The Snowflake Fractal

Exploring Fractal Patterns In Our Lives

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Introduction

The world around us seems random. Each and every cloud that we see in the sky appears to be random, and it is said that no two snowflakes are ever the same. This is what we think of as chaos theory — a branch of mathematics that focuses on systems that have completely random states of disorder and randomness.

Recent advancements in science and mathematics have shown that the clouds are in fact not completely random, and while it is not proven that no two snowflakes are never the same, all snowflakes do share similar chaotic patterns and repetition loops. New discoveries in the branch of chaos have shown that chaotic complex systems are not defined by completely random states of disorder and randomness, but by underlying patterns, feedback loops, repetition, self-similarity, and self-organization. The simultaneous combinations of complex repeating patterns and feedback loops result in what we call fractals.

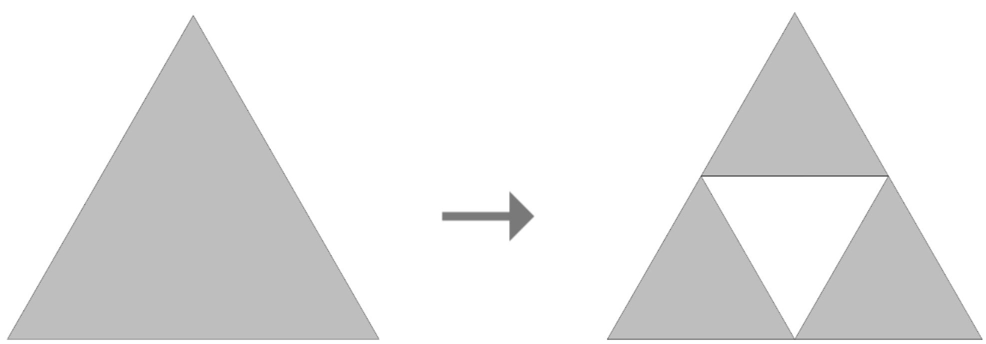
When you zoom in on a picture on your computer, it gets blurry and pixelated. But what if the picture never gets blurry or pixelated? Instead, the more you zoom in on the picture, the same emerging pattern appears in greater and greater detail. This is what a fractal is — essentially, an object that can be infinitely zoomed in will result in a greater detail of a certain pattern.

The term fractal was coined by Benoit Mandelbrot, who was working for the computer giant company IBM. Mandelbrot then inputted equations derived from famous mathematicians on repeating patterns into computers to be analyzed over and over again and discovered a set of numbers known as the Mandelbrot set, which is an infinite geometric figure of a fractal.

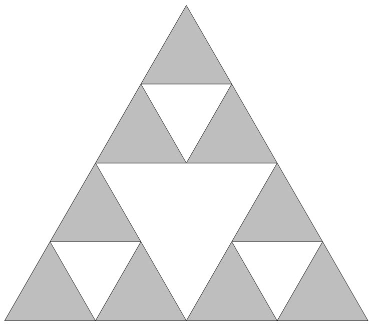
While there are an infinite number of fractals, I will be focusing on basic fractals to give an introductory exploration of fractals, as fractals found in real life are often overly complex and difficult to understand. The following fractals, although very basic, can result in very interesting theories and mathematical concepts.

