# The Burnside's Lemma

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## Abstract

Burnside's Lemma is a mathematical tool that is used to help solve questions related to counting the number of distinct objects with respect to symmetry. It gives a formula to count the number of distinct objects - which means that objects that can be symmetric when a rotation or reflection is applied to it do not count as a distinct object. This paper will first provide an overview of the Burnside's Lemma formula and explain the key concepts of the formula. Then, this paper will include three "investigations" that will demonstrate the use of Burnside's Lemma in a mathematical field.


## 1. Introduction

To first demonstrate the purpose of Burnside's Lemma, let's assume that there is a square. The four sides of this square can be colored individually. How many colorings of a square using two colors is possible? The answer will be $2^4$ since there are two different colors and four possible sides to color. In total, there are 16 possible colorings of a square. However, by rotating or reflecting the squares, many of the colorings would be the same. This is where Burnside's Lemma comes to play. Burnside's Lemma is a result of the mathematics field called group theory. In a general sense, group theory is the mathematics of symmetry. Burnside's Lemma will allow the finding of distinct objects, in which when a reflection or a rotation is applied to the object, the object cannot be symmetric to the other objects.

## 2. The Burnside's Lemma Formula

The Burnside's Lemma "gives a way to count the number of *orbits* of a finite set *acted on* by a finite group." The Burnside's Lemma theorem states:

*Let* $G$ *be a finite group that acts on set X. Secondly, let* $X/G$ *be the set of* orbits *of X. Finally, for any element* $g \in G$*, let* $X^g$ *be the set of points of* $X$ *which are* fixed *by* $g$ *:*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

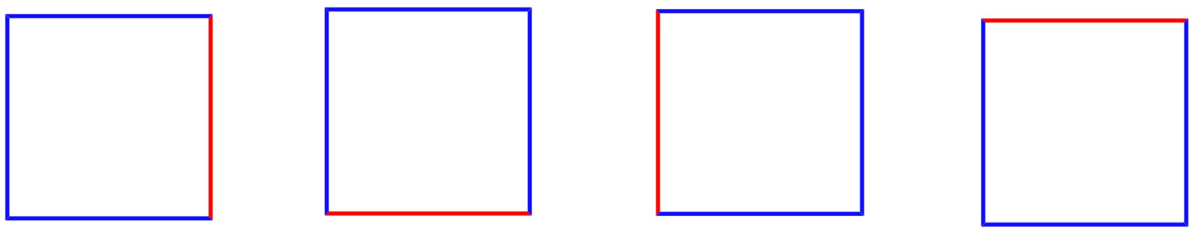
$X^g = \{x \in X : g \cdot x = x\}$. *Then the equation would be:*

At first, this seems very confusing, but when the equation is simplified, it essentially says that the number of *orbits* is equal to the average number of *fixed points* of $G$. (The terms *orbits* and *fixed points* will be explained later). The number of *orbits* is represented by the left side of the equation and the average number of *fixed points* of $G$ is represented by the right side of the equation. The number of *orbits* will always be a whole number, since having a fraction of an *orbit* is impossible. If there are zero elements of $X$, only then will there be zero *orbits*.

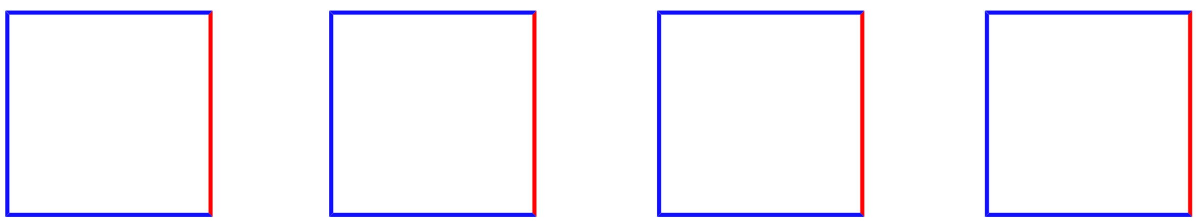
## 3. Background Information on the Burnside's Lemma

### 3.1. Orbits

A key term that is required to understand Burnside's Lemma is the term *orbits.* An *orbit* consists of a group of elements such that when one element can be "turned" to become symmetrical to any other element in the *orbit*. For example, all of these squares are elements of $X$.

