Mini project for the course
“Modern Control theory”

“Consensus of multi-agent systems with finite-time observation”

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Introduction

During last decades, systems of multiple agents have attracted the attention of the scientific society. Standard problems in multi-agents systems include consensus problems, formation control, and flocking theory **[1][2]**. Each one of these problems has its own applications, and its own challenges. However, the main idea of consensus control is to provide control laws for the agents that ensures an agreement between the agents states. On other words, the main idea of consensus control is to ensure the convergence of the differences between the agents states to zero, no matter what the actual value of the agents states are. **[3]** shows a theoretical framework for analysis of consensus algorithms for multi-agent networked systems.

Many control methods are suggested in literatures to deal with consensus problem of multiagent systems considering different scenarios, for example **[4][5][6]**. However, there is a noticeable interest in event triggered control of multiagent systems, because of its effectiveness in reduction of communication expenses and control cost **[7][8]**.

On the other side, finite time stability is an attractive domain of research. This type of stability imposes some advantages over the traditional asymptotic stability. In some practical applications the main concern is the behavior of systems over finite time interval. Most real neural systems only operate over finite time intervals. Moreover, finite time stability shows a better performance in the presence of uncertainties and disturbances **[9]**. In the domain of state estimation, finite-time observation is a simple way to realize the separation principle, the condition under which the control and the observation algorithms can be designed and analyzed independently **[10]**.

Finite time stability using implicit Lyapunov function (ILF) is considered recently in **[11][12][13]**. Finite time stability conditions in these works are obtained in the form of Linear matrix inequalities (LMIs), which provide simple constructive schemes for tuning the control parameters to prescribe the required convergence time and/or to ensure stability and robustness to perturbations of a certain magnitude. Finite time observation using implicit Lyapunov function (ILF) is considered in **[10]**, where finite time stability of the observation error is obtained in the form of Linear matrix inequalities (LMIs).

In this work we study consensus of multi-agent system with finite time control observation which is presented in **[14]**, where they construct a finite time observer using ILF (based on **[10]**) for states of each agent. Then the observations are transmitted between the agents at some triggered instants in the communication topology to achieve the consensus. In the rest of this work, we at first show a brief summarization of the definitions and theorems that are important for us in this work, then we show a detailed numerical example, checking of finite time observation proposed in **[14][10]**. Then, a check of input to state stability (ISS) property of the observation system (based on **[10]**) is considered numerically. Next, the performance with presence of delay is checked, which is based on note **[15]**. Finally, a conclusion is drawn with summarization of the results.

Theoretical part

- **Finite time stability using implicit Lyapunov function**

As the suggested observer of the system is design using implicit Lyapunov function, it worth to start with a definition of finite time stability and a representation of a theorem that provide finite time of a dynamic system. In this section, we change the traditionally used state vector notation from x to e, for the sake of alignment with the rest of the report, where the described definition and theorem will be applied in the context of the observation error dynamic.

Consider the system defined by

$$\dot{e}(t) = f(t, e(t)) \quad t \in \mathbb{R}_+, e(0) = e_0 \tag{1}$$

Where $e(t) \in \mathbb{R}^n$ and $f: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. Then the definition of finite time stability of system (1) is as follows:

Definition 1 (Definition 1 of [13]): The origin of system (1) is said to be globally uniformly finite-time stable if it is uniformly Lyapunov stable and finite-time attractive. On other words, for any solution $e(t, e_0)$ with $e_0 \in \mathbb{R}^n$, there exists a non-negative number $T(e_0) \geq 0$ such that $e(t, e_0) = 0$ for all $t \geq T(e_0)$. The non-negative number $T(e_0)$ is called the settling time.

Theorem 1 (Theorem 4 of [13]): if there exists a continuous function

$$\Phi: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}: (V, e) \to \Phi(V, e)$$

Satisfying the conditions

(C1) Φ is continuously differentiable outside the origin.

(C2) For any $e \in \mathbb{R}^n \setminus \{0\}$ there exists $V \in \mathbb{R}_+$ such that $\Phi(V, e) = 0$.

(C3) Let $\Omega = \{(V, e) \in \mathbb{R}_+ \times \mathbb{R}^n : \Phi(V, e) = 0\}$ then $\lim\limits_{\substack{e \to 0 \\ (V,e)\in\Omega}} V = 0^+$, $\lim\limits_{\substack{V \to 0^+ \\ (V,e)\in\Omega}} \| e \| = 0$ and $\lim\limits_{\substack{\|e\| \to 0 \\ (V,e)\in\Omega}} V = +\infty$.

(C4) $\dfrac{\partial \Phi(V, e)}{\partial V} < 0$ for all $V \in \mathbb{R}_+$ and $e \in \mathbb{R}^n \setminus \{0\}$.

(C5) There exists $c > 0$ and $0 < \mu \leq 1$ such that $\dfrac{\partial \Phi(V, e)}{\partial e} f(t, e) \leq cV^{1-\mu} \dfrac{\partial \Phi(V, e)}{\partial V}$ for all $(V, e) \in \Omega$.

then the origin of the system (1) is globally uniformly finite-time stable and $T(e_0) \leq \dfrac{V_0^\mu}{c\mu}$, where $\Phi(V_0, e_0) = 0$.

- **Model of the system**

The system considered is a high order multiagent system with N identical agents, where the dynamic of the agents is presented by

$$i = 1, 2, ..., N : \begin{cases} \dot{x}_i(t) = \mathbf{A}x_i(t) + \mathbf{B}u_i(t) \\ y_i(t) = \mathbf{C}x_i(t) \end{cases} \tag{2}$$

Where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{n\times 1}$$

$x_i(t) \in \mathbb{R}^n$, $y_i(t) \in \mathbb{R}$, and $u_i(t) \in \mathbb{R}$. The pair $(\mathbf{A}, \mathbf{C})$ are assumed to be observable. Then according to (**Lemma 11 in [10]**), there exists a nonsingular transformation Θ , such that:

$$\Theta\mathbf{A}\Theta^{-1} = \mathbf{F}\tilde{\mathbf{C}} + \tilde{\mathbf{A}}, \quad \mathbf{C}\Theta^{-1} = c_0 \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}}_{\tilde{\mathbf{C}}} = c_0\tilde{\mathbf{C}} \tag{3}$$

Where

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & \mathbf{A}_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{A}_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

And c_0 is a constant, and $\mathbf{F} \in \mathbb{R}^{n\times 1}$, and $\mathbf{A}_{j-1,j}$ are the entries of matrix $\tilde{\mathbf{A}}$ for $j = 2, ..., n$. On other words, the dynamic of the agents can be treated as the chain of integrators dynamics. The algorithm to find this transformation is shown in (**Appendix of [10]).**

- **Finite-time observer**

In this section, we represent the finite-time observer for the agents, while in the next section we show the proposed event triggered consensus control. In fact, in the original work **[14]**, both finite-time observer design and event triggered consensus control design are presented at one step, where they use Kronecker product notation ($\otimes$) to represent the implicit Lyapunov function for all the agents. Then they prove the finite time observation and the consensus stability in (**theorem 1 of [14]**). However, for the sake of simplicity, we divided the design process into two steps as the observer uses only local information for each agent. Moreover, the proof of finite time observation followed in (**theorem 1 of [14]**) is similar to proof of finite time observation in (**theorem 13 of**

[10]), where they only considered a system of a one dynamic (single agent), that is identical to the agent dynamic represented in (2).

The observer models of the agents are suggested by **[14]** as follows

$$i = 1, 2, ..., N : \dot{\hat{x}}_i(t) = \mathbf{A}\hat{x}_i(t) + \mathbf{B}u_i(t) - \phi(y_i - C\hat{x}_i(t)) \tag{4}$$

Where $\hat{x}_i(t) \in \mathbb{R}^n$ is the state vector of the observer i, and the function ϕ is

$$\phi : \mathbb{R} \to \mathbb{R}^n : \zeta \to \begin{cases} 0 & : \zeta = 0 \\ \Theta^{-1}(\Psi_{\bar{\upsilon}}(|\frac{c_0}{\alpha\zeta}|)\mathbf{H} - \mathbf{F})\frac{\zeta}{c_0} & : \zeta \neq 0 \end{cases} \tag{5}$$

Where c_0, and the matrices Θ, $\mathbf{F}$ are defined in (3). The matrix $\mathbf{H} \in \mathbb{R}^n$, which is an observation vector, and $\alpha > 0$, which is an observation constant, are determined by solving a group of LMIs to provide the finite time stability of the observer. Also, the vector $\bar{\upsilon}$ is defined as $\bar{\upsilon} = \begin{pmatrix} \frac{\mu}{1+(n-1)\mu} & \cdots & \frac{n\mu}{1+(n-1)\mu} \end{pmatrix}^T$. And $\Psi_{\bar{\upsilon}}(.)$ is defined as follows:

$$\Psi_{\bar{\upsilon}}(\theta) = diag\{\theta^{\frac{\mu}{1+(n-1)\mu}}, ..., \theta^{\frac{n\mu}{1+(n-1)\mu}}\} \tag{6}$$

According to (3) and the observer dynamic (4), the observation error for agents is:

$$i = 1, 2, ..., N : \dot{e}_i(t) = (\tilde{\mathbf{A}} + \Psi_{\bar{\upsilon}}(|\frac{1}{\alpha\tilde{\mathbf{C}}e_i(t)}|)\mathbf{H}\tilde{\mathbf{C}})e_i(t) \tag{7}$$

The finite time observation of the suggested observed is ensured by means of the following theorem.

Theorem 2 (first part of theorem 1 of [14]): Consider the MAS (2) with the observation system (4). If there exist some positive constants $\alpha > 0, \beta > 0, \rho > 0$, the matrices $\mathbf{P}$ and $\mathbf{S}$ such that the following inequalities hold:

$$\begin{pmatrix} \mathbf{P}(\tilde{\mathbf{A}} + \mathbf{H}\tilde{\mathbf{C}}) + (\tilde{\mathbf{A}} + \mathbf{H}\tilde{\mathbf{C}})^T\mathbf{P} + \rho\mathbf{P} + \beta(\mathcal{H}_\upsilon\mathbf{P} + \mathbf{P}\mathcal{H}_\upsilon) & \mathbf{P} \\ \mathbf{P} & -\rho\mathbf{S} \end{pmatrix} \leq 0 \tag{8}$$

$$\mathbf{P} > 0, \ \mathbf{S} \geq 0 \tag{9}$$

$$\mathbf{P} \geq \alpha^2\tilde{\mathbf{C}}^T\tilde{\mathbf{C}}, \ \mathcal{H}_\upsilon\mathbf{P} + \mathbf{P}\mathcal{H}_\upsilon > 0 \tag{10}$$

$$\mathbf{H}^T\Xi(v)\mathbf{S}\Xi(v)\mathbf{H} \leq \alpha^2 \quad : \forall v \in [0,1] \tag{11}$$

Where $\mathbf{P},\mathbf{S}\in\mathbb{R}^{n\times n}$, $\upsilon=\begin{pmatrix}1 & 1-\dfrac{\mu}{1+(n-1)\mu} & \cdots & 1-\dfrac{(n-1)\mu}{1+(n-1)\mu}\end{pmatrix}^T$, $\mathcal{H}_\upsilon=diag\{\upsilon_1,...,\upsilon_n\}$, and $\Xi(v)=\theta(\Psi_{\bar{\upsilon}}(\theta^{-1})-I_n)$. Then the observation system with achieves the finite-time observation.

***Note*:** It can be seen that the inequalities (8-11) don't represent linear matrix inequalities which make the solving them considering the unknown matrices $\mathbf{P},\mathbf{H},\mathbf{S}$ and the unknown observation gain α is difficult using LMI solvers. However, (**theorem 13 of [10]**) has a better representation of LMIs (8-11). We can find similar representation by defining the following unknown matrices

$$\begin{aligned} Y &= \mathbf{P}H \ : Y\in\mathbb{R}^n \\ X &= \alpha^2 \ : X\in\mathbb{R} \end{aligned} \tag{12}$$

Then using (12) and a design parameter $\tau\geq 1$we have the following:

(8) can be rewritten as $\begin{pmatrix}\mathbf{P}\tilde{\mathbf{A}}+Y\tilde{\mathbf{C}}+\tilde{\mathbf{A}}\mathbf{P}+\tilde{\mathbf{C}}^TY^T+\rho\mathbf{P}+\beta(\mathcal{H}_\upsilon\mathbf{P}+\mathbf{P}\mathcal{H}_\upsilon) & \mathbf{P} \\ \mathbf{P} & -\rho\mathbf{S}\end{pmatrix}\leq 0$ (13)

(9) can be rewritten as $\mathbf{P}>0,\ \mathbf{S}\geq 0, X>0$ (14)

(10) can be rewritten as $\mathbf{P}\geq\tilde{\mathbf{C}}^TX\tilde{\mathbf{C}},\ \mathcal{H}_\upsilon\mathbf{P}+\mathbf{P}\mathcal{H}_\upsilon>0$ (15)

(11) can be obtained if $\begin{pmatrix}\tau X & Y^T \\ Y & \mathbf{P}\end{pmatrix}\geq 0,\ \ \Xi(v)\mathbf{S}\Xi(v)\leq\dfrac{1}{\tau}\mathbf{P}\ :\forall v\in[0,1]$ (16)

One can see that (13-16) are the same LMIs shown in (**theorem 13 of [10]**), Which are feasible for sufficiently small $\mu>0$ as stated by (**Corollary 14 of [10]**). Moreover, (16) can be solved with chosen a proper grid of samples $v_j=\dfrac{j}{m}$ where $j\in\{0,1,...,m\}$ and ($m+1$) is the number of samples. This true due to the smoothness of $\Xi(v)$ with respect to the interval $]0,1]$.

