Sums of Two Fractional Powers

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1 Multiple Hits

Fix $\theta \in \mathbb{R}_{>1}$, for the equation $m_1^\theta + n_1^\theta = m_2^\theta + n_2^\theta$ where $m_1, n_1, m_2, n_2 \in \mathbb{Z}_{\geqslant 0}$.

Proposition 1. *If $m_1 = \max\{m_1, m_2, n_1, n_2\}$, then $n_1 = \min\{m_1, m_2, n_1, n_2\}$.*

Proof. Suppose otherwise. Since $m_1 = \max\{n_1, m_1, n_2, m_2\}$ and $n_1 \neq \min\{m_1, m_2, n_1, n_2\}$, we have $n_1 > \min\{m_2, n_2, m_1\} = \min\{m_2, n_2\} \Rightarrow n_1^\theta > \min\{m_2^\theta, n_2^\theta\}$. We also have $m_1^\theta \geqslant \max\{m_2^\theta, n_2^\theta\}$. Hence, $m_1^\theta + n_1^\theta > \min\{m_2^\theta, n_2^\theta\} + \max\{m_2^\theta, n_2^\theta\} = m_2^\theta + n_2^\theta$. Contradiction. Hence, we know that $n_1 = \min\{m_2, n_2, m_1, n_1\}$. □

Proposition 2. *Given $m_0 < m_1 < m_2$, $\dfrac{m_1^\theta - m_0^\theta}{m_1 - m_0} \leqslant \dfrac{m_2^\theta - m_1^\theta}{m_2 - m_1}$.*

Proof. Fact: $\phi(x) = x^\theta$ is a convex function. By the definition of convex function, we have

$$\phi\left((1-t)\,a + tb\right) \leqslant (1-t)\,\phi\,(a) + t\phi\,(b)$$

for all $0 < t < 1$ and $a, b > 0$. Now we substitute $a = m_0$, $b = m_2$, $t = \dfrac{m_1 - m_0}{m_2 - m_0}$ and get

$$\begin{aligned}
(m_2 - m_0)\,\phi(m_1) &\leqslant (m_2 - m_1)\,\phi\,(m_0) + (m_1 - m_0)\,\phi\,(m_2) \\
m_2\phi\,(m_1) - m_2\phi\,(m_0) + m_1\phi\,(m_0) &\leqslant m_1\phi\,(m_2) - m_0\phi\,(m_2) + m_0\phi\,(m_1) \\
m_2\phi\,(m_1) - m_2\phi\,(m_0) - m_1\phi\,(m_1) + m_1\phi\,(m_0) &\leqslant m_1\phi\,(m_2) - m_0\phi\,(m_2) - m_1\phi\,(m_1) + m_0\phi\,(m_1) \\
(\phi\,(m_1) - \phi\,(m_0))\,(m_2 - m_1) &\leqslant (\phi\,(m_2) - \phi\,(m_1))\,(m_1 - m_0) \\
\frac{\phi\,(m_1) - \phi\,(m_0)}{m_1 - m_0} &\leqslant \frac{\phi\,(m_2) - \phi\,(m_1)}{m_2 - m_1} \\
\frac{m_1^\theta - m_0^\theta}{m_1 - m_0} &\leqslant \frac{m_2^\theta - m_1^\theta}{m_2 - m_1}
\end{aligned}$$

□

Proposition 3. *If $m_1 > m_2$, $n_1 < n_2$, then $m_1 + n_1 \leqslant m_2 + n_2$.*

Proof. WOLOG, assume $n_1 < n_2 < m_2 < m_1$. Then by Proposition 1 we have $\dfrac{n_2^\theta - n_1^\theta}{n_2 - n_1} \leqslant \dfrac{m_2^\theta - n_2^\theta}{m_2 - n_2} \leqslant \dfrac{m_1^\theta - m_2^\theta}{m_1 - m_2} \Rightarrow \dfrac{n_2^\theta - n_1^\theta}{n_2 - n_1} \leqslant \dfrac{m_1^\theta - m_2^\theta}{m_1 - m_2}$. Since $n_2^\theta - n_1^\theta = m_1^\theta - m_2^\theta \neq 0$, we get $n_2 - n_1 \geqslant m_1 - m_2 \Rightarrow m_2 + n_2 \geqslant m_1 + n_1$. □

