Estimating Timber

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March 29, 2015

1 Introduction

Estimating the total amount of timber in a forest is a popular subject of Forest Measurement. This paper gives a mathematical explanation of the "Variable-radius Plot Sampling" method assuming the trees are uniformly distributed.

Section 2 states the rules of applying the method, clarifies notations and gives an important conjecture; section 3 illustrates the rationale behind this method; section 4 analyzes the probability of a tree hiding from being counted. We have designed a program to model the forest, as codes being provided in the appendix.

In this paper, we assume all trees are perfect cylinders, are uniformly distributed and their board feet are directly proportional to their basal area. Hence, it is crucial to find the total basal area of trees in the forest.We use geometric approaches to get the multiplier that is applied to get the total basal area, and improve the drawback of the method by minimizing the error propagation.

2 Counting Trees with the Prism

Figure 2.1: Using a wedge prism relascope.

As is illustrated by Figure 2.1, the portion of the trunk seen through the prism is offset to one side. We say the tree is counted if its image in the prism overlaps its tree trunk, and not counted if the tree looks disconnected seen through the prism.

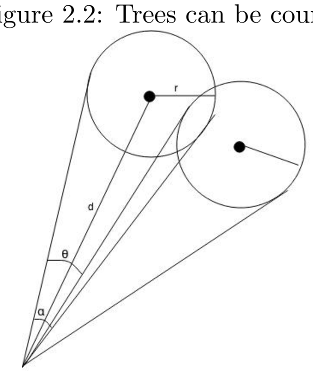
Notations. In this paper, we use: θ to denote the angle of the prism, d to denote the distance from the center of a tree to the plot center, r to denote the radius of the tree trunk, B to denote the Basal Area,

S to denote the sample area, P to denote the plotting area of a tree which covers the region where the tree can be counted, and TB to denote Total Basal Area.

We made the following conjecture based on the above method:

Conjecture 1. *Assuming $d > \sqrt{2} \cdot r$, a tree will be counted if and only if $\frac{r}{d} \geqslant \sin\frac{\theta}{2}$.*

Figure 2.2: Trees can be counted.



Proof. As shown by the above graph, α is the angle between the two lines that cross the viewing point and are tangent to the stem of the tree.

$d > \sqrt{2} \cdot r$ gives us $\alpha \in (0, \frac{\pi}{2})$[1]. Assume $\theta \in (0, \frac{\pi}{2})$:

The tree is counted if and only if $\alpha \geqslant \theta$, which is equivalent to: $\frac{\alpha}{2} \geqslant \frac{\theta}{2}$

Since θ, $\alpha \in (0, \frac{\pi}{2})$, then $\sin\frac{\alpha}{2} \geqslant \sin\frac{\theta}{2}$

Because $\sin\frac{\alpha}{2} = \frac{r}{d}$,

Plugging into the in equation gives us: $\frac{r}{d} \geqslant \sin\frac{\theta}{2}$

□

Now we assume θ and the plot center are fixed.

3 Getting the Total Basal Area

In this section, we make the following assumption:

- The forest is bounded and we know the area of the forest.
- We can look through the whole forest.
- There are finitely many trees in the forest.
- The trees of each size are uniformly distributed in the forest.

[1] if $d \leqslant \sqrt{2} \cdot r$, it follows that, then the tree must be counted since $\alpha \geqslant \frac{\pi}{2} > \theta$.

Note that since there are finite many trees in the forest, then the variety of trees' size is finite as well. And we regard two trees have the same size if they have the same r.

Using the following steps to calculate TB:
Step 1: Choose a sample area.
Step 2: Calculate the sum of basal areas of trees in the sample area.

$$B = \sum_{i=1}^{n} B_i$$

where n is the number of trees in the sampling region.
Step 3: Calculate the density of B in the sample area (by the assumption of uniform distribution), then multiply it by the total area of the forest to get TB, which denotes the total basal area:

$$TB = \frac{B}{\text{Sample Area}} \cdot \text{Forest Area}$$

To prove the correctness of the method that uses the prism , first we consider the case when all trees are of the same size. Since all the trees have the same size, they have the same r_i and the same $B_i = \pi \cdot r_i^2$ Let N denote the total number of tree in the forest. Since the trees are uniformly distributed, given any sampling region we have

$$E(N) = n \cdot \frac{\text{Forest Area}}{S},$$

Then

$$E(TB) = E(N) \cdot B = n \cdot \frac{\text{Forest Area}}{S} \cdot B.$$

This methods works for an arbitrary sampling region.