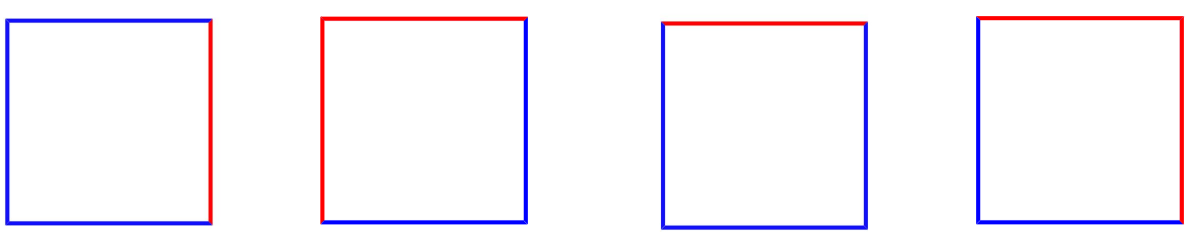


Each square, as shown above, looks different. But when all of the squares are rotated or reflected over a specific way, all of these squares will appear to be the same, as shown below.



Since any of these squares (which are elements of $X$) can be rotated or reflected to be symmetric to any other square in this example, the squares are all part of one *orbit.* The action of rotating or reflecting the squares is stated as “ the action of an element of $G$. ”

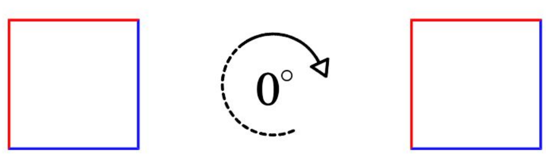
Another example is demonstrated here. This time, however, there are two squares with one type of pattern and two squares with another type of pattern.



All of these squares, again, are elements of $X$. Let's call the squares, from left to right, one through four. When you rotate square three 90° clockwise, it will become symmetric to square one. Likewise, when you rotate square four 90° counter-clockwise, it will also become symmetric to square two. However, there is no possible way to rotate or reflect square one such that it will become symmetrical to square two. Similarly, there is no possible way to rotate square two such that it will become symmetrical to square three. This means that squares two and four cannot be in the same *orbit* as squares one and three since these squares cannot be symmetrical to each other. In the end, there are two total *orbits* possible.

### 3.2. <u>Fixed Point</u>

Another terminology required to understand Burnside's Lemma is the concept of *fixed* or *fixed by.* To use Burnside's Lemma, the total amount of objects that is fixed by each rotation and each reflection must be known. So what does fixed point mean? The formal definition of a *fixed point* is that for an element $g \in G$, a *fixed point* of $X$ is an element $x \in X$ such that $g.x = x$. Essentially, it says that $x$, which consists of the objects, will be unaffected, or unchanged by the action of an element $G$, such as a rotation or a reflection. The object will look exactly the same. For example, rotating a square by 0° clockwise will not change how the square appears. The square is unaffected by that specific rotation, which means that all possible squares rotated by 0° will become a fixed point.

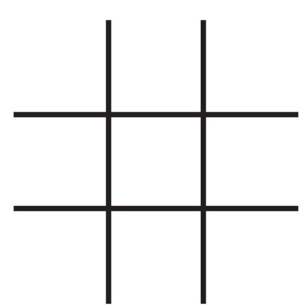


For example, how many fixed points are there for a square when reflected by 0° and 90°? The square has four sides, and each side can be colored with either red or blue. Now, if we take all the possible configurations of a square and rotate every configuration by 0°, all the squares will still look identical to its original configuration. Because the original squares look identical to the squares after a rotation of 0°, all the squares become a fixed point. Since there are $2^4$ possible different colored squares, there will be $2^4$, or 16 fixed points for a rotation of 0°.

