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DC Programming: Overview

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Abstract. Mathematical programming problems dealing with functions, each of which can be represented as a difference of two convex functions, are called DC programming problems. The purpose of this overview is to discuss main theoretical results, some applications, and solution methods for this interesting and important class of programming problems. Some modifications and new results on the optimality conditions and development of algorithms are also presented.

Key Words. DC functions, DC programming, global optimization, nonconvex programming, optimality conditions.

1. Introduction

In the field of nonconvex optimization, DC programming plays an interesting and important part because of its theoretical aspects as well as its wide range of applications. A function is called DC if it can be represented as the difference of two convex functions. Mathematical programming problems dealing with DC functions are called DC programming problems. It is known that the set of DC functions defined on a compact convex set of $\mathbb{R}^n$ is dense in the set of continuous functions on this set. Therefore, in principle, every continuous function can be approximated by a DC function with any desired precision. Moreover, every C^2-function is a DC function. Although DC representations are available for important function classes, finding such a representation for an arbitrary DC function is a hard open problem.

The purpose of this overview is to present essential results on theory, applications, and solution methods of DC programming in the sense of global optimization. We restrict ourselves to deterministic approaches and to the class of DC programming problems dealing with DC functions, each

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of which has an explicit DC representation on a space of finite dimension.

In Section 2, we recall some key properties of DC functions, and define a general DC programming problem and its canonical form. Section 3 deals with typical examples of DC programming problems, which can be modeled in a natural way. Global optimality conditions and duality of DC programming problems are presented in Section 4. Based upon the reciprocity principle concept of Tikhonov (Ref. 1), we establish a global optimality condition for a special DC programming problem, called generalized canonical DC program, from which some known optimality criteria as well as some new optimality conditions can be derived. Duality of DC programming is presented thereafter.

Section 5 is devoted to selected implementable solution methods for DC programming problems. In global optimization, there are two classes of algorithms which are applied extensively and belong to the most promising tools in many cases. The first class contains algorithms of the branch-and-bound type; the second class, called outer-approximation algorithms, is based on the idea of successive approximation of a convex feasible set by a sequence of polyhedral convex sets containing it. In many situations, a suitable combination between branch-and-bound algorithms and outer-approximation algorithms leads to more efficient procedures. The algorithms presented in Section 5 include pure branch-and-bound algorithms, pure outer-approximation algorithms, and combinations thereof.

2. DC Functions and DC Programming Problems

In this section, we recall without proof some key properties of DC functions, and give a general form as well as some special cases of DC programming problems. For proofs of these properties, we refer e.g. to Ref. 2.

2.1. DC Functions.

Definition 2.1. Let C be a convex subset of $\mathbb{R}^n$. A real-valued function $f: C \to \mathbb{R}$ is called DC on C, if there exist two convex functions $g, h: C \to \mathbb{R}$ such that f can be expressed in the form

$$f(x) = g(x) - h(x). \tag{1}$$

If $C = \mathbb{R}^n$, then f is simply called a DC function. Each representation of the form (1) is said to be a DC decomposition of f. We call a function $f: \mathbb{R}^n \to \mathbb{R}$ locally DC, if, for every $x_0 \in \mathbb{R}^n$, there exists $\epsilon > 0$ such that f is

DC on the ball

$$B(x_0, \epsilon) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq \epsilon\}.$$

Examples of DC functions include indefinite quadratic functions of the form xQx (for each $n \times n$ matrix Q, there exist two positive semidefinite $n \times n$ matrices A, B such that $xQx = xAx - xBx$), inner products

$$xy = (1/4)(\|x+y\|^2 - \|x-y\|^2),$$

and the square of the distance function $d_M : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_M(x) = \inf\{\|x-y\| : y \in M\},$$

where M is any nonempty closed subset of $\mathbb{R}^n$,

$$\begin{aligned} d_M^2(x) &= \inf\{\|x-y\|^2 : y \in M\} \\ &= \|x\|^2 + \inf\{-\|x\|^2 + \|x-y\|^2 : y \in M\} \\ &= \|x\|^2 - \sup\{2xy - \|y\|^2 : y \in M\}. \end{aligned}$$

The following proposition shows that the class of DC functions is closed under many operations encountered frequently in optimization.

Proposition 2.1. Let f and f_i, $i = 1, \ldots, m$, be DC functions. Then, the following functions are also DC:

(i) $\sum_{i=1}^{m} \lambda_i f_i(x)$, $\lambda_i \in \mathbb{R}$, $\quad i = 1, \ldots, m$,

(ii) $\max_{i=1,\ldots,m} f_i(x)$, $\quad \min_{i=1,\ldots,m} f_i(x)$,

(iii) $|f(x)|$, $\quad f^+(x) := \max\{0, f(x)\}$, $\quad f^-(x) := \min\{0, f(x)\}$,

(iv) $\prod_{i=1}^{m} f_i(x)$, $\quad$ where $\prod$ is the product symbol.

It is worth noting that the above results can be proved constructively; i.e., for each of the functions defined above, an explicit DC decomposition can be constructed. For example, consider the function

$$f(x) = \max\{f_i(x) = g_i(x) - h_i(x) : i = 1, \ldots, m\},$$

with g_i and h_i being convex functions, $i = 1, \ldots, m$. Since, for each $i = 1, \ldots, m$,

$$f_i = g_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} h_j - \sum_{j=1}^{m} h_j,$$

we obtain the following DC decomposition of f:

$$f=g-h, \qquad g=\max_{i=1,\ldots,m}\left\{g_i+\sum_{\substack{j=1\\j\neq i}}^{m} h_j\right\}, \qquad h=\sum_{j=1}^{m} h_j. \tag{2}$$

A main result concerning the recognition of DC functions goes back to Hartman (Ref. 3).

Proposition 2.2. Every locally DC function is DC.

The key result of Hartman leads to the following important consequences (Refs. 2, 4).

Proposition 2.3.

(i) Every function f: $\mathbb{R}^n \to \mathbb{R}$ whose second partial derivatives are continuous everywhere is DC.

(ii) Let C be a compact convex subset of $\mathbb{R}^n$. Then, every continuous function on C is the limit of a sequence of DC functions which converges uniformly on C; i.e., for any continuous function c: $C \to \mathbb{R}$ and for any $\epsilon > 0$, there exists a DC function f: $C \to \mathbb{R}$ such that $|c(x)-f(x)| \leq \epsilon$, $\forall x \in C$.

(iii) Let f: $\mathbb{R}^n \to \mathbb{R}$ be DC, and let g: $\mathbb{R} \to \mathbb{R}$ be convex. Then, the composite function $(g \circ f)(x) = g(f(x))$ is DC.

2.2. DC Programming Problems. Programming problems dealing with DC functions are called DC programming problems. The general form of DC programming problems considered in this paper is given by

$$\min\{f_0(x): x \in X, f_i(x) \leq 0, i=1, \ldots, m\}, \tag{3}$$

where $f_i = g_i - h_i$, $i=0, \ldots, m$, are DC functions and X is a closed convex subset of $\mathbb{R}^n$. The problem

$$\min\{c(x): x \in X, \psi(x) \leq 0\}, \tag{4}$$

where c is a linear function, $X \subset \mathbb{R}^n$ is closed convex, and ψ is a concave function, is often called a canonical DC program. Ever DC programming problem of the form (3) can be transformed into the canonical form (4) as follows.

By using an additional variable x_{n+1}, problem (3) is rewritten as

$$\min\{x_{n+1}: x \in X, x_{n+1} \in \mathbb{R}, f_0(x) - x_{n+1} \leq 0, f_i(x) \leq 0, i=1, \ldots, m\}.$$

Next, define a DC function f: $\mathbb{R}^{n+1} \to \mathbb{R}$ by

$$f(x, x_{n+1}) = \max\{(f_0(x) - x_{n+1}), f_1(x), \ldots, f_m(x)\},$$

and let $f = g - h$ be a DC decomposition determined as in (2). Finally, using another additional variable x_{n+2}, and setting

$$z = \{x, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+2},$$

$$Z = \{z \in \mathbb{R}^{n+2}: x \in X, g(x, x_{n+1}) - x_{n+2} \leq 0\},$$

$$\psi(z) = x_{n+2} - h(x, x_{n+1}),$$

we obtain the canonical DC program

$$\min\{c(z) = x_{n+1}: z \in Z, \psi(z) \leq 0\}.$$

For the establishment of optimality conditions and for the development of solution methods for problem (4), in many cases, the linear function c can be replaced by a convex function. In what follows, we call problem (4) a generalized canonical DC program, if the function c is convex.

