CS390 Computational Game Theory and Mechanism Design July 3, 2013

Lecture 2, Part 1

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1 Order independence

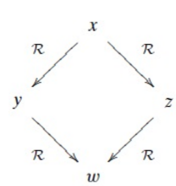
Consider a Normal-Form game as follow:

	L	R
T	0,0	1,0
M	2,0	2,1
B	1,0	0,1

When doing the *Iterated Elimination of Strictly Dominated Strategies*, we may have two different ways:

$$\begin{array}{ccc} & \{T,M,B\} \times \{L,R\} & \\ & \swarrow \quad \searrow & \\ \{T,M\} \times \{L,R\} & & \{M,B\} \times \{L,R\} \\ \downarrow & & \downarrow \\ \{M\} \times \{L,R\} & & \{M,B\} \times \{R\} \\ \downarrow & & \downarrow \\ \{M,R\} & & \{M,R\} \end{array}$$

We notice that the order of elimination does not matter. We will prove it as a theorem. The following lemma tells us that there always exist w satisfying

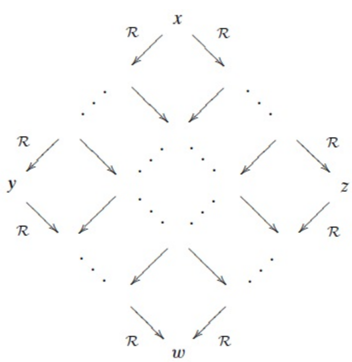


Lemma 1 (Diamond Property) *For all $x, y, z \in S$, if y and z are two different results of doing IESD on x, then there exists $w \in S$ such that w is the result of doing IESD on either x or y.*

Proof. Let's focus on pure strategies. In fact, there won't be anything new to do when proving the mixed strategy version but just turn into calculating probability distributions. Since y and z are different sets, there exists $y' \in y, y' \notin z$. Namely, y' is strictly dominated by some z'_0. $z'_0 \succ y'$ in x, so $z'_0 \succ y'$ in y. If $z'_0 \in y$, then we can eliminate y' from y. Otherwise, there exist $z'_1 \succ z'_0$, and check whether z_1 is in y. If not, do the same step until get $y' \prec z'_0 \prec z'_1 \prec \cdots \prec \hat{z}, \hat{z} \in y$. The strategy profile $\hat{z}$ must exist since y is finite. So we can always eliminate y' from y.
Analogously, we can eliminate all elements in $y \otimes z$. Let $w = y \cap z$, then w is exactly the set we want to get. □

Theorem 2 (Order independence) *The sets of strategy profiles surviving two arbitrary elimination orders are identical.*

Proof. All the necessary intuition is contained in the following picture.



If you start from a set of strategy profiles x and get two different sets y and z, you can always eliminate some strategy profiles from both of them and get the same set w. □

2 Nash equilibrium and iterated elimination of strictly dominated strategies

The first relation to show is that any pure strategy in a Nash equilibrium is not strictly dominated. To show that, we will introduce a lemma from [OR94, p33].

Lemma 3 *Let $G = (N, S, u)$ be a finite normal-form game. Then $a^* \in \times_{i \in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G if and only if for every player $i \in N$, every pure strategy in supp(a_i^*) is a best response to a_{-i}^*.[1]*

Theorem 4 *If s_i appears in a Nash equilibrium, then it is not strictly dominated.*

Proof. Let σ be a Nash equilibrium and s_i be a pure strategy of player i. If s_i appears in the Nash equilibrium, that's to say $\sigma_i(s_i) > 0$, then according to lemma 3, $s_i \in B_i(\sigma_{-i})$.

If $\exists \hat{\sigma}_i \succ s_i$, then $u_i(s_i, \sigma_{-i}) < u_i(\hat{\sigma}_i, \sigma_{-i})$, which contradicts the argument that s_i is the best response to σ_{-i}.

Therefore, s_i is not strictly dominated. $\square$

The next to show is that the support of a Nash equilibrium survives IESD.

Theorem 5 *If S^K survives iterated elimination of strictly dominated strategies and σ is a Nash equilibrium, then $supp(\sigma) \subseteq S^K$.*

Proof. Prove by induction:

Initial step: σ is a Nash equilibrium of the game (N, S^0, u). $supp(\sigma)$ is a subset of the set of all the pure strategy profiles, so $supp(\sigma) \in S^0$.