How many fixed points would there be for a square when reflected by 90°? Well, if a square is rotated by 90° (rotation is clockwise), the top side will become the right side, the right side will become the bottom side, and so on. How can the square be colored so that a rotation of 90° would not affect its appearance? The only possible way for the square to be identical after the rotation to its original configuration is if all the sides of the squares are the same color. Thus, there are two fixed points for a square with a rotation of 90° - an all blue square or an all red square. In total, there are 18 fixed points for this square.

## 4. Investigation One - Tic-Tac-Toe

Remember Tic-Tac-Toe, the game we used to play when we were a child? Even this simple Tic-Tac-Toe board could be used to demonstrate Burnside's Lemma.



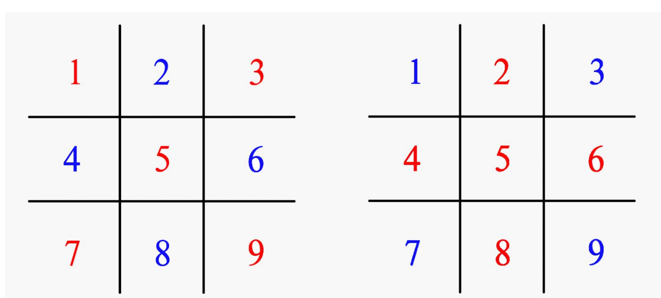
A young child playing this game might ask, how many different ways can this Tic-Tac-Toe board be filled up using 5 X’s and 4 O’s, and a board is not considered different when it can be obtained through a rotation of another board? Also, the board has no win conditions, which means the board can have three X’s or three O’s in a row. Initially, many people would think that there are $\frac{9!}{5!4!}$ ways to fill this board, since there are nine positions and nine objects (5 X’s and 4 O’s.)

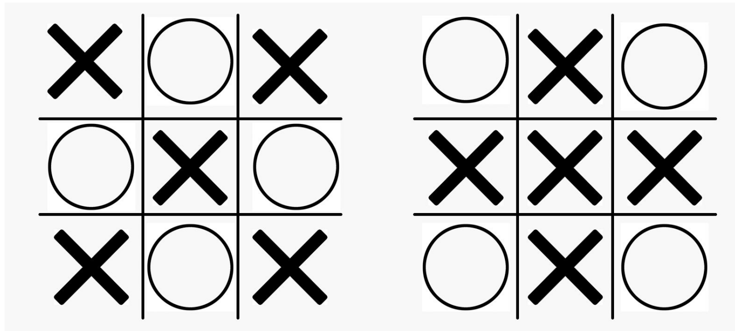
| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

In the first position, there are nine possible objects to fill it with; in the second position, there are eight possible objects to fill it with, and so on. This results in $9!$. However, the X’s and O’s can be interchanged without changing the grid itself, because all the X’s are indistinguishable with each other, and all the O’s are indistinguishable with each other. Therefore, these 5 of those X’s can be swapped around with each other, and 4 of those O’s can be swapped around with each other. This results in $\frac{9!}{5!4!}$, or 126. Unfortunately, it’s not that easy. This answer of 126 includes rotations and reflections, which is what we do not want.

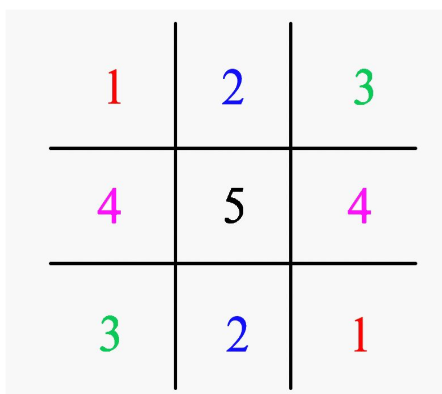
To solve this problem, we need to know how many different rotations we can apply onto our grid. We can rotate the board 0°, 90°, 180°, and 270°, for a total of four rotations that we can apply onto our Tic-Tac-Toe grid. All the different grids that we can get after applying the rotations are the orbits for that grid. Now if we take one grid from each orbit, we should essentially get all the different grids including rotations.