3. Some DC Programming Problem Models

DC programming has a wide field of applications. In this section, we discuss some typical problems which can be modeled as DC programming problems.

3.1. Programming Problem with Constraints of Complementarity Type. Consider the programming problem

$$\min\{c(x): x \in C\}, \tag{5}$$

where c: $\mathbb{R}^n \to \mathbb{R}$ is a continuous function and C a closed convex subset of $\mathbb{R}^n$, with the following additional constraints:

$$x \geq 0, \qquad h(x) \geq 0, \qquad xh(x) = 0, \tag{6}$$

where h: $\mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector function. Problem (6) is usually called the complementarity problem; as well known, problem (5 and 6) is central to many problems including engineering design, economic equilibrium, and multilevel game-theoretic problems; see e.g. Refs. 5, 6. Assume that the components h_i of h, $i = 1, \ldots, n$, are concave functions, and define a function

$$\psi(x) = \sum_{i=1}^{n} \min\{h_i(x); x_i\}.$$

Then obviously, ψ is a concave function, and the system (6) is equivalent to

$$x \geq 0, \qquad h(x) \geq 0, \qquad \psi(x) \leq 0.$$

Thus, letting

$$X = \{x \in \mathbb{R}^n : x \in C, x \geq 0, h(x) \geq 0\},$$

we can formulate problem (5) with the additional constraints (6) equivalently in the form

$$\min\{c(x) : x \in X, \ \psi(x) \leq 0\},$$

which is a DC programming problem or a generalized canonical DC program, whenever c is DC or convex, respectively.

3.2. Bridge Location Problems. A bridge of minimum length has to be built between two convex islands M and D (problem P1), and between a convex island M and the shore of a convex lake L (problem P2), respectively. DC programming models to these problems are

$$\text{(P1)} \quad \min\{d_M^2(x) : x \in D\}$$

and

$$\text{(P2)} \quad \min\{d_M^2(x) : x \in R \setminus \operatorname{int} L\},$$

respectively, where R is any convex region around the lake L. Notice that d_M^2 is a DC function, as seen in Section 2.1.

3.3. Design Centering Problem. The design centering problem can be formulated as follows (Refs. 7–10). Let D and K be two compact subsets of $\mathbb{R}^n$ having nonempty interiors, and assume that $0 \in \operatorname{int} K$. Further, let $r_D : \mathbb{R}^n \to \mathbb{R}$ be a function defined by

$$r_D(x) = \begin{cases} \max\{r : x + rK \subset D\}, & \text{if } x \in D, \\ 0, & \text{if } x \notin D. \end{cases} \tag{7}$$

Find a point $x^* \in D$ and a number r^* such that

$$r^* = r_D(x^*) = \max\{r_D(x) : x \in D\}. \tag{8}$$

It can be shown (see the above mentioned references) that problem (8) can be solved by linear programming techniques if K and D are polyhedral sets. For the case that K is convex and D has the form

$$D = D_0 \cap D_1 \cap \cdots \cap D_m,$$

where D_0 is a convex set and, for each $i \geq 1$, D_i is the complement of an open convex set, the function r_D is a DC function, so that (8) is a DC programming problem.

3.4. Location Problem. An aftersale service has to be located in a convex set X of the plane $\mathbb{R}^2$ in order to serve k customers located at points $c_j \in X, j = 1, \ldots, k$. When the service is located at $x \in X$, its attraction to customer j is described by a function $g_j(d_j(x))$, where $d_j(x) = \|x - c_j\|$ is the distance from c_j to x. Each function g_j is supposed to be convex. Choosing the location with maximal total attraction amounts to solving the problem

$$\max\left\{f(x) = \sum_{j=1}^{k} g_j(d_j(x)) : x \in X\right\}. \tag{9}$$

In view of Proposition 2.3(iii), the objective function f in (9) is a DC function, since each function $g_j(d_j(x))$ is DC. Several formulations and generations of the above problem can be found e.g. in Refs. 11–13.

3.5. Packing Problem. The well known packing problem is the problem of finding the maximum radius of n equal and nonoverlapping circles which can be packed into the unit square; see e.g. Refs. 14–16. This problem is equivalent to the scattering of n points z^i, $i = 1, \ldots, n$, in the unit square $C \subset \mathbb{R}^2$, such that the minimum distance between any two of them becomes as large as possible. The mathematical programming model to this problem is the following:

$$\max_{z^1, \ldots, z^n \in C} \min\{\|z^i - z^k\| : 1 \leq i < k, i < k \leq n\}. \tag{10}$$

Define

$$x = (z_1^1, \ldots, z_1^n, z_2^1, \ldots, z_2^n) \in \mathbb{R}^{2n},$$

$$E = \{x \in \mathbb{R}^{2n} : 0 \leq x_j \leq 1, j = 1, \ldots, 2n\},$$

$$J = \{1, \ldots, 2n\},$$

$$J_{ik} = \{i, k, n+i, n+k\}.$$

Then, problem (10) can be formulated as follows:

$$\max_{x\in E}\min\left\{(x_i-x_k)^2+(x_{n+i}-x_{n+k})^2:\ 1\le i<k,\ i<k\le n\right\}$$

$$=\max_{x\in E}\min\left\{\left[(x_i-x_k)^2+(x_{n+i}-x_{n+k})^2-2\sum_{j=1}^{2n}x_j\right]\right.$$

$$\left.+2\sum_{j=1}^{2n}x_j:\ 1\le i<k,\ i<k\le n\right\}$$

$$=\max_{x\in E}\left\{2\sum_{j=1}^{2n}x_j^2+\min\left\{-\left(2\sum_{j\in J\setminus J_{ik}}x_j^2+(x_i+x_k)^2\right.\right.\right.$$

$$\left.\left.\left.+(x_{n+i}+x_{n+k})^2\right):\ 1\le i<k,\ i<k\le n\right\}\right\}$$

$$=\max_{x\in E}\left\{2\sum_{j=1}^{2n}x_j^2-\max\left\{\left(2\sum_{j\in J\setminus J_{ik}}x_j^2+(x_i+x_k)^2\right.\right.\right.$$

$$\left.\left.\left.+(x_{n+i}+x_{n+k})^2\right):\ 1\le i<k,\ i<k\le n\right\}\right\}.$$

The objective function of the last problem is the DC function $g(x)-h(x)$, where

$$g(x)=2\sum_{j=1}^{2n}x_j^2,$$

$$h(x)=\max\left\{\left(2\sum_{j\in J\setminus J_{ik}}x_j^2+(x_i+x_k)^2+(x_{n+i}+x_{n+k})^2\right):\ 1\le i<k,\ i<k\le n\right\}.$$

3.6. Optimization over Efficient Sets. Consider the following multiple-objective linear programming problem:

$$\max c^ix,\ i=1,\dots,p,\qquad \text{s.t. } x\in X, \tag{11}$$

where X is a polytope (bounded polyhedral set) in $\mathbb{R}^n$, and $c^i\in\mathbb{R}^n\setminus\{0\}$, $i=1,\dots,p$. The vectors c^i are called the criterion vectors of problem (11).

Let C be the $p\times n$ matrix having rows $c^1,\dots,c^p$. A point $y\in X$ is called an efficient solution of problem (11), if there is no point $x\in X$ such that $Cx\ge Cy$ and $Cx\ne Cy$. Letting E_X denote the set of all efficient solutions of problem (11), and letting $c:\mathbb{R}^n\to\mathbb{R}$ be a convex function, we consider the following optimization problem over the efficient set E_X:

$$\min\{c(y): y \in E_X\}. \tag{12}$$

Optimization over the efficient set is one of the most important and interesting approach in multiple-criteria optimization; see e.g. Refs. 17–23. Let $e = (1, \ldots, 1) \in \mathbb{R}^n$, and let $b: X \to \mathbb{R}$ be a function defined by

$$b(y) = \max\{eC(x-y): Cx \geq Cy, x \in X\}. \tag{13}$$

Then obviously, b is concave on X satisfying

$$b(y) \geq 0, \qquad \text{for } y \in X,$$

and y is an efficient solution iff $b(y) = 0$. Thus, problem (12) can be formulated as

$$\min\{c(y): b(y) \leq 0, y \in X\}, \tag{14}$$

which is a generalized canonical DC programming problem.

4. Optimality Conditions and Duality

4.1. Definitions and Notations. We recall in this section some definitions and notations used often in convex analysis and optimization; we introduce some additional notations which are used for the establishment of optimality conditions thereafter.