To find the nontrivial solutions i.e. $(m_1, n_1) \neq (m_2, n_2) \neq (n_1, m_1)$, there are two cases for $\theta \in \mathbb{R}_{>1}$:

Case 1. $\theta \in \mathbb{Q}_{>1}$:

Since $\theta \in \mathbb{Q}$, WOLOG, suppose $\theta = \dfrac{p}{q}$ for some $p \in \mathbb{Z}_{\geqslant 0}, q \in \mathbb{Z}_{>0}$ such that $\gcd(p, q) = 1$, then $m^{\theta} = \sqrt[q]{m^p}$. Since $\theta \in \mathbb{Z}_{>1}$, it's possible to find out the nontrivial solutions of the pairs (m_1, n_1) and (m_2, n_2).

$$m_1^{\theta} + n_1^{\theta} = m_1^{\frac{p}{q}} + n_1^{\frac{p}{q}} = m_2^{\frac{p}{q}} + n_2^{\frac{p}{q}} = m_2^{\theta} + n_2^{\theta}.$$

Note that $m^{\theta} + n^{\theta}$ is not necessary to be rational, and it's impractical to calculate the nontrivial solutions which are irrational, but we still could find out some nontrivial solutions such that $m^{\theta} + n^{\theta} \in \mathbb{Z}$.

Next, we set up some algorithm to check if the pairs (m_1, n_1) and (m_2, n_2) are nontrivial solutions for the equation.

Example 1. Let $\theta = 2$.

Apply the transposition to n_1 and n_2, we have $m_1^2 - n_2^2 = m_2^2 - n_1^2$, then $(m_1 - n_2)(m_1 + n_2) = (m_2 - n_1)(m_2 + n_2)$. In this case, we can first find the divisor for some integers. Now we pick a number that have at least two pairs of divisors of which sum are even integers.

e.g. If we pick $105 = 21 \times 5 = 15 \times 7 = 35 \times 3$. Take the pair $(21, 5)$ and $(15, 7)$. we have $\begin{cases} m_1 + n_2 &= 21 \\ m_1 - n_2 &= 5 \end{cases}$, $\begin{cases} m_2 + n_1 &= 15 \\ m_2 - n_1 &= 7 \end{cases}$, then $\begin{cases} (m_1, n_1) &= (13, 4) \\ (m_2, n_2) &= (11, 8) \end{cases}$.

Once we collect the acceptable pairs of solutions for p=2, put them into one set. If we want to compute the multiplier of 2, we only need to consider the pairs in the set we have.

Similarly, to check the pairs of solutions (m_1, n_1) and (m_2, n_2) with some sufficiently large integer θ, since $\theta = \prod_{i=1}^{r} p_i^{n_i}$ for some $p_i \in \mathbb{Z}_{>1}$ prime, then

$$m_1^{\theta} + n_1^{\theta} = \left(\left(m_1^{p_1^{n_1}}\right)^{p_2^{n_2}}\right)^{\cdots} + \left(\left(n_1^{p_1^{n_1}}\right)^{p_2^{n_2}}\right)^{\cdots}.$$

We only need to check if the m_1, n_1, m_2, n_2 match the solution we computed for $m_1^{p_i^{n_i}}, n_1^{p_i^{n_i}}, m_2^{p_i^{n_i}}, n_2^{p_i^{n_i}}$ $\forall i \in (1, r) \cap \mathbb{Z}$.

Since $m_1, n_1, m_2, n_2 \in \mathbb{Z}_{\geqslant 0}$ are in perfect q-power, then, WOLOG, suppose $m_1 = a^q$, $n_1 = b^q$ for some $a, b \in \mathbb{Z}_{\geqslant 0}$.

Claim 1. *If* $\dfrac{p}{q} \in \mathbb{Q}_{>1} \backslash \mathbb{Z}$, *then* $q \geqslant 2$.