Next, the number of grids that are fixed by for each rotation must be calculated. Basically, a grid is fixed by when a rotation does nothing to the grid - it will look exactly the same. Let’s start with rotations. If the 0° rotation is applied onto the grid, all 126 of the possible grids will be fixed, since a 0° rotation is essentially a “do-nothing” rotation and therefore, will not change how the grid would look like. Now, how can the grid look the same if a 90° rotation is applied onto it? Well, when we turn the grid, the corners will have to be the same. Additionally, the outer-middle slots will also have to be the same. Thus, there are only two ways for a grid to be fixed by when a 90° rotation is applied onto it.





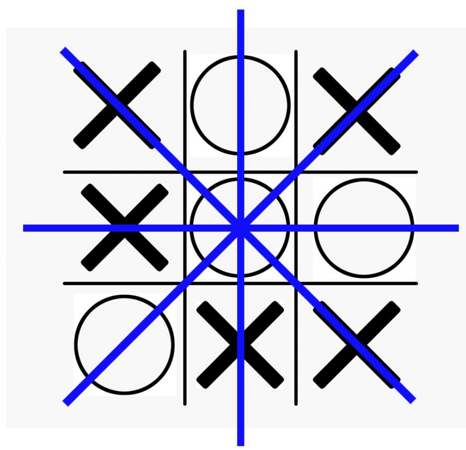
Because a 90° clockwise rotation is essentially the same as a 270° counterclockwise rotation, the 270° rotation will also have two fixed grids. Now, for the 180° rotation. How can the grid look the same if a 180° is applied onto it? A 180° rotation is the same as a reflection over the vertical and horizontal axis. This means that the opposite corners will have to be the same, as well as the opposite sides of the grid will have to be the same, like so:



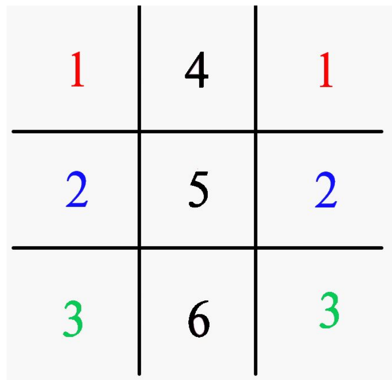

Eventually, we can work out that there are six possible fixed grids for the 180° rotation. Rotating each of these grids by 180° will have no effect on the appearance of that grid. Now all possible fixed grids are found. According to Burnside's Lemma formula, it states that the number of orbits is equal to the average number of fixed points. Adding up the number of fixed points, we get $126+2+2+6$, which is 136 fixed grids. To take the average, we divide 136 by 4, and we get 34. Since each orbit essentially represents a different grid, by counting the number of orbits, we get all the different possible boards that can be filled up with 5 X's and 4 O's, not including rotations of the board. Thus, our answer is 34.

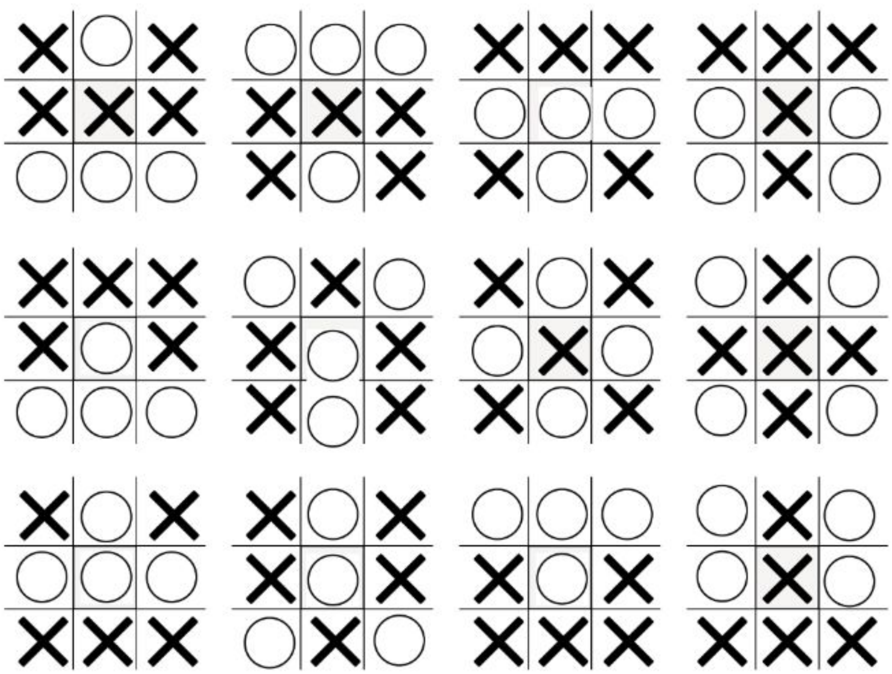
## 5. Investigation Two - Tic-Tac-Toe Part Two