At this point and before moving to the consensus control, we comment that the prove of **theorem 2** is carried out considering implicit Lyapunov function of the form:

$$\Phi(V(t),e(t))=e^T(t)[I_N\otimes(\Psi_\upsilon(V^{-1}(t))\mathbf{P}\Psi_\upsilon(V^{-1}(t)))]e(t)-1 \tag{17}$$

Where $\otimes$ is Kronecker product, $e=\begin{pmatrix}e_1^T & e_2^T & \cdots & e_N^T\end{pmatrix}^T$, and $\Psi_\upsilon(.)$ is defined as in (6), with the vector $\upsilon=\begin{pmatrix}1 & 1-\dfrac{\mu}{\mu+(n-1)\mu} & \cdots & 1-\dfrac{(n-1)\mu}{\mu+(n-1)\mu}\end{pmatrix}^T$ instead of $\bar{\upsilon}$. Then the conditions (C1-C3) of **Theorem 1** is proven to be satisfied considering inequalities results form the maximum and minimum eigenvalue of the matrices $\mathbf{P}$ and $\Psi_\upsilon(V^{-1}(t))$. Then the second inequality of (10) is used to prove that (C4) of **Theorem 1** is satisfied. Finally, the other inequalities in of **Theorem 2** are used to prove that

$$\frac{\partial\Phi(V,e)}{\partial e}\dot{e} \le \beta V^{1-\frac{\mu}{1+(n-1)\mu}} \frac{\partial\Phi(V,e)}{\partial V} \tag{18}$$

Which provide finite time observation with settling time

$$T(e_0) \le \frac{V_0^{\frac{\mu}{1+(n-1)\mu}}}{\beta(\frac{\mu}{1+(n-1)\mu})} \tag{19}$$

Where $\Phi(V_0, e_0) = 0$.

- **Consensus control and event-triggered protocol**

In this domain the graph that connects between the agents G is considered undirected and connected. The Laplacian matrix of the graph is defined as:

$$L = [l_{ij}] = \begin{cases} \sum_{k\in\mathcal{N}_i} a_{ik} & : i = j \\ -a_{ij} & : i \ne j \ and \ (i,j) \in \mathcal{E} \\ 0 & : i \ne j \ and \ (i,j) \notin \mathcal{E} \end{cases} \tag{20}$$

Where a_{ij} is the weight of the connection $(i, j) \in \mathcal{E}$, and $\mathcal{E}$ is a group pairs (i, j) that represents the connection between the agents, such that the pair (i, j) means that agent (i) can receive information of agent (j). $\mathcal{N}_i$ is neighbors group of agent (i). According to **[16]**, the Laplacian matrix defined in (20) is symmetric positive semidefinite with simple eigenvalue equal zero, when the graph is undirected and connected. On other words, L has only one zero eigenvalue and the other eigenvalue are strictly positive.

The proposed control law for the consensus is as follows:

$$i = 1,2,...,N : u_i(t) = \mathbf{e}^{-\eta(t-t_k)} K \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{x}_j(t_k) - \hat{x}_i(t_k)) \quad t \in [t_k, t_{k+1}[\tag{21}$$

Where t_k is a trigger instant that is chosen based on a trigger mechanism we will show later, and $\hat{x}_i(t_k)$ is the observed state vector of agent (i) at trigger instant t_k. $\eta > 0$ and $K \in \mathbb{R}^{1\times n}$ is gain matrix to be designed.

To define the trigger function, it is needed to define the following:

- Consensus error for agent (i):

$$\tilde{x}_i(t) = x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(t) \quad : \tilde{x}_i(t) \in \mathbb{R}^n \tag{22}$$

- Consensus error vector for all the agents:

$$\tilde{x}(t) = \left(\tilde{x}_1^T(t) \quad \tilde{x}_2^T(t) \quad \cdots \quad \tilde{x}_N^T(t) \right)^T \quad : \tilde{x}(t) \in \mathbb{R}^{(N*n)} \tag{23}$$

- Error between actual local observed consensus error and local consensus error based on observed state at trigger instant t_k :

$$\omega_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{x}_j(t) - \hat{x}_i(t)) - \mathbf{e}^{-\eta(t-t_k)} \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{x}_j(t_k) - \hat{x}_i(t_k)) \quad t \in [t_k, t_{k+1}[\tag{24}$$

$$\omega(t) = \left(\omega_1^T(t) \quad \omega_2^T(t) \quad \cdots \quad \omega_N^T(t) \right)^T \tag{25}$$

Then, the trigger function is defined as follows:

$$f(\omega(t), \tilde{x}(t)) = \| \omega(t) \| - \frac{\sigma\varepsilon}{2\| \mathcal{P}\mathbf{B}\mathbf{B}^T\mathcal{P} \|} \| \tilde{x}(t) \| - \frac{\varepsilon_0}{2\| \mathcal{P}\mathbf{B}\mathbf{B}^T\mathcal{P} \|} \tag{26}$$

Where $\varepsilon, \varepsilon_0 > 0$, $\sigma \in]0,1[$, and $\mathcal{P}$ is a symmetric positive matrix to be determined. Then, the trigger instants are time instants that corresponds to $f(\omega(t), \tilde{x}(t)) = 0$.

Theorem 3 (second part of theorem 1 of [14]): Consider the MAS (2) with the finite time observation system (4) ,where conditions of theorem 2 are satisfied. Then the trigger function in (26) with the control input (21), will ensure an exponential convergence of the consensus error vector $\tilde{x}(t)$ to a bounded region around the origin giving that:

$$\mathcal{P}\mathbf{A} + \mathbf{A}^T\mathcal{P} - 2\lambda_2(L)\mathcal{P}\mathbf{B}\mathbf{B}^T\mathcal{P} + \varepsilon I_n < 0, \mathcal{P} > 0 \tag{27}$$

Where $\lambda_2(L)$ is the first strictly positive eigenvalue of the Laplacian matrix, and $K = \mathbf{B}^T\mathcal{P}$ in (21). And the consensus error vector will be bounded in following region:

$$M = \{\tilde{x}(t) \in \mathbb{R}^{(N*n)} : \| \tilde{x}(t) \| \leq \frac{\varepsilon_0 \lambda_{\max}(\mathcal{P})}{(1-\sigma)\varepsilon\lambda_{\min}(\mathcal{P})}\} \tag{28}$$

It worth to mention that inequality (27), can be rewritten in the form of linear matrix inequality as follows:

$$\begin{pmatrix} 2\lambda_2(L)\mathbf{B}\mathbf{B}^T - \mathbf{A}X - X\mathbf{A}^T & \sqrt{\varepsilon}X \\ \sqrt{\varepsilon}X & I_n \end{pmatrix} > 0, X > 0 \tag{29}$$

Where $X = \mathcal{P}^{-1}$.

Before moving to the practical part, it worth to mention the following definitions, and expressions:

- Vector of states of all agents:

$$x(t) = \begin{pmatrix} x_1^T(t) & x_2^T(t) & \cdots & x_N^T(t) \end{pmatrix}^T \quad : x(t) \in \mathbb{R}^{(N*n)} \tag{30}$$

- Vector of observed states for all agents:

$$\hat{x}(t) = \begin{pmatrix} \hat{x}_1^T(t) & \hat{x}_2^T(t) & \cdots & \hat{x}_N^T(t) \end{pmatrix}^T \quad : \hat{x}(t) \in \mathbb{R}^{(N*n)} \tag{31}$$

- Vector of control input for all agents:

$$u(t) = \begin{pmatrix} u_1(t) & u_2(t) & \cdots & u_N(t) \end{pmatrix}^T \quad : u(t) \in \mathbb{R}^{N} \tag{32}$$

According to definitions (30-32) together with definition (22-25), expressions for vector quantities for all agents together can be obtained using Kronecker product notations as follows:

$$\tilde{x}(t) = ((I_N - \frac{1}{N}\mathbf{1}_{N\times N}) \otimes I_n)x(t) \tag{33}$$

$$u(t) = -\mathbf{e}^{-\eta(t-t_k)}(L \otimes K)\hat{x}(t_k) \tag{34}$$

$$\omega(t) = -(L \otimes I_n)\hat{x}(t) + \mathbf{e}^{-\eta(t-t_k)}(L \otimes I_n)\hat{x}(t_k) = (L \otimes I_n)(-\hat{x}(t) + \mathbf{e}^{-\eta(t-t_k)}\hat{x}(t_k)) \quad t \in [t_k, t_{k+1}[\tag{35}$$

Implementation

To check the effectiveness of the suggested control method we simulate a MAS system of five agents where, the agents dynamic:

$$i = 1, 2, ..., 5 : \begin{cases} \dot{x}_i(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathbf{A}} x_i(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{B}} u_i(t) \\ y_i(t) = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{\mathbf{C}} x_i(t) \end{cases} \tag{36}$$

And the communication graph between the agent is shown in **Figure 1**. The agents model is observable because:

$$Rank\begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \end{pmatrix} = Rank\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 2 \tag{37}$$

Then the bock decomposition can be applied to get the equivalent chain of integrators system. The equivalent chain of integrator system is obtained with the following transformation:

$$\Theta = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \quad , \quad \mathbf{C}\Theta^{-1} = \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} c_0 & 0 \end{pmatrix} \; , \; \Theta\mathbf{A}\Theta^{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \tag{38}$$

Because $\Theta\mathbf{A}\Theta^{-1}$ has the chain of integrators form we can just choose $\mathbf{F} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$, and get:

$$\tilde{\mathbf{C}} = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad , \quad c_0 = 1 \quad , \quad \tilde{\mathbf{A}} = \Theta \mathbf{A} \Theta^{-1} - \mathbf{F}\tilde{\mathbf{C}} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \tag{39}$$

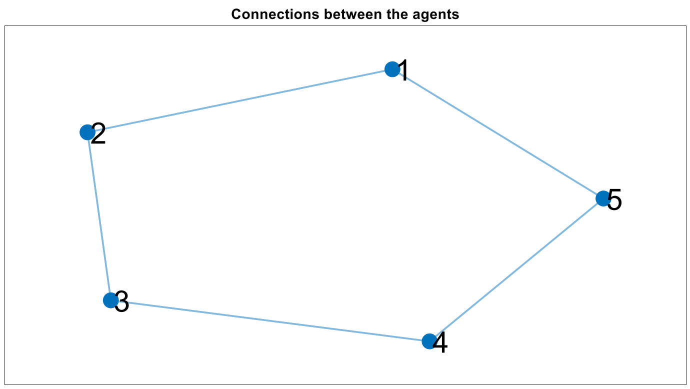

Figure 1: Connections graph between agents

The instructions, by which the model of the agent is defined, and the transformation matrix, are calculated is shown in **Script 1**. It also shows the initial state vectors for the agents

$$x_1(0) = \begin{pmatrix} 1 & 2 \end{pmatrix}, x_2(0) = \begin{pmatrix} -1 & 2 \end{pmatrix}, x_3(0) = \begin{pmatrix} 3 & -2 \end{pmatrix}, x_4(0) = \begin{pmatrix} -2 & 3 \end{pmatrix}, x_5(0) = \begin{pmatrix} 6 & 0 \end{pmatrix} \tag{40}$$

And the parameters (on_off) which is used to add or delete the noise of the system, (s) which is the variance of the additive noise on the measurement channel, and T_d which is the value of the delay on the measurement channel.

```
%% Definition of connections between the agents
N=5; %number of agents
G1=[1 1 2 3 4];
G2=[2 5 3 4 5];
G=graph(G1,G2);
% Agent model
A=[0 1;0 0];
C=[1 1];
B=[0;1];
rank([C;C*A;C*A*A]) % Check observability
%initial values
x1_0=[1;2];
x2_0=[-1;2];
x3_0=[3;-2];
x4_0=[-2;3];
x5_0=[6;0];
on_off=0;%parameter to add noise or delete it
s=sqrt(0.1);%noise standard deviation
Td=0.5%delay value
%% transformation to get chain of integrator form
TITA=[1 1;0 -1];
Ac=TITA*A/TITA;
Cc=C/TITA;
c0=Cc(1);
F=Ac(:,1);
C_tilda=[1 0];
A_tilda=[0 Ac(1,2);0 0];
```

Script 1: Multiagent system model and transformation calculation.