Proof. Suppose $q < 2$, since $q \in \mathbb{Z}_{>0}$, then $q = 1$. Note that since $p \in \mathbb{Z}_{\geqslant 0}$, then $\dfrac{p}{q} \in \mathbb{Z}$, meets the contradiction. Therefore, $q \geqslant 2$. □

Since we suppose that $\theta = \dfrac{p}{q} > 1$ and $q \geqslant 2$, then $p > 2$. Note that $m_1^{\theta} + n_1^{\theta} = (a^q)^{\theta} + (b^q)^{\theta} = a^p + b^p$.

We want to calculate the nontrivial solutions in perfect q-power.

Claim 2. *For fix* $\theta = \dfrac{p}{q} \in \mathbb{Q}_{>1}$*. If there exists some nontrivial solutions for* $(m_2, n_2) = (c^q, d^q)$ *where* $c, d \in \mathbb{Z}_{>0}$*, then* $(m_1, n_1) \in \mathbb{Z}^2_{>0}$*.*

Proof. Assume one of m_1 and n_1 is zero. WOLOG, suppose $m_1 = 0$, then $m_1^\theta + n_1^\theta = n_1^\theta = b^p = m_2^\theta + n_2^\theta = c^p + d^p$. By the Fermat's Last Theorem, since $p > 2$, we know that $(m_2, n_2) \notin \mathbb{Z}^2_{>0}$, meets the contradiction. Therefore, $(m_1, n_1) \in \mathbb{Z}^2_{>0}$. $\square$

Therefore, after decomposing p, by the method above, we can find out the perfect q-power solutions for the sum of two rational powers.

Remark 2. For $\theta = \dfrac{p}{q} \in \mathbb{Q}$, define $P = \{g \in G \mid |g| = p'\}$ where, one decompose prime of p, $p' \in \mathbb{Z}_{>0}$. If $G = \mathbb{Z}_n$, then $[m_1], [n_1], [m_2], [n_2] \in P$.

Regarding the statement above, the numbers we need to consider become even less by using the Euler's function $\phi(n)$.

Case 2. $\theta \in (\mathbb{R}\backslash\mathbb{Q})_{>1}$:

Though there is no general formula for the irrational power, we still can get a good approximation by rational powers. For example, $\pi = 3.14159265\cdots$, the infinite sequence $\{s_n\} = \{3, 3.1, 3.14, 4.141, \dots\}$, we see that $s_n \to \pi$.

Lemma 3. If $s_n \to \theta$, then $\lim\limits_{s_n \to \theta} m^{s_n} = m^\theta$.

Therefore, we can determine the approximate amount of its solutions by the method in next section.

2 An Analysis on Gaps

Let $OA'B' = \{(x, y) : x^\theta + y^\theta \leqslant 1\}$. Fix X and let $OAB = \{(x, y) : x^\theta + y^\theta \leqslant X\}$. Denote the area of OAB by S_{OAB} and the area of $OA'B'$ by $S_{OA'B'}$.

Since the region OAB can be obtained by doing a linear transformation on $OA'B'$ and the determinant of this linear transformation is $X^{\frac{2}{\theta}}$, $S_{OAB} = X^{\frac{2}{\theta}} \cdot S_{OA'B'}$. Let $S_{OAB} = C_\theta$ where C_θ is a constant that depends on θ.

Let's first calculate C_θ. Taking the Taylor expansion of the integrand at zero with respect to m^θ, we have

$$\left(1 - m^\theta\right)^{\frac{1}{\theta}} = 1 + \sum_{i=1}^{\infty} (-1)^i \frac{\prod_{j=1}^{i} \left(\frac{1}{\theta} - j\right) m^{i\theta}}{i!}$$

Hence,

$$\begin{aligned} C_\theta &= \int_0^1 \left(1 - m^\theta\right)^{\frac{1}{\theta}} dm \\ &= \left(m + \sum_{i=1}^{\infty} (-1)^i \frac{\prod_{j=1}^{i} \left(\frac{1}{\theta} - j\right) m^{i\theta+1}}{i!\,(i\theta + 1)} \right) \Bigg|_{m=0}^{1} \\ &= 1 + \sum_{i=1}^{\infty} (-1)^i \frac{\prod_{j=1}^{i} \left(\frac{1}{\theta} - j\right)}{i!\,(i\theta + 1)} \end{aligned}$$

