Satellite Dish Azimuth and Elevation Analysis

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I. Introduction

II. NWZ Frame Analysis

Let $\beta \in (0, 2\pi]$ and $\eta \in [-\pi/2, \pi/2]$ denote the azimuth and elevation of the satellite, respectively, regarding the antenna dish NWZ reference frame.

$$x = r\cos\eta\cos\beta \tag{1}$$

$$y = r\cos\eta\sin\beta \tag{2}$$

$$z = r\sin\eta \tag{3}$$

where

$$r = \sqrt{x^2+y^2+z^2} \tag{4}$$

$$\eta = \arcsin\left(z/r\right) \tag{5}$$

$$\beta = \arctan 2\left(y, x\right) \tag{6}$$

It is important to notice that when the satellite is precisely at the zenith ($x = y = 0$ or $\eta = \pi/2$), the azimuth is not well defined. Such pathological case will be later considered.

A. Speed

$$\begin{aligned}\dot{r} &= \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y} + \frac{\partial r}{\partial z}\dot{z} \\ &= \nabla r \cdot \boldsymbol{v} \\ &= \hat{\boldsymbol{r}} \cdot \boldsymbol{v}\end{aligned} \tag{7}$$

B. Speed of XY-Projection

Let $s = \sqrt{x^2+y^2}$

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial x}\dot{x} + \frac{\partial s}{\partial y}\dot{y} \\ &= \frac{x\dot{x}+y\dot{y}}{s}\end{aligned} \tag{8}$$

C. Elevation Rate ($\eta \neq \pi/2$)

Considering

$$\sin\eta = \frac{z}{r} \quad \text{and} \quad \cos\eta = \frac{s}{r}, \tag{9}$$

then

$$\begin{aligned}\cos\eta\cdot\dot{\eta} &= \frac{\dot{z}}{r} - z\frac{\dot{r}}{r^2} \\ \sin\eta\cdot\dot{\eta} &= -\frac{\dot{s}}{r} + s\frac{\dot{r}}{r^2}\end{aligned} \tag{10}$$

which leads to

$$\cos^2\eta\cdot\dot{\eta} = \frac{s\dot{z}}{r^2} - sz\frac{\dot{r}}{r^3}$$

$$\sin^2\eta\cdot\dot{\eta} = -\frac{\dot{s}z}{r^2} + sz\frac{\dot{r}}{r^3}$$

Thus

$$\dot{\eta} = \frac{s\dot{z}-\dot{s}z}{r^2} \tag{11}$$

D. Azimuth Rate ($\eta \neq \pi/2$)

Considering

$$\sin\beta = \frac{y}{\sqrt{x^2+y^2}} \quad \text{and} \quad \cos\beta = \frac{x}{\sqrt{x^2+y^2}} \tag{12}$$

Then

$$\begin{aligned}\cos\beta\cdot\dot{\beta} &= \frac{d}{dt}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{\dot{y}\sqrt{x^2+y^2} - y\frac{d}{dt}\sqrt{x^2+y^2}}{x^2+y^2} \\ \sin\beta\cdot\dot{\beta} &= -\frac{d}{dt}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{-\dot{x}\sqrt{x^2+y^2} + x\frac{d}{dt}\sqrt{x^2+y^2}}{x^2+y^2}\end{aligned} \tag{13}$$

Since $\frac{d}{dt}\sqrt{x^2+y^2} = \frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}$

$$\cos\beta\cdot\dot{\beta} = \frac{\dot{y}}{\sqrt{x^2+y^2}} - \frac{y}{x^2+y^2}\frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}$$

$$\sin\beta\cdot\dot{\beta} = \frac{-\dot{x}}{\sqrt{x^2+y^2}} + \frac{x}{x^2+y^2}\frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}$$

then

$$x\cdot\dot{\beta} = \dot{y} - \frac{y}{x^2+y^2}\left(x\dot{x}+y\dot{y}\right)$$

$$y\cdot\dot{\beta} = -\dot{x} + \frac{x}{x^2+y^2}\left(x\dot{x}+y\dot{y}\right)$$