Definition 4.1. Let $\varphi: \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be an arbitrary function. The set

$$\text{epi } \varphi = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}: \varphi(x) - t \leq 0\} \tag{15}$$

is called the epigraph of φ.

Let X be a subset of $\mathbb{R}^n$. We denote

$$\text{epi } \varphi | X = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}: x \in X, \varphi(x) - t \leq 0\}. \tag{16}$$

Definition 4.2. Let $\varphi: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function, $\bar{x} \in \mathbb{R}^n$ such that $\varphi(\bar{x}) < +\infty$, and $\epsilon \geq 0$. A vector $s \in \mathbb{R}^n$ is called an ϵ-subgradient of φ at $\bar{x}$ if it satisfies

$$\varphi(x) \geq \varphi(\bar{x}) + s(x - \bar{x}) - \epsilon, \qquad \forall x \in \mathbb{R}^n. \tag{17}$$

The set of all ϵ-subgradients of φ at $\bar{x}$ is called the ϵ-subdifferential of φ at $\bar{x}$ and is denoted by $\partial_\epsilon \varphi(\bar{x})$. If $\epsilon = 0$, then s and $\partial\varphi(\bar{x})$ are simply called the subgradient and subdifferential of φ at $\bar{x}$, respectively.

Let X be a subset of $\mathbb{R}^n$. We denote

$$\partial X_\epsilon \varphi(\bar{x}) = \{s \in \mathbb{R}^n : \varphi(x) \geq \varphi(\bar{x}) + s(x - \bar{x}) - \epsilon, \forall x \in X\}. \tag{18}$$

Definition 4.3. Given an arbitrary function $\varphi: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the function $\varphi^*: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\varphi^*(y) = \sup\{yx - \varphi(x) : x \in \mathbb{R}^n\}, \tag{19}$$

is called the conjugate function of φ.

Definition 4.4.

(i) For any set $X \subset \mathbb{R}^n$, the function

$$I_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X, \end{cases} \tag{20}$$

is called the indicator function of X.

(ii) Let X be any convex subset of $\mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$ and $\epsilon \geq 0$. The set

$$N_\epsilon(X, \bar{x}) = \{d \in \mathbb{R}^n : d(x - \bar{x}) \leq \epsilon, \forall x \in X\} \tag{21}$$

is called the ϵ-normal cone to X at $\bar{x}$. The 0-normal cone is simply called the normal cone and is denoted by $N(X, \bar{x})$.

(iii) For any convex set $X \subset \mathbb{R}^n$, the function

$$I_X^*(y) = \sup\{yx : x \in X\} \tag{22}$$

is called the support function of X.

Remark 4.1. By definition, the function I_X is convex whenever the set X is convex. The function φ^* is convex for every φ. Moreover, if φ is convex, then $\varphi^{**} = \varphi$ (Ref. 24).

4.2. Optimality Conditions. We begin the establishment of optimality conditions in DC programming by recalling the reciprocity principle concept of Tikhonov in Ref. 1, originally introduced for nonconvex optimization problems and ill-posed variational problems. Consider a pair of programming problems given by

$$\omega_\delta^* = \inf\{\omega(z) : z \in Z, \psi(z) \leq \delta\}, \tag{23}$$

$$\psi_\eta^* = \inf\{\psi(z) : z \in Z, \omega(z) \leq \eta\}, \tag{24}$$

where $Z \subseteq \mathbb{R}^p$ and ω, ψ are finite functions on $\mathbb{R}^p$. Let Ω_δ^* and Ψ_η^* be the sets of optimal solutions of problems (23) and (24), respectively.

Definition 4.5. We say that problems (23) and (24) are reciprocal if $\Omega_\delta^* = \Psi_\eta^*$; in this case, we also say that the reciprocity principle holds.

Proposition 4.1. See Ref. 1. If $Z = \mathbb{R}^p$, $\omega(z) = \|z\|$, and $\eta = \omega_\delta^*$, then the reciprocity principle holds for any continuous function $\psi(z)$ whenever $\{z \in Z: \psi(z) \leq \delta\} \neq \emptyset$.

The following almost obvious result is a key for deriving optimality conditions (Ref. 25).

Proposition 4.2. Assume that, in problems (23) and (24), it holds that $\eta = \omega_\delta^*$ and $\psi_\eta^* = \delta$. Then, the reciprocity principle holds for every set Z and arbitrary functions $\omega(z)$, $\psi(z)$.

Proof. Let z^* be any optimal solution of (23). Then, we have

$$z^* \in Z, \qquad \omega(z^*) = \omega_\delta^* = \eta, \qquad \psi(z^*) \leq \delta = \psi_\eta^*,$$

which imply that z^* is an optimal solution of (24). Thus, $\Omega_\delta^* \subseteq \Psi_\eta^*$. The relation $\Omega_\delta^* \supseteq \Psi_\eta^*$ is shown analogously. □

Next, we give some optimality conditions for optimization problems dealing with DC functions, which can be derived immediately from Proposition 4.2. For our purpose, we introduce the following concept of robust subsets in $\mathbb{R}^p$.

Definition 4.6. Let $Z \subseteq \mathbb{R}^p$, $f: \mathbb{R}^p \to \mathbb{R}$, and $\delta \in \mathbb{R}$. The set $S(Z, f, \delta) = \{z \in Z: f(z) \leq \delta\}$ is said to be robust if $S(Z, f, \delta) = \mathrm{cl}(\{z \in Z: f(z) < \delta\})$, where, for each set S, $\mathrm{cl}(S)$ denotes the closure of S.

Lemma 4.1. If in problems (23) and (24) the function ω is convex, and if the following condition is fulfilled:

$$\exists z^0 \in Z: \omega(z^0) < \omega^*, \tag{25}$$

then the set $S(Z, \omega, \eta)$ is robust for each $\eta \geq \omega^*$.

Proof. From (25) and the convexity of ω, it follows that, for each $\eta \geq \omega^*$, it holds that

$$\mathrm{cl}\,(\{z \in \mathbb{R}^p: \omega(z) < \eta\}) = \{z \in \mathbb{R}^n: \omega(z) \leq \eta\},$$

$$Z \cap \{z \in \mathbb{R}^p: \omega(z) < \eta\} \neq \varnothing.$$

It is obvious that

$$\begin{aligned} \mathrm{cl}\,(\{z \in Z: \omega(z) < \eta\}) &= \mathrm{cl}\,(Z \cap \{z \in \mathbb{R}^p: \omega(z) < \eta\}) \\ &\subseteq Z \cap \mathrm{cl}\,(\{z \in \mathbb{R}^p: \omega(z) < \eta\}) \\ &= Z \cap \{z \in \mathbb{R}^p: \omega(z) \leq \eta\} \\ &= S(Z, \omega, \eta). \end{aligned}$$

To show the relation $\supseteq$, let

$$z^1 \in S(Z, \omega, \eta) = Z \cap \{z \in \mathbb{R}^p: \omega(z) \leq \eta\},$$

and let (z^0, z^1) and $[z^0, z^1]$ be the open and closed line segments, respectively. Then,

$$(z^0, z^1) \subset Z \cap \{z \in \mathbb{R}^n: \omega(z) < \eta\},$$

and hence,

$$[z^0, z^1] \subset \mathrm{cl}\,(Z \cap \{z \in \mathbb{R}^p: \omega(z) < \eta\});$$

i.e.,

$$z^1 \in \mathrm{cl}\,(\{z \in Z: \omega(z) < \eta\}). \qquad \square$$

The following optimality condition for the generalized canonical DC program is a corollary of Proposition 4.2. Notice that the generalized canonical DC program defined in Section 2.2 is a special case of problem (23), where $\delta = 0$, and the functions ω and $-\psi$ are convex.