Proposition 1. *Given all the trees are identical, the plotting area of a tree can be used as a sample region. In fact, the region is a disk.*

Proof. It is shown the proof of next theorem. □

Now, with $P = S$, the equation can be written as:
$TB = \dfrac{n \cdot B}{P} \cdot \text{Forest Area}$

Theorem 2. $\dfrac{B}{P} = \sin^2 \dfrac{\theta}{2}$.

Proof. Pick an arbitrary tree with radius r. By *conjecture 1*, for the tree to be counted, it has to satisfy the condition that $d \leqslant \dfrac{r}{\sin \frac{\theta}{2}}$. This shows that the plotting area of a fixed tree is a disk with radius $\dfrac{r}{\sin \frac{\theta}{2}}$. So $P = \pi(\dfrac{r}{\sin \frac{\theta}{2}})^2$, $\dfrac{B}{P} = \dfrac{\pi \cdot r^2}{\pi(\frac{r}{\sin \frac{\theta}{2}})^2} = \sin^2 \dfrac{\theta}{2}$. This is a constant that only depends on the angle of prism θ.

□

Now we consider the case when trees have finitely different sizes.

Theorem 3. *The expected Total Basal Area in the forest can be expressed as follows, regardless of the variety in tree sizes:*

$$E(TB) = n \cdot \textit{Forest Area} \cdot \sin^2 \frac{\theta}{2}$$

Proof. Since there are only finitely many sizes of trees, let $\{B_1, B_2, ..., B_m\}$ be the set of all possible Basal Areas. Let N_i be the total number of trees of size r_i and n_i be the number counted trees of size r_i, and TB_i be the total basal area covered by trees of size r_i. Then

$$TB = \sum_{i=1}^{m} TB_i.$$

Since trees of each size are distributed uniformly and we can see all the trees, TB_is can be obtained by applying *Proposition 1.* Then we have:

$$\begin{aligned} E(TB) &= \sin^2 \frac{\theta}{2} \cdot \text{Forest Area} \cdot \sum_{i=1}^{m} n_i \\ &= n \cdot \sin^2 \frac{\theta}{2} \cdot \text{Forest Area} \end{aligned}$$

□

4 The "Hidden Tree" Problem

This method is not perfectly plausible because, one can't include all countable trees in a sample area if some trees are hidden behind closer trees. We here give an analysis on how sparse the trees have to be in order to minimize the expected number of hidden trees and to make this estimate more precise.

Now we assume that each countable tree is either completely blocked or can be seen as a whole. If a tree has a "blocking region", then these regions don't intersect with each other, because otherwise part of the tree that is farther away must be blocked by the closer tree:

Trees are blocked by the farthest visible tree in that direction. Let's denote the number of visible countable trees by N. It's also the number of countable trees that we can see in all directions.Then the set of all the blocked countable trees can be partitioned into several subsets, with each subset of trees blocked by a farthest visible tree in some direction.

Let S denote the sum of all blocked countable trees, S_i denote the number of trees blocked by tree i. We have $S = \sum_{i=1}^{N} S_i$. Hence $E(S) = \sum_{i=1}^{N} E(S_i)$.

Next we will show that $E(S_i)$ is a function of random variables R and D where R is the radius of the tree and D is the distance of the tree from plot center. Now suppose $R = r_0$ and $D = d_0$.