Finally

$$\dot{\beta} = \begin{cases}\dfrac{\dot{y}}{x} - \dfrac{y}{x}\dfrac{x\dot{x}+y\dot{y}}{x^2+y^2}, & x \neq 0 \\ \dfrac{-\dot{x}}{y}, & x = 0\end{cases} \tag{14}$$

or

$$\dot{\beta} = \frac{x\dot{y}-\dot{x}y}{x^2+y^2} \tag{15}$$

III. Optimal control

During a pass, assume that the satellite rises over the horizon at the instant $t_{\text{sr}} \in \mathbb{R}$ and that it sets at $t_{\text{ss}} > t_{\text{sr}}$. Thus, the ground station antenna should be able to follow the satellite during the interval $I = [t_{\text{sr}}, t_{\text{ss}}]$. Moreover, let $\hat{\boldsymbol{r}}_{\text{sat}} \in \mathcal{C}^1\left(I; \mathbb{R}^{3\times 1}\right)$, that maps $I \ni t \mapsto \hat{\boldsymbol{r}}_{\text{sat}}(t) \in \mathbb{R}^{3\times 1}$, denote the unit vector from ground station in the satellite direction. Mathematically, it can be said that the ground station antenna should track $\hat{\boldsymbol{r}}_{\text{sat}}$ as close as possible.

Consider that the azimuth $\beta \in [0, 2\pi)$ and elevation $\eta \in [0, \pi/2]$ of the antenna can be controlled throughout an input signal $\mathbf{u} \in \mathcal{C}^0\left(I; \mathbb{R}^{n\times 1}\right)$ by the state law

$$\dot{\mathbf{x}} = \boldsymbol{\Phi}\left(t, \mathbf{x}, \mathbf{u}\right) \tag{16}$$

while constrained by a set (i-indexed) of state-control restrictions on the form

$$G_i\left(t, \mathbf{x}, \mathbf{u}\right) \leqslant 0 \tag{17}$$

where the state is define by $\mathbf{x} = \begin{bmatrix} \eta & \beta & \dot{\eta} & \dot{\beta} \end{bmatrix}^{\mathrm{T}}$.

Consider $D = [0, \pi/2] \times [0, 2\pi)$ and let $\hat{\boldsymbol{r}}_{\text{ant}} : D \to \mathbb{R}^{3\times 1}$, that maps $D \ni (\eta, \beta) \mapsto \hat{\boldsymbol{r}}_{\text{ant}}(\eta, \beta) \in \mathbb{R}^{3\times 1}$, be the antenna pointing direction, which is defined by

$$\hat{\boldsymbol{r}}_{\text{ant}}\left(\eta, \beta\right) = \begin{bmatrix} \cos\eta \cdot \cos\beta \\ \cos\eta \cdot \sin\beta \\ \sin\eta \end{bmatrix}. \tag{18}$$

In this scenario, the antenna control unit (ACU) should be able to keep $\hat{\boldsymbol{r}}_{\text{ant}}$ as close as possible to $\hat{\boldsymbol{r}}_{\text{sat}}$ by minimising the cost functional

$$J\left(\mathbf{x}\right) = \int_I \left\|\hat{\boldsymbol{r}}_{\text{ant}} - \hat{\boldsymbol{r}}_{\text{sat}}\right\|^2 d\tau. \tag{19}$$

As established in [**?**, Troutman, Chapter 10], the solution is to consider the lagrangian muiltipliers $\boldsymbol{\lambda}$ and μ_i and the hamiltonian

$$h\left(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \mu\right) = g\left(t, \mathbf{x}\right) + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{\Phi}\left(t, \mathbf{x}, \mathbf{u}\right) + \sum_i \mu_i G_i\left(t, \mathbf{x}, \mathbf{u}\right), \tag{20}$$

where the lagragian is define by

$$g\left(t, \mathbf{x}\right) = \left\|\hat{\boldsymbol{r}}_{\text{ant}}\left(\eta\left(t\right), \beta\left(t\right)\right) - \hat{\boldsymbol{r}}_{\text{sat}}\left(t\right)\right\|^2. \tag{21}$$

(See conditions for unique solution in Troutman, section 10.1)