AB] + \mathbb{E}[B^2].$$
By definition of conditional variance,
$$\mathrm{Var}(X \mid \mathscr{P}) = \mathbb{E}\big[(X - \mathbb{E}[X \mid \mathscr{P}])^2 \mid \mathscr{P}\big] = \mathbb{E}[A^2 \mid \mathscr{P}].$$
Taking expectations and using the tower property,
$$\mathbb{E}\big(\mathrm{Var}(X \mid \mathscr{P})\big) = \mathbb{E}\big(\mathbb{E}[A^2 \mid \mathscr{P}]\big) = \mathbb{E}[A^2].$$
Next, we show that $\mathbb{E}[AB] = 0$. Using conditional expectation,
$$\mathbb{E}[AB] = \mathbb{E}\big(\mathbb{E}[AB \mid \mathscr{P}]\big).$$
Since B is $\mathscr{P}$-measurable, we can take it outside:
$$\mathbb{E}[AB \mid \mathscr{P}] = B\,\mathbb{E}[A \mid \mathscr{P}].$$
But
$$\mathbb{E}[A \mid \mathscr{P}] = \mathbb{E}\big[X - \mathbb{E}[X \mid \mathscr{P}] \,\big|\, \mathscr{P}\big] = \mathbb{E}[X \mid \mathscr{P}] - \mathbb{E}[X \mid \mathscr{P}] = 0.$$
Hence $\mathbb{E}[AB \mid \mathscr{P}] = B \cdot 0 = 0$, and therefore
$$\mathbb{E}[AB] = \mathbb{E}\big(\mathbb{E}[AB \mid \mathscr{P}]\big) = 0.$$
We now identify $\mathbb{E}[B^2]$. Note that
$$B = \mathbb{E}[X \mid \mathscr{P}] - \mathbb{E}[X],$$
so B is exactly the centered version of $\mathbb{E}[X \mid \mathscr{P}]$. Moreover,
$$\mathbb{E}[B] = \mathbb{E}\big(\mathbb{E}[X \mid \mathscr{P}]\big) - \mathbb{E}[X] = \mathbb{E}[X] - \mathbb{E}[X] = 0$$
by the law of total expectation. Thus,
$$\mathbb{E}[B^2] = \mathrm{Var}\left(\mathbb{E}(X \mid \mathscr{P})\right).$$
Hence,
$$\boxed{\mathrm{Var}(X) = \mathbb{E}[A^2] + 2\mathbb{E}[AB] + \mathbb{E}[B^2] = \mathbb{E}\big(\mathrm{Var}(X \mid \mathscr{P})\big) + \mathrm{Var}\left(\mathbb{E}(X \mid \mathscr{P})\right).}$$

Exercise 6

We toss n times a coin and consider a gambling which consis ts on scoring 1 point when Heads appears or losing 2 points otherwise. The probability of having Heads with the coin is p. Let Y_i be the random variable representing the scored point on the i-th toss only and let X_n be the cumulated scored point after the n first tosses with X_0 Let $(\mathcal{F}_n)_{n\geq 0}$ with $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$ be the natural filtration of the process $(Y_n)_{n\geq 1}$.

1. Write X_n as a function of $Y_1, \cdots, Y_n$.
2. Determine the value of p which makes the process $(X_n)_{n\geq 0}$ a $\mathcal{F}_n$ martingale.
3. Determine the value of p which makes the process $(X_n)_{n\geq 0}$ a $\mathcal{F}_n$ sur-martingale.
4. Determine the value of p which makes the process $(X_n)_{n\geq 0}$ a $\mathcal{F}_n$ sub-martingale.

Solution

1. We toss the coin n times. For each toss $k \geq 1$ we define the *score* at time k by

$$Y_k = \begin{cases} 1, & \text{if the } k\text{-th toss is Head,} \\ -2, & \text{if the } k\text{-th toss is Tail.} \end{cases}$$

Let $(\mathcal{F}_n)_{n\geq 0}$ be the natural filtration,

$$\mathcal{F}_n = \sigma(Y_1, \dots, Y_n), \quad \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

The total gain after n tosses is

$$X_n = Y_1 + \cdots + Y_n = \sum_{k=1}^{n} Y_k,$$

which is clearly $\mathcal{F}_n$-measurable.

