

Assignment 2: Single Qubits

Assignment Due: Friday 16th September

Solutions can be handwritten on a separate sheet of paper, typed or done on a tablet. You can print this, write the solutions on it, and then scan and upload it.

Send the completed assignment to tclarke@asesoftware.com If you have any questions or difficulties, please do reach out to the same email.

Challenge Questions are Optional

Question 1. ___ does not play dice

A dice has 6 faces numbered 1,2,3,4,5,6. For this question we will count from 1. Rather than a 2-level system like a qubit, this is a 6-level system. We'll call each state by the number on the face. For instance $|3\rangle$ is the state of the dice with face with 3 up.

When we toss the dice, let's say it's in the superposition state similar to $|+\rangle$ for the qubit.

$$|+_6\rangle = \frac{1}{\sqrt{6}}(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle)$$

What is the expectation value of the dice?

Hint: rather than computing a 6x6 matrix-vector product, you can do the sum

$$\langle N \rangle = \sum_{j=1}^6 j |\langle j | +_6 \rangle|^2$$

Question 2. Challenge question 1: Turn the dice

With the coin, turning it over can be modelled with the X-gate. Such that

$$\hat{X} |h\rangle = |t\rangle$$

For the dice, turning it over corresponds to putting the face at the bottom on top. Since opposite faces of the coin add up to 7, this means

$$|1\rangle \rightarrow |6\rangle$$

$$|2\rangle \rightarrow |5\rangle$$

$$|3\rangle \rightarrow |4\rangle$$

And the reverse for $|4\rangle, |5\rangle, |6\rangle$

Write down the 6x6 matrix that corresponds to the operator \hat{F} such that:

$$\hat{F} |j\rangle = |k\rangle$$

With j and k being the numbers on opposite faces

Question 3. Challenge question 2: Superposition: A matter of perspective

The states $|0\rangle$ & $|1\rangle$ are not generally considered superpositions. They correspond to our coin being heads or tails, or the 0 or 1 classical bits. But $|0\rangle$ & $|1\rangle$ are superpositions if you measure in a different basis.

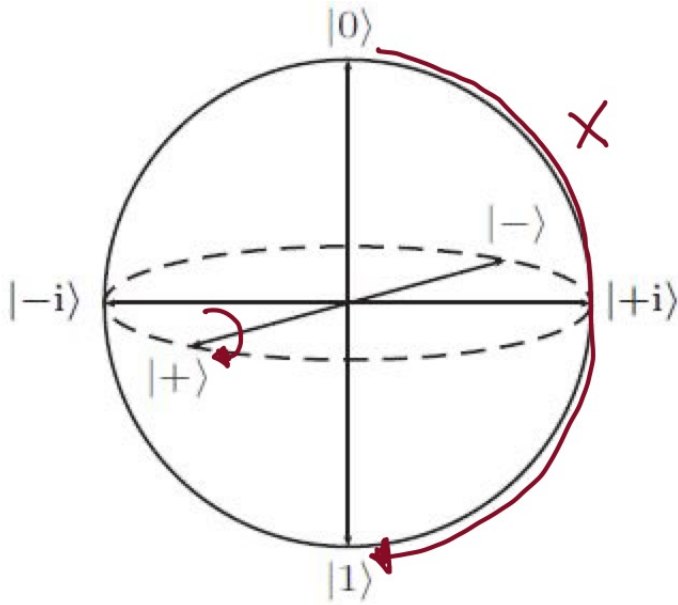
Take the X-basis, (X-axis on the Bloch sphere) given by

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Express $|0\rangle$ & $|1\rangle$ in terms of $|+\rangle$ & $|-\rangle$. Interpret your result using the Bloch sphere or otherwise

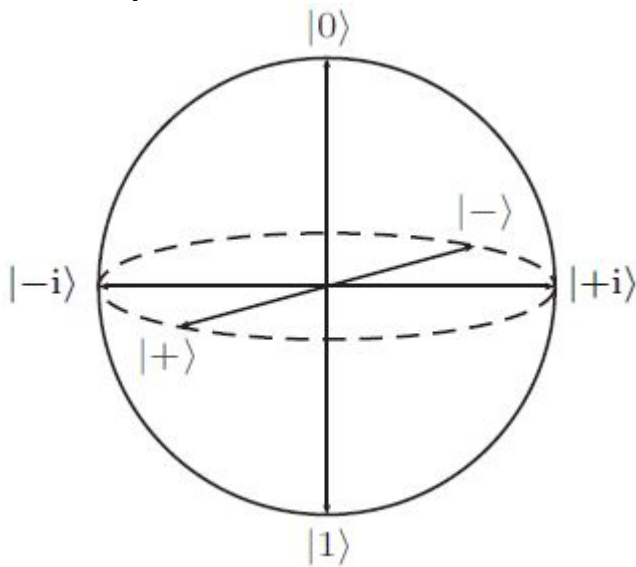
Question 4. On the Bloch Sphere, the X gate can be represented as a rotation of a 180° (π) as shown below .