Proposition 4.3. Assume that, in the generalized canonical DC program

$$\omega^* = \inf\{\omega(z): z \in Z, \psi(z) \leq 0\}, \tag{26}$$

the set $S(Z, \psi, 0) = \{z \in Z: \psi(z) \leq 0\}$ is robust and condition (25) is fulfilled. Then, a feasible point z^* of (26) is an optimal solution if and only if

$$0 = \inf\{\psi(z): z \in Z, \omega(z) \leq \omega(z^*)\}. \tag{27}$$

Proof. Since $\omega(z^*) \geq \omega^*$, it follows that the set $S(Z, \omega, \omega(z^*))$ is robust, by Lemma 4.1. Therefore, we have

$$\begin{aligned} &\inf\{\psi(z): z\in Z, \omega(z) \leq \omega(z^*)\} \\ &= \inf\{\psi(z): z\in Z, \omega(z) < \omega(z^*)\} = 0, \end{aligned}$$

which implies, in view of the robustness of the set $S(Z, \psi, 0)$,

$$\begin{aligned} \omega(z^*) &\leq \inf\{\omega(z): z\in Z, \psi(z) < 0\} \\ &= \inf\{\omega(z): z\in Z, \psi(z) \leq 0\} \\ &= \omega^*. \end{aligned}$$

Thus, in (27), we must have $\omega(z^*) = \omega^*$; hence, by Proposition 4.2, problems (26) and (27) are reciprocal. □

An important class of DC optimization problems is the following:

$$\omega^* = \inf\{g(x) - h(x): x\in X\}, \tag{28}$$

where g and h are two convex functions on $\mathbb{R}^n$, and X is a closed convex subset of $\mathbb{R}^n$. The next result gives an optimality condition for problem (28).

Proposition 4.4. Assume that problem (28) is solvable. Then, a point $x^* \in X$ is an optimal solution to it if and only if there is $t^* \in \mathbb{R}$ such that

$$0 = \inf\{-h(x) + t: x\in X, t\in \mathbb{R}, g(x) - t \leq g(x^*) - t^*\}. \tag{29}$$

Proof. Using an additional variable t, we define the problem

$$\omega^* = \inf\{g(x) - t: x\in X, t\in \mathbb{R}, -h(x) + t \leq 0\}. \tag{30}$$

Obviously, $x^* \in X$ is an optimal solution of (28) if and only if there is $t^* \in \mathbb{R}$ such that (x^*, t^*) is an optimal solution of (30). Let

$$p = n+1, \quad Z = X \times \mathbb{R}, \quad \omega(z) = g(x) - t, \quad \psi(z) = -h(x) + t.$$

Then, it is easy to see that the assumptions in Proposition 4.3 are fulfilled, so that problems (30) and (29) are reciprocal. □

Based on the above optimality condition, an algorithm for solving DC programming problems of type (28) is developed in Section 5.3. Using the notation (16), we obtain the following geometric optimality condition for problem (28).

Proposition 4.5. A point $x^* \in X$ is an optimal solution of problem (28) if and only if

$$\operatorname{epi} \bar{g} | X \subset \operatorname{epi} h | X, \tag{31}$$

where $\bar{g}(x) = g(x) - (g(x^*) - h(x^*))$.

Proof. We show that condition (31) is equivalent to condition (29). As seen in the proof of Proposition 4.4, problems (29) and (30) are reciprocal. Therefore, letting (x^*, t^*) be an optimal solution of problem (30), we have $h(x^*) = t^*$ in (29). Thus from (29), it follows that

$$h(x) - t \leq 0,$$

for each $(x, t) \in X \times \mathbb{R}$ satisfying

$$\bar{g}(x) - t = g(x) - (g(x^*) - t^*) - t \leq 0,$$

which implies that condition (31) holds. Next, from (31), it follows that

$$0 \leq \inf\{-h(x) + t : x \in X, t \in \mathbb{R}, g(x) - t \leq g(x^*) - h(x^*)\}.$$

Since the point (x^*, t^*), with $x^* \in X$, $t^* = h(x^*)$, is a feasible solution of the above problem, condition (29) follows. □

Another form of condition (31) is the following condition, where we use the notation (18).

Proposition 4.6. A point $x^* \in X$ is an optimal solution of problem (28) if and only if

$$\partial X_\epsilon h(x^*) \subset \partial X_\epsilon g(x^*), \qquad \forall \epsilon \geq 0. \tag{32}$$

Proof. Suppose that x^* is not an optimal solution of (28). By Proposition 4.5, this is equivalent to saying that condition (31) does not hold; i.e., there is $\bar{x} \in X$ such that $(\bar{x}, \bar{g}(\bar{x})) \notin \operatorname{epi} h | X$. Since $\operatorname{epi} h | X$ is a closed convex set, there exists a nonvertical hyperplane separating the point $(\bar{x}, \bar{g}(\bar{x}))$ from $\operatorname{epi} h | X$; i.e., there exists $s \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$ such that

$$h(x) \geq s(x - x^*) - \gamma, \qquad \forall x \in X, \tag{33a}$$

$$\bar{g}(\bar{x}) = g(\bar{x}) - (g(x^*) - h(x^*)) < s(\bar{x} - x^*) - \gamma. \tag{33b}$$

Let

$$\epsilon = \gamma + h(x^*).$$

Then from (33), we have

$$\epsilon \geq 0,$$

$$h(x) \geq h(x^*) + s(x - x^*) - \epsilon, \qquad \forall x \in X,$$

$$g(\bar{x}) < g(x^*) + s(\bar{x} - x^*) - \epsilon,$$

which imply that condition (32) does not hold. □

Remark 4.2.

(i) For the case $X = \mathbb{R}^n$, condition (32) reads

$$\partial_\epsilon h(x^*) \subset \partial_\epsilon g(x^*), \qquad \forall \epsilon > 0. \tag{34}$$

This is a well known result in DC optimization and was first proved by Hiriart-Urruty (Refs. 26–28).

(ii) If $g \equiv 0$, then problem (28) is usually called a concave programming problem. By using the indicator function of X in (20), this problem can be rewritten as

$$\inf\{I_X(x) - h(x) : x \in \mathbb{R}^n\}. \tag{35}$$

Thus, for this case, condition (32) becomes

$$\partial_\epsilon h(x^*) \subset N_\epsilon(X, x^*), \qquad \forall \epsilon \geq 0, \tag{36}$$

where $N_\epsilon(X, x^*)$ is defined in (21). It can be shown (Ref. 29) that condition (36) is equivalent to the following conditions:

$$h(x^*) = \sup\{I_X^*(y) - h^*(y) : y \in \mathbb{R}^n\}, \tag{37}$$

$$\partial h(z) \subset N(X, z), \qquad \forall z \in \{x : h(x) = h(x^*)\}. \tag{38}$$

Condition (37) is established based on the duality to problem (35); see e.g. Refs. 30–34. Condition (38) needs an additional assumption,

$$\inf\{h(x) : x \in X\} < h(x^*),$$

and is due to Strekalovski (Refs. 35, 36).

4.3. Duality in DC Programming. Consider DC programming problems given in the form

$$\inf\{g(x) - h(x) : x \in \mathbb{R}^n\}, \tag{39}$$

where $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $h: \mathbb{R}^n \to \mathbb{R}$ are two convex functions. Notice that problems of the form

$$\inf\{g(x) - h(x): x \in X\},$$

where X is a convex subset of $\mathbb{R}^n$, can be rewritten in the form (39) by setting

$$g = g + I_X,$$

where I_X is the indicator function of X. Let g^* and f^* be the conjugate functions of g and h, respectively, and let

$$Y = \{y \in \mathbb{R}^n: h^*(y) < +\infty\}. \tag{40}$$

The programming problem

$$\inf\{h^*(y) - g^*(y): y \in Y\} \tag{41}$$

is called the Fenchel–Rockafellar dual problem of (39). Before establishing the relationship between problems (39) and (41), we give some useful formula for computing the values of conjugate functions; see Ref. 37.

Lemma 4.2.

(i) Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \partial h(\bar{x})$. Then,

$$h^*(\bar{y}) = \bar{y}\bar{x} - h(\bar{x}). \tag{42}$$

(ii) Let $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function, and let $h: \mathbb{R}^n \to \mathbb{R}$ be a convex function. Further, let $f = g - h$. Then, the conjugate function f^* of f is given by

$$f^*(z) = \sup\{g^*(y+z) - h^*(y): y \in Y\}, \tag{43}$$

where Y is defined in (40).

Proof.