2. We look for p such that $(X_n)_{n\geq 0}$ is an $(\mathcal{F}_n)$-martingale, i.e.

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n \quad \text{a.s.}$$

We write $X_{n+1} = X_n + Y_{n+1}$, hence

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n + Y_{n+1} \mid \mathcal{F}_n] = X_n + \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n].$$

Since Y_{n+1} is independent of $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ (independent tosses),

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[Y_{n+1}] = \mathbb{E}[Y_1].$$

Now

$$\mathbb{E}[Y_1] = 1 \cdot p + (-2) \cdot (1-p) = p - 2 + 2p = 3p - 2.$$

Therefore

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n + (3p - 2).$$

For the martingale property we need

$$X_n + (3p-2) = X_n \quad \Rightarrow \quad 3p - 2 = 0 \quad \Rightarrow \quad p = \frac{2}{3}.$$

Hence (X_n) is a martingale iff $p = \dfrac{2}{3}$.

3. (X_n) is a super-martingale if

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n.$$

From above,

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n + (3p-2),$$

so the condition becomes

$$X_n + (3p-2) \leq X_n \quad \Longleftrightarrow \quad 3p - 2 \leq 0 \quad \Longleftrightarrow \quad p \leq \frac{2}{3}.$$

Thus (X_n) is an $(\mathcal{F}_n)$-super-martingale iff $p \leq \dfrac{2}{3}$.

4. (X_n) is a sub-martingale if
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n.$$
Using again
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n + (3p-2),$$
we get
$$X_n + (3p-2) \geq X_n \iff 3p-2 \geq 0 \iff p \geq \frac{2}{3}.$$
Thus (X_n) is an $(\mathcal{F}_n)$-sub-martingale iff $p \geq \dfrac{2}{3}$.

Exercise 7

Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables with the same distribution $\mathcal{N}(0,\, \sigma^2)$ where $\sigma > 0$. Set $\mathscr{F}_n = \sigma(Y_1, \cdots, Y_n)$ and $X_n = Y_1 + \cdots + Y_n$.

1. Show that
$$\mathbb{E}\left[e^{uY_1}\right] = e^{\frac{u^2\sigma^2}{2}}$$
2. Let $Z_n^u = \exp\left(uX_n - n\frac{u^2\sigma^2}{2}\right)$. Show that , for any $u \in \mathbb{R}, (Z_n^u)_{n\geq 1}$ is an $\mathscr{F}$-martingale.
3. Show that for any $u \in \mathbb{R}, (Z_n^u)_{n\geq 1}$ converges almost surely toward a random variable Z_∞^u which is finite.

Solution

1. Show that $\mathbb{E}(e^{uY_1}) = e^{\frac{u^2\sigma^2}{2}}$.

We know that $(Y_n)_{n\geq 1}$ is a sequence of independent random variables with
$$Y_k \sim \mathcal{N}(0, \sigma^2), \quad k \geq 1.$$
In particular, Y_1 has density
$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right), \qquad y \in \mathbb{R}.$$
For $u \in \mathbb{R}$,
$$\mathbb{E}\left[e^{uY_1}\right] = \int_{\mathbb{R}} e^{uy} f(y)\, dy = \int_{\mathbb{R}} e^{uy} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)\, dy.$$
Combine equation
$$uy - \frac{y^2}{2\sigma^2} = -\frac{1}{2\sigma^2}\left(y^2 - 2u\sigma^2 y\right) = -\frac{1}{2\sigma^2}\left[(y-u\sigma^2)^2 - u^2\sigma^4\right] = -\frac{(y-u\sigma^2)^2}{2\sigma^2} + \frac{u^2\sigma^2}{2}.$$
Hence
$$\begin{aligned}\mathbb{E}(e^{uY_1}) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-u\sigma^2)^2}{2\sigma^2} + \frac{u^2\sigma^2}{2}\right)\, dy \\ &= e^{\frac{u^2\sigma^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-u\sigma^2)^2}{2\sigma^2}\right)\, dy.\end{aligned}$$
The integral is 1 (density of $\mathcal{N}(u\sigma^2, \sigma^2)$).

Therefore

$$\boxed{\mathbb{E}(e^{uY_1}) = e^{\frac{u^2\sigma^2}{2}}.}$$

2. Let $Z_n^u = \exp\Big(uX_n - \frac{nu^2\sigma^2}{2}\Big)$. Show that $(Z_n^u)_{n\geq 1}$ is an $(\mathcal{F}_n)$-martingale, where $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ and $X_n = Y_1 + \dots + Y_n$.