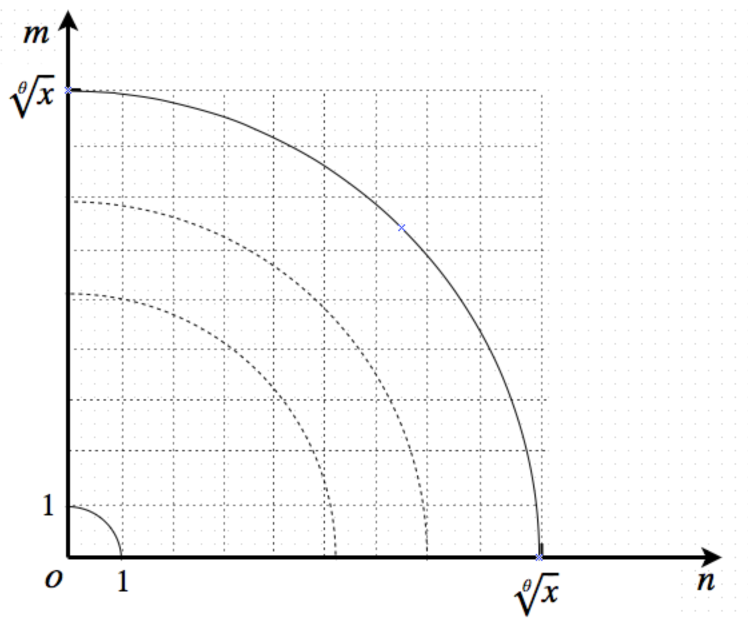
Therefore,

$$S_{OAB} = X^{\frac{\theta}{2}}\left(1+\sum_{i=1}^{\infty}(-1)^i\,\frac{\prod_{j=1}^{i}\left(\frac{1}{\theta}-j\right)}{i!\,(i\theta+1)}\right).$$

Fact 4. *For X large, S_{OAB} is a good approximation to the number of pairs of (m,n) such that $m^\theta + n^\theta \leqslant X$.*

If (m,n) satisfies $m^\theta + n^\theta \leqslant X$, then $[m,m+1)\times[n,n+1)$ must intersect with OAB. Each square $[m,m+1)\times[n,n+1)$ has area 1. For X large, most of the squares $[m,m+1)\times[n,n+1)$ such that $m^\theta + n^\theta \leqslant X$ are completely covered by the region OAB. The rest part of OAB that is not in the squares it covers will be covered by the remaining squares. This actually shows that the area of OAB is a lower bound for the area of $\bigcup_{m^\theta+n^\theta\leqslant X}[m,m+1)\times[n,n+1)$. However, since the fraction of the region $\bigcup_{m^\theta+n^\theta\leqslant X}[m,m+1)\times[n,n+1)$ not covered by OAB is small when X get large. We can take the S_{OAB} as a good approximation of the area of the region $\bigcup_{m^\theta+n^\theta\leqslant X}[m,m+1)\times[n,n+1)$ which equals the number of integer pairs of (m,n) that satisfies $m^\theta + n^\theta \leqslant X$ when X is large.

Figure 2.1: Approximating the Number of Pairs of (m,n) with S_{OAB}



Fact 5. *The growth rate of the number of solutions with respect to X is approximately the growth rate of S_{OAB} for X large.*

This fact follows from Fact 3. $\dfrac{dS_{OAB}}{dX} = \dfrac{d(X^{\frac{2}{\theta}})C_\theta}{dX} = C_\theta\dfrac{2}{\theta}X^{(\frac{2}{\theta}-1)}$. Hence for $\theta > 2$, $\dfrac{2}{\theta}-1<0$ and the asymptomatic growth rate of the number of integer solutions (m,n) to $m^\theta + n^\theta \leqslant X$ decreases as X get larger. For $\theta = 2$, $\dfrac{2}{\theta}-1=0$ and the growth rate of the number of integer solutions shouldn't change very much X. For $\theta < 2$, $\dfrac{2}{\theta} - 1 > 0$ and the number of solutions will grow faster as X gets larger. Now let's consider all the y's such that $y = m^\theta + n^\theta$. We arrange them in the increasing order.