Sierpinski Triangle

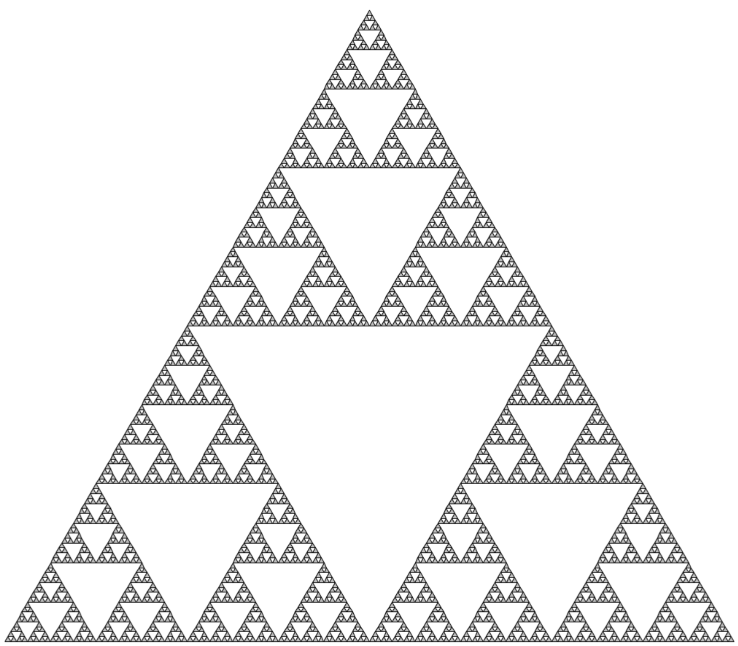
One very famous example of a fractal is Sierpinski's Triangle. Although the name of the idea may be unfamiliar, many people recognize this pattern. To create Sierpinski's Triangle, take a basic equilateral triangle and apply the following function: whenever there is an equilateral triangle, break it up into four smaller equilateral triangles, and remove the middle upside-down equilateral triangle.



Let's call the triangle on the left the 0th iteration ($\Delta\iota_0$), and the triangle on the right the 1st iteration ($\Delta\iota_1$). In $\Delta\iota_1$, there now exist three equilateral triangles, which are shaded in gray (think of all the white equilateral triangles as being taken out of the larger triangle). Now, apply the function continuously to get the 2nd iteration.



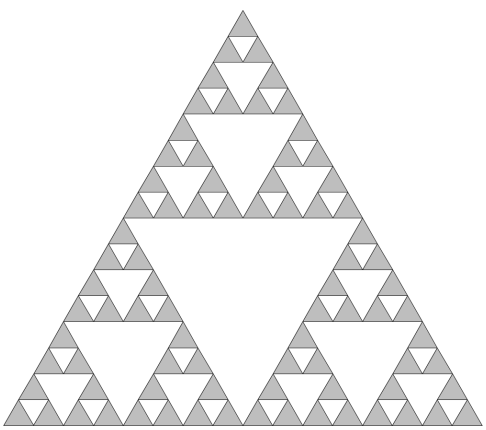
This can be repeated an infinite number of times, which would result in the nth iteration: $\Delta\iota_n$ or $\Delta\iota_\infty$.



In this nth iteration, you can see that the triangle replicates itself in smaller subsets. For further exploration into this topic, we can determine the number of triangles that exist after applying the function infinitely. We can also determine the area of the Sierpinski Triangle (only the gray parts).

Sierpinski's Triangle Exploration I:

The first exploration would be to determine the number of triangles that exist after applying the function infinitely. Let's start from the beginning: $\Delta\iota_0$ has one gray triangle. Moving onto the next iteration, $\Delta\iota_1$ has three gray triangles and one white triangle. Next, $\Delta\iota_2$ has nine gray triangles with four white triangles, where one of the four white triangles is four times larger than the other three white triangles. $\Delta\iota_3$ has 27 gray triangles and one large white triangle, three medium white triangles, and nine smaller white triangles, and thus, a total of thirteen white triangles. $\Delta\iota_4$ has 81 gray triangles and 40 white triangles, consisting of one large, three medium, nine small, and 27 tiny triangles.



$Figure\ of\ \Delta\iota_4$

Iteration	Gray Triangles	White Triangles

0	1	0
1	3	1
2	9	4
3	27	13
4	81	40

We can clearly see that the gray triangles increase by a factor of 3 starting with 1 when $\iota = 0$. Thus, the number of gray triangles can be found using the following formula: $G = 3^{\iota}$. For the white triangles, you can see a pattern emerging such that it takes the previous number of white triangles and adds to that value by $3^{\iota-1}$. Thus, the number of white triangles can be found using the following formula: $W_{\iota} = W_{\iota-1} + 3^{\iota-1}$, which can be rewritten as $W_{\iota} = 3^0 + 3^1 + 3^2 +... 3^{\iota-1}$, where 3^0 is $\Delta\iota_1$. So, if you wanted to know the number of white triangles in the 4th iteration, then $W_4 = 3^0 + 3^1 + 3^2 + 3^3 = 40$.