To solve the LMIs (13-16), we choose the following constants:

$$\beta = 0.1 \ , \ \mu = 0.4 \ , \ \rho = 0.6 \ , \ \tau = 1.2 \tag{41}$$

The instruction use to solve the LMIs (13-16) is shown in **Script 2**. As a result, we get the following matrices:

$$\begin{gathered} \mathbf{P} = \begin{pmatrix} 6.7690 & 4.2092 \\ 4.2092 & 7.6933 \end{pmatrix}, \ \mathbf{S} = \begin{pmatrix} 66.8517 & 24.6332 \\ 24.6332 & 37.5851 \end{pmatrix} \ , \ Y = \begin{pmatrix} -3.3761 \\ 2.1642 \end{pmatrix} \\ X = 4.4269 \ , \ \alpha = \sqrt{X} = 2.1040 \ , \ \mathbf{H} = \mathbf{P}^{-1} Y = \begin{pmatrix} -1.0211 \\ 0.8400 \end{pmatrix} \end{gathered} \tag{42}$$

To worth to mention that to solve the LMI ($\Xi(v)\mathbf{S}\Xi(v) \le \frac{1}{\tau}\mathbf{P} \ : \forall v \in [0,1]$) we use the recommendation of **[10]** , where we replaced it by the group of LMIs ($\Xi(\frac{j}{100})\mathbf{S}\Xi(\frac{j}{100}) \le \frac{1}{\tau}\mathbf{P}$ $: j \in \{0,1,...,100\}$) , then checked that it is satisfied with more accurate sampling step which is equal to (10^{-5}). The maximum eigenvalue of the matrix $(\Xi(v)\mathbf{S}\Xi(v) - \frac{1}{\tau}\mathbf{P})$ is shown in **Figure 2**, where it is shown to be always negative.

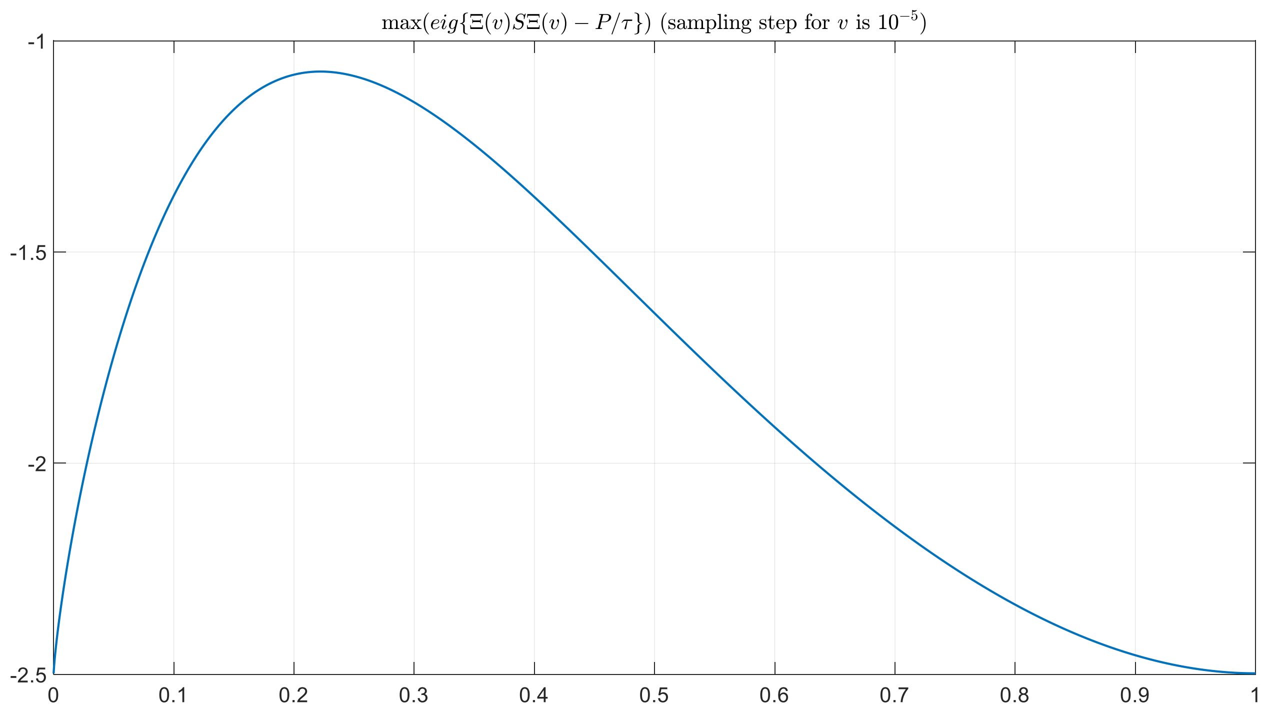

Figure 2: maximum eigenvalue of the matrix $(\Xi(v)S\Xi\ (v) - \frac{P}{\tau})$

```
%% Solution of LMI to for finite time stability of the observer
%Parameters
beta=0.1;
mui=0.4;
row=0.6;
tou=1.2;

Hv=[1 0;0 1-mui/(1+mui)];
E=@(x)(([x^(1-mui/(1+mui)) 0;0 x^(1-2*mui/(1+mui))]-x*eye(2)));

clear LMIs
P=sdpvar(2,2);
S=sdpvar(2,2);
Y=sdpvar(2,1);
alpha_2=sdpvar(1,1);% it is named as X in the report
LMIs=[P>=eye(2)*0.1];
LMIs=LMIs+[S>=eye(2)*0.1];
LMIs=LMIs+[alpha_2>=0.1];
LMIs=LMIs+[[P*A_tilda+Y*C_tilda+A_tilda'*P+C_tilda'*Y'+row*P+beta*(Hv*P+P*Hv),P;P,-
row*S]<=0];
LMIs=LMIs+[(P-C_tilda'*alpha_2*C_tilda)>=0];
LMIs=LMIs+[(Hv*P+P*Hv)>=eye(2)];
LMIs=LMIs+[[tou*alpha_2,Y';Y,P]>=0];
for v=0:0.01:1
LMIs=LMIs+[(E(v)*S*E(v)-P/tou)<=0];
end
solvesdp(LMIs);
P=double(P)
S=double(S)
Y=double(Y)
alpha_2=double(alpha_2)
alpha=sqrt(alpha_2)
H=P\Y
%Check that ((E(v)*S*E(v)-P/tou)<=0) for more accurate grid
o=zeros(100001,1);
i=0;
for v=0:1e-5:1
    i=i+1;
    o(i)=max(eig((E(v)*S*E(v)-P/tou)));
end
figure();
plot([0:1e-5:1],o,'LineWidth',2);
grid on;
title('$$\max (eig\{ \Xi (v)S\Xi (v) - P/\tau \} )$$ (sampling step for $$v$$ is $$10^{-
5}$$)', 'Interpreter', 'LaTeX','FontSize',20);
set(gca,'FontSize',20)
```

Script 2: Instructions to solve the LMIs that define the observer.

To solve the LMIs that define the consensus control law, we choose $\varepsilon = 0.1$, and as a result we find:

$$\mathcal{P} = \begin{pmatrix} 0.9896 & 0.9896 \\ 0.9896 & 2.6389 \end{pmatrix},\ K = \mathbf{B}^T \mathcal{P} = \begin{pmatrix} 0.9896 & 2.6389 \end{pmatrix} \tag{43}$$

Instruction to solve the LMIs that define the consensus control law is shown in **Script 3**.

```
%% Solution of LMI to for consensus convergence
%Parameters
ep=0.1;
L=full(laplacian(G));%Laplacian matrix of the graph
Lambda2=[0 1 0 0 0]*eig(L);%second eigenvalue of the laplacian matrix
X=sdpvar(2,2);
clear LMIs;
LMIs=[X>=0];
LMIs=LMIs+[[2*Lambda2*B*B'-A*X-X*A',sqrt(ep)*X;sqrt(ep)*X,eye(2)]>=0];
solvesdp(LMIs);
P2=inv(double(X));
K=B'*P2; %Feedback gain
```

Script 3: Instructions to solve the LMIs that define consensus control.

For the trigger function we choose the following parameters $\varepsilon_0 = 0.1$, $\sigma = 0.98$, and we also choose the constant $\eta = 0.1$ for exponential speed by which the control signal vanishes over the interval $[t_k, t_{k+1}[$. Definition of the trigger function constants is shown in **Script 4**, where also there is shown the definition of the matrices $((I_N - \frac{1}{N}\mathbf{1}_{N\times N})\otimes I_n)$, $(L\otimes K)$, and $(L\otimes I_n)$, that simplify the model in Simulink.

```
%% Matrices and parameters used to define the triggering function
ep0=0.1;
sigma=0.98;
eta=0.1;
L_kron_eye2=kron(L,eye(2));
M=(5*eye(5)-ones(5,5))/5;
M_kron_eye2=kron(M,eye(2));
L_kron_K=kron(L,K);
c1=sigma*ep/(2*norm(P2*B*B'*P2))
c2=ep0/(2*norm(P2*B*B'*P2))
```

Script 4: Definition of trigger function constants and some matrices to simplify the model in Simulink.

Before moving to Simulink model, we represent the instruction that are used to calculate the maximum expected settling time of the observer for each agent which is shown in **Script 5**. The calculation of the maximum expected settling time is done using expression (19), where the calculation of the initial value of implicit Lyapunov function is done using (**Algorithm 10 of [13]**) .The following settling times are found:

$$\text{maximum expected settling time of the first agent observer } \max(T_1) = 73.3233\,\text{sec} \tag{44}$$

$$\text{maximum expected settling time of the second agent observer } \max(T_2) = 65.9445\,\text{sec} \tag{45}$$

$$\text{maximum expected settling time of the third agent observer } \max(T_3) = 64.5271\,\text{sec} \tag{46}$$

$$\text{maximum expected settling time of the fourth agent observer } \max(T_4) = 77.1057\,\text{sec} \tag{47}$$

$$\text{maximum expected settling time of the fifth agent observer } \max(T_5) = 76.7439\,\text{sec} \tag{48}$$

```
%% Maximum time for finite time observation
e0=[1;2];%for first agent
%e0=[-1;2];%for second agent
%e0=[3;-2];%for third agent
%e0=[-2;3];%for fourth agent
%e0=[6;0];%for fifth agent
f=@(x)(e0'*[x^(-1) 0;0 x^(-1+mui/(1+mui))]*P*[x^(-1) 0;0 x^(-1+mui/(1+mui))]*e0-1);
Vmin=0.000005;
a=0.001;
b=20;
for i=1:1:1000
if(f(b)>0)
    a=b;b=2*b;
elseif (f(a)<0)
    b=a;a=max(a/2,Vmin);
else
    c=(a+b)/2;
    if(f(c)<0)
        b=c;
    else
        a=max(c,Vmin);
    end
end
if (abs(b-a)<0.000001)
    break;
end
end
V=b;
Te0=V^(mui/(1+mui))/(mui/(1+mui)*beta)
```

Script 5: Calculation for settling time.

Simulink model of the system is shown in **Figure 5**. Agent model is shown in **Figure 3**, while observer model is shown in **Figure 4**.

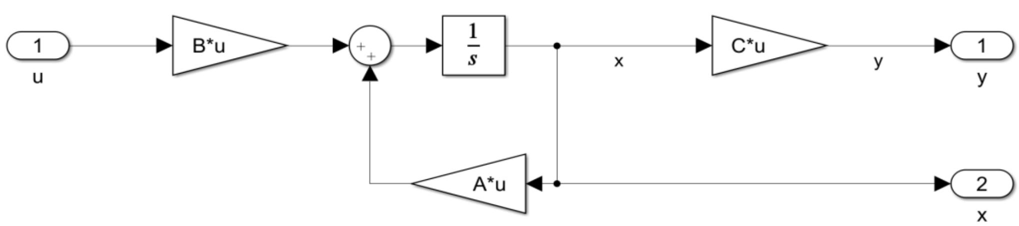

Figure 3: Agent model.

