Sums of Two Fractional Powers

Investigate sums of two fractional powers $m^\theta + n^\theta$ where m, n run over nonnegative integers, for fixed real values $\theta > 1$, with emphasis on non-integer values of θ

Some initial questions to consider to begin exploration follow. Many of following questions are formulated for one value of θ, but one may consider several different values of θ to get an idea how the answers to questions might depend on θ. The range $1 < \theta < 3$ might be a good place to start, for example $\theta = 3/2$ or $5/2$.

1. (Multiple hits) Determine information about which values of θ give a multiple solution
$$m_1^\theta + n_1^\theta = m_2^\theta + n_2^\theta$$
(Here $(m_1, n_1) \neq (m_2, n_2)$.) What can you say about the set of all such θ having a multiple solution?

2. (Growth rate) Arrange the set of all numbers $y = m^\theta + n^\theta$ in increasing order,
$$0 = y_0 \leq y_1 \leq y_2 \leq \cdots$$
(The values y_i may occur with multiplicity.) How many solutions are there with $1 \leq y \leq X$ for large X? Determine an asymptotic growth rate for this number if possible. What is the expected gap size $z_i = y_{i+1} - y_i$ below X?

3. (Gap distribution) What does the distribution of all the gap sizes up to X look like? How does it vary with different θ? (This problem might be explored using the computer, with the initial goal to come up with guesses what these distributions might look like. Compare plots of the distributions.)

4. (Gap Sizes) For a given θ, as X grows, how does the largest gap size behave, as a function of X? How does the smallest gap size behave, as a function of X?

5. (Variant: Ceiling and floor functions) One can get a problem with integer values by using the floor function (or ceiling function). That is, study the set $S(\theta)$ of all integers of the form
$$N := \lfloor m^\theta \rfloor + \lfloor n^\theta \rfloor,$$
for integer $(m, n) \geq 0$. For this set, one can ask all the same questions as above.

Note. (This project is designed to parallel "Sums of Cubes" but be distinct from it. Not all questions above may have nice answers; discovering whether there might be a nice answer is part of the exploration.)