However, we can see that all the gray triangles will always end up to be the same size, but the white triangles have different sizes. Based on how the function is defined, every successively smaller white triangle is four times smaller than the white triangle it came from. We can modify the the equation for counting the white triangles — so if each successively smaller white triangle is four times smaller than the white triangle where it is derived, then we can rewrite $W_{\iota} = 3^0 + 3^1 + 3^2 +... 3^{\iota-1}$ as $W_{e\iota} = 4^{(\iota-1)}3^0 + 4^{(\iota-2)}3^1 + 4^{(\iota-3)}3^2 +... 4^0 3^{\iota-1}$. Ultimately, the smallest white triangle will be congruent to any gray triangle, so that means this would result in the number of white triangles with the same area as the gray triangles. So, if you wanted to know the number of *equivalent* white triangles in the 4th iteration, then

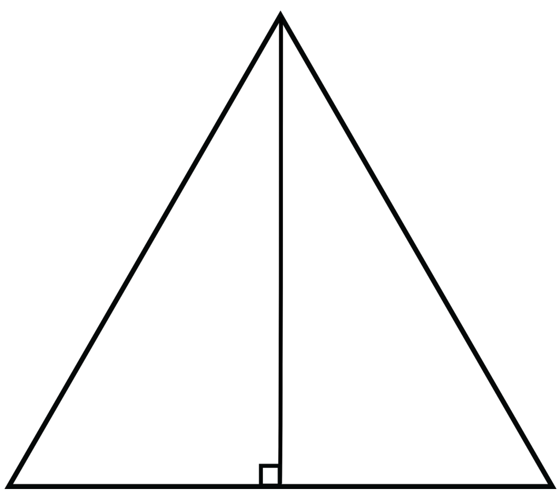
$W_{e4} = 4^3 3^0 + 4^2 3^1 + 4^1 3^2 + 4^0 3^3 = 175$. If we calculate G_4, we get 81 gray triangles. This means that the total number of equivalent triangles in the entire original triangle is $81 + 175 = 256$. Thus, we can determine the number of total equivalent triangles within the entire original triangle using the formula: $G_\iota + W_{e\iota}$. By doing this process for the other iterations, we can recapitulate our data table.

Iteration	Gray Δ	White Δ	Equivalent White Δ	Total Δ
0	1	0	0	1
1	3	1	1	4
2	9	4	7	16
3	27	13	37	64
4	81	40	175	256

From the table, we can see that the total number of equivalent triangles are just powers of four. We can rewrite the total number of equivalent triangles as 4^ι. Therefore, $4^\iota = G_\iota + W_{e\iota}$, which means that $4^\iota - G_\iota = W_{e\iota}$, and plugging in the formula we derived for $W_{e\iota}$ and G_ι leads to this formula: $4^\iota - 3^\iota = 4^{(\iota-1)}3^0 + 4^{(\iota-2)}3^1 + 4^{(\iota-3)}3^2 + ... 4^0 3^{\iota-1}$. If you set $\iota = 3$, this is essentially the difference of cubes formula, where $4^3 - 3^3 = (4 - 3)(4^2 + 4 \times 3 + 3^2)$, thereby demonstrating that fractal patterns can in fact prove a special case of the difference of the cubes formula. Additionally, this also proves the special case of the binomial theorem, where x and y are constants.

Sierpinski's Triangle Exploration II:

The second exploration would be to find the area of the gray triangles given it has undergone n iterations. From the first exploration, we discovered that the total number of triangles can be found using 4^{ι}. We also found that the total number of gray triangles can be found using 3^{ι}. Thus, the area of the gray triangles can be found by multiplying the area of original triangle by $\frac{3^{\iota}}{4^{\iota}}$. The area of the original triangle can be calculated by splitting the equilateral triangle into two 30-60-90 triangles, as shown below.