Suppose the intersecting surface of the farthest visible countable tree is C_0. C_0 is the assumed to be a circle. Denote θ_0 to be angle of two distinct rays from the plot center which are tangent to the disk C_0,

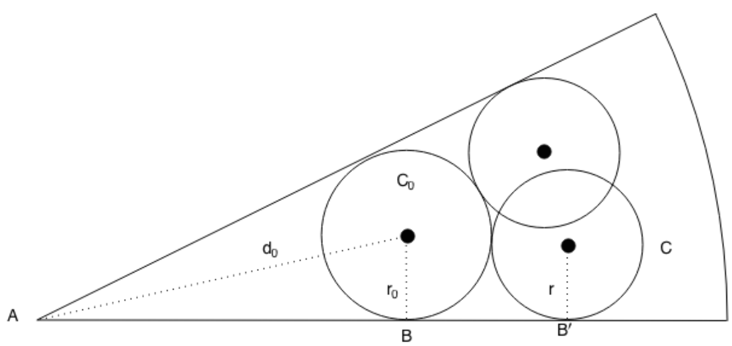
θ to be angle of rays from the plot center that are tangent to disk C. Denote the hidden tree's radius to be r and its intersecting surface to be C. Then

$$\sin\left(\frac{\theta}{2}\right) \leqslant \frac{r}{d} \leqslant \sin\left(\frac{\theta_0}{2}\right).$$

As tree trunks don't intersect, all the hidden trees fall into the following two cases:

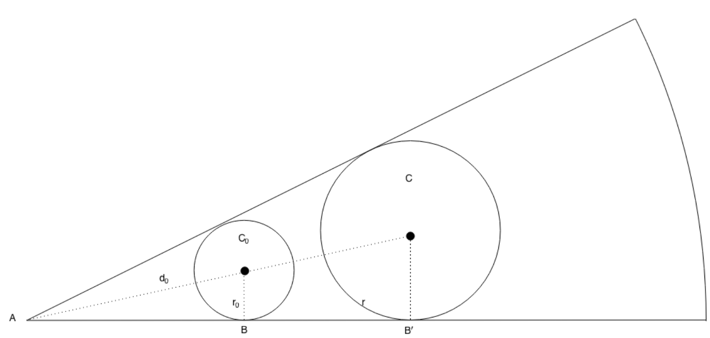
Case 1. C can be tangent to C_0.

Figure 4.1: Case 1.



Case 2. C cannot be tangent C_0.

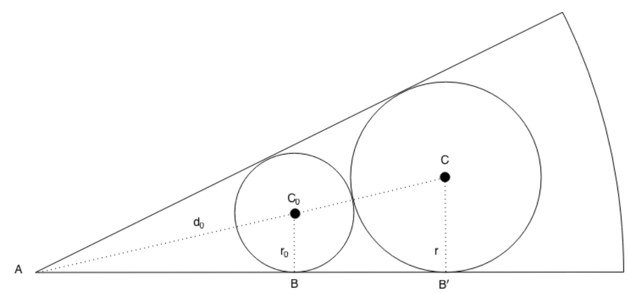
Figure 4.2: Case 2.



Claim. *C intersects C_0 if and only if* $r \leqslant \dfrac{d_0 \cdot r_0 + r_0^2}{d_0 - r_0}$.

Proof. Note that the largest acceptable C is the one that is tangent to C_0 as well as the two tangent lines of C_0.

Figure 4.3: The largest tangent tree.



Due to symmetry, C's center, C_0's center and the plot center are on on the angle bisector of θ_0.

$$\sin\left(\frac{\theta_0}{2}\right) = \frac{r_0}{d_0} = \frac{r}{r + r_0 + d_0}.$$

Therefore $r = \dfrac{r_0 d_0 + r_0^2}{d_0 - r_0}$ and $\theta_0 = 2\arcsin\left(\dfrac{r_0}{d_0}\right)$. Since $\triangle ABO$ and $\triangle AB'O'$ are right triangles, then the hypotenuse, $AC_0 = d_0$ is greater than $BC_0 = r_0$. Hence $d_0 - r_0 > 0$. $\square$