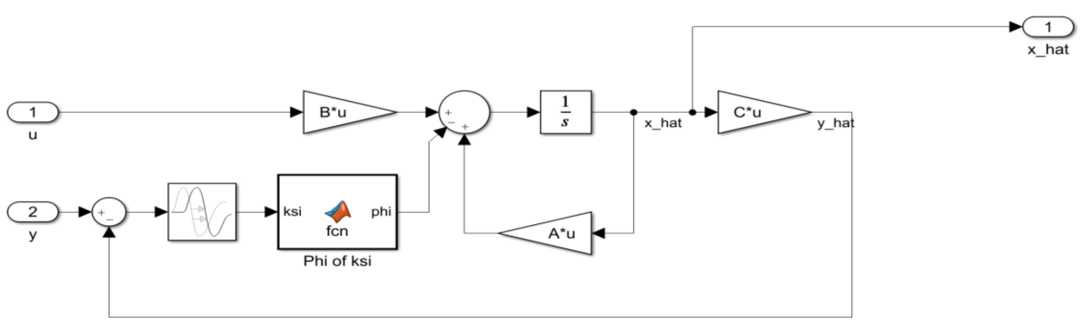

Figure 4: Observer model

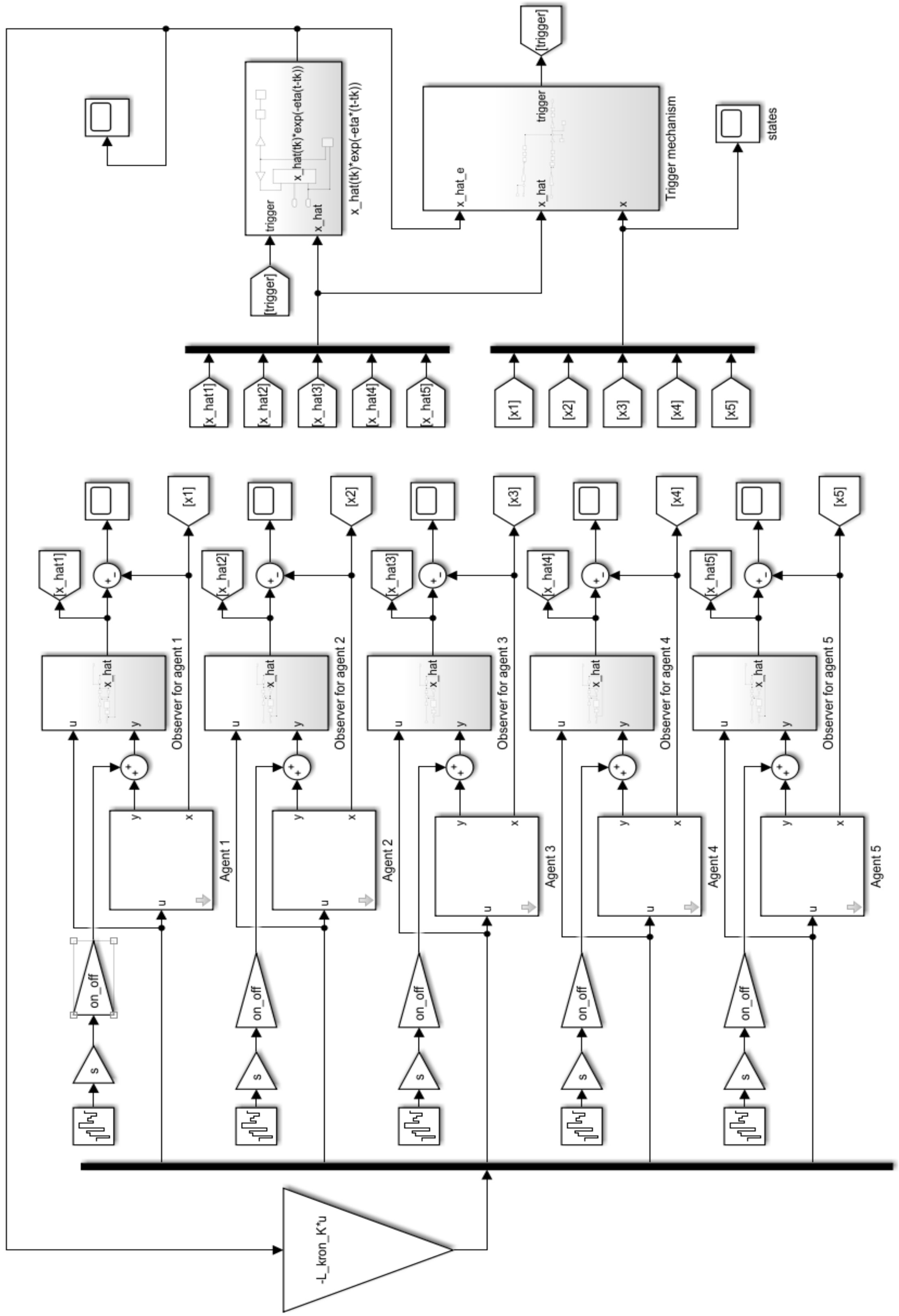

Figure 5: Simulation model of the system in Simulink.

The instructions that are used in MATLAB function ($\phi(\zeta)$) in the observer model are shown in ***Script 6***.

```
function phi = fcn(ksi,c0, alpha, TITA, mui, H, F)
if(ksi~=0)
    x=abs(c0)/alpha./abs(ksi);
    phi=TITA\([x^(mui/(1+mui)) 0;0 x^(2*mui/(1+mui))]*H-F)*ksi/c0;
else
    phi=[0;0];
end
```

Script 6 Calculation of function (ϕ (ζ)) in observer model.

The trigger mechanism block is shown in **Figure 6**, where we considered that the trigger function is equal zero when its value become less than 10^{-9}. The definition of the constants c_1, c_2 can be found in **Script 4**. The block that computes $\mathbf{e}^{-\eta(t-t_k)}\hat{x}(t_k)$ is shown in **Figure 7**.

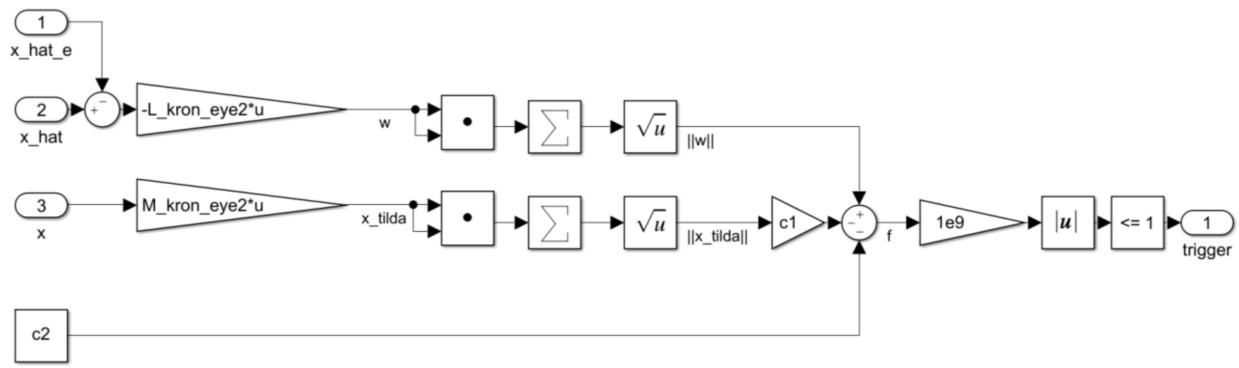

Figure 6: Trigger mechanism block.

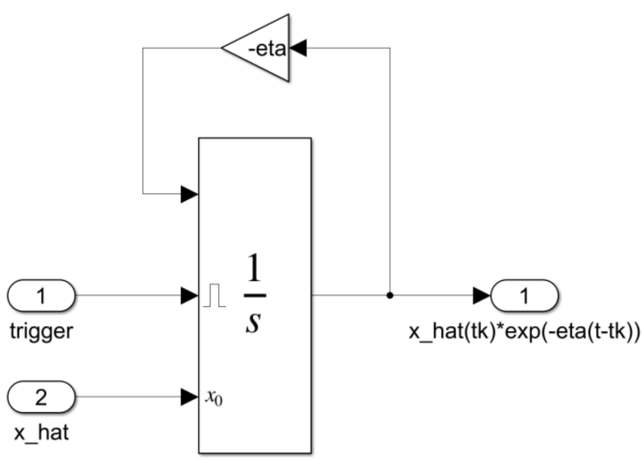

Figure 7: Block computes $e^{-\eta(t-t_k)}\,\hat{x}(t_k)$

Simulation results

- **Noise-free and delay-free simulation**

In this section, we simulate the system without any noise and without any delay in the output channel. Simulation results of the observer error for the first and second state of the agents are shown in **Figure 8** and **Figure 9**, respectively.

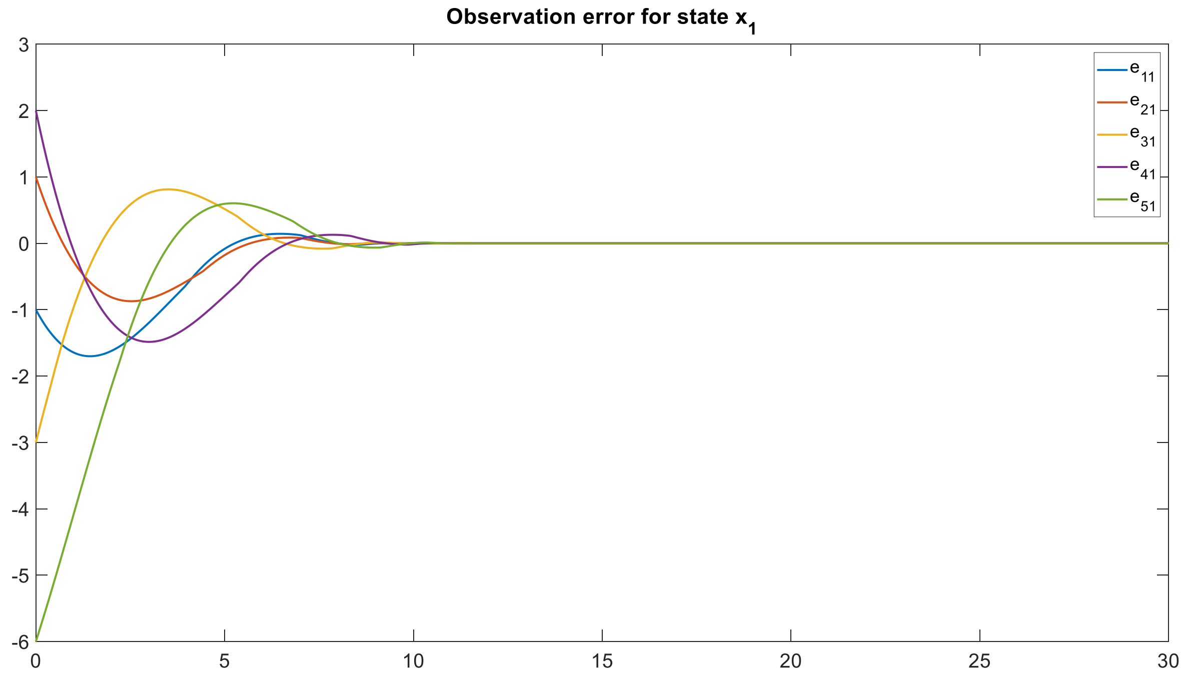

Figure 8: observation error for the first state.

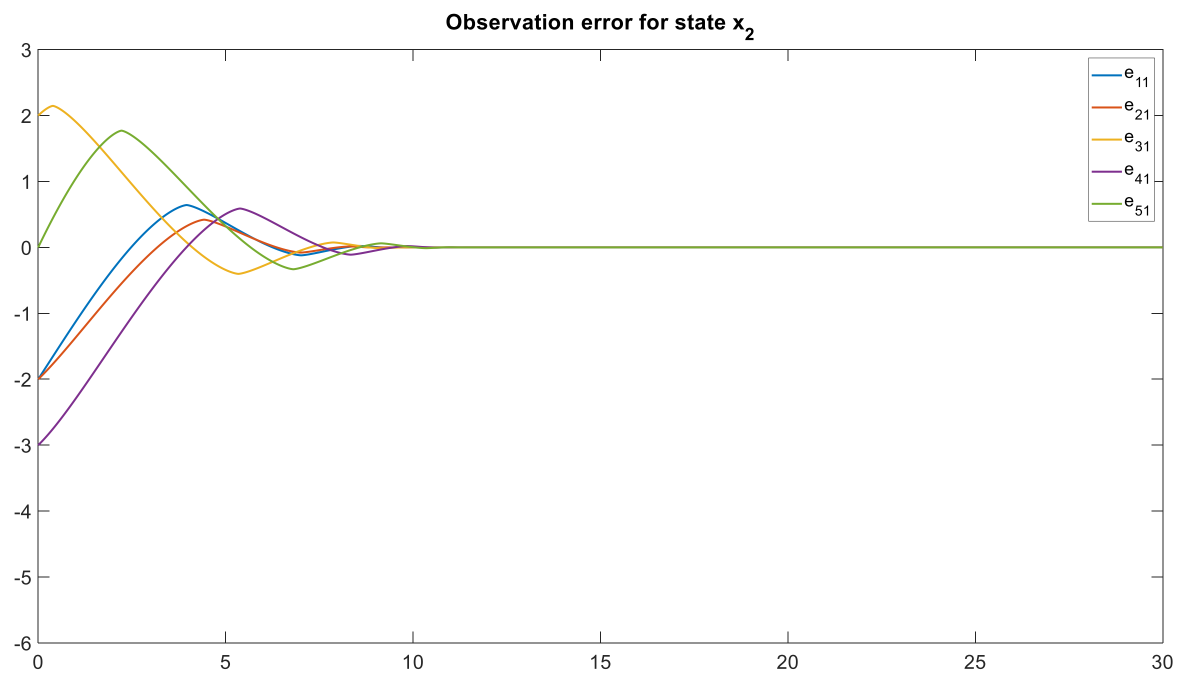

Figure 9: observation error for the second state.