(i) Since

$$h(x) \geq h(\bar{x}) + \bar{y}(x - \bar{x}), \qquad \forall x \in \mathbb{R}^n,$$

i.e.,

$$\bar{y}x - h(x) \leq \bar{y}\bar{x} - h(\bar{x}), \qquad \forall x \in \mathbb{R}^n,$$

it follows that

$$h^*(\bar{y}) = \sup\{\bar{y}x - h(x): x \in \mathbb{R}^n\} \leq \bar{y}\bar{x} - h(\bar{x}).$$

This implies that

$$h^*(\bar{y}) = \bar{y}\bar{x} - h(\bar{x}).$$

(ii) By the definition of conjugate functions, it follows that

$$\begin{aligned} f^*(z) &= \sup\{zx - (g(x) - h(x)) : x \in \mathbb{R}^n\} \\ &= \sup\{((y+z)x - g(x)) - (yx - h(x)) : x \in \mathbb{R}^n\}, && \forall y \in \mathbb{R}^n, \\ &\geq \sup\{(y+z)x - g(x)) : x \in \mathbb{R}^n\} - \sup\{yx - h(x)) : x \in \mathbb{R}^n\}, && \forall y \in \mathbb{R}^n, \\ &\geq \sup\{g^*(y+z) - h^*(y) : y \in Y\}. \end{aligned}$$

Next, suppose that

$$f^*(z) > \sup\{g^*(y+z) - h^*(y) : y \in Y\};$$

i.e., there exists $\bar{x} \in \mathbb{R}^n$ such that

$$z\bar{x} - (g(\bar{x}) - h(\bar{x})) > \sup\{g^*(y+z) - h^*(y) : y \in Y\}. \tag{44}$$

Let $\bar{y}$ be a subgradient of the finite convex function h at $\bar{x}$. In view of (42), we have

$$h^*(\bar{y}) = \bar{y}\bar{x} - h(\bar{x}).$$

From this, it follows that

$$\begin{aligned} z\bar{x} - (g(\bar{x}) - h(\bar{x})) &= ((\bar{y}+z)\bar{x} - g(\bar{x})) - (\bar{y}\bar{x} - h(\bar{x})) \\ &\leq g^*(\bar{y}+z) - h^*(\bar{y}), \end{aligned}$$

which is a contradiction to (44). □

Using Lemma 4.2, we obtain following results on the relationship between problems (39) and (41).

Theorem 4.1.

(i) If problem (39) has an optimal solution, then it holds that

$$\inf\{g(x) - h(x) : x \in \mathbb{R}^n\} = \inf\{h^*(y) - g^*(y) : y \in Y\}. \tag{45}$$

(ii) The Fenchel–Rockafellar dual problem of (41) is problem (39).

Proof.

(i) By definition of f^*, we have

$$f^*(0) = \sup\{-f(x): x \in \mathbb{R}^n\},$$

i.e.,

$$\inf\{f(x): x \in \mathbb{R}^n\} = -f^*(0).$$

From (43), it follows that

$$\begin{aligned}\inf\{f(x): x \in \mathbb{R}^n\} &= -f^*(0)\\ &= -\sup\{g^*(y) - h^*(y): y \in Y\}\\ &= \inf\{h^*(y) - g^*(y): y \in Y\}.\end{aligned}$$

(ii) This follows from the fact that $h^{**} = h$ and $g^{**} = g$, because the functions h and g are convex as mentioned in Remark 4.1. □

The formula (45) was first given by Pshenichnyi (Ref. 38). Related results on duality can be found in Refs. 30–34, and 39. Since h^* and g^* are convex functions, problem (41) is exactly a DC programming problem of type (39). Moreover, the symmetry property also appears in the relationship between the optimal solutions of these problems in the following sense.

Proposition 4.7.

(i) If x^* is an optimal solution of problem (39), then each $y^* \in \partial h(x^*)$ is an optimal solution of problem (41).

(ii) If y^* is an optimal solution of problem (41), then each $x^* \in \partial g^*(y^*)$ is an optimal solution of problem (39).

Proof. We show (i). Assertion (ii) is shown analogously. Let x^* be any optimal solution of (39), and let $y^* \in \partial h(x^*)$. Then, it follows from (32) that $y^* \in \partial g(x^*)$. From (42), we have

$$h^*(y^*) = y^* x^* - h(x^*), \qquad g^*(y^*) = y^* x^* - g(x^*).$$

Therefore,

$$h^*(y^*) - g^*(y^*) = g(x^*) - h(x^*),$$

which implies by (45) that y^* is an optimal solution of problem (41). □

5. Solution Methods

This section is devoted to methods for solving some classes of DC programming problems.

At this point, it should be noticed that the canonical DC program belongs to the class of reverse convex problems involving reverse convex constraints of the form $f(x) \geq 0$, where f is convex. The class of reverse convex problems plays an important part in optimization. For applications and solution methods of this class, we refer to Refs. 40–52.

As shown in Section 2, each DC programming problem can be transformed into a canonical DC program. However, in general, it is not adequate to speculate that solving a DC programming problem could be reduced to solving a canonical DC program. The reason is quite simple: for the establishment of algorithms for the canonical DC program, one often needs assumptions, which cannot be fulfilled when transforming a DC programming problem of the general form into canonical form.

Thus, as mentioned in Section 1, we intend to present only some types of algorithms which are really implementable and can be applied to problems dealing with DC functions whose DC decompositions are given explicitly.

5.1. A Branch-and-Bound Algorithm. The following algorithm is designed to solve DC programming problems given in the form

$$\min\{f_0(x): x \in X, f_i(x) \leq 0, i=1, \dots, m\}, \tag{46}$$

where

$$f_i = g_i - h_i, \qquad i=0, \dots, m,$$

are DC functions and $X \subset \mathbb{R}^n$ is a polytope (bounded polyhedral set), defined by

$$X = \{x \in \mathbb{R}^n : Ax + b \leq 0\}, \tag{47}$$

A and b being matrix and vector of appropriate sizes. We denote by F the feasible set of problem (46). The algorithm to be presented here belongs to the branch-and-bound type and can be formally outlined as follows.

Algorithm 5.1.

Initialization.

(i) Construct a simplex $S^1 \supseteq X$;
(ii) compute a lower bound $\mu(S^1)$ for f_0 on $S^1 \cap F$;
(iii) determine a set $Q(S^1) \subset S^1 \cap F$ [note that $Q(S^1)$ can be empty];
(iv) set $Q^1 = Q(S^1)$, $\gamma_1 = \min\{f_0(x): x \in Q^1\}$, where γ_1 is an upper bound for the optimal value of the underlying problem, $\gamma_1 = +\infty$ if $Q^1 = \emptyset$;
(v) choose a point $\xi^1 \in Q^1$ such that $f_0(\xi^1) = \gamma_1$;
(vi) set $\mu_1 = \mu(S^1)$, $\mathscr{R}^1 = \{S^1\}$, $k=1$.

Iteration k.

(i) If $\gamma_k = \mu_k$, then stop: ξ^k is an optimal solution and γ_k is the optimal value; otherwise, divide S^k into r simplices $S_1^k, \ldots, S_r^k$ satisfying $\bigcup_{i=1}^r S_i^k = S^k$, $\operatorname{int} S_i^k \cap \operatorname{int} S_j^k = \varnothing$, for $i \neq j$;

(ii) for each $i = 1, \ldots, r$, compute a lower bound $\mu(S_i^k)$ for f_0 on $S_i^k \cap F$ satisfying $\mu(S_i^k) \geq \mu_k$, and determine a set $Q(S_i^k) \subset S_i^k \cap F$ [note that $Q(S_i^k)$ can be empty];

(iii) set $Q^{k+1} = Q^k \cup \{Q(S_i^k): i = 1, \ldots, r\}$, and $\gamma_{k+1} = \min\{f_0(x): x \in Q^{k+1}\}$;

(iv) choose $\xi^{k+1} \in Q^{k+1}$ such that $f_0(\xi^{k+1}) = \gamma_{k+1}$;

(v) set $\mathscr{R}^{k+1} = \mathscr{R}^k \setminus \{S^k\} \cup \{S_i^k: i = 1, \ldots, r\}$;

(vi) delete all $s \in \mathscr{R}^{k+1}$ such that $\mu(S) \geq \gamma_{k+1}$, and rename the remaining set $\mathscr{R}^{k+1}$, i.e., set $\mathscr{R}^{k+1} = \{S \in \mathscr{R}^{k+1}: \mu(S) < \gamma_{k+1}\}$;

(vii) if $\mathscr{R}^{k+1} = \varnothing$, set $\mu_{k+1} = \gamma_{k+1}$; otherwise, set $\mu_{k+1} = \min\{\mu(S): S \in \mathscr{R}^{k+1}\}$;

(viii) choose $S^{k+1} \in \mathscr{R}^{k+1}$ such that $\mu(S^{k+1}) = \mu_{k+1}$;

(ix) set $k \leftarrow k+1$, and return to (i).

For the implementation the above branch-and-bound scheme, we have to specify three basic operations:

(a) subdivision of simplices;
(b) estimation of lower bounds;
(c) computation of upper bounds.

These basic operations are discussed below.