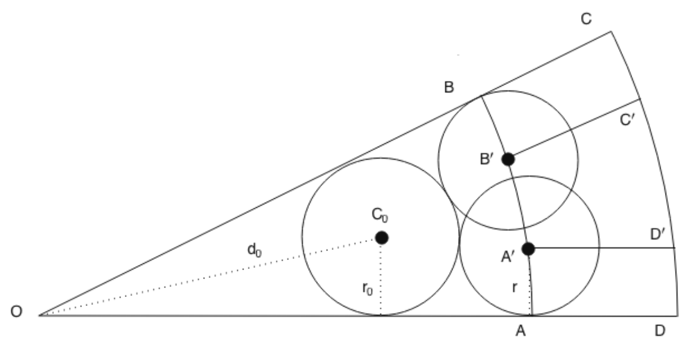
Next, we need to calculate the restriction of the area that the trees there can be fully blocked. First, all the trees with intersecting surface in radius r should be in the sample area. With the given value θ_0, r_0, d_0, r, we know that the radius of the plotting area, denoted by d, is equal to $\dfrac{r}{\sin\left(\frac{\theta}{2}\right)}$. Secondly, since we assume the trees are fully blocked, then the distance between their centers and the rays need to be greater than the sum of the radius, which is the shadowed area, $A'B'C'D'$ in the Figure 4.3. Denote the shadowed area to be S. We need to calculate S with given values.

Consider the two cases we mentioned above:

Case 1. C is tangent to C_0.

Denote the vertices with new alphabets.

Figure 4.4: Case 1.



Let $\angle AOA' = \theta_1$, $\angle DOD' = \theta_2$, take the altitude of $\triangle AOA'$ and $\triangle DOD'$ due to the vertex A' and D', such that $A'P_1 \perp OA$ and $D'P_2 \perp OD$, then we have

$$\begin{array}{rcccl} \sin\theta_1 & = & \dfrac{A'P_1}{OA} & = & \dfrac{r}{d_0+r_0+r} \\ \sin\theta_2 & = & \dfrac{D'P_2}{OD} & = & \dfrac{r}{d} \end{array} \Rightarrow \begin{cases} \theta_1 = \sin^{-1}\left(\dfrac{r}{d_0+r_0+r}\right) \\ \theta_2 = \sin^{-1}\left(\dfrac{r}{d}\right) \end{cases}.$$

Since $\triangle AOA' \cong \triangle BOB'$ and $\triangle COC' \cong \triangle DOD'$, then $\angle AOA' = \angle BOB' = \theta_1, \angle COC' = \angle DOD'$. Hence $\angle A'OB' = \theta_0 - 2\theta_1$ and $\angle C'OD' = \theta_0 - 2\theta_2$. Therefore, the area of the pencil $OA'B'$ is $\pi\left(d_0+r_0+r\right)^2 \cdot \dfrac{\theta_0-2\theta_1}{2\pi}$ and the area of the pencil $OC'D'$ is $\pi d^2 \cdot \dfrac{\theta_0-2\theta_2}{2\pi}$, denoted by A_1 and A_2. Note that $S = A_2 - A_1 - S_{\triangle A'OD'} - S_{\triangle B'OC'}$, then we need to calculate the area of these two triangle. It's sufficient to show that $\triangle A'OD' \cong \triangle B'OC'$, WOLOG, suppose $S' = S_{\triangle A'OD'} = S_{\triangle B'OC'}$. Note that one edge of $\triangle A'OD'$, $OD' = d$, and the altitude due to this edge is $OA' \cdot \sin\left(\angle A'OD'\right) = \left(d_0+r_0+r\right)\sin\left(\angle AOA' - \angle DOD'\right) = \left(d_0+r_0+r\right)\sin\left(\theta_1-\theta_2\right)$. Therefore, $S' = \dfrac{d\left(d_0+r_0+r\right)\sin\left(\theta_1-\theta_2\right)}{2}$.

