Sums of Two Fractional Powers: Proposal

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1 Multiple Hits

Fix $\theta \in \mathbb{R}_{>1}$, for the equation $m_1^\theta + n_1^\theta = m_2^\theta + n_2^\theta$ where $m_1, n_1, m_2, n_2 \in \mathbb{Z}_{\geqslant 0}$.

Since $\theta \in \mathbb{Q}_{>1}$, WOLOG, suppose $\theta = \dfrac{p}{q}$ for some $p \in \mathbb{Z}_{\geqslant 0}, q \in \mathbb{Z}_{>0}$ such that $\gcd(p,q) = 1$, then $m^\theta = \sqrt[q]{m^p}$.

Since $\theta \in (\mathbb{R} \backslash \mathbb{Q})_{>1}$, $m^\theta = \exp(\theta \ln(m))$.

Proposition 1. *If* $m_1 = \max\{m_1, m_2, n_1, n_2\}$*, then* $n_1 = \min\{m_1, m_2, n_1, n_2\}$.

Proof. Since $m_1, m_2, n_1, n_2 \in \mathbb{Z}_{\geqslant 0}, \theta \in \mathbb{R}_{>1}$ and $m_1 = \max\{m_1, m_2, n_1, n_2\}$, then $m_1^\theta > n_1^\theta, m_2^\theta, n_2^\theta$. Since $m_1^\theta + n_1^\theta = m_2^\theta + n_2^\theta$, then $n_1^\theta < m_2^\theta, n_2^\theta < m_1^\theta$. Hence $n_1 = \min\{m_1, m_2, n_1, n_2\}$. □

Proposition 2. *Given* $m_0 < m_1 < m_2$, $\dfrac{m_1^\theta - m_0^\theta}{m_1 - m_0} \leqslant \dfrac{m_2^\theta - m_1^\theta}{m_2 - m_1}$.

Proof. Fact: $\phi(x) = x^\theta$ is a convex function. By the definition of convex function, we have

$$\phi((1-t)a + tb) \leqslant (1-t)\phi(a) + t\phi(b)$$

for all $0 < t < 1$ and $a, b > 0$. Now we substitute $a = m_0$, $b = m_2$, $t = \dfrac{m_1 - m_0}{m_2 - m_0}$ and get

$$\begin{aligned}
(m_2 - m_0)\phi(m_1) &\leqslant (m_2 - m_1)\phi(m_0) + (m_1 - m_0)\phi(m_2) \\
m_2\phi(m_1) - m_2\phi(m_0) + m_1\phi(m_0) &\leqslant m_1\phi(m_2) - m_0\phi(m_2) + m_0\phi(m_1) \\
m_2\phi(m_1) - m_2\phi(m_0) - m_1\phi(m_1) + m_1\phi(m_0) &\leqslant m_1\phi(m_2) - m_0\phi(m_2) - m_1\phi(m_1) + m_0\phi(m_1) \\
(\phi(m_1) - \phi(m_0))(m_2 - m_1) &\leqslant (\phi(m_2) - \phi(m_1))(m_1 - m_0) \\
\frac{\phi(m_1) - \phi(m_0)}{m_1 - m_0} &\leqslant \frac{\phi(m_2) - \phi(m_1)}{m_2 - m_1} \\
\frac{m_1^\theta - m_0^\theta}{m_1 - m_0} &\leqslant \frac{m_2^\theta - m_1^\theta}{m_2 - m_1}
\end{aligned}$$

□

Proposition 3. *If* $m_1 > m_2$*,* $n_1 < n_2$*, then* $m_1 + n_1 \leqslant m_2 + n_2$.

Proof. Without loss of generality, assume $n_1 < n_2 < m_2 < m_1$. Then by proposition 1 we have $\dfrac{n_2^\theta - n_1^\theta}{n_2 - n_1} \leqslant \dfrac{m_2^\theta - n_2^\theta}{m_2 - n_2} \leqslant \dfrac{m_1^\theta - m_2^\theta}{m_1 - m_2} \Rightarrow \dfrac{n_2^\theta - n_1^\theta}{n_2 - n_1} \leqslant \dfrac{m_1^\theta - m_2^\theta}{m_1 - m_2}$. Since $n_2^\theta - n_1^\theta = m_1^\theta - m_2^\theta \neq 0$, we get $n_2 - n_1 \geqslant m_1 - m_2 \Rightarrow m_2 + n_2 \geqslant m_1 + n_1$. □

2 An Analysis on Gaps

The equation $m_1^\theta + n_1^\theta = m_2^\theta + n_2^\theta$ can be written as:

$$m_1^\theta - m_2^\theta = n_2^\theta - n_1^\theta$$

If we define $G(x, y) = y^\theta - x^\theta$, then this problem can be changed to finding two same "Gaps" between two different pairs of integers.
We are going to analyze the distribution of $G(n, n+1)$ fixing θ.