- Ex.1

1)

We know that :

$Var(x) = E[x^2] - E[x]^2 = \sigma^2$ **(1)**

$\frac{1}{n}\sum^{n}{}_{\iota=1} x_i = M$ **(2)**

$\frac{1}{n}\sum^{n}{}_{\iota=1}(x_i - M)^2 = \frac{1}{n}\sum^{n}{}_{\iota=1} x_i{}^2 - nM^2$**(3)**

$E[M] = \mu$**(4)**

$Var(M) = \sigma^2$

Starting from:

$W_n{}^2 = \frac{1}{n}\sum^{n}{}_{\iota=1}(x_i - \mu)^2$

We can say that:

$E[W_n{}^2] = E[\frac{1}{n}\sum^{n}{}_{\iota=1}(x_i - \mu)^2] = \frac{1}{n}\sum^{n}{}_{\iota=1} E[(x_i - \mu)^2] =$

$= \frac{1}{n}\sum^{n}{}_{\iota=1}(E[x_{i^2}] - 2E[\mu x_i] + E[\mu^2]) = \frac{1}{n}\sum^{n}{}_{\iota=1}(E[x_{i^2}] - 2\mu E[x_i]) + n\mu^2 =$

$= \frac{1}{n}(\sum^{n}{}_{\iota=1} E[x_{i^2}] - 2n\mu^2 + n\mu^2) =$

$= \frac{1}{n}(n\sigma^2 + n\mu^2 - 2n\mu^2 + n\mu^2) = \sigma^2$

which proves that $W_n{}^2$**is unbiased estimator of** σ^2

Now starting from:

$S_n{}^2 = \frac{1}{n-1}\sum^{n}{}_{\iota=1}(x_i - \frac{1}{n}\Sigma^{n}{}_{\iota=1} x_i)^2$

We can say that:

$E[S_n{}^2] = E[\frac{1}{n-1}\sum^{n}{}_{\iota=1}(x_i - \frac{1}{n}\sum^{n}{}_{\iota=1} x_i)^2] = \frac{1}{n-1}\sum^{n}{}_{\iota=1} E[(x_i - M)^2]$

from (3)

$\frac{1}{n-1}\sum^{n}{}_{\iota=1}E[(x_i - M)^2] = \frac{1}{n-1}\sum^{n}{}_{\iota=1}E[x_i{}^2] - nE[M^2] =$

$= \frac{1}{n-1}\sum^{n}{}_{\iota=1}E[x_i{}^2] - \frac{n}{n-1}(\mu^2 + \frac{\sigma^2}{n}) = \frac{n}{n-1}(\sigma^2 + \mu^2) - \frac{n}{n-1}(\frac{\sigma^2}{n} + \mu^2) = \sigma^2$

which proves that $S_n{}^2$is also unbiased estimator of σ^2

2)

To prove that $(W_n{}^2)^{\propto}{}_{i=1}$, $(S_n{}^2)^{\propto}{}_{i=2}$ are consistent we should prove that $Var((W_n{}^2)^{\propto}{}_{i=1})$ and $Var((S_n{}^2)^{\propto}{}_{i=2})$ are almost zero, as n approximates infinity

and this is true because:

$Var((W_n{}^2)^{\propto}{}_{i=1}) = \frac{1}{n}\sum^{n}{}_{\iota=1}Var(x_i - \mu)^2$

$Var(S_n{}^2) = \frac{1}{n-1}\sum^{n}{}_{\iota=1}Var((x_i - \frac{1}{n}\Sigma^{n}{}_{\iota=1}x_i)^2)$

as n approximates infinity $\frac{1}{n}$ becomes zero so both of them are equal to zero

3)

To prove that $W_n{}^2 = \frac{1}{n}\sum^{n}{}_{\iota=1}(x_i - \mu)^2$ is more efficient that $S_n{}^2 = \frac{1}{n-1}\sum^{n}{}_{\iota=1}(x_i - \frac{1}{n}\Sigma^{n}{}_{\iota=1}x_i)^2$ we should prove that $Var(W_n{}^2) \prec Var(S_n{}^2)$

as we already proved ,they are both unbiased estimators of σ^2

$Var(W_n{}^2) = \frac{1}{n}\Sigma^{n}{}_{\iota=1}Var(x_i - \mu)^2 = \frac{1}{n}(\sigma_4 - \sigma^4)$

$Var(S_n{}^2) = \frac{1}{n-1}\sum^{n}{}_{\iota=1}Var((x_i - \frac{1}{n}\Sigma^{n}{}_{\iota=1}x_i)^2)$

knowing that:

$S_n{}^2 = \frac{1}{2n(n-1)}\sum^{n}{}_{\iota=1}\sum^{n}{}_{j=1}(x_i - x_j)$

$Var(S_n{}^2) = Cov(S_n{}^2, S_n{}^2) = \frac{1}{4n^2(n-1)^2}\sum^{n}{}_{\iota=1}\sum^{n}{}_{j=1}\sum^{n}{}_{k=1}\sum^{n}{}_{l=1}(x_i - x_j)^2(x_k - x_l)^2$

We have four cases:

$2n^3 - n^2$ **terms if** $Cov((x_i - x_j)^2(x_k - x_l)^2) = 0$ **if i=j or k=l**

$n(n-1)(n-2)(n-3)$ **terms if** $Cov((x_i - x_j)^2(x_k - x_l)^2) = 0$ **if i,j,k are distinct**

$2n(n-1)$ **terms if** $Cov((x_i - x_j)^2(x_k - x_l)^2) = 2\sigma_4 - 2\sigma^4$ **if i!=j and {k,l}={i,j}**

$4n(n-1)(n-2)$ **terms if** $Cov((x_i - x_j)^2(x_k - x_l)^2) = \sigma_4 - \sigma^4$ **if i!=j and k!=l and** $|(\{i,j\} \wedge \{k,l\})|$

Substituting gives the result:

$Var(S_n{}^2) = \frac{1}{n}(\sigma_4 - \frac{n-3}{n-1}\sigma^4)$

$Var(S_n{}^2) - Var(W_n{}^2) = \frac{1}{n}(\sigma_4 - \frac{n-3}{n-1}\sigma^4) - \frac{1}{n}(\sigma_4 - \sigma^4) = \sigma_4 - \sigma^4 \succ 0$

$Var(S_n{}^2) \succ Var(W_n{}^2)$

4)

The asymptotic relative efficient of $W_{n=1}{}^2$**to**$S_{n=2}{}^2$**is:**

$lim_{n\to\propto}(Eff(W_{n=1}{}^2 t, S_{n=2}{}^2)) = lim_{n\to\propto}(\frac{Var(S_n{}^2)}{Var(W_n)}) = lim_{n\to\propto}(\frac{\frac{1}{n}(\sigma_4 - \frac{n-3}{n-1}\sigma^4)}{\frac{1}{n}(\sigma_4 - \sigma^4)}) = lim_{n\to\propto}\frac{(n-1)\sigma_4 - (n-3)\sigma^4}{(n-1)(\sigma_4 - \sigma^4)} = 1$