In Investigation One, only rotations could affect the Tic-Tac-Toe grid. However, what if we take in the possibility of reflections and rotations? For a Tic-Tac-Toe grid, only four reflections can be applied onto it - a reflection over the vertical axis, horizontal axis, and the two diagonals of $y = x$ and $y = -x$ - assuming that the middle of the Tic-Tac-Toe grid is centered at the origin of a coordinate plane.

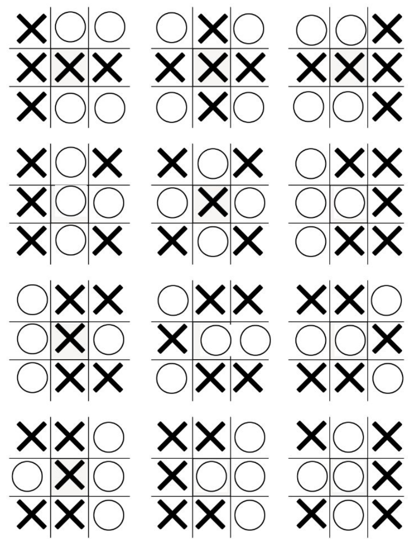


Let's find all the fixed grids for a vertical reflection. For a vertical reflection (or a reflection over the line $x = 0$, assuming that the middle of the grid is centered at the origin in the coordinate plane), the left and right sides of the grid have to be the same. The images below demonstrates this:

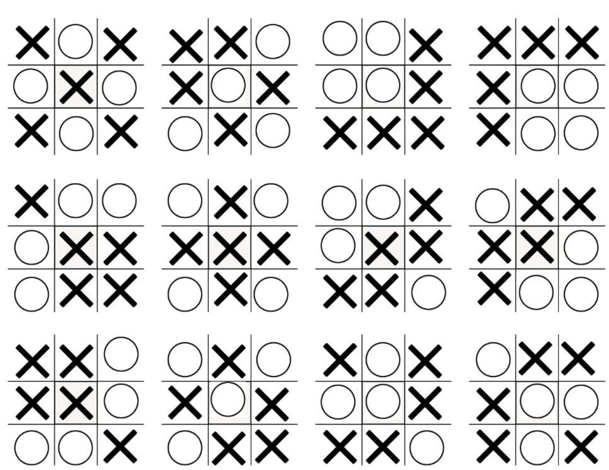


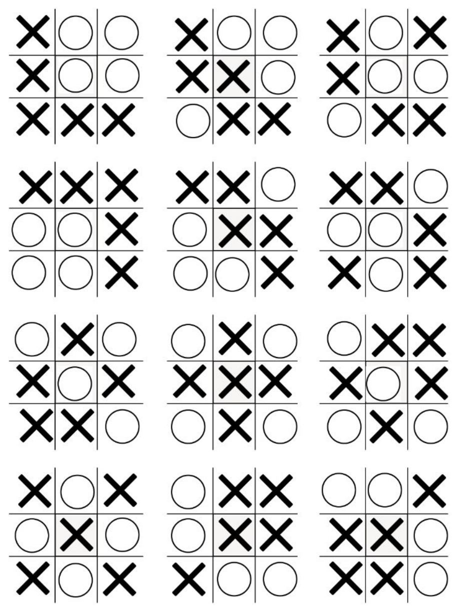
There are a total of 12 fixed grids when a vertical reflection has been applied to a Tic-Tac-Toe grid. The number of fixed grids also needs to be calculated for the horizontal reflection. However, we do not need to do so much work to find fixed grids when a horizontal reflection is applied onto a Tic-Tac-Toe board. When a vertical line is rotated by 90°, it will become a horizontal line. Therefore, to find the number of fixed grids for a horizontal reflection, we can just rotate the image above by 90°, as such:



Now, we have to find the amount of fixed grids for a reflection over the diagonals. We can find that there are also 12 additional fixed grids for each reflection over the diagonals. Again, because the diagonals are actually the lines $y = x$ and $y = -x$, if any of those two lines are rotated by 90°, it becomes identical to the other line. Therefore, we can find the number of fixed grids for just $y = -x$, shown below...