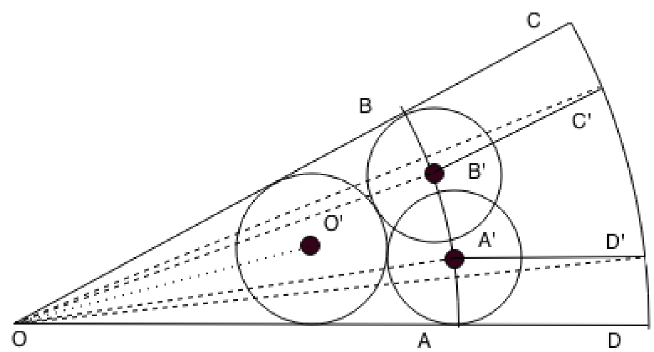
Hence

$$\begin{aligned} S &= \pi d^2 \cdot \frac{\theta_0-2\theta_2}{2\pi} - \pi\left(d_0+r_0+r\right)^2 \cdot \frac{\theta_0-2\theta_1}{2\pi} - 2\left(\frac{d\left(d_0+r_0+r\right)\sin\left(\theta_1-\theta_2\right)}{2}\right) \\ &= \frac{d^2\left(\theta_0-2\theta_2\right)}{2} - \frac{\left(d_0+r_0+r\right)^2\left(\theta_0-2\theta_1\right)}{2} - d\left(d_0+r_0+r\right)\sin\left(\theta_1-\theta_2\right) \\ &= d^2\left(\sin^{-1}\left(\frac{r_0}{d_0}\right) - \arcsin\left(\frac{r}{d}\right)\right) - \left(d_0+r_0+r\right)^2\left(\arcsin\left(\frac{r_0}{d_0}\right) - \arcsin\left(\frac{r}{d_0+r_0+r}\right)\right) - \\ &\quad d\left(d_0+r_0+r\right)\left(\left(\frac{r}{d_0+r_0+r}\right)\sqrt{1-\left(\frac{r}{d}\right)^2} - \left(\frac{r}{d}\right)\sqrt{1-\left(\frac{r}{d_0+r_0+r}\right)^2}\right) \end{aligned}$$

Case 2. C is not tangent to C_0.

Note that $S_{\triangle C'OD'} = S_{\triangle COD} - S_{\triangle OO'C'} - S_{\triangle OO'D'}$.

Figure 4.5: Case 2.



It's similar to solve the shadowed area in case 2. Denote the area of two pencil $OA'B'$ and $OC'D'$ to be A_1 and A_2, then $S = A_2 - A_1 - S_{\triangle C'OO'} - S_{\triangle D'OO'}$, $OO' = \dfrac{r}{\sin\left(\frac{\theta_0}{2}\right)}$, and $\angle A'OB' = \angle C'OD' = 2\arcsin\left(\dfrac{r}{d}\right)$, then

$$\begin{cases} A_1 & = \pi\left(OO'\right)^2 \cdot \dfrac{\angle A'OB'}{2\pi} = \left(\dfrac{r}{\sin\left(\frac{\theta_0}{2}\right)}\right)^2 \cdot \left(\dfrac{2\arcsin\left(\frac{r}{d}\right)}{2}\right) = \left(\dfrac{rd_0}{r_0}\right)^2 \left(\arcsin\left(\dfrac{r}{d}\right)\right) \\ A_2 & = \pi d^2 \cdot \dfrac{\angle C'OD'}{2\pi} = d^2 \arcsin\left(\dfrac{r}{d}\right) \\ S_{\triangle C'OO'} & = OC' \cdot OO' \cdot \sin\left(\dfrac{\angle COD - \angle C'OD'}{2}\right) = \left(\dfrac{drd_0}{r_0}\right)\left(\dfrac{r_0}{d_0}\sqrt{1-\left(\dfrac{r}{d}\right)^2} - \dfrac{r}{d}\sqrt{1-\left(\dfrac{r_0}{d_0}\right)^2}\right) \end{cases}.$$

Hence $S = \left(\dfrac{rd_0}{r_0}\right)^2 \left(\sin^{-1}\left(\dfrac{r}{d}\right)\right) - d^2\sin^{-1}\left(\dfrac{r}{d}\right) - 2\left(\dfrac{drd_0}{r_0}\right)\left(\dfrac{r_0}{d_0}\sqrt{1-\left(\dfrac{r}{d}\right)^2} - \dfrac{r}{d}\sqrt{1-\left(\dfrac{r_0}{d_0}\right)^2}\right)$.

