CS390 Computational Game Theory and Mechanism Design June 30, 2013

Problem Set 1

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Problem 1 (No collaborator.)

1. There is only one that

$$\sigma = (\frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S, \frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S)$$

For any other σ_{-i},

$$u_1(\sigma_1, \sigma_{-1}) = u_2(\sigma_2, \sigma_{-2}) = 0$$

First, we'll prove that, for σ_2, σ_1 is the best response. Suppose there is another $\sigma_1' = aR + bP + cS$, $a + b + c = 1$.

$$u_1'(\sigma_1', \sigma_2) = -1/3a + 1/3a - 1/3b + 1/3b - 1/3c + 1/3c = 0$$

Therefore,

$$\sigma_1 \in \operatorname*{argmax}_{\sigma_1'} u_1(\sigma_1', \sigma_{-1})$$

By semmetry,

$$\sigma_2 \in \operatorname*{argmax}_{\sigma_2'} u_1(\sigma_2', \sigma_{-2})$$

So σ is a Nash Equilibrium.
Next, we'll prove that, for any other $\sigma' \neq \sigma$, σ' is not a Nash Equilibrium.
Assume that $\sigma_1' = aR + bP + cS$, $a + b + c = 1$, $\sigma_1' \neq \sigma_1$. Then we'll assume that $\sigma_2' = cR + aP + bS$.

$$\begin{aligned} u_1 &= -a^2 + ab - b^2 + bc - c^2 + ac \\ &= -\frac{1}{2}(a-b)^2 - \frac{1}{2}(a-c)^2 - \frac{1}{2}(b-c)^2 \end{aligned}$$

Since $\sigma_1' \neq \sigma_1$, then we can suppose $a \neq b$ W.O.L.G. Therefore,

$$u_1 < 0 = u_1(\sigma_1, \sigma_2')$$

σ_1' can not be part of a NE. By semmetry, $\sigma_2' = aR + bP + cS$, $a + b + c = 1$, $\sigma_2' \neq \sigma_2$. σ_2 can not be part of a NE.
Therefore, there is only one Nash Equilibrium in an Rock-Paper-Scissors game.

2. $\sigma = (\frac{2}{3}B + \frac{1}{3}S, \frac{1}{3}B + \frac{2}{3}S)$ is a NE.
   We'll prove that σ_1 is the best response for σ_2. We have

$$u_1(\sigma_1, \sigma_{-1}) = \frac{2}{3}$$

   Assume that $\sigma_1' = aB + bS$, $a + b = 1$.

$$u_1' = \frac{1}{3}a \cdot 2 + \frac{2}{3}b \cdot 1 = \frac{2}{3}(a+b) = \frac{2}{3}$$

   Then

$$\sigma_1 \in \operatorname*{argmax}_{\sigma_1'} u_1(\sigma_1', \sigma_{-1})$$

   By semmetry

$$\sigma_2 \in \operatorname*{argmax}_{\sigma_2'} u_2(\sigma_2', \sigma_{-2})$$

   So σ is a Nash Equilibrium.

3. There are three Equilibria $\sigma = (B, B)$ or $\sigma = (S, S)$, and the mixed one given above.
   Suppose there exist another $\sigma' = (aB + bS, a'B + b'S)$ is a Nash Equilibrium.

$$\begin{aligned} u_1 &= 2aa' + bb' \\ u_2 &= aa' + 2bb' \end{aligned}$$

   Suppose that $2a' > b'$, we have

$$\begin{aligned} u_1 &= b'(a+b) + (2a' - b')a \\ &= b' + (2a' - b')a \end{aligned}$$

   For fixed a', b', $a = 1, b = 0$ is the best response. So we have $u_2 = a'$, to get the bigges utility. $\sigma = (B, B)$
   Suppose that $2a' < b'$,

$$\begin{aligned} u_1 &= 2a'(a+b) + (b - 2a')b \\ &= 2a' + (b - 2a')b \end{aligned}$$

   For fixed a', b', $a = 0, b = 1$ is the best response. Then we have $u_2 = 2b'$, $\sigma = (S, S)$
   When $2a' = b'$, $a' + b' = 1$, $a' = \frac{1}{3}, b' = \frac{2}{3}$. From the proof above, we have $a = \frac{2}{3}, b = \frac{1}{3}$.
   So there are three Equilibria for BoS game.