$$0 = y_0 \leqslant y_1 \leqslant y_2 \leqslant y_3 \ldots .$$

Here we allow duplicates y_i. Notice that $C_\theta \dfrac{2}{\theta} X^{(\frac{2}{\theta}-1)}$ is approximately the growth rate of the size of set $\{y : y = m^\theta + n^\theta \text{ and } y \leqslant X\}$ with respect to X. Hence the reciprocal of $C_\theta \dfrac{2}{\theta} X^{(\frac{2}{\theta}-1)}$, $C_\theta \dfrac{\theta}{2} X^{1-\frac{2}{\theta}}$ evaluated at y_i is approximately the change of X so that the next number of y_i, y_{i+1}, will be included in the set $\{y : y = m^\theta + n^\theta \text{ and } y \leqslant X\}$ which is approximately the gap between y_i and y_{i+1}. Let $z_i = y_{i+1} - y_i$. From our conclusions about the growth rate, we know that for $\theta > 2$, z_i is expected to get larger as y_i gets larger. For $\theta = 2$, z_i should scatter around some constant and for $\theta < 2$, z_i is supposed to get smaller as y_i gets larger.

We verified our conclusion against all the $y_i \leqslant 250^\theta$ for θ equaling 1.5, 2, 2.5 with experiments. Instead of directly analyzing z_i, we do a convolution operation on the set of z_is by taking the moving average of every 100 z_is. We present the results of our experiments below.