For $n \geq 1$,

$$X_{n+1} = X_n + Y_{n+1},$$

hence

$$\begin{aligned} Z_{n+1}^u &= \exp\left(uX_{n+1} - \frac{(n+1)u^2\sigma^2}{2}\right) = \exp\left(uX_n - \frac{nu^2\sigma^2}{2}\right)\exp\left(uY_{n+1} - \frac{u^2\sigma^2}{2}\right) \\ &= Z_n^u \cdot \exp\left(uY_{n+1} - \frac{u^2\sigma^2}{2}\right). \end{aligned}$$

Taking conditional expectation w.r.t. $\mathcal{F}_n$,

$$\mathbb{E}(Z_{n+1}^u \mid \mathcal{F}_n) = Z_n^u \, \mathbb{E}\left(\exp\left(uY_{n+1} - \frac{u^2\sigma^2}{2}\right) \Bigm| \mathcal{F}_n\right).$$

Since Y_{n+1} is independent of $\mathcal{F}_n$,

$$\mathbb{E}\left(\exp\left(uY_{n+1} - \frac{u^2\sigma^2}{2}\right) \Bigm| \mathcal{F}_n\right) = \mathbb{E}\left(\exp\left(uY_{n+1} - \frac{u^2\sigma^2}{2}\right)\right) = e^{-\frac{u^2\sigma^2}{2}}\, \mathbb{E}(e^{uY_{n+1}}).$$

Using sol 1

$$\mathbb{E}(e^{uY_{n+1}}) = e^{\frac{u^2\sigma^2}{2}},$$

so the factor is

$$e^{-\frac{u^2\sigma^2}{2}} \cdot e^{\frac{u^2\sigma^2}{2}} = 1.$$

Hence

$$\mathbb{E}(Z_{n+1}^u \mid \mathcal{F}_n) = Z_n^u.$$

Moreover, Z_n^u is $\mathcal{F}_n$-measurable and nonnegative, so

$$\boxed{(Z_n^u)_{n\geq 1} \text{ is an } (\mathcal{F}_n)\text{-martingale.}}$$

3. Show that for any $u \in \mathbb{R}$, $(Z_n^u)_{n\geq 1}$ converges almost surely toward a finite random variable Z_∞^u.

First, from sol 2: (Z_n^u) is a nonnegative martingale. Also,

$$\mathbb{E}(Z_n^u) = \mathbb{E}\left(\exp\left(uX_n - \frac{nu^2\sigma^2}{2}\right)\right) = \left(\mathbb{E}(e^{uY_1})\right)^n e^{-\frac{nu^2\sigma^2}{2}} = \left(e^{\frac{u^2\sigma^2}{2}}\right)^n e^{-\frac{nu^2\sigma^2}{2}} = 1,$$

so $\sup_n \mathbb{E}(Z_n^u) = 1 < +\infty$.

By Doob's martingale convergence theorem for nonnegative (sub-)martingales, there exists an a.s. finite random variable Z_∞^u such that

$$Z_n^u \xrightarrow[n\to\infty]{\text{a.s.}} Z_\infty^u.$$

Therefore

$$\boxed{Z_n^u \to Z_\infty^u \text{ almost surely, with } 0 \leq Z_\infty^u < \infty \text{ a.s.}}$$

Exercise 8

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(X_k)_{k=0,\ldots,n}$ be a sequence of random variables. Let $(\mathcal{G}_k)_{k=0,\ldots,n}$ be the filtration generated by the process $(Y_k)_{k=1,\ldots,n}$ with $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and let $\bar{\mathbb{P}}$ be the new probability defined as

$$d\bar{\mathbb{P}} = Z_n d\mathbb{P}, \qquad \text{with} \qquad Z_n = \exp\left(\sigma(Y_1 + \cdots + Y_n) - \frac{n\sigma^2}{2}\right)$$