Subdivision of Simplices. In what follows, we assume that X is of full dimension, so that it can be packed into an n-simplex $S^1 \subset \mathbb{R}^n$. At each iteration of the algorithm, an n-simplex S has to be divided into r n-subsimplices $S_1, \ldots, S_r$ such that

$$\bigcup_{i=1}^r S_i = S, \qquad \operatorname{int} S_i \cap \operatorname{int} S_j = \varnothing, \text{ for } i \neq j.$$

The following radial subdivision is used often in global optimization. Let S be an n-simplex having $n+1$ vertices $v^1, \ldots, v^{n+1}$, and let $v \in S \setminus \{v^1, \ldots, v^{n+1}\}$. Then, v is uniquely described by

$$v = \sum_{i=1}^{n+1} \lambda_i v^i, \ \lambda_i \geq 0, \ i = 1, \ldots, n+1, \qquad \sum_{i=1}^{n+1} \lambda_i = 1.$$

For each i such that $\lambda_i > 0$, an n-simplex S_i is defined having the vertex set $\{v^1, \ldots, v^{n+1}\} \setminus \{v^j\} \cup \{v\}$, and we say that S is divided into finitely many subsimplices S_i by a radial subdivision using a point v.

For the convergence proofs of branch-and-bound algorithms, the most useful characterization of a partition process is the concept of exhaustiveness. A nested subsequence of partition sets $\{S^k\}$, $S^k \supset S^{k+1}$, $\forall k$, is called exhaustive if S^k shrinks to a unique point as $k \to \infty$, i.e.,

$$\bigcap_{k=1}^{\infty} S^k = \{x\}.$$

Within a branch-and-bound algorithm, a partitioning process is called exhaustive if every nested subsequence of partition sets generated throughout the algorithm is exhaustive.

For several kinds of simplicial subdivisions and typical examples of exhaustive partition processes such as simplicial bisection, we refer to Thoai and Tuy (Ref. 53), Horst, Pardalos, and Thoai (Ref. 2), Horst and Tuy (Ref. 54).

Lower Bound. Let S be an n-simplex generated throughout the algorithm having $n+1$ vertices $v^1, \ldots, v^{n+1}$. Assume that S is generated from the simplex S' as described above, and assume that a lower bound $\mu(S')$ of f_0 over the set $S' \cap F$ is at hand; recall that F is the feasible set of problem (46). We intend to compute a lower bound $\mu(S)$ of f_0 over the set $S \cap F$.

Let v^0 be a point of S, e.g.,

$$v^0 = [1/(n+1)](v^1 + \cdots + v^{n+1});$$

for each $i = 0, \ldots, m$, let s^i be a subgradient of g_i at v^0. Further, let U and $\bar{U}$ be the $n \times (n+1)$ matrix with columns $v^1, \ldots, v^{n+1}$ and the $(n+1) \times (n+1)$ matrix with columns $(v^1, 1), \ldots, (v^{n+1}, 1)$, respectively; i.e.,

$$U = (v^1 \cdots v^{n+1}), \qquad \bar{U} = \begin{bmatrix} v^1 & \cdots & v^{n+1} \\ 1 & \cdots & 1 \end{bmatrix}.$$

Then, for each $i = 0, \ldots, m$, the affine function

$$l_i(x) = (x - v^0)s^i + g_i(v^0) = s^i x + (g_i(v^0) - s^i v^0)$$

is an underestimator of the convex function g_i, and the affine function

$$\varphi_i(x) = -\sum_{j=1}^{n+1} h_i(v^j)\bar{U}_j^{-1}(x, 1),$$

where $\bar{U}_j^{-1}$ is the jth row of the matrix $\bar{U}^{-1}$, is an underestimator of the concave function $-h_i$. Usually, φ_i is called the convex envelope of $-h_i$ over

S. Thus, a lower bound of f_0 over the set $S \cap F$ can be computed by solving the following linear program:

$$\min\{l_0(x) + \varphi_0(x) : x \in X \cap S,\ l_i(x) + \varphi_i(x) \leq 0,\ i = 1, \ldots, m\}. \tag{48}$$

By setting

$$x = \sum_{j=1}^{n+1} \lambda_j v^j = U\lambda, \tag{49}$$

the linear program (48) can be formulated equivalently as the following linear program in the variables $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$:

$$\min \quad \sum_{j=1}^{n+1} (s^0 v^j - h_0(v^j))\lambda_j, \tag{50a}$$

$$\text{s.t.} \quad (AU)\lambda + b \leq 0, \tag{50b}$$

$$\sum_{j=1}^{n+1} (s^i v^j - h_i(v^j))\lambda_j + (g_i(v^0) - s^i v^0) \leq 0,\ i = 1, \ldots, m, \tag{50c}$$

$$\sum_{j=1}^{n+1} \lambda_j = 1, \tag{50d}$$

$$\lambda_j \geq 0,\ j = 1, \ldots, n+1. \tag{50e}$$

Finally, letting $\bar{\mu}(S)$ be the optimal value of problem (50), if it exists, we can compute a lower bound by

$$\mu(S) = \begin{cases} \max\{\mu(S'); (\bar{\mu}(S) + (g_0(v^0) - s^0 v^0)), & \text{if } \bar{\mu}(S) < +\infty, \\ +\infty, & \text{otherwise.} \end{cases} \tag{51}$$

Note that we have $\bar{\mu}(S) < +\infty$ whenever the feasible set of problem (50) is nonempty. If problem (50) has no feasible solution, then it follows immediately that $S \cap F = \varnothing$.

For the case $\bar{\mu}(S) < +\infty$, let $\lambda(S)$ denote an optimal solution of (50). The point

$$x(S) = U\lambda(S) \tag{52}$$

will be used to establish the convergence of the algorithm.

Upper Bounds. For each simplex S generated throughout the algorithm satisfying $\mu(S) < +\infty$, we try to find feasible points of problem (46) contained in S. This can be done by checking a subset of S, e.g., $\{v^0, v^1, \ldots, v^{n+1}\}$ and $x(S)$. A collection of feasible points found in S is denoted by $Q(S)$. We set $Q(S) = \varnothing$ whenever no feasible point is found after checking a selected subset of S. Throughout the algorithm, more and more feasible points can be found; thereby, the upper bound for the optimal value of problem (46) can be improved iteratively.

Convergence. If the algorithm does not terminate after finitely many iterations, it generates at least one infinite decreasing sequence $\{S^q\}$ of simplices, i.e., $S^{q+1} \subset S^q$, $\forall q$. For this case, the following result is of fundamental interest in the investigation of the convergence of the algorithm.

Theorem 5.1. Assume that Algorithm 5.1 generates an infinite subsequence of simplices $\{S^q\}$, such that $S^{q+1} \subset S^q$ for all q and $\lim_{q\to\infty} S^q = \bigcap_{q=1}^{\infty} S^q = \{x^*\}$. Then, x^* is an optimal solution of problem (46).

Proof. For each simplex S^q, let v^{qj}, $j=0,\ldots,n+1$, x^q, l_i^q, $i=0,\ldots,m$, and φ_i^q, $i=0,\ldots,m$, stand for v^j, $j=0,\ldots,n+1$, $x(S)$, l_i, $i=0,\ldots,m$, and φ_i, $i=0,\ldots,m$, respectively, which are determined according to $S=S^q$. By construction, it holds that, for each q,

$$l_i^q(v^{q0}) = g_i(v^{q0}), \qquad i=0,\ldots,m,$$

$$\varphi_i^q(v^{qj}) = -h_i(v^{qj}), \qquad i=0,\ldots,m \quad \text{and} \quad j=1,\ldots,n+1.$$

Since $\{v^{qj}\}, j=0,\ldots,n+1$, and $\{x^q\}$ have x^* as a common limit point, it follows, that for each $i=0,\ldots,m$,

$$\begin{aligned}
&\lim_{q\to\infty} \{l_i^q(x^q) + \varphi_i^q(x^q)\} \\
&= \lim_{q\to\infty} \{l_i^q(v^{q0}) + \varphi_i^q(v^{q1})\} \\
&= \lim_{q\to\infty} \{g_i(v^{q^0}) + h_i(v^{q^1})\} \\
&= g_i(x^*) - h_i(x^*).
\end{aligned} \tag{53}$$

From the computation of the lower bounds [see (51)], we have

$$\mu^q = \mu(S^q) \geq l_0(x^q) + \varphi_0(x^q), \qquad \text{for all } q.$$

Therefore, it follows from (53) that

$$\mu^* = \lim_{q\to\infty} \mu^q \geq g_0(x^*) - h_0(x^*).$$

On the other hand, since

$$x^q \in X, \qquad l_i^q(x^q) + \varphi_i^q(x^q) \leq 0, \qquad i=1,\ldots,m, \quad \text{for all } q,$$

recall that x^q is an optimal solution of problem (48) according to S^q], it follows from the closedness of X and from (53) that

$$x^* \in X, g_i(x^*) - h_i(x^*) \leq 0, \qquad i = 1, \ldots, m,$$

which implies that x^* is feasible to problem (46). Thus, as μ^* is a lower bound of the optimal value of problem (46), x^* is an optimal solution of this problem. □

Theorem 5.1 is the key to following properties of Algorithm 5.1.