Figure 2.2: $\theta = \dfrac{3}{2}$

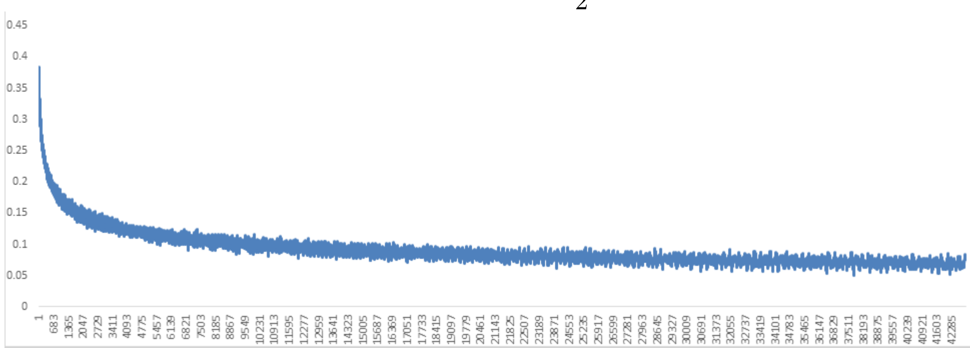


Figure 2.3: $\theta = 2$

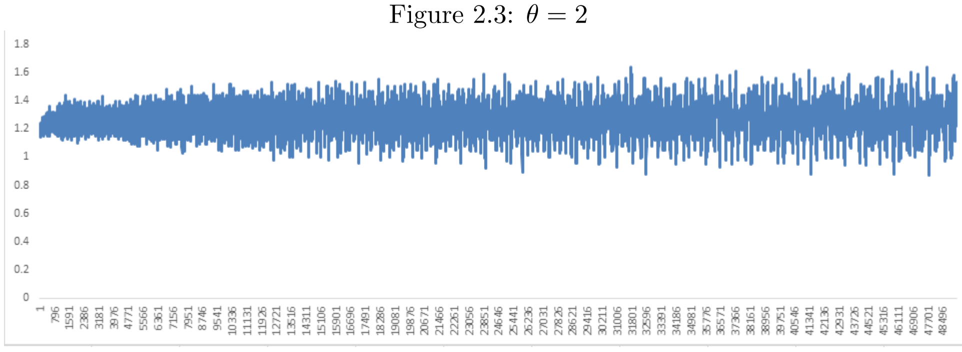
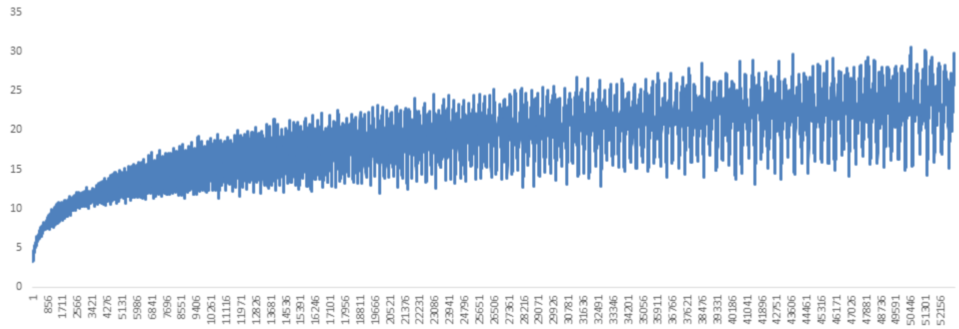


Figure 2.4: $\theta = \dfrac{5}{2}$



The experiment results support our conclusions.

Fix an arbitrary $\epsilon > 0$. Consider the interval $[x, x+\epsilon]$. For $\theta > 2$, since the gap between y_i and y_{i+1} will get bigger as y_i gets bigger. So it's reasonable to predict that there will be less and less y_i that falls in to this interval $[x, x+\epsilon]$. Equivalently, as X gets larger, there will be less and less pairs of (m, n) that satisfies $X \leqslant m^\theta + n^\theta \leqslant X + \epsilon$. However, for $\theta < 2$, the gap between y_i and y_{i+1} will get smaller as y_i increases and the growth rate of size of the set $\{(m, n) : m^\theta + n^\theta \leqslant X\}$ will get larger as x get larger. So it's reasonable to predict that the number of pairs of (m, n) that satisfies $X \leqslant m^\theta + n^\theta \leqslant X + \epsilon$ will get larger as X get larger. Based on these two predictions, we end this section with the following two conjectures.

Conjecture 6. *For $1 < \theta < 2$, the number of nontrivial solution to the equation $m_1^\theta + n_1^\theta = m_2^\theta + n_2^\theta$ is infinite.*

Conjecture 7. *For $\theta > 2$, the number of nontrivial solution to the equation $m_1^\theta + n_1^\theta = m_2^\theta + n_2^\theta$ is finite.*

A Code

```cpp
//
//  main.cpp
//  389p3
//
//  Created by Ruizhi Deng on 4/18/15.
//  Copyright (c) 2015 Ruizhi Deng. All rights reserved.
//

#include <iostream>
#include <iomanip>
#include <vector>
#include <cmath>
#include <algorithm>
#include <fstream>
#include <sstream>
#include <map>
using namespace std;

int main(int argc, const char * argv[]) {
    ofstream output;
    ostringstream* ost = new ostringstream;
    *ost << setprecision(15);
    output.open("/Users/wisdondeng/Desktop/gap3_with0.csv");
    double power = 3.0/2;
    vector<double>* result = new vector<double>;

    if (!output.good()) {
        exit(1);
    }
    for (int i = 0; i < 250; ++i) {
        for (int j = 0; j < 250; ++j) {
            result->push_back(pow(double(i), power) + pow(double(j), power));
        }
    }
    sort(result->begin(), result->end());
    auto it = result->begin();
    auto prev = it;
    double op;
    double upper_bound = pow(250, power);
    while (it!=result->end() && (*it) <= upper_bound) {/* make sure the current value is smaller than 250^\theta*/
        ++it;
        op = *it - *prev;
        *ost << *prev << "," << op <<"\n";
        ++prev;
    }
    output << ost->str();
    output.close();
    delete ost;
    delete result;
    return 0;
}
```