1. Show that the process $(Z_k)_{k=0,\ldots,n}$ defined for every $k \in 0, \ldots, n$ by $Z_k = \mathbb{E}_{\mathbb{P}}(Z_n|\mathcal{G}_k)$ is a martingale.
2. Determine $\mathbb{E}_{\mathbb{P}}(X_{k+1}|\mathcal{G}_k)$ for all $k \in \{0, \ldots, n-1\}$. Deduce the value oof μ such that $(X_k)_{k=0,\ldots,n}$ is a martingale.
3. Show that
$$\mathbb{E}_{\mathbb{P}} = x_0 e^{\mu}$$
4. We consider the process $(\bar{X}_k)_{k=0,\ldots,n}$ defined for every $k \in \{0, \ldots, n\}$ by $\bar{X}_k = e^{-\mu k/n} X_k$. Show that $(\bar{X}_k)_{k=0,\ldots,n}$ is $\mathbb{P}$-martingale.
5. Determine the valuer of σ such that $(\bar{X}_k)_{k=0,\ldots,n}$ is $\bar{\mathbb{P}}$-martingale.

Exercise 9

A stochastic process $(M_n)_{n\geq 0}$ to said to be with independent increments if for any n, the random variable $M_{n+1} - M_n$ is independent from $\mathscr{F} = \sigma(M_0, \ldots, M_n$. A real random variable M is square integrable if $\mathbb{E}(M^2)$ is this $\mathbb{E}(M^2) < +\infty$.

1. Let $(M_n)_{n\geq 0}$ be a square integrable with independent increments.
   (a) Show that $\mathbb{E}[M_n] = \mathbb{E}[M_k]$ for every $n \leq k$ and verify that $cov(M_n, M_k) = Var(M_n)$. We recall that for a given random variables X and Y $cov(X, Y) = E[XY] - E[X]E[Y]$.
   (b) We set $\sigma_0^2 = Var(M_0)$ and for $n \geq 0, \sigma_k^2 = Var(M_k - M_{k-1})$. Show that for every $n \geq 0$ $Var(M_n) = \sum_{k=0}^{n} \sigma_k^2$.
2. Let $(\langle M_n \rangle)_{n\geq 0}$ be the compensator of the sub-martingale $(M_n^2)_{n\geq 0}$, defined by $\langle M \rangle_0 = 0$ and for any $n \geq 1$ by $\langle M \rangle_n = \langle M_n \rangle = \langle M_{n-1} \rangle + \mathbb{E}[M_n^2 - M_{n-1}^2|\mathscr{F}_{n-1}]$
   a Show that
   $$\mathbb{E}[M_n^2 - M_{n-1}^2|\mathscr{F}_{n-1}] = \mathbb{E}[(M_n - M_{n-1})^2|\mathscr{F}_{n-1}]$$
   b Determine $\langle M \rangle_n$.
   c Deduce that the process defined for every $n \geq 1$ by $M_n^2 - \sum_{k=1}^{n} \sigma_k^2$ is an $\mathscr{F}_n$-martingale.
3. Now, let $(M_n)_{n\geq 0}$ be a gaussian process, means: for any $n \geq 0$ the vector $(M_0, \cdots, M_n)$ is a gaussian vector. Suppose that $(M_n)_{n\geq 0}$ is a martingale.
   (a) Show that for any $k = 0, \ldots, n$,we have $\mathbb{E}(M_k(M_{n+1} - M_n)) = 0$.
   (b) Deduce that $(M_n)_{n\geq 0}$ is with independent increments.
   (c) Show that for any fixed $\theta \in \mathbb{R}$ the process
   $$Z_n^\theta = e^{\theta M_n - \theta^2 \langle M_n \rangle/2}$$
   is martingale. Is this process converge almost surely ?

Exercise 10

Let $(S_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}$: $S_0 = 0, S_n = U_1 + \cdots + U_n$ where the $U_i \in \{x, y\}$ are independent with the same distribution means, $0 < \mathbb{P}(U_i = x) = p < 1, \mathbb{P}(U_i = -y) = 1-p$, for any $i \in \{1, \cdots, n\}$. We define $(\mathscr{F}_n)_{n\geq 0}$ as the filtration generated by $(S_n)_{n\geq 0}$: $\mathscr{F}_n = \sigma(S_0, \cdots, S_n)$ for every $n \geq 1$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

I We set $x = 1, y = 2$. Determine the value of p such that $(S_n)_{n\geq 1}$ is sub-martingale.

1 Let $\mu = \mathbb{E}(U_1)$ and let $X_n = S_n - n\mu$ for every $n \geq 0$.

(a) Show that $(X_n)_{n\geq 0}$ is a martingale.