starting from the top at $|0\rangle$, draw out the path traced by the qubit from the sequence of gates H, Z, H
Where H is the Hadamard gate, Z is the 180° (π) rotation around the Z-axis of the Bloch sphere and X is the not gate

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

Label each path with the gate. Compare your result to the example of the X gate [You may wish to use this blank diagram]



Question 5. By considering the effect on $|0\rangle$ $|1\rangle$, or by doing matrix multiplication, calculate the following states:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

1) $XZ|1\rangle$

2) $XH|+\rangle$

3) $XYZ|0\rangle$

Question 6. Using matrix multiplication, writing out the state vector, or otherwise, show that

$$HZH = X$$

If the previous question is done correctly this should verify your result.

Question 7. All single qubit gates can be described as unitary operators in the following form

$$U(\theta, \varphi, \lambda) = \begin{bmatrix} \cos(\theta/2) & -e^{i\lambda} \sin(\theta/2) \\ e^{i\varphi} \sin(\theta/2) & e^{i(\varphi+\lambda)} \cos(\theta/2) \end{bmatrix}$$

By choosing suitable values of $(\theta, \varphi, \lambda)$ show how the following gates can be obtained:

1) $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

2) $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

3) $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

4) $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

Question 8. Local vs global phase

Any quantum state can be written as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$$

We can factor out a global phase- the phase common to all the states- and remove it. For instance the global phase $e^{i\Theta}$ can be removed like

$$\begin{aligned} e^{i\Theta}|\psi\rangle &= e^{i\Theta}(\cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle) \\ &= |\psi\rangle \end{aligned}$$

For the following states, factor out the global phase (if there is any) and write down the angles (θ, φ) that describe their position on the Bloch Sphere

- 1) $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$
- 2) $|\psi\rangle = \sqrt{\frac{1}{3}}i|1\rangle + \sqrt{\frac{2}{3}}|0\rangle$ Hint: you can multiply everything by a global phase to move the i to the $|1\rangle$

Question 9. Challenge Question 4: Why global phase doesn't matter

In the class, I said we could ignore global phases. In this question we will explore that idea a little.

- 1) Consider a generic quantum state $|\psi\rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle$ and the same state with a global phase added $|\psi_G\rangle = e^{i\Phi}\cos(\alpha)|0\rangle + e^{i\Phi}\sin(\alpha)|1\rangle$.

The probability of measuring $|\psi\rangle$ from $|\psi_G\rangle$ is given by the inner product as $|\langle\psi_G|\psi\rangle|^2$. Calculate this probability.

- 2) What if we had a local (or relative) phase difference instead, a phase shift between $|0\rangle$ & $|1\rangle$. Let's define this state $|\psi_R\rangle = \cos(\alpha)|0\rangle + e^{i\varphi}\sin(\alpha)|1\rangle$
Compute the inner product of $|\langle\varphi_L|\varphi\rangle|^2$
- 3) We'll choose $\alpha = \pi/4$ such that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\psi_L\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$ What is the value of θ such that the inner product from (2) is zero?

Question 10. Challenge question 5: The Pauli matrices

The Pauli matrices satisfy the relations

$$\sigma_j\sigma_k = \delta_{j,k}I + i\epsilon_{jkl}\sigma_l$$

Where σ_j or σ_k can be the gates $\{\sigma_x = X, \sigma_y = Y, \sigma_z = Z\}$. $\delta_{j,k}$ is the Kronecker-Delta $\delta_{j,k} = 1$ if $j = k$ (i.e. the two matrices are the same) or 0 if $j \neq k$. I is the identity matrix, ϵ_{jkl} is the **Levi-Civita symbol** = 1 if the pair j, k are in the order X,Y,Z. For example $\sigma_j\sigma_k = XY$ would have $\epsilon_{jkl} = 1$ but $\sigma_j\sigma_k = ZY$ (reverse order) would have $\epsilon_{jkl} = -1$

Show that the gates X, Y satisfy this by computing the matrix products

- 1) XX
- 2) XY
- 3) YX

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