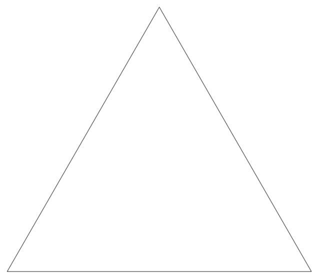
Utilizing the 30-60-90 special triangle ratios, you can get that the altitude of the triangle is $x\sqrt{3}$ and the base of the triangle is $2x$. Given that the area of the triangle is $\frac{1}{2}bh$, we can figure out that the area of the equilateral triangle is $\frac{1}{2}(2x)(x\sqrt{3})$, which is $x^{2}\sqrt{3}$. Since x is half the length of the side of the equilateral triangle, then $(\frac{1}{2}x)^{2}\sqrt{3}$, so $\frac{x^{2}\sqrt{3}}{4}$, where x is the length of the side of the triangle. Then, the area of the gray triangles can be found using the formula: $\frac{\sqrt{3}}{4}x \times \frac{3^{\iota}}{4^{\iota}}$. So if we want to find the area of the gray triangles in the fourth iteration, given that the original triangle has an area of one, we would get that $1 \times \frac{3^4}{4^4} = 0.3164\sim$. Furthermore, if we want to find the area of the gray triangles in the nth iteration, given that the original triangle has an area of one, we would take the limit as ι approaches infinity: $\lim_{\iota \to \infty} \frac{3^\iota}{4^\iota}$, then we would get that the area of the gray triangles would actually approach zero.

Snowflake Fractal

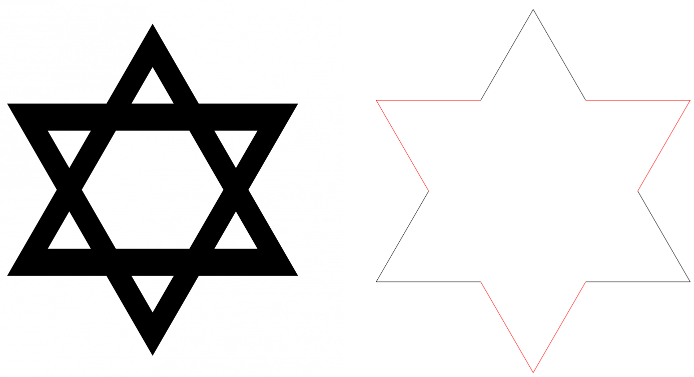
An example of naturally occurring fractals in nature can be found in snowflakes. Although it is said that no two snowflakes have the exact same pattern, many have extremely similar patterns. We can also generate our own unique snowflake, which is known as Koch's Snowflake.

The function for generating the snowflake fractal is defined by the following: everytime there is a straight line, divide it into three equal pieces, and use the middle of the three pieces to extend an equilateral triangle out of it. This can be visualized with the following diagram:

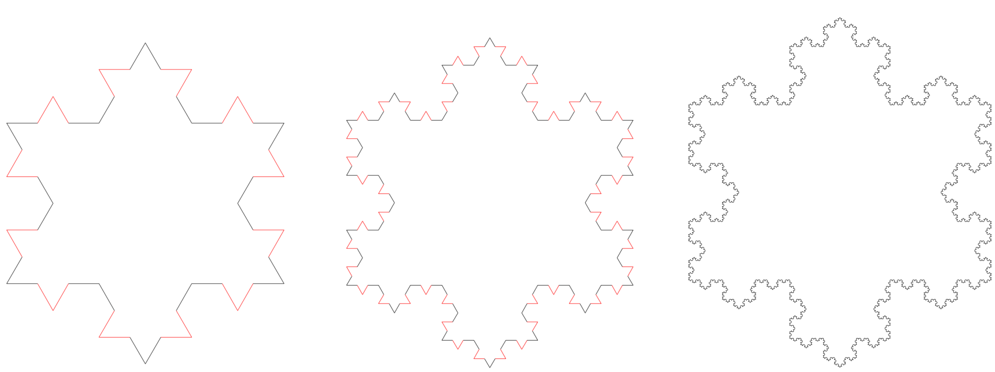
The Koch Snowflake emerges when this function applied onto an equilateral triangle nth times. Starting with the 0th iteration, the equilateral triangle is… an equilateral triangle.