Under uniform distribution, the probability of trees in one size (intersection surface areas are the same) can be fully shadowed is equal to the ratio of the shadowed area to the total area (sample area).

Since the sample area is πd^2, denoted by SA, then the probability is $\dfrac{S}{SA}$ in general form.

$$P(r) = \begin{cases} 0 & \text{if } r < \dfrac{r_0 \sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} \\ \dfrac{d^2\left(\arcsin\left(\frac{r_0}{d_0}\right) - \sin^{-1}\left(\frac{r}{d}\right)\right) - (d_0+r_0+r)^2\left(\arcsin\left(\frac{r_0}{d_0}\right) - \sin^{-1}\left(\frac{r}{d_0+r_0+r}\right)\right)}{\pi r^2/\sin^2\frac{\theta}{2}} - & \text{if } \dfrac{r_0\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} \leqslant r < \\ \dfrac{d\,(d_0+r_0+r)\left(\left(\frac{r}{d_0+r_0+r}\right)\sqrt{1-\left(\frac{r}{d}\right)^2} - \left(\frac{r}{d}\right)\sqrt{1-\left(\frac{r}{d_0+r_0+r}\right)^2}\right)}{\pi r^2/\sin^2\frac{\theta}{2}} & \dfrac{d_0\cdot r_0 + r_0^2}{d_0 - r_0} \\ \dfrac{\left(\frac{rd_0}{r_0}\right)^2\frac{\theta}{2} - d^2\frac{\theta}{2} - 2\left(\frac{drd_0}{r_0}\right)\left(\frac{r_0}{d_0}\sqrt{1-\left(\frac{r}{d}\right)^2} - \frac{r}{d}\sqrt{1-\left(\frac{r_0}{d_0}\right)^2}\right)}{\pi r^2/\sin^2\frac{\theta}{2}} & \\ & \text{if } r \geqslant \dfrac{d_0\cdot r_0 + r_0^2}{d_0 - r_0} \end{cases}.$$

Hence, given a kind of tree with radius r_j and number N_j, the expectation of number of countable trees blocked by this tree is just $\sum_{j=0}^{n_0} E_j$ where n_0 denotes tree sizes. $E_j = N_j \cdot P(r_j)$.

Notice that the radius of R has the same distribution for every tree. Given that the trees are uniformly distributed, R determines the upper bound of D and the distribution of D in the following way. Fix $R = r$. The cumulative function for D is

$$F_D(d) = \begin{cases} 0 & \text{if } d \leqslant 0 \\ \dfrac{d^2\cdot\sin^2\frac{\theta}{2}}{r^2} & \text{if } 0 < d \leqslant \dfrac{r}{\sin\frac{\theta}{2}} \\ 1 & \text{if } d > \dfrac{r}{\sin\frac{\theta}{2}} \end{cases}.$$

Hence the density distribution function of R is

$$f_D(d) = \begin{cases} 0 & \text{if } d \leqslant 0 \\ \dfrac{\sin^2\frac{\theta}{2}\cdot 2d}{r^2} & \text{if } 0 < d \leqslant \dfrac{r}{\sin\frac{\theta}{2}} \\ 0 & \text{if } d > \dfrac{r}{\sin\frac{\theta}{2}} \end{cases}.$$

With $D = d$, the expectation of S_i is just

$$\begin{aligned} E_D(R = r_0) &= \int_0^{r_j/\sin\frac{\theta}{2}} \sum_{j=0}^{n_0} E_j \\ &= \sum_{j=0}^{n_0} \int_0^{r_j/\sin\frac{\theta}{2}} E_j f_D(d_0)\,dd_0. \end{aligned}$$