Induction step: Suppose σ is a Nash equilibrium of the game (N, S^k, u) and $supp(\sigma) \in S^k$ $(0 \leq k \leq K-1)$. From theorem 4 we know $\forall s_i$ that satisfies $\sigma_i(s_i) > 0$ is not strictly dominated, so s_i stays in S_i^{k+1}. Therefore, $supp(\sigma) \in S^{k+1}$. Since σ is a Nash equilibrium on the set s^k, σ is still a Nash equilibrium on its subset s^{k+1} because considering the fact that no new profile appears in the subset S_{k+1}, for any player i, σ_i is still the best response to σ_{-i}.

Therefore, $supp(\sigma) \in S^K$.

$\square$

3 Never-best responses

Definition 1 *A strategy s_i is a never-best-response if $\forall \sigma_i \in \Delta(S_i), s_i \notin B_i(\sigma_i)$.*

Theorem 1 *A strategy s_i is strictly dominated if and only if it is a never-best-response.*

Proof.
First, we prove that *if a strategy s_i is strictly dominated, it is a never-best-response.*

Because s_i is strictly dominated,

$$\exists \sigma_i, s.t. \quad \forall s_{-i} : \ u_i(s_i, s_{-i}) < u_i(\sigma_i, s_{-i})$$

$$\forall \sigma_i \ \in \ \delta(S_i) \ u_i(s_i, \sigma_{-i}) = \sum_{s_{-i}} \ \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i})$$

$$< \sum_{s_{-i}} \sigma_{-i}(s_{-i})u_i(\sigma_i, s_{-i}) = u_i(\sigma_i, \sigma_{-i})$$

so we can see that $s_i \notin B_i(\sigma_{-i})$, that is to say s_i is a never-best-reponse.

Second, we prove that *if a strategy s_i is a never-best-response, it is strictly dominated.*

We can construct a 2-player zero-sum game.
Assign:

player i is player 1 and player -i is player 2;
$A_1 = S_1 \setminus S_i$ $A_2 = S_{-i}$;
$v_1(s_i', s_{-i}) = u_i(s_i', s_{-i}) - u_i(s_i, s_{-i})$; $v_2(s_i', s_{-i}) = -v_1$

Because s_i is a never-best-response,so:

$$\forall \sigma_{-i} \ \in \ \delta(S_{-i}) = \delta(A_2), \ \exists s_i' \neq s_i, \ s.t. u_i(s_i', \sigma_{-i}) \ > \ u_i(s_i, \sigma_{-i})$$

$$\implies v_1(s_1', \sigma_{-i}) \ > \ 0$$

$$\implies \forall \sigma_- i \ \in \ \delta(A_2), \ max_{\sigma_1 \in \delta(A_1)} v_1(\sigma_1, \sigma_{-i}) \ > \ 0$$

$$\implies min_{\sigma_{-i} \ in \delta(A_2)} \ max_{\sigma_1 \ in \delta(A_1)} v_1(\sigma_1, \sigma_{-i}) \ > \ 0$$

$$\implies max_{\sigma_1 \ in \delta(A_1)} \ min_{\sigma_{-i} \ in \delta(A_2)} v_1(\sigma_1, \sigma_{-i}) \ > \ 0$$

Let $\sigma_1^* \in argmax_{\sigma_1} \ min_{\sigma_- i} v_1(\sigma_1, \sigma_{-i})$

$$\implies min_{\sigma_- i \in \delta(A_2)} \ v_1(\sigma_1^*, \sigma_{-i}) \ > \ 0$$

$$\implies \forall \sigma_{-i} \ \in \ \delta(A_2) \ v_1(\sigma_1^*, \sigma_{-i}) \ > \ 0$$

$$\implies \forall s_{-i} \ \in \ S_{-i} \ v_1(\sigma_1^*, s_{-i}) \ > \ 0$$

$$= u_1(\sigma_1^*, s_{-i}) - u_1(s_1, s_{-i}) \ > \ 0$$

$$\implies s_i \ \prec \ \sigma_1^* \in \delta(A_1) = \delta(S_1 s_i)$$

That is to say, s_i is strictly dominated. $\square$

4 Acknowledgment

Section 1 is written by Yanqing Peng. Section 2 is written by Zhuoyue Zhao. Section 3 is written by Jiachen Shi.

References

[1] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)

[3] Jing Chen and Silvio Micali. *The order independence of iterated dominance in extensive games.* Theoretical Economics 8, 125–163, 2013.