The observation error for the first state and second state using logarithmic scale are shown in **Figure 10** and **Figure 11**, respectively.

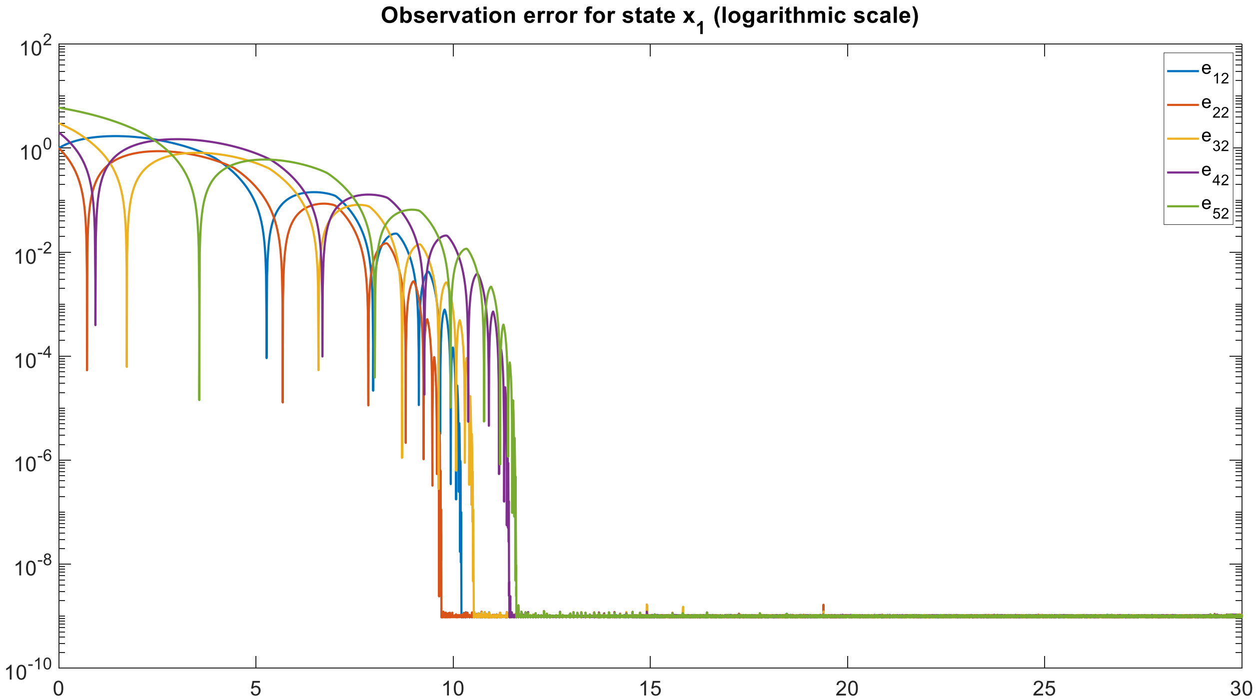

Figure 10: observation error for the first state (logarithmic scale).

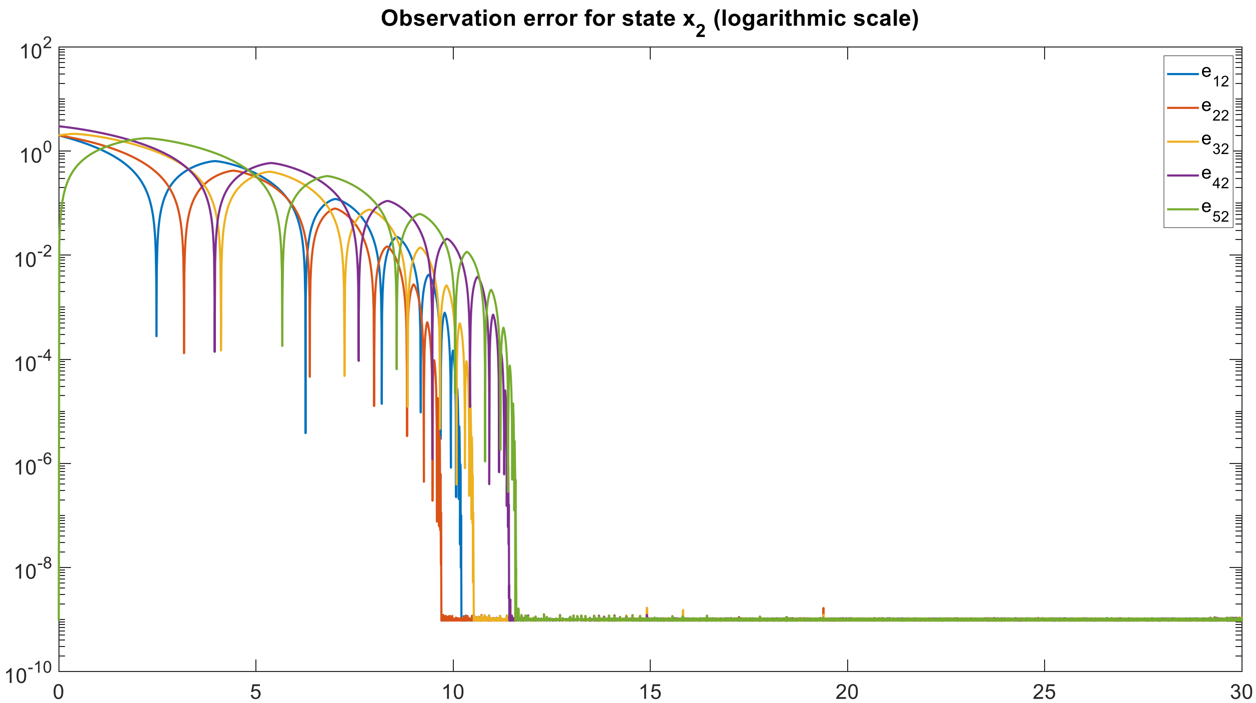

Figure 11: observation error for the second state (logarithmic scale).

we can see that the finite time observation is achieved with settling times less than the maximum settling time were shown in (44-48). We can see from **Figure 10** and **Figure 11** that settling time is less than 15 sec, while we found previously that $\max(T_1) = 73.3233\sec$, $\max(T_2) = 65.9445\sec$, $\max(T_3) = 64.5271\sec$, $\max(T_4) = 77.1057\sec$ and $\max(T_5) = 76.7439\sec$. Trigger instants are shown in **Figure 12**.

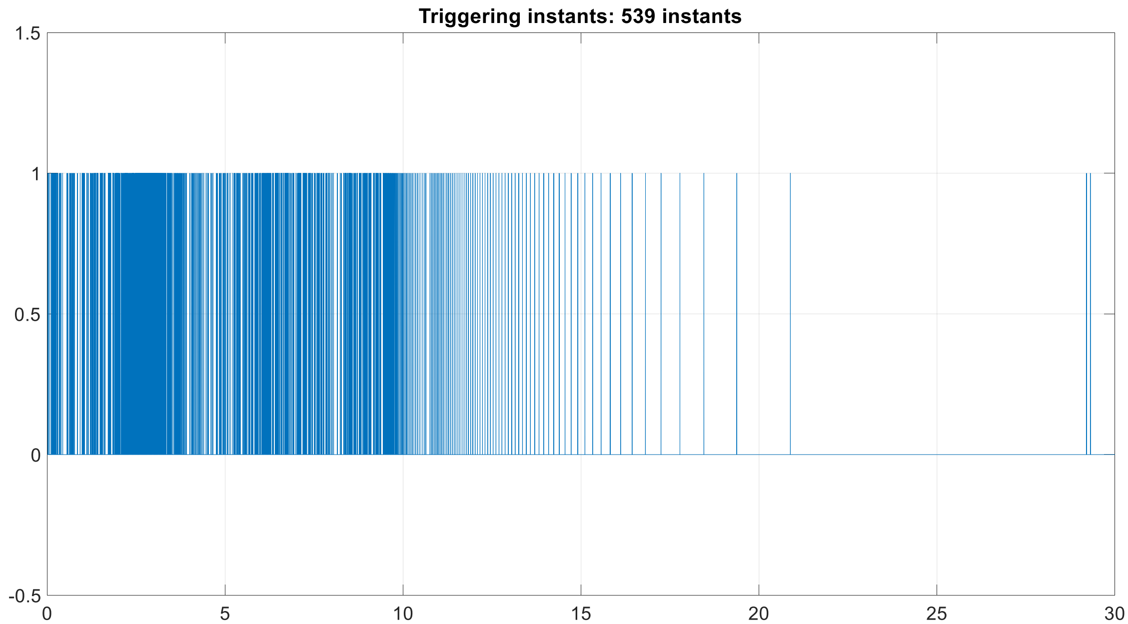

Figure 12: Trigger instants.

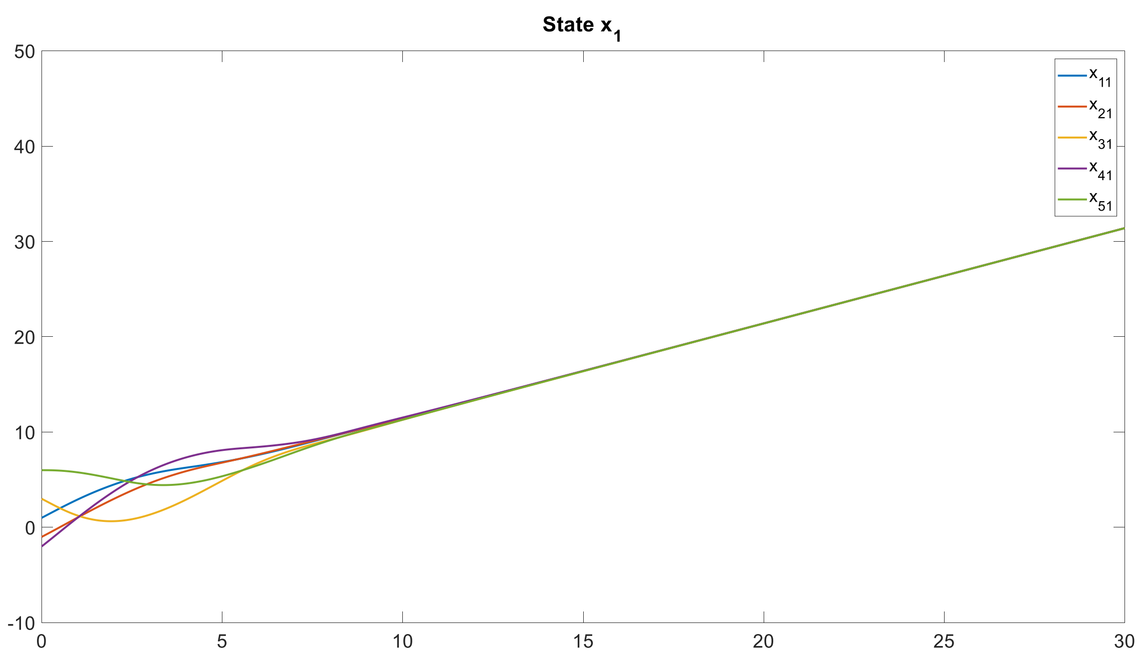

Figure 13: First state of the agents.