(b) Let $a > 0$ and let τ be a stopping time defined by

$$\tau = \inf\{n \geq 0, |X_n| > a\}$$

Show that for all $n \geq 0, \mathbb{E}(S_{n\wedge\tau})$

(c) Deduce that $\mathbb{E}(S_\tau) = \mu\mathbb{E}(\tau)$

2 We set $\sigma^2 = Var(U_1)$ and $Y_n = (S_n - n\mu)^2 - n\sigma^2$ for all $n \geq 0$. Show that $(Y_n)_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$-martingale.

II Suppose now $x = y = 1$.

(a) Let $Z_n = (q/p)^{S_n}$. Show that $(Z_n)_{n\geq 0}$ is a positive martingale.

(b) Show that

$$\mathbb{P}\left(\sup_{n\geq 0} Z_n \geq (q/p)^k\right) \leq \left(\frac{p}{q}\right)^k.$$

(c) Deduce that

$$\mathbb{P}\left(\sup_{n\geq 0} S_n \geq k\right) \leq \left(\frac{p}{q}\right)^k.$$

Exercise 11

We consider a binomial model with N period on an underlying asset S with initial value $S_0 = s_0 > 0$ a constant value. We suppose that any time $n \in \{1, \cdots, N\}$, the value S_n of the asset may go up or go down and the asset take upward value uS_{n-1} and the downward value dS_{n-1}, with $0 < d < r+1 < u$, where r is the interest and where S_{n-1} is the value asset at time $n-1$. We suppose we may write down for every $n \geq 1$.

$$S_n = S_{n-1}Y_n$$

where $(Y_n)_{n\geq 1}$ is independent and identical distributed sequence of random variable with

$$Y_n = \begin{cases} u, & \text{with probability } \tilde{p}, \\ d, & \text{with probability } \tilde{q} = 1 - \tilde{p}, \end{cases}$$

with $\tilde{p} = \frac{1+r-d}{u-d}$. We define the updated value $(\tilde{(}S_n)_{n\geq 0})$ of the asset process $(S_n)_{n\geq 0}$ by

$$\bar{S}_n = \frac{1}{(1+r)^n} S_n, \qquad \forall n \geq 0$$

and denote by $(\mathcal{F}_n)_{n\geq 0}$ the filtration generated by $(Y_n)_{n\geq 1}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$

1. Show that $(\bar{S}_n)_{n\geq 0}$ is a martingale will response to $(\mathcal{F}_n)_{n\geq 0}$.

2. We define the process $(V_n)_{n\geq 0}$ by
$$V_n = \Delta_n S_n + (1+r)(V_{n+1} - \Delta_n S_{n-1})$$
where $(\Delta_n)_{n\geq 0}$ is predictable with respect to $(\mathcal{F}_n)_{n\geq 0}$. Let
$$\bar{V}_n = \frac{1}{(1+r)^n} V_n, \qquad \forall n \geq 0$$
   a Show that $(\bar{V})_{n\geq 0}$ is an $\mathcal{F}_n$-martingale.
   b We suppose that $V_N = \max(K - S_N, 0)$ for $K > 0$ and $N \geq 1$. Show that
$$V_0 = \frac{1}{(1+r)^N} \mathbb{E}[\max(K - S_N, 0)]$$
   c Compute explicitly V_0 when $N = 1$ and $dS_0 < K < uS_0$.

Exercise 12

We consider a gambling where we toss a coin and gain 2 points if Heads appears and loss 2 points when Tails appears. We denote by p the probability that Head appears. Let's X_n denotes our score at the n-th toss of the coin and let $(Y_i)_{i\geq 1}$ be independent and identical distributed sequence of random variables such that

$$Y_i = \begin{cases} 2, & \text{with probability } p, \\ -2, & \text{with probability } 1-p \end{cases}$$

1. Show that $(X_n)_{n\geq 0}$ is a Markov chain and determine its transition matrix.
2. Compute the following quantities: $\mathbb{P}(X_0 = 0, X_1 = 2, X_2 = 4, X_3 = 6)$ and $\mathbb{P}(X_0 = 0, X_1 = -2, X_2 = 0)$.

