CS390 Computational Game Theory and Mechanism Design July 3, 2013

Lecture 2, Part 2

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In this class we talked about Extensive form game with perfect information and its Nash equilibrium, subgame perfect equilibrium and the one deviation property.

1 Extensive form game with perfect information

Definition 1 *An extensive form game is tuple* (N, T, r, D, Z, P, u).

$N = \{1, 2, \cdots, n\}$ *is set of players.*

T *is a tree(game tree).*

r *is the root.*

D *is the set of non-terminal nodes(decision nodes).*

Z *is the set of terminal nodes.*

P *is called player function:* $D \to N$ $\forall x \in D$ $P(x) = i$ *means* i *acts at* x.

$\boldsymbol{u}$ *is the utility function.* $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$ *where* $u_i : Z \to \mathbb{R}$.

Now we introduce some basic concepts of extensive form game.

Action

$\forall x \in D$ $A(x)$ is set of actions(i.e. set of children of node x).

Strategy

Strategy of i S_i maps each x with $P(x) = i$ to an action in $A(x)$.

History

$\forall x \in D \cup Z$ history leading to x $h(x)$ is the sequence of actions leading to x.

$h(x) = \perp$ denotes an empty string.

Example 1

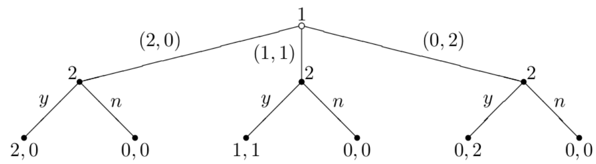

Figure 1: example 1

$N = \{1, 2\}$

$S_1 = \{L, M, R\}$

$S_2 = \{ll'l'', ll'r'', lr'l'', lr'r'', rl'l'', rl'r'', rr'l'', rr'r''\}$

Given strategy profiles $s = (L, lr'l'')$, $h(s)$ is path selected by s.

$u(s)$ is utility at terminal nodes.

The game follows illustrates an important point: given some strategy profile s, the path selected by s, namely $h(s)$, may meet a "dead end" halfway.

Example 2

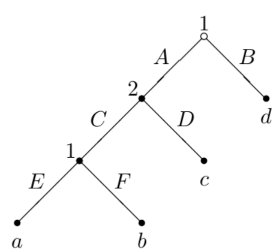

Figure 2: example 2

$S_1 = \{AE, AF, BE, BF\}$

While after player 1 chosen action B, he can never choose action E and F, BE and BF is still a eligible strategy of player 1.

2 Nash equilibrium

Definition 2 *A strategy profile $s^\star$ is a Nash equilibrium if* $\forall i \forall s_i \in S_i \; u_i\left(s_i^\star, s_{-i}^\star\right) \geqslant u_i\left(s_i, s_{-i}^\star\right)$

Now we introduce the Normal Form representation $\left(N, S, (u)_{i\in N}\right)$ of an extensive form game. We will illustrate it by the example above.

1/2	C	D
AE	1,2	0,1
AF	3,0	0,1
BE	2,2	2,2
BF	2,2	2,2

It's easy to notice that $s^\star$ is a Nash equilibrium of the extensive form game iff it is a Nash equilibrium of the Normal Form representation.

The unsatisfactory of Nash equilibrium in extensive form game

Sometimes the Nash equilibrium in extensive form game lacks plausibility.

Example 3

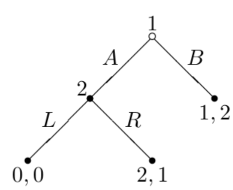

Figure 3: example 3

In this game, both (A, R) and (B, L) is Nash equilibrium. The reason that (B, L) is a Nash equilibrium is that $u_1(A, L) = 0 < u_1(B, L)$ and $u_2(B, R) = u_2(B, L)$. However, if player 1 chooses action A, then player 2(rational) would choose action R over L because he can obtain higher payoff by doing so. The equilibrium is sustained by the "threat" of player 2 to chooses L if player 1 chooses A. In fact, if we change the utility $\boldsymbol{u}$(the end of AL)$= (-\infty, \infty)$, the equilibrium still holds. Therefore, the Nash equilibrium (B, L) is not so persuasive.

3 Subgame perfect equilibrium

Motivated by the unsatisfactory of Nash equilibrium, we introduce a better equilibrium: subgame perfect equilibrium. First, we define the notion of subgame.

Definition 3 *The subgame after history h is an extensive game*

$$G(h) = (N, T|_h, r, D|_h, Z|_h, P|_h, u|_h)$$

The notion of subgame can be illustrated by the notion of subtree.

Strategy of subgame

$S_i|_h : \forall x \in D|_h$ such that $P|_h(x) = i, S_i|_h(x) = s_i(x)$.

Definition 4 *A strategy profile $s^\star$ is a subgame perfect equilibrium(SPE) if $\forall i \forall$non-terminal history h leading to x with $P(x) = i$, we have $u_i\left(s_i^\star|_h,\ s_{-i}^\star|_h\right) \geqslant u_i\left(s_i, s_{-i}^\star|_h\right)$ for any $s_i \in S_i|_h$. Equivalently, $s^\star$ is a subgame perfect equilibrium if $\forall h : s_i^\star|_h$ is a Nash equilibrium in subgame $G(h)$.*

Lemma 1 *(The one deviation property) $\forall$finite horizon extensive game Γ with perfect information, $s^\star$ is a SPE of $\Gamma \iff \forall i \forall h$ leading to x where $P(x) = i$, $\forall s_i \in S_i|_h$ satisfying s_i and $s_i^\star|_h$ differ only at x, $u_i(s_i^\star|_h, s_{-i}^\star|_h) \geqslant u_i(s_i, s_i^\star|_h)$.*

Proof. The "$\Longrightarrow$" part is trivial.

As for the "$\Longleftarrow$" part, we can prove it by contradiction. Assume that $s^\star$ satisfies the one deviation property but it is not a SPE. Then $\exists i,\ h',\ i$ can deviate profitably in $G(h')$. $\exists s_i \in S_i|_{h'}$, so that $u_i(s_i', s_{-i}^\star|_{h'}) > u_i(s_i^\star, s_{-i}^\star|_{h'})$.

Without loss of generality, s_i' is the strategy with the least nodes satisfying $s_i'(x) \neq s_i^\star$ for at most l nodes where $l =$height of $G(h)$. It is profitable deviation, which means that it can result to better outcome. Then take the **longest** path $\hat{h}$ so that $s_i'(\hat{x}) \neq s_i^\star|_{h(\hat{x})}$. Then $s_i'|_{\hat{h}}$ is a deviation from $s_i^\star|_{\hat{h}}$. Thus, we can claim that $s_i'|_{\hat{h}}$ is a profitable deviation in $G(\hat{h})$. Define $\hat{s}_i$, and it is profitable with less nodes. Then it contradicts the assumption. □

References

[1] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)