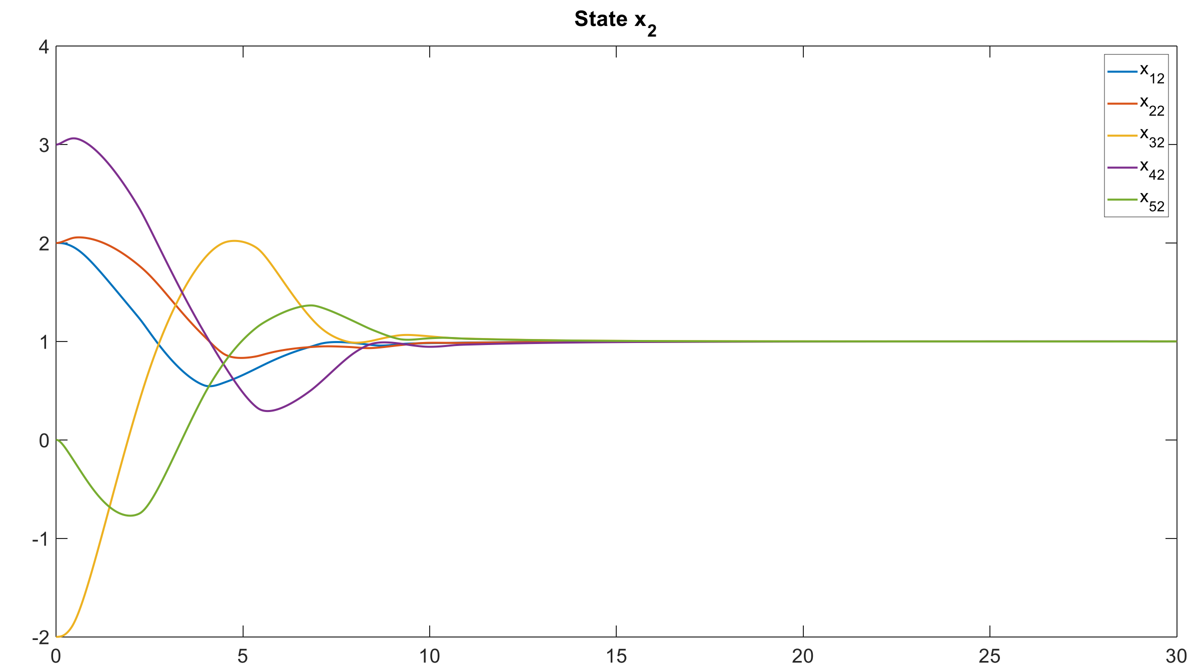

Figure 14: Second state of the agents.

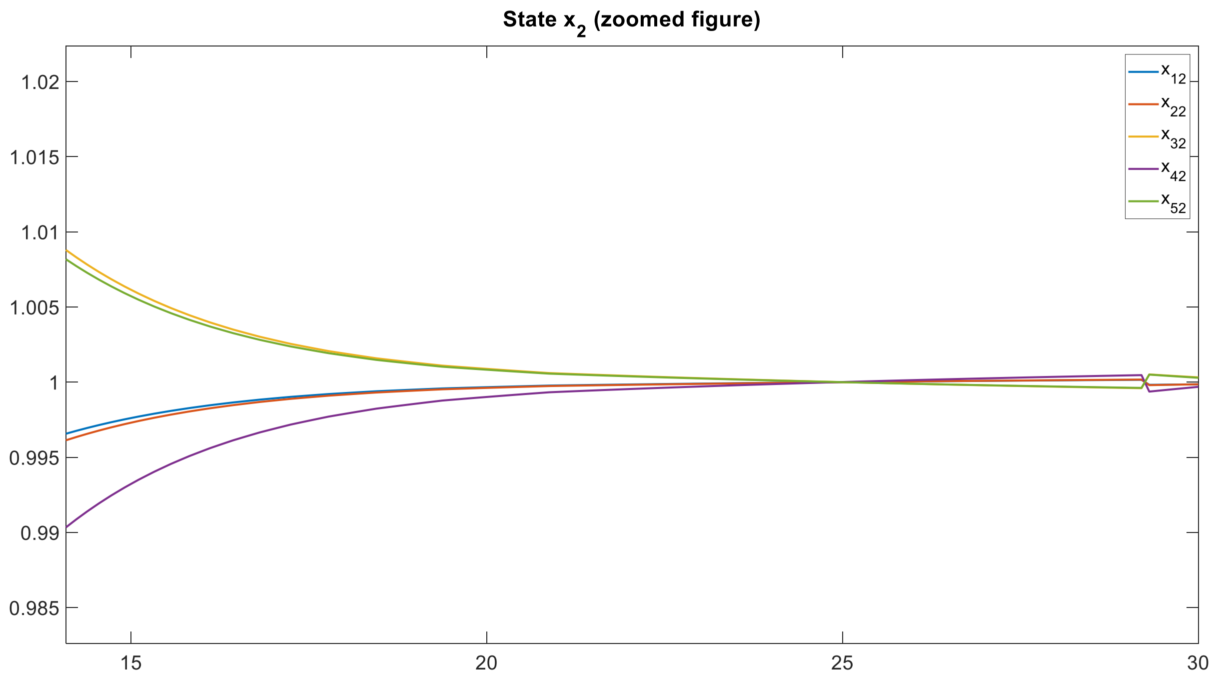

Figure 15: second state of the agents (zoomed figure)

The first state and second state variables of the agents are shown in **Figure 13** and **Figure 14** respectively. The second state variable of the agents is shown in **Figure 15** with more focus on the steady state region. We can see that the consensus error converges to a bounded region around zero consensus error. Simulation results of the observer error for the first and second state of the agents are shown in and, respectively.

- **Simulation with additive noise in output channel**

In this section, we consider the existence of an additive noise in the output channels for the agents. As we choose $\mu = 0.4$, then according to (**Corollary 18 of [10]**), the error dynamic of the observer is ISS with respect to additional perturbations in the measurement channel. We choose the value band-limited white noise with variance $s^2 = 0.1$.

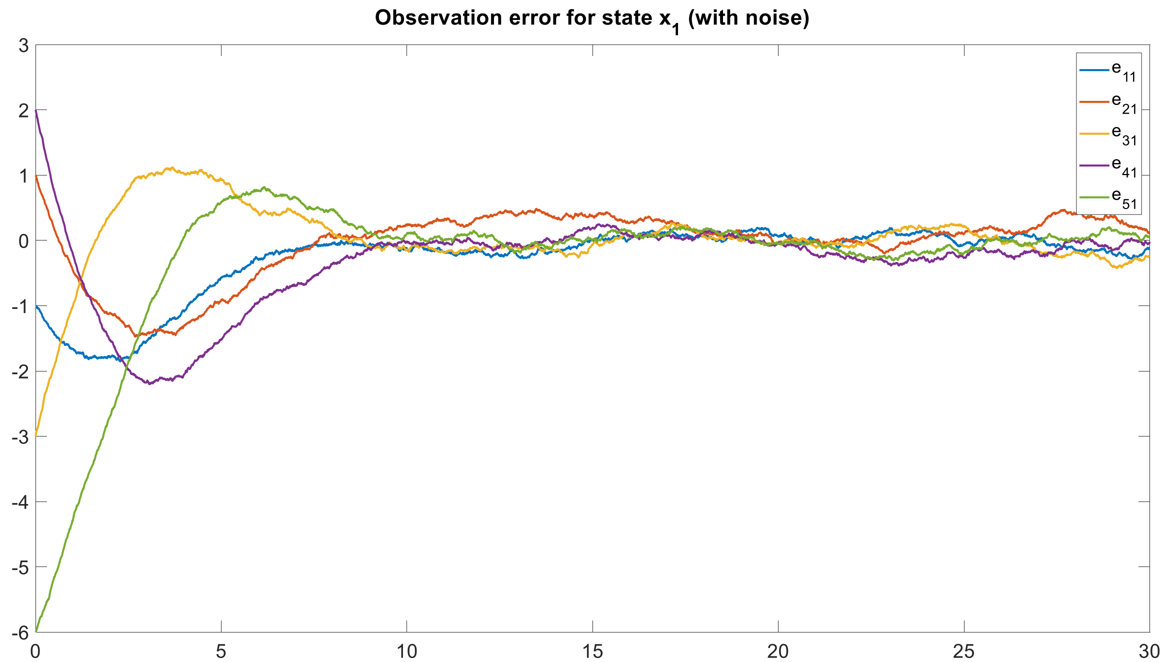

Figure 16: observation error for the first state (with noise).

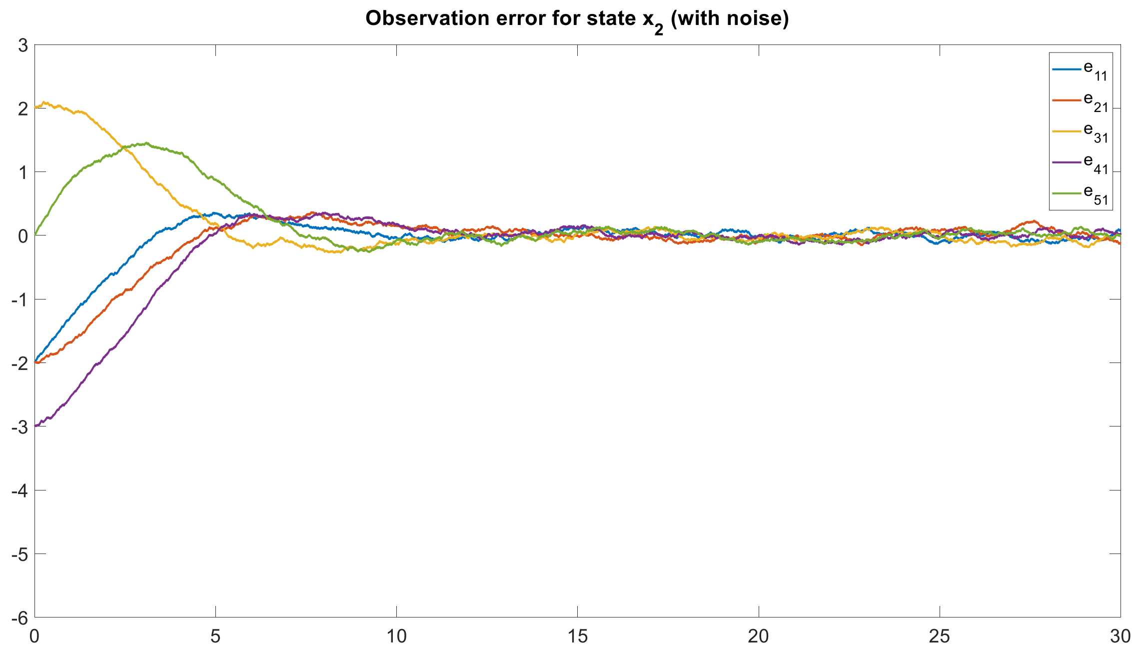

Figure 17: observation error for the second state (with noise).

We can see that the system is robust to the noise and the observation error is bounded around zero.

Trigger instants with additive noise in the output channel are shown in **Figure 18**. Comparing with noise-free case, that was shown in **Figure 12**, we can see a huge increases in the communication instants between the agents.

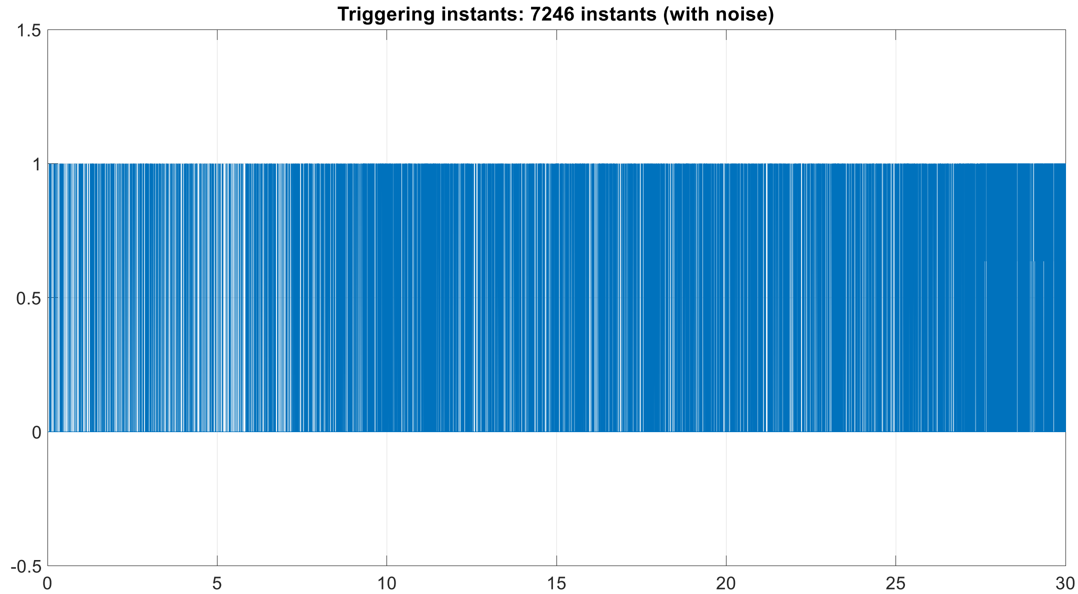

Figure 18: Trigger instants (with noise).