… and rotate the image by 90° to find the number of fixed grids for a reflection over the line $y = x$.

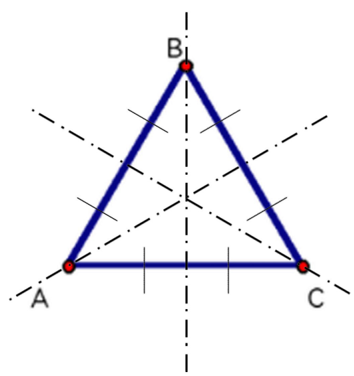


In total, there are 48 grids that were fixed under reflections. In the previous investigation, there were 136 grids that were fixed under rotations. Next, we add up the total number of grids fixed by *both* reflections and rotations, which equals to 184. To find the average of 184, we divide 184 by 8 to get 23. This means that there are 23 different Tic-Tac-Toe boards using 5 X's and 4 O's, not including rotations and reflections.

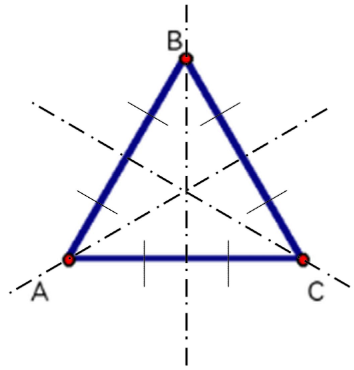
## 6. <u>Investigation Three - Equilateral Triangles</u>

So far, we have only used squares. Squares are very easy to use in the applications of Burnside's Lemma. This is because it is relatively easy to see that a square has a possibility of being the same when it is rotated by 90°, 180°, 270°, and 360° (also 0°), or when the square is reflected by the lines $y = x$, $y = -x$, $x = 0$, and $y = 0$. This investigation will now involve the use of other shapes, such as the equilateral triangle.

This question comes from the 2006 AIME II contest, and the reason why I decided to choose this topic as my research paper. The question asks: *There is an equilateral triangle created out of four smaller equilateral triangles. Each of the smaller equilateral triangles can be completely colored in using six different colors. Two large triangles are considered distinguishable if it is not possible to place one on the other using translations, reflections, and rotations. How many distinguishable large equilateral triangles can be formed?*

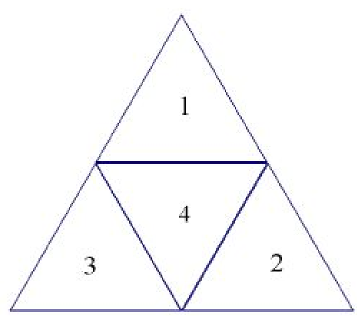



To begin, we can essentially ignore translations and only focus on rotations and reflections. How can a equilateral triangle be rotated and reflected such that it is possible to have it appear the same? The equilateral triangle can be rotated by 120°, 240°, and 360°. Additionally, the equilateral triangle can be reflected from the lines that are created from one corner to the midpoint of the opposite side of the corner. The image below shows the possible reflections.



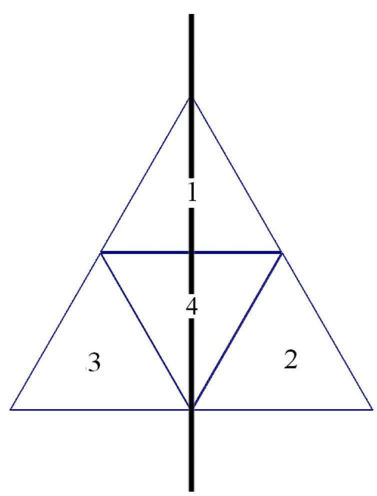

For the 360° rotation, every single possible equilateral triangle possible will become fixed - for a total of $6^4$ fixed triangles. This is because a 360° rotation is basically a 0° rotation, which means that all possible triangles will look exactly the same. For the 120° rotation, we realize that triangle 1 moves onto triangle 2, triangle 2 moves onto triangle 3, and triangle 3 moves onto triangle 1. Triangle 4 will remain at the same place.