Exercise 13

Let $(X_n)_{n\geq 0}$ be a Markov chain defined on $E = \{1, 2, 3, 4\}$ with transition matrix
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1. Show that P^2 is a transition matrix
2. Let $j \in E$ and define the number of passage of the Markov Chain at the state j by
$$N_j = \sum_{k=0}^{+\infty} \mathbf{1}_{\{X_k = j\}}$$
Compute $E_i(N_1)$ for all $i \in E$
3. Let τ_3 be the shopping time (the first passage time at state 3) defined by $\tau_3 = \inf\{n \geq 0 : X_n = 3\}$. Compute $\mathbb{P}_i(\tau_3 < +\infty)$ for any $i \in E$
4. Show that the Markov Chain is irreducible and recurrent.
5. Determine the invariante probability of the Markov Chain.
6. Deduce for any $i \in E$, the almost sure limit of $\frac{1}{n}\sum_{k=0}^{n-1} \nVdash_{\{X_k=i\}}$.
7. Compute the almost sure limit of $\frac{1}{n}\sum_{k=0}^{n-1} X_k^2$.

Exercise 14

A mobile moves randomly on $\mathbb{Z}$ following Markov Chain with transition matrix P with components

$$P(i,j) = \begin{cases} p, & \text{if } j = i+1 \\ 1-p, & \text{if } j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

with $0 < p < 1$.

1. Let $(Z_n)_{\geq 1}$ be an independent and identical distributed sequence of random variables such that $\mathbb{P}(Z_n = 1) = p, \mathbb{P}(Z_n = -1) = 1-p$. We set $X_0 = 0$ and $X_n = Z_1 + \cdots + Z_n$. Show that $(X_n)_{n\geq 0})$ is a Markov Chain starting at 0 with transition matrix P.
2. What is the limit of $lim_{n\to+\infty}\frac{1}{n}X_n$? If $p \neq 1/2$, the chain is recurrent or transient ?
3. Set $Y_i = \frac{1}{2}(Z_i + 1)$. What is the distributions of Y_i and $T_n = \frac{1}{2}(X_n + n)$. Compute $P^{(n)}(0,0) = \mathbb{P}(X_n = 0)$ when n is even and when it is odd.
4. If $p = \frac{1}{2}$, show that the Markov Chain is recurrent. Is it positive recurrent ?
5. We suppose that $p = \frac{1}{2}$, and for $a, b > 0$ we define the stopping times τ_{-a} and τ_b by

   $$\tau_{-a} = \inf\{n \geq 0, X_n = -a\} \quad \text{and} \quad \tau_b = \inf\{\tau_b \geq 0, X_n = b\}$$

   and set $\tau = \tau_{-a} \wedge \tau_b$.

   a Show that the process defined by $S_n = X_n^2 - n$, for all $n \geq 0$ is martingale.

   b Show that $\mathbb{E}(S_\tau) = 0$ and deduce the values of $\mathbb{P}(\tau_{-a} < \tau_b$ and $\mathbb{P}(\tau_{-a} > \tau_b$.

Exercise 15

Let $(X_n)_{n\geq 0}$ be a Markov Chain with state space $E = \{1,2,3,4\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix}$$

1. Calculate the probability that, starting from $i \in E$, the hitting time of $A = \{2,3\}$ is finite: $\mathbb{P}_i(\tau_A < +\infty)$ for all $i \in E$ where

   $$\tau_A = \inf\{= n \geq 0, X_n \in A\}$$

2. Calculate the expectation of the time until reaching A starting from $i \in E : \mathbb{E}_i(\tau_A)$
3. Compute $\mathbb{P}_i(\tau_1 < \tau_2)$ for $i \in E$.

Exercise 16

Consider the total growth process $(X_n)_{n\geq 0}$ defined by $X_0 = 0$ and for every $n \geq 1$ by

$$X_n = Y_1 + \cdots + X_n$$

where $(Y_n)_{n\geq 1}$ is independent and identical distributed sequence of the random variable valued in $\{-d, e\}, d, e > 0$ satisfying for any $n \geq 1, \mathbb{P}(Y_n = -d) = q$ and $\mathbb{P}(Y_n = e) = p$ with $q = 1 - p$. Let $(\bar{X}_n)_{n\geq 0}$ be a stochastic process define by $\bar{X}_0 = 0$ and for any $n \geq 1$, by

$$\bar{X}_n = \bar{Y}_1 + \cdots + \bar{Y}_n \quad \text{with} \quad \bar{Y}_k = \frac{1}{e+d}(2Y_k + d - e), \quad k \geq n$$