First and second state variables of the agents are shown in **Figure 19** and **Figure 20**, respectively. We can see that the area around zero consensus error is bigger than the one for noise-free case.

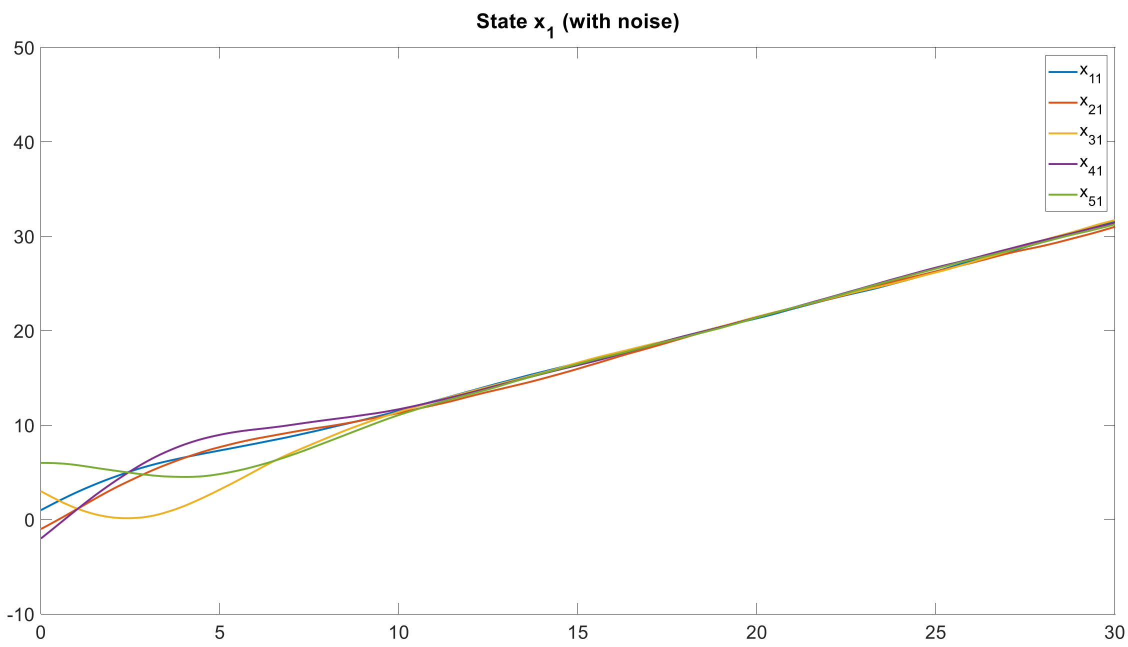

Figure 19: Second state variable (with noise).

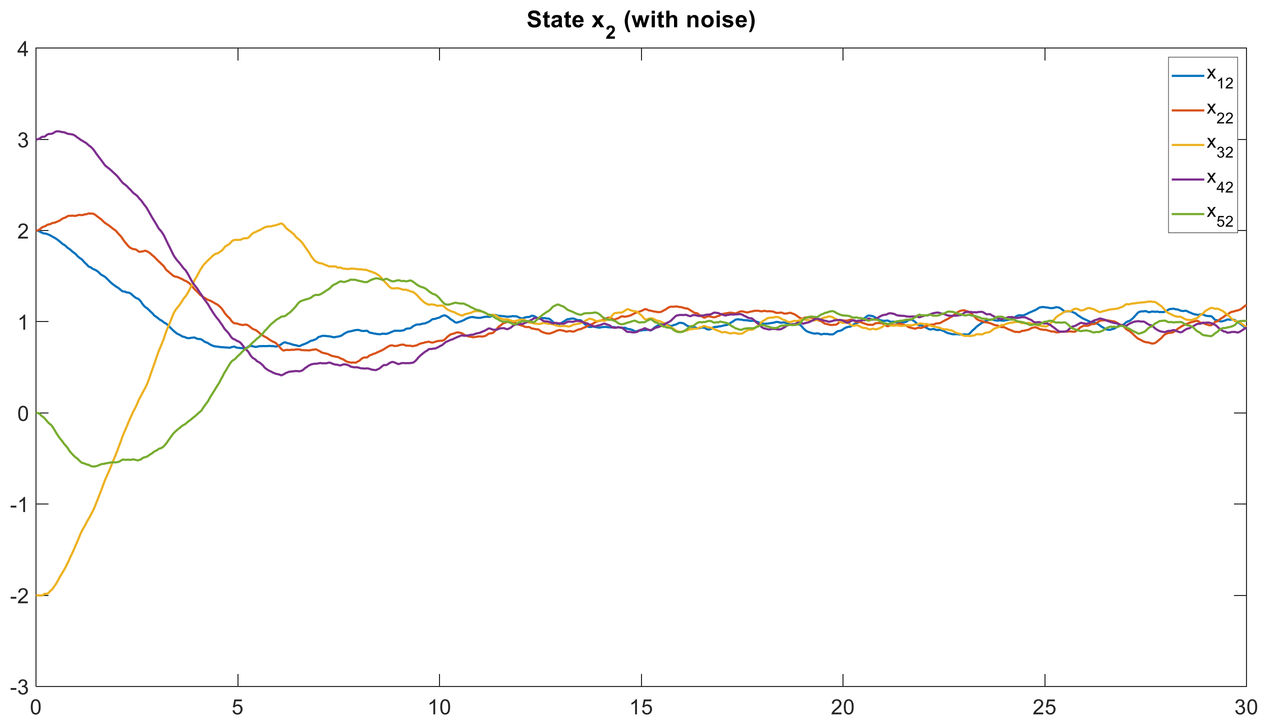

Figure 20: Second state variable (with noise).

- **Simulation with delay in the output channel**

Even the delay case is not considered neither in the work that design the observer for the multiagent system **[14]**, nor in the original work that consider the design of a finite time observer for dynamic systems **[10]**, we can comment with the following:

*The observer error dynamic (7) has the form $\dot{e} = f(e)$. According to **[10]**, the dynamic is r-homogenous of degree ($\frac{-\mu}{1+(n-1)\mu} < 0$). Then according to (**Lemma 4 of [15]**), where they focused on delay robustness for homogeneous systems with negative, we can find that for any $\Delta > 0$, there is $0 < T_0 < +\infty$ such that the observer error dynamic is globally asymptotically stable with respect to Banach space $B_{\Delta}^{T_d}$ for any delay $0 \le T_d \le T_0$. It is important to mention that this is true due to the fact the observer is globally asymptotically stable at the origin with delay-free case.*

We choose delay value equal to $T_d = 0.5\sec$. Observation error for the first state and second state are shown in **Figure 21** and **Figure 22**, respectively. However, for better representation we show the observation error in phase portrait in **Figure 23**. We can see that the errors converges to certain region around the origin as we expected based on (**Lemma 4 of [15]**).

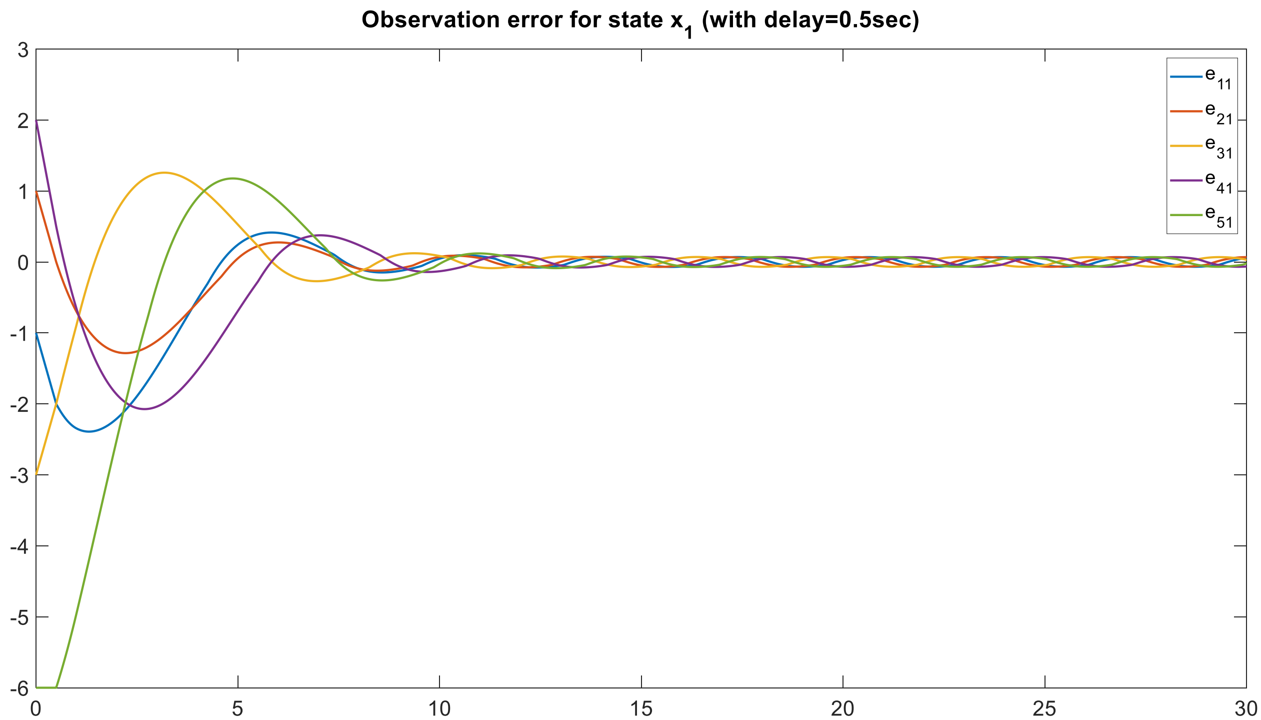

Figure 21: Observation error for first state variable (with delay=0.5sec)

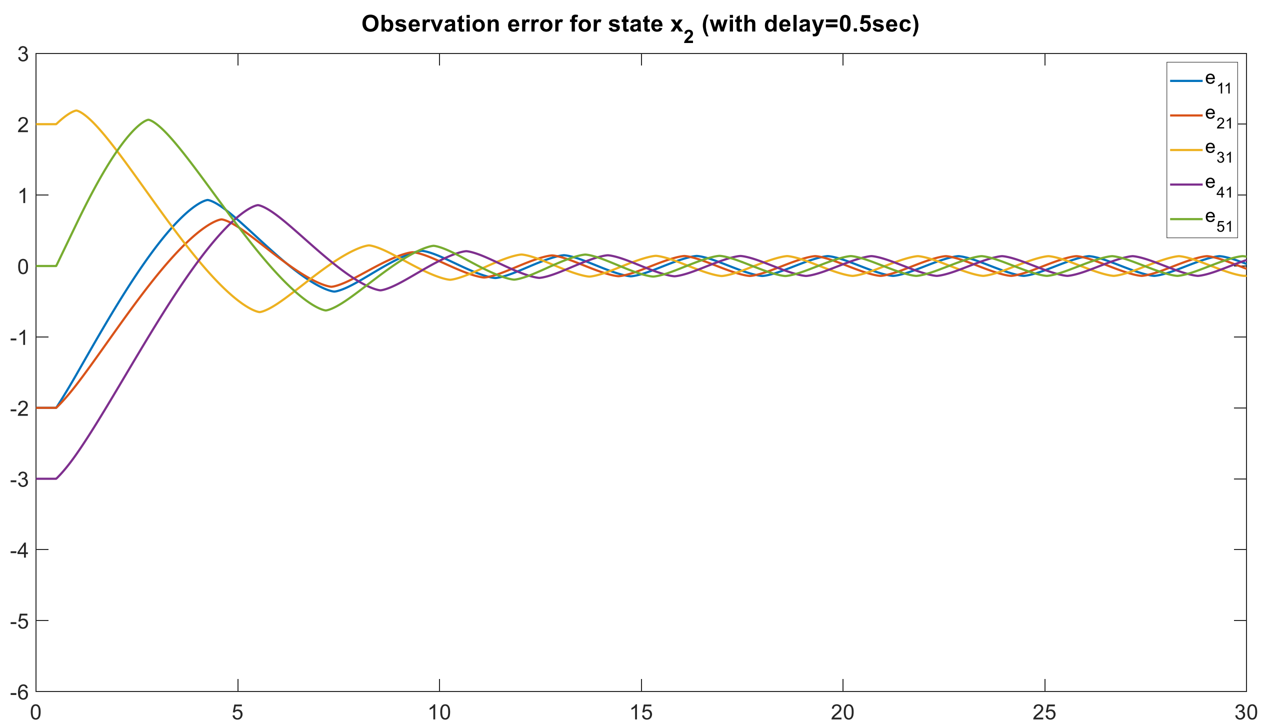

Figure 22: Observation error for first state variable (with delay=0.5sec)