Now we specifically focus on the second case when the blocked trees can't be tangent to the blocking tree, given a type of tree of radius r_j with number N_j, the expectation of number of this type of tree blocked by a tree of size r_0 with distance d_0 to the sampling point is $\int_0^{\min\left(\frac{r_0^2+r_0\cdot d}{d-r_0}, r_j/\sin\frac{\theta}{2}\right)} E_j \cdot f_D(d)\ dd_0$ since d_0 has to be both smaller than $\dfrac{r_0}{\sin\frac{\theta}{2}}$ for the blocking tree to be countable and smaller than $\dfrac{r_0^2+r_0\cdot d}{d-r_0}$ for it to be in the second case. We have

$$\begin{aligned}\int_0^{\min\left(\frac{r_0^2+r_0\cdot d}{d-r_0}, r_j/\sin\frac{\theta}{2}\right)} E_j \cdot f_D(d)\ dd_0 \leqslant& \int_0^{r_j/\sin\frac{\theta}{2}} E_j \cdot f_D(d)\ dd_0 \\ =&\frac{1}{4\pi}+\frac{4}{3\pi}+\cos^3\frac{\theta}{2}\sin\frac{\theta}{2}-\frac{\theta r_j}{2\pi\sin\frac{\theta}{2}}-\frac{2r_j\cos\frac{\theta}{2}}{\pi}.\end{aligned}$$

This is an upper bound of the expectation of the number trees blocked by one tree in the second case. Since this result doesn't depends on r_0, the expectation of the number of all the blocked countable trees of radius r_j can be obtained by multiplying it with an N. This result also indicates that we can make the upper bound of the expectation smaller by choosing a prism with an appropriate angle.

For the first case, the blocked countable trees are of relatively small size compared with the blocked countable trees in the second case. Hence it's reasonable to assume that the the probability for an r-radius tree to be blocked should depend heavily on the radius of the blocking tree and the distance from the blocking tree to the sampling point. Since $d > d_0 + r_0 + r$ we have

$$\begin{aligned}P(r) =&\frac{d^2\left(\arcsin\left(\frac{r_0}{d_0}\right)-\arcsin\left(\frac{r}{d}\right)\right)-(d_0+r_0+r)^2\left(\arcsin\left(\frac{r_0}{d_0}\right)-\arcsin\left(\frac{r}{d_0+r_0+r}\right)\right)}{\pi r^2/\sin^2\frac{\theta}{2}}- \\ &\frac{d\,(d_0+r_0+r)\left(\left(\frac{r}{d_0+r_0+r}\right)\sqrt{1-\left(\frac{r}{d}\right)^2}-\left(\frac{r}{d}\right)\sqrt{1-\left(\frac{r}{d_0+r_0+r}\right)^2}\right)}{\pi r^2/\sin^2\frac{\theta}{2}} \\ =&\frac{(d^2-(d_0+r_0+r)^2)\arcsin\left(\frac{r_0}{d_0}\right)-d^2\sin^{-1}\left(\frac{r}{d}\right)+(d_0+r_0+r)\sin^{-1}\left(\frac{r}{d_0+r_0+r}\right)}{\pi r^2/\sin^2\frac{\theta}{2}}- \\ &\frac{d\,(d_0+r_0+r)\left(\left(\frac{r}{d_0+r_0+r}\right)\sqrt{1-\left(\frac{r}{d}\right)^2}-\left(\frac{r}{d}\right)\sqrt{1-\left(\frac{r}{d_0+r_0+r}\right)^2}\right)}{\pi r^2/\sin^2\frac{\theta}{2}}.\end{aligned}$$

Since $\dfrac{r_0}{d_0} \leqslant \dfrac{r}{d}$, we have

$$\begin{aligned}P(r) \leqslant&\frac{(d_0+r_0+r)^2\left(\arcsin\left(\frac{r}{d_0+r_0+r}\right)-\arcsin\left(\frac{r_0}{d_0}\right)\right)}{\pi r^2/\sin^2\frac{\theta}{2}}- \\ &\frac{d\,(d_0+r_0+r)\left(\left(\frac{r}{d_0+r_0+r}\right)\sqrt{1-\left(\frac{r}{d}\right)^2}-\left(\frac{r}{d}\right)\sqrt{1-\left(\frac{r}{d_0+r_0+r}\right)^2}\right)}{\pi r^2/\sin^2\frac{\theta}{2}}.\end{aligned}$$