Let E be the set of values taken by $(\bar{X}_n)_{n\geq 0}$ and let $-b, a \in E$ with $a, b > 0$. Define the stopping time τ by

$$\tau = \tau_{-b} \wedge \tau_a \quad \text{with } \tau_l = \inf\left\{k \geq 0, \bar{X}_k = l\right\}, \quad \text{for} l \in \{-b, a\}$$

1. Is the process $(\bar{X})_{n\geq 0}$ a martingale with respect to its natural filtration ? Justify your answer.
2. We set $A_n = \left\{-b < \bar{X}_n < a\right\}$ for any $n \geq 0$. Prove that
$$\mathbb{P}(\tau = +\infty) = \mathbb{P}\left(\bigcup_{n\geq 0} A_n\right) \leq \lim_{n\to+\infty} \mathbb{P}(A_n).$$
3. Use the Central Limit Theorem 1 and the result 2 to show that τ is finite almost surely.
4. We define the process $(Z_n)_{n\geq 0}$ by $Z_n = (q/p)^{\bar{X}_n}$, for every $n \geq 0$. Show that there exists a constant $C > 0$ such that $\sup_{n\geq 0} \mathbb{E}(|Z_{n\wedge\tau}|) < C$
5. Show that the process $(Z_n)_{n\geq 0}$ is a martingale with respect to the natural filtration of the process $(\bar{X}_n)_{n\geq 0}$ and that $\mathbb{E}(Z_\tau) = 1$
6. Show that
$$\mathbb{P}(\tau_{-b} < \tau_a) = \frac{1 - (q/p)^a}{(q/p)^{-b} - (q/p)^a} \quad \text{and} \quad \mathbb{P}(\tau_a < \tau_{-b}) = \frac{(q/p)^{-b} - 1}{(q/p)^{-b} - (q/p)^a}$$

Exercise 17

Let $(Z_n)_{n\geq 1}$ be a sequence of independent random variables such that $\mathbb{P}(Z_n = 1) = p, \mathbb{P}(Z_n = -1) = 1 - p$. Define the random walk $(X_n)_{n\geq 0}$ by $X_0 = x \in \mathbb{Z}$ and $X_n = Z_1 + \cdots + Z_n$. Let $a, b \in \mathbb{Z}$ be such that $a < x < b$ and let τ_y be the stopping time defined by $\tau_y = \inf\{n \geq 0, X_n \geq y\}$, for $y \in \mathbb{Z}$.

$$\mathbb{P}_x(\tau_b < \tau_a) = \begin{cases} \dfrac{\left(\frac{q}{p}\right)^{x-a} - 1}{\left(\frac{q}{p}\right)^{b-a} - 1}, & \text{if } p \neq q, \\[2ex] \dfrac{x-a}{b-a}, & \text{if } p = q. \end{cases} \tag{1}$$

1. We define $\phi(x, l) = \mathbb{P}_x(\tau_l < \tau_0)$, for $0 < x < l \in \mathbb{Z}$. Show that $\mathbb{P}_x(\tau_b < \tau_a) = \phi(x - a, b - a)$, for $a < x < b$. What are the values of $\phi(0, l)$ and $\phi(l, l)$?
2. Suppose that $a = 0$ and $b = l$ and define $\delta(x) = \phi(x + 1, l) - \phi(x, l)$ for $x \in \mathbb{N}^*$.
   (a) Show that for all $x \in \mathbb{N}^*, p\delta(x) = q\delta(x - 1)$.

(b) Deduce that $\delta(x) = \left(\frac{q}{p}\right)^x \delta(0)$.

(c) We suppose that $q \neq p$

i. Show that $\delta(0) = \frac{\frac{q}{p}-1}{\left(\frac{q}{p}\right)^l-1}$

ii. Using the previous parts show that $\phi(x,l) = \frac{\left(\frac{q}{p}\right)^x-1}{\left(\frac{q}{p}\right)^l-1}$

(d) We suppose that $p = q$. Show that in this case $\delta(0) = 1/l$ and that $\phi(x,l) = x/l$.

3. Deduce the equation (1).

4. Let $x > 0$. Observing that $\mathbb{P}_x(\tau_0 < +\infty) = \lim_{l\to+\infty} \mathbb{P}_x(\tau_0 < \tau_l)$, show that

$$\mathbb{P}_x(\tau_0 < +\infty) = \begin{cases} (q/p)x, & \text{if } p > 1/2, \\ 1, & \text{if } p \leq 1/2. \end{cases} \tag{2}$$