Corollary 5.1. Assume that, in Algorithm 5.1, the partition process is exhaustive. Then, the algorithm has following convergence properties:

(i) if the algorithm is infinite, and it generates an infinite subsequence $\{Q(S^q)\}$ satisfying $Q(S^q) \neq \emptyset$ for all q, then each accumulation point of $\{\xi^q\}$ is an optimal solution of problem (46);
(ii) if the algorithm is infinite, then each accumulation point of $\{x^k\}$ is an optimal solution of problem (46);
(iii) if the feasible set F of problem (46) is empty, then Algorithm 5.1 terminates after finitely many iterations.

Proof. To show property (i), let ξ^* be an accumulation point of $\{\xi^q\}$; note that accumulation points exist because of the compactness of X. Then, it is easy to see that an infinite nested subsequence $\{S^p\}$ of $\{S^q\}$ exists such that $\lim_{p \to \infty} \xi^p = \xi^*$. Let

$$x^* = \lim_{p \to \infty} S^p = \bigcap_{p=1}^{\infty} S^p,$$

and for each p, let

$$y^p \in Q(S^p).$$

Then,

$$\lim_{p \to \infty} y^p = x^*,$$

and by Theorem 5.1, x^* is an optimal solution of problem (46). Since

$$\gamma^p = f_0(\xi^p) \leq f_0(y^p), \qquad \text{for all } p,$$

we have

$$f_0(\xi^*) \leq f_0(x^*),$$

which implies that ξ^* is also an optimal solution of problem (46).

Properties (ii) and (iii) follow immediately from Theorem 5.1. □

ϵ-Optimal Solutions and Heuristic Subdivision Rules. For the case that $F \neq \emptyset$, a point $x^* \in F$ is called an ϵ-optimal solution of problem (46) with a prescribed number $\epsilon > 0$ if

$$f_0(x^*) - \epsilon |f_0(x^*)| \leq f_0(x), \qquad \forall x \in F, f_0(x^*) \neq 0,$$

$$f_0(x^*) - \epsilon \leq f_0(x), \qquad \forall x \in F, f_0(x^*) = 0.$$

In order to determine an ϵ-optimal solution, Algorithm 5.1 can be modified as follows. At each iteration k, the operation

"delete all $S \in \mathscr{R}^{k+1}$ such that $\mu(S) \geq \gamma_{k+1}$"

is replaced by

"delete all $S \in \mathscr{R}^{k+1}$ such that

$\mu(S) \geq \gamma_{k+1} - \epsilon\gamma_{k+1}$, if $\gamma_{k+1} \neq 0$, $\gamma_{k+1} < +\infty$,

or $\mu(S) \geq \gamma_{k+1} - \epsilon$, if $\gamma_{k+1} = 0$".

From Corollary 5.1, it is easy to see that the modified algorithm terminates always after finitely many iterations yielding an ϵ-optimal solution.

From a practical point of view, we propose the following simplicial subdivision rules called λ-bisection and λ-radial subdivision, respectively. The convergence of the algorithm when using these subdivision rules is not guaranteed; however, computational results indicate that, in most cases, these rules are most promising for implementing the algorithm.

λ-Bisection. Let

$$S = [v^1, \ldots, v^{n+1}]$$

be the simplex which is divided at the present iteration, and let

$$\lambda(S) = (\lambda_1(S), \ldots, \lambda_{n+1}(S))$$

be an optimal solution of the linear program in (50) according to S. Then, the simplex S is divided into two subsimplices by using the point

$$v = (1/2)(v^{i_1} + v^{i_2}),$$

where i_1 and i_2 are chosen by

$$\lambda_{i_1}(S) = \max\{\lambda_i(S) : \lambda_i(S) > 0\},$$

$$\lambda_{i_2}(S) = \max\{\lambda_i(S) : \lambda_i(S) > 0, i \neq i_1\}.$$

λ-Radial Subdivision. Let $\delta > 0$ be a small number; a suitable choice is e.g. $\delta = 1/2n^2$. If

$$\min\{\lambda_i(S) : \lambda_i(S) > 0\} > \delta,$$

then a radial subdivision using the point $x(S)$ is performed, where $x(S)$ is defined in (52) according to S. Otherwise, a λ-bisection is performed.

5.2. Combination of Branch-and-Bound Scheme and Outer-Approximation Method. The algorithm that we are going to present in this section is a typical example for the idea of combining a conical branch-and-bound scheme with an outer-approximation method; it was originally established for concave minimization problems (Refs. 53–56).

We consider a DC programming problem of the form

$$\min\{f(x)=g(x)-h(x): x\in X\}, \tag{54}$$

where g and h are finite convex functions and X is a convex subset of $\mathbb{R}^n$. Moreover, we assume that the convex set X is compact having a nonempty interior and that a point $y^0 \in \operatorname{int} X$ is available.

By using the additional variable t, problem (54) can be transformed into an equivalent form,

$$\min\{\varphi(z)=\varphi(x, t)=t-h(x): x\in X, g(x)-t\leq 0\}. \tag{55}$$

Problem (55) deals with the minimization of a concave function over a convex set of $\mathbb{R}^{n+1}$. It is known that, from an optimal solution $z^*=(x^*, t^*)$ of problem (55), one obtains an optimal solution x^* of the original DC program (54).

The main idea of a conical algorithm for problem (55) can be expressed briefly as follows.

Let K and P be a convex polyhedral cone and a convex polyhedral set of $\mathbb{R}^{n+1}$, respectively, such that K contains the set of all optimal solutions and P contains the feasible set D of problem (55). Based on the sets D, K, P and the objective function $t-h(x)$, we determine an upper bound γ of the optimal value of problem (55) and a lower bound μ of

$$\varphi(z)=t-h(x)$$

over the set $P\cap K$, which yields simultaneously a lower bound of $\varphi(z)$ over $D\cap K$, and a feasible point $z=(x, t)\in D$ satisfying

$$\varphi(z)=t-h(x)=\gamma.$$

If $\gamma=\mu$, then we are done: the point $z=(x, t)$ is obviously an optimal solution of problem (55) with optimal value γ. Otherwise, we divide the cone K into a finite number of convex polyhedral subcones $K_1, \ldots, K_r$ and construct a convex polyhedral set $\bar{P}$ such that $P\supseteq\bar{P}\supseteq D$. For each $i=1, \ldots, r$, a lower bound μ_i of φ over $K_i\cap\bar{P}$ is computed, and we obtain a new lower bound $\bar{\mu}$ of φ over $K \cap D$ by setting

$$\bar{\mu} = \min\{\mu_i : i = 1, \ldots, r\}.$$

Throughout the bound estimation, new feasible points can be generated, among those a new (better) upper bound of the optimal value is computed. The procedure is continued in this way until an upper bound is found that coincides with a lower bound over $K \cap D$.

The algorithm is described below.

Algorithm 5.2.

Initialization.

(i) Construct a cone K, a set P as described above;
(ii) set $K^1 = K$, $P^1 = P$;
(iii) set $\gamma_1 = f(y^0) = g(y^0) - h(y^0)$, $y^0 \in \operatorname{int} X$;
(iv) choose $z^1 = (x^1, t^1) \in \arg \gamma_1$, i.e., $z^1 \in D$ and $\varphi(z^1) = t^1 - h(x^1) = \gamma_1$;
(v) compute a lower bound $\mu_1 = \mu(K^1)$ of φ over $K^1 \cap P^1$;
(vi) set $\mathscr{R}^1 = \{K^1\}$, $k = 1$.

Iteration k.