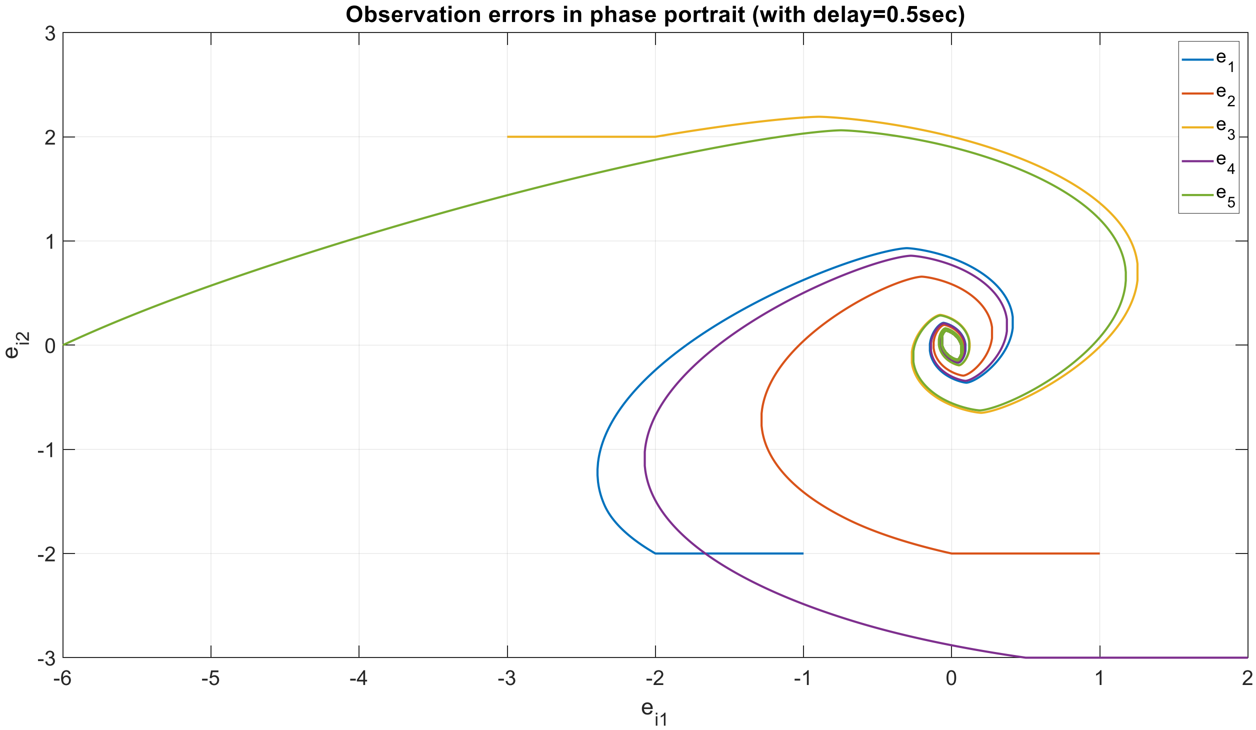

Figure 23: observation error in phase portrait (with delay=0.5 sec)

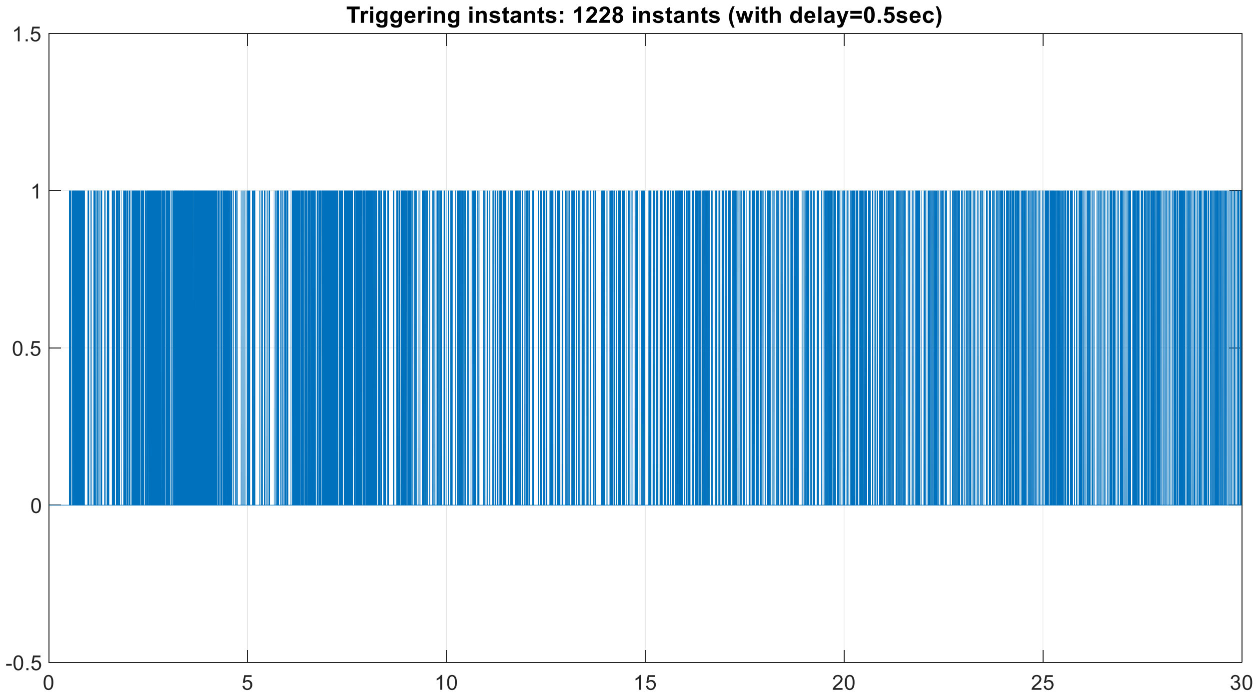

Figure 24: Trigger instants (with delay=0.5 sec)

Trigger instants are shown in **Figure 24**. we can see an increase in the trigger events comparing with the noise free case.

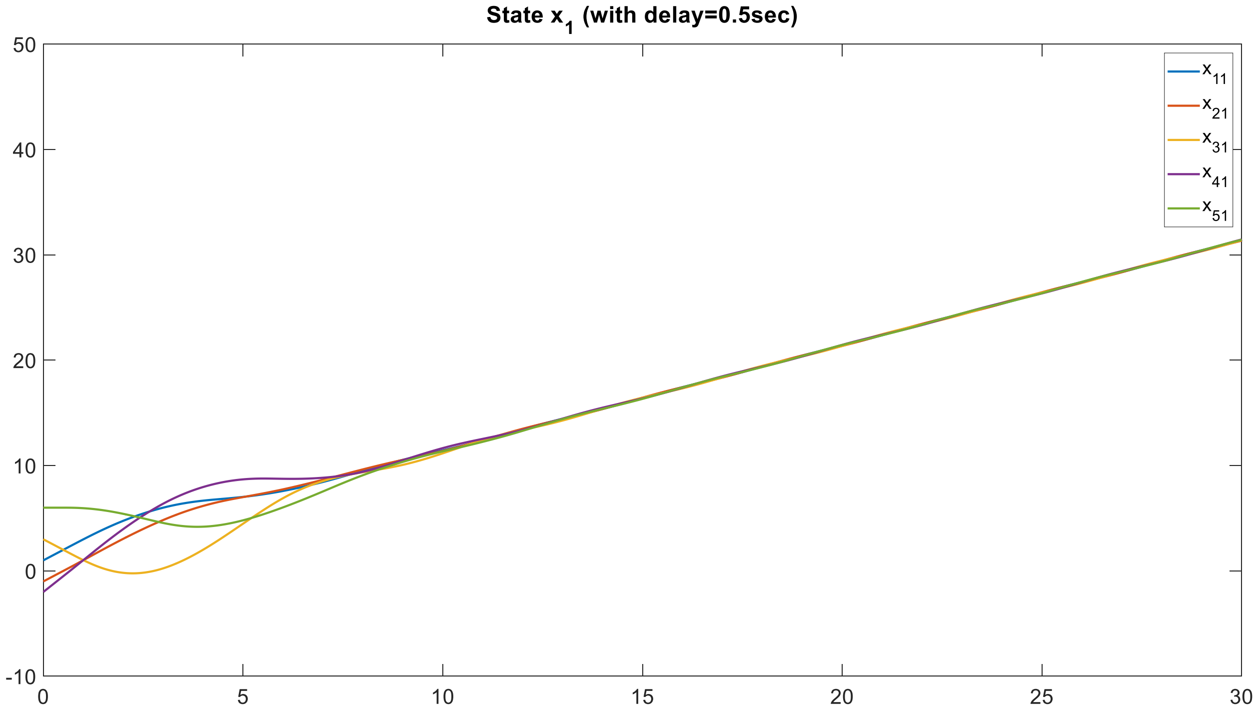

Figure 25: first state variable of the agents (with delay =0.5 sec)

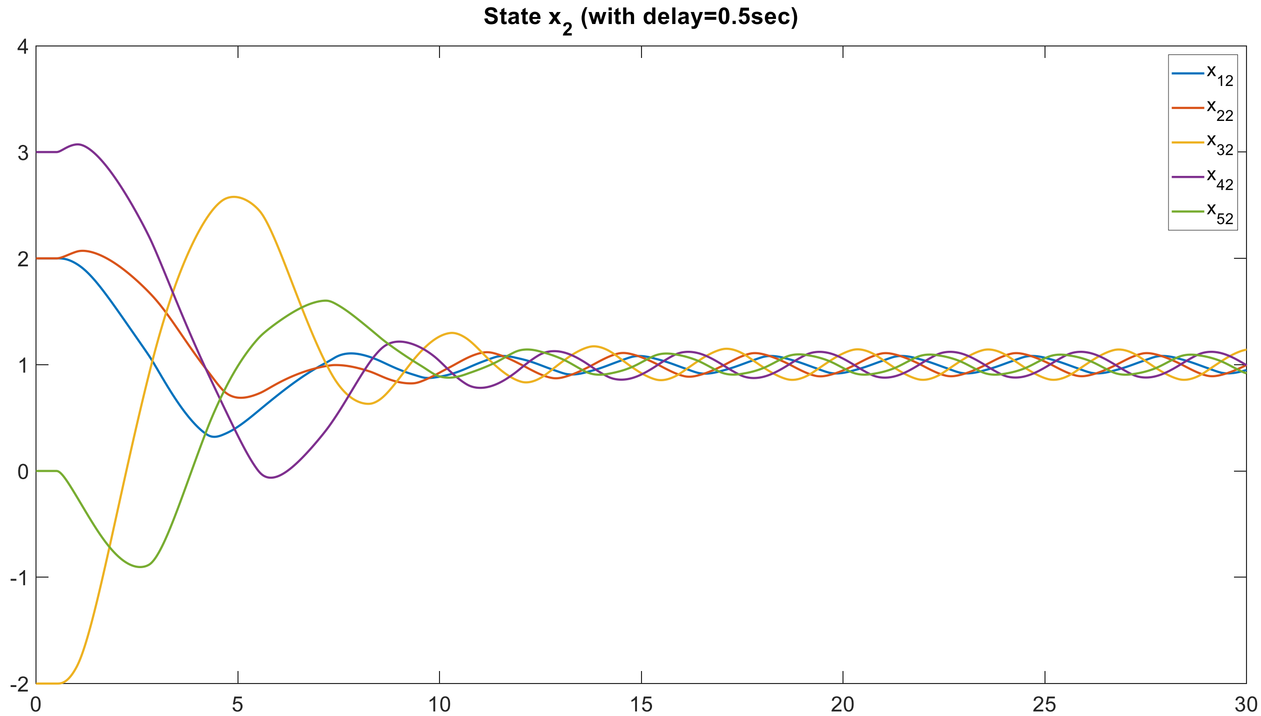

Figure 26: first state variable of the agents (with delay =0.5 sec)

First state and second state of the agents are shown in **Figure 25** and **Figure 26**, respectively. We can see that the area around zero consensus error is bigger than the one for noise-free case.

Conclusion

At the end of this work, we summarize the work that we did, where we covered the topic of finite time observer design of multiagent system, and event-triggered consensus control of the multiagent system based on exchanging the estimated states. We check the performance of the suggested observer and controller experimentally. We tested robustness of the observer for noise and for delay in output channel.

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