4. We'll construct the game $G = \langle N, S, u \rangle$.

- $N = \{1, 2\}$
- $S = S_1 \times S_2$
  $S_i = \mathbb{N}$
- $u_i = \begin{cases} s_i & s_i \text{ is the maximum of the two numbers players choose} \\ 0 & \text{other} \end{cases}$

Suppose player 1 chooses number a, then he can also choose any number larger than a and the number player 2 choose, to get a better uitlity. By semmetry, there is also no best response for player 2 when given choice of player 1.
Therefore, G is a game with no Nash Equilibrium.

Problem 2 (No collaborator.)

1. $G = \langle N, S, u \rangle$.
   - $N = \{1, 2, 3 \ldots, n\}$ denoting the players.
   - $S = S_1 \times S_2 \times \ldots \times S_n$
     $S_i = \mathbb{N}$ denoting the prices each player give.
   - $u_i(s_1, s_2, \ldots, s_n) = \begin{cases} v_i - s_i & s_i \text{ is the highest price among the others} \\ 0 & \text{other} \end{cases}$

   The equilibriua are in the form of $\sigma = \{s_1, s_2, s_3, \ldots, s_n\}$, $v_1 \geq s_1 \geq v_2 - 1$, $\exists s_i, i \neq 1, s_i = s_1$, $\forall i \geq 2$, $0 < s_i \leq s_1$, $s_i \in \mathbb{N}$. We'll prove that in an equilibrium, player 1 always gets the object by contradiction.
   Suppose that player 1 does not get the object. So that there must exist a player i, $s_i > v_1$, or by making the bid less that or equal to v_1, player 1 can get the object having utility larger than or equal to 0.
   By the definition of the game $u_i = v_i - s_i < v_i - v_1 < 0$, the strategy is worse than giving a price less than s_1 getting profit of 0.
   Therefore, player 1 always gets the object in a Nash Equilibrium.

2. We'll define the notion of weakly dominance as follows.
   s_i weakly dominates s_i' iff $u_i(s_i, S_{-i}) \geq u_i(s_i', S_{-i})$.
   Then, we'll prove that v_i is a weakly dominated strategy. Suppose that player i gets the object, then he'll pay the price $p \leq v_i$. So he gets the utility ≥ 0. Choosing any higher price then v_i won't change his utility. If he chooses $s_i < v_i$, if $s_i < p$, he'll lose the object having utility 0. If $s_i = p$, if he still has an earlier label then all people bidding p. He'll get the object and still having the utility $v_i - p$. Or he'll lose the object having utility 0. If $s_i > p$, the utility won't change.

We'll prove that player 1 gives the bid of v_2 and player 2 gives the bid of $v_1 + 1$ when other players' bids are all less than v_2 is a equilibrium and player 2 gets the objects.
We can see that the only way player 1 gets the object is by bidding larger than $v_1 + 1$ and he'll get the utility less than 0, or he'll always gets the utility 0. So v_2 is the rational choice for him to make.
While for player 2, he'll always gets the utility 0 so this is a equilibrium for him. And for the others, they'll always get 0, and if they want the object they need to get the utility less than 0. Therefore, it is a equilibrium while player 2 gets the object.

Problem 3 (No collaborator.)

First, consider the situation that player 1 choose 100. Suppose the sum of others is S, $S < 14 \times 100$. Since the average is at least 100, player 1 has utility 0. To have some profit,

$$\frac{S + s_1}{15 \times 3} = s_1$$

He can choose any number closer to $\frac{S}{44} < 100$. Therefore, we can eliminate 100 from his strategy set.
Similarly, we can eliminate 100 from any othes' strategy set.
Then 99 become the biggest number that any one can choose. We have $S < 14$ Similarly, it can make no profits. And we can choose some less s_i closer to $\frac{S}{44}$. We can eliminate 99 from everyone's strategy set.
Then comes $98, 97 \ldots$.
Finally we have only 1 left in our strategy set. So for everyone the equilibrium is like everyone chooses 1.

References

[1] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)