Since triangles 1, 2, and 3 will map onto each other, these triangles will need to be the




same color to be fixed. Additionally, triangle 4 can be any color to be fixed, since it does not move. We can determine that there are $6^2$ ways for the triangle to be fixed under a 120° rotation. There are six colors for triangle 4 to possibly be, and six colors for triangles 1, 2, and 3 to possibly be. Finally, a 240° rotation will have the same procedure. We realize that triangle 1 moves onto triangle 3, triangle 2 moves onto triangle 1, and triangle 3 moves onto triangle 2. Triangle 4 will remain at the same place. Again, there are $6^2$ ways for the triangle to be fixed under a 240° rotation.

For reflections, we realize that all the lines that are created from one point to the opposite side's midpoint can be mapped onto each other. This situation is exactly the same as the situation in Investigation Two. We only need to find the fixed triangles for a reflection of a line, and just multiply the result by the number of reflections. In this case, there are three reflections, so we multiply the number of fixed triangles by three. Let's use the reflection of the top point to the bottom side's midpoint, since it is the easiest to visualize.




When we reflect the equilateral triangle by this vertical line, we can see that small equilateral triangles 1 and 4 will remain at the same place, but small equilateral triangles 2 and 3 will swap. Thus, triangles 2 and 3 will have to be the same color, while triangles 1 and 4 can be any color for the entire equilateral triangle to be fixed under this reflection. This means that there are $6^3$ fixed triangles under this reflection. Next, we multiply $6^3$ by 3 to find the total amount of fixed triangles under all three reflections.

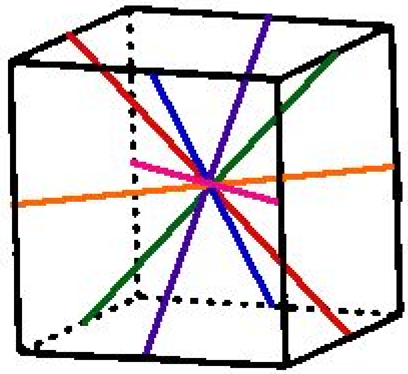
Finally, we use the Burnside's Lemma formula. We first add up all fixed triangles, which is: $6^4 + 6^2 \times 2 + 6^3 \times 3$. This results in 2016 fixed triangles. To find the average, we divide 2016 by 6, which results in 336. There are 336 distinguishable large equilateral triangles that can be

so $2016/6 = \boxed{336}$ distinguishable ways to color the triangle.

formed. To check our work, we can go on the math contest website, and behold:

## 7. Further Research

While Burnside's Lemma is a course that is taught in the first year of college, I only touched on the very basics of this specific topic within the larger group of group theory (no pun intended). In this research paper, I only touched upon the applications of Burnside's Lemma on two-dimensional shapes. However, Burnside's Lemma can also be applied on three-dimensional shapes - although it becomes immensely more complicated. For example, a two-dimensional square can only have four lines of symmetry. In a three-dimensional cube, there are a multitude of "lines" of symmetry. In a three-dimensional level, it is actually an plane of symmetry, which already makes it more complicated. Additionally, there are nine planes of symmetry in a cube compared to just four lines of symmetry in a square. And that's only reflections! Likewise, a cube would also have nine rotations of symmetry, as shown below.



Therefore, I decided not to research Burnside's Lemma with three dimensional shapes due to its complexity, and I was not sure if others could wrap their mind around all the different rotations and reflections. Another research I would like to do in the future is the Polya's Enumeration Theorem. While Burnside's Lemma gives us the number of distinct objects (orbits), it does not include additional information about the type of configuration of the orbits. However, Polya's Enumeration Theorem is a much more complicated theorem to study and would take a much longer time to understand as a 9th grader.

## 8. **References**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