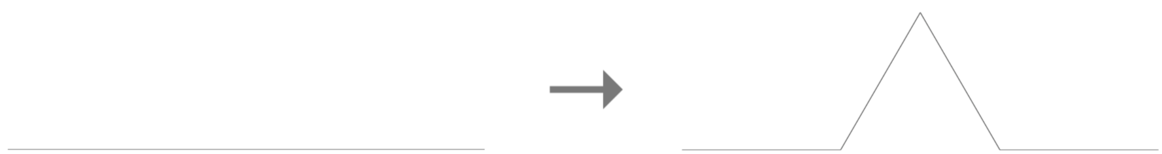
Then, moving onto the 1st iteration, the equilateral triangle becomes like the Star of David. Interestingly enough, ancient Jews were fascinated with triangles and fractals, which is in part why the Star of David resembles the 1st iteration of the Koch Snowflake.



As the shape progresses into the 2nd, 3rd, nth iteration, it appears to look more like a snowflake.



From the figures above, when the snowflake progresses into the 2nd, 3rd, 4th, and even to its nth iteration, it appears that the area it gains for each successive iteration decreases, but the perimeter grows larger and larger. Starting from the beginning, let's look at one side of the equilateral triangle when it progresses into the 1st iteration.



Everytime the function that defines the Koch Snowflake is applied, the original line segment is divided into three equal parts. Then, the middle part of the original line segment becomes a base for an equilateral triangle that will extend out of it, and since it is an equilateral triangle, each side length will be a third of the original line segment. Thus, the perimeter of the line segment after one iteration will be $\frac{4}{3}$ times larger.

So if the original triangle had a perimeter of s, each iteration would increase the perimeter by $\frac{4}{3}$. Thus, the first iteration would result in a perimeter of $\frac{4}{3}s$. The second iteration would result in a perimeter of $(\frac{4}{3})(\frac{4}{3})s$. The third iteration would be $(\frac{4}{3})(\frac{4}{3})(\frac{4}{3})s$, and we can see that the perimeter would be $(\frac{4}{3})^{\iota}s$, where ι is the iteration. Thus, as the number of iterations approach infinity (meaning that $\iota \to \infty$), the perimeter will turn out to be infinite.

However, the area will not follow the same end behavior, meaning that the area will not approach infinity. This can be proven quite easily — if a circle is drawn around the Koch Snowflake, the nth iteration would never extend out of the region drawn by the circle. Since the circle has a finite area, it can be assumed that the Koch Snowflake will thus also have a finite area. Furthermore, the equilateral triangle that extends outwards in each successive iteration becomes smaller and smaller, and as it approaches the nth iteration, the increase in area due to the addition of equilateral triangles for each successive iteration approaches zero, and is thus negligible in increasing the size of the overall shape.

Real Life Applications

In the 1990s, a radio astronomer named Nathan Cohen created a new antenna he called the fractal antenna. Cohen wanted to create a compact antenna and came up with the idea of building miniature antennas on the actual antenna — similar to how a fractal works. This type of fractal antenna worked surprisingly well and had the potential to receive data from various signals. This was due to the unique characteristics of fractals, where the more the antenna repeated itself, the more signals it could pick up. This made it superior to the other designs of antennas that were employed in the 1990s.

This is one of the many examples of the use of fractal mathematics in the real world. Fractals are found in many aspects of nature, whether they are sea shells or lightning bolts. In fact, MIT scientists recently discovered that chromatin, which essentially organizes DNA, is a fractal. Fractals are also found in the foods you eat. Think of a whole chunk of broccoli — each smaller division of the broccoli head is just a smaller version of the original broccoli. Cloud formation can also be studied through the use of fractals, and helping with cancer research by helping to identify the growth patterns of mutated cells. In fact, some believe that everything in existence is a fractal. Starting from the universe, it is subdivided into millions of supercluster galaxies, each with billions of galaxies. Inside those galaxies, trillions of stars can be found with billions of planets. Among those planets, Earth contains continents, cities, and humans — and within each human, there are millions of cells, each with their own atoms composed of electrons, protons, and neutrons. Ultimately, scientists are using fractal mathematics and physics to aid them in their search for smaller subatomic particles. Perhaps, there is an infinite number of subdivisions: we can always zoom into (or out) and there will always be a set of unique patterns that make up the world around us.

Works Cited

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