Applying $d > d_0 + r_0 + r$ to both the first half and the second half of the equation, we can have

$$\begin{aligned}
P(r) \leqslant & \frac{(d_0 + r_0 + r)^2 \left(\arcsin\left(\frac{r}{d_0+r_0+r}\right) - \arcsin\left(\frac{r_0}{d_0}\right)\right)}{\pi r^2 / \sin^2 \frac{\theta}{2}} \\
\leqslant & \frac{\sin^4 \frac{\theta}{2} \left(\arcsin\left(\frac{r}{d_0+r_0+r}\right) - \arcsin\left(\frac{r_0}{d_0}\right)\right)}{\pi}.
\end{aligned}$$

All the upper bound we find depends on depends on both r_0 and d_0. However, this upper bound still indicates that the probability of missing blocked trees might be made smaller in the following two ways:

1. choosing a prism with a smaller offset angle.

2. choosing a sample point where trees around are small.

When we decrease the offset angle of the prism, θ, $\sin^4 \dfrac{\theta}{2}$ goes down and this upper bound goes down. Actually, it can actually be arbitrarily small as we can choose a prism with a small offset angle. When choose sampling points where trees around are small, r_0 should also be small. As a result, $\arcsin\left(\dfrac{r}{d_0 + r_0 + r}\right)$ and $-\arcsin\left(\dfrac{r_0}{d_0}\right)$ are both small which make the upper bound small.

5 Conclusion

In this project, we have fully explained the steps of making an estimation of timber using "Variable-radius Plot Sampling" method; given a formalized proof of the multiplier that is used to obtain the final result; discovered a serious drawback of the method; calculated an upper bound for the probability of error, and improved the method by suggesting on the choose of prism and the plot center.

The mathematical model is derived based on the important assumption that trees are uniformly distributed in the forest. However, it is barely true in practice. The analysis under complex distributions remains for future exploration.

A Code

```python
from __future__ import division

from visual import *
from visual.graph import *

import math,random


width = 300
height = 300
obv_point = [[vector(0,0,0)]*5 for i in range(5)]

for i in range(5):
    for j in range(5):
        obv_point[i][j] = vector(width*(1/10+i/5),height*(1/10+j/5),0)

print obv_point


scene = display(title='Estimating Timber',
                width=width, height=height,
                center=obv_point[2][2], background=color.white)


number = 20000    #number of trees
theta = 12       #angle of prism
threshold = math.tan(theta*pi/360)
print threshold

r_min = 0.10    #min radius of trees
r_max = 0.75    #max radius of trees
h_fix = 10      #fix the height of trees be 10 meters

trees = []      #set of trees

tree_empty = cylinder(pos = vector(0,0,0),
                      axis = vector(0,0,0),
                      radius = 0)


def set_up_field(number):
    count = 0
    for i in range(number+1):
        if (i == 0):
            trees.append(tree_empty)
        else:
            pos = vector(random.uniform(0,height),random.uniform(0,width),0)
            r = random.uniform(r_min,r_max)
            if (mag(pos-trees[i-1].pos) < r+trees[i-1].radius):
                trees.append(tree_empty)
                count += 1
            else:
                trees.append(cylinder(pos = pos,
                              axis = vector(0,0,h_fix),
                              radius = r,
                              color = color.orange))
    return count

ava_trees = number - set_up_field(number)
```

```
print ava_trees




def countable(distance,radius):
    output = 0
    if (radius/distance > threshold):
        output += 1
    elif (radius/distance == threshold):
          output += 1/2
    else:
          output += 0
    return output

def count_tree(vec):
    count_1 = 0
    for i in range(1,ava_trees):
          d = mag(vec-trees[i].pos)
          r = trees[i].radius
          count_1 += countable(d,r)
    return count_1

for i in range(5):
    for j in range(5):
        print count_tree(obv_point[i][j])
```