(i) If $\mu_k = \gamma_k$, then stop: $z^k = (x^k, t^k)$ is an optimal solution and γ_k is the optimal value of problem (55); otherwise, divide K^k into r subcones $K_1^k, \ldots, K_r^k$ satisfying

$$\bigcup_{i=1}^{r} K_i^k = K^k, \qquad \operatorname{int} K_i^k \cap \operatorname{int} K_j^k = \varnothing;$$

(ii) construct a polyhedral set $\bar{P}^k$ such that $P^k \supseteq \bar{P}^k \supseteq D$;
(iii) compute lower bounds $\mu(K_i^k)$ of φ over $K_i^k \cap \bar{P}^k$, $i = 1, \ldots, r$;
(iv) compute γ_{k+1} and $z^{k+1} = (x^{k+1}, t^{k+1}) \in \arg \gamma_{k+1}$ by using all newly generated feasible points;
(v) set $\mathscr{R}^{k+1} = \mathscr{R}^k \setminus \{K^k\} \cup \{K_1^k, \ldots, K_r^k\}$;
(vi) delete all $K \in \mathscr{R}^{k+1}$ such that $\mu(K) \geq \gamma_{k+1}$, and rename the remaining set $\mathscr{R}^{k+1}$, i.e., set $\mathscr{R}^{k+1} = \{K \in \mathscr{R}^{k+1} : \mu(K) < \gamma_{k+1}\}$;
(vii) if $\mathscr{R}^{k+1} = \varnothing$, set $\mu_{k+1} = \gamma_{k+1}$; otherwise, set $\mu_{k+1} = \min\{\mu(K) : K \in \mathscr{R}^{k+1}\}$;
(viii) choose $K^{k+1} \in \mathscr{R}^{k+1}$ satisfying $\mu(K^{k+1}) = \mu_{k+1}$;
(ix) set $P^{k+1} = \bar{P}^k$, $k = k + 1$, and return to (i).

Clearly, Algorithm 5.2 consists of three basic operations:

(a) construction of a starting cone containing the set of optimal solutions and the division of a cone at each iteration;

(b) the estimation of bounds;
(c) the construction of a decreasing sequence of convex polyhedral sets containing the feasible set.

These basic operations are discussed below.

Construction of a Starting Cone and Conical Division. First, we show how to construct a starting cone containing the set of all optimal solutions of problem (55).

Let $S^1 \subset \mathbb{R}^n$ be an n-simplex containing the compact convex set X (several possibilities for constructing such simplex can be found e.g. in Ref. 54), and let T^1 be the n-simplex in $\mathbb{R}^{n+1}$ defined by

$$T^1 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in S^1, t = \bar{t}\}, \tag{56}$$

where

$$\bar{t} = \max\{g(x) : x \in S^1\}. \tag{57}$$

Note that $\bar{t}$ is simply obtained by the comparison of $g(x)$ at the vertices of the simplex S^1. Further, let

$$z^0 = (y^0, t^0), \qquad \text{where } t^0 \text{ is a number satisfying } t^0 > \bar{t}. \tag{58}$$

Then, we have the following proposition.

Proposition 5.1. The cone $K^1 = K(T^1) = K(z^0, T^1) \subset \mathbb{R}^n \times \mathbb{R}$, generated by $n+1$ rays emanating from z^0 and passing through the vertices of the simplex T^1, respectively, contains the set of all optimal solutions of problem (55).

Proof. Since $S^1 \supset X$, it follows by the construction of K^1, that

$$\begin{aligned} K^1 &\supset \{(x, t) : x \in S^1, g(x) - t = 0\} \\ &\supset \{(x, t) : x \in X, g(x) - t = 0\}. \end{aligned}$$

This implies that K^1 contains the optimal solution set of problem (55), since each optimal solution (x^*, t^*) must satisfy $x^* \in X, g(x^*) = t^*$. □

The starting cone constructed by Proposition 5.1 is a polyhedral convex cone having $n+1$ edges. Obviously, this structure is the simplest which a cone in $\mathbb{R}^n \times \mathbb{R}$ can have to contain the set of all optimal solutions of problem (55). At each iteration of Algorithm 5.2, a cone is divided into a finite number of subcones. It is natural that the suitable structure of K^1 should be kept for every subcone generated throughout the algorithm. A simple way to do this is the following classical radial subdivision.

Let K be any cone vertexed at z^0 and having $n+1$ edges which pass through the $n+1$ vertices of an n-simplex $T=[v^1, \ldots, v^{n+1}] \subseteq T^1$, respectively; and let $u \in K$ be a point that does not lie on an edge of K. Further, let v be the intersection point of T and the ray emanating from z^0, passing through u. By a radial subdivision of the simplex T using the point v, we obtain subsimplices $T_1, \ldots, T_r$, $2 \leq r \leq n+1$. The cone K is accordingly divided into r subcones $K_1, \ldots, K_r$, where $K_i = K(z^0, T_i)$, for each $i = 1, \ldots, r$, is the cone generated by $n+1$ rays emanating from z^0 and passing through the vertices of the simplex T_i. We say that the collection $\{K_1, \ldots, K_r\}$ forms a radial subdivision of the cone K by using the point u. A nested subsequence $\{K^q\}$, $K^q \supset K^{q+1}$, $\forall q$, is called exhaustive if the intersection $\bigcap_{q=1}^{\infty} K^q$ is a ray (a halfline emanating from z^0). A conical division process is called exhaustive if every nested subsequence of cones generated throughout the algorithm is exhaustive.

Lower Bound Estimation. Let

$$T=[v^1, \ldots, v^{n+1}] \subset T^1, \qquad K=K(z^0, T)$$

as in the previous section. Further, let P be a polyhedral set, containing the feasible set D of problem (55), defined by

$$P=\{z=(x, t) \in \mathbb{R}^n \times \mathbb{R}: Az \leq b\},$$

A and b being matrix and vector of appropriate sizes.

A way to determine a lower bound of the objective function over the set $K \cap P$ (which is also a lower bound of the objective function over $K \cap D$, since $P \subset D$) is proposed in Refs. 53 and 56 based on an idea given in Ref. 57. Here, we recall briefly this procedure.

For our purpose, we need the concept of γ-extensions. Let φ: $\mathbb{R}^\nu \rightarrow \mathbb{R}$ be a concave function, $z \in \mathbb{R}^\nu$, and let γ and θ_1 be real numbers satisfying $\gamma \leq \varphi(z)$ and $\theta_1 > 0$. A point $\bar{z} \in \mathbb{R}^\nu$ is called a γ-extension of z in the direction d with respect to φ if

$$\bar{z}=z+\theta d, \qquad \text{with } \theta=\min\{\theta_1; \sup\{\lambda: \varphi(z+\lambda d) \geq \gamma\}\}.$$

From the concavity of φ, the existence of a γ-extension of any given point z in any given direction d is guaranteed whenever $\gamma \leq \varphi(z)$.

Next, assume that an upper bound $\bar{\gamma}$ of the optimal value of problem (55) is at hand. For each $i=1, \ldots, n+1$, let z^i be the point where the ith edge of K intersects the boundary ∂D of D (the existence of such intersection points is guaranteed, since $z^0 \in \operatorname{int} D$), and let $\bar{z}^i$ be the γ-extension of z^i in the direction $z^i - z^0$, where

$$\gamma = \min\{\bar{\gamma}, \varphi(z^1), \ldots, \varphi(z^{n+1})\}.$$

Denote by Y the matrix with columns $\bar{z}^1 - z^0, \ldots, \bar{z}^{n+1} - z^0$, and define

$$c^* = \max\left\{c(\lambda) = \sum_{i=1}^{n+1} \lambda_i : AY\lambda \le b - Az^0, \lambda \ge 0\right\}. \tag{59}$$

Then, a lower bound $\mu(K) = \mu(K, P)$ of φ over $K \cap P$ is given by

$$\mu(K) = \mu(K, P) = \begin{cases} \gamma, & \text{if } c^* \le 1, \\ \min\{\varphi(\hat{z}^1), \ldots, \varphi(\hat{z}^{n+1})\}, & \text{else,} \end{cases} \tag{60}$$

where

$$\hat{z}^i = c^*(\bar{z}^i - z^0) + z^0 = c^* \bar{z}^i + (1 - c^*) z^0, = 1, \ldots, n+1.$$

Remark 5.1. Let K' be a cone such that K is generated by a subdivision of K'. Then clearly, we should set $\mu(K) = \mu(K')$ whenever the number $\mu(K)$ computed by (60) is less than $\mu(K')$.

Construction of the Polytope $\bar{P}$. At each iteration of the algorithm, a polyhedral set $\bar{P}$ is constructed satisfying $P \supseteq \bar{P} \supseteq D$. This operation is performed as follows. Let K be the cone which is divided at the present iteration, and let $\hat{z}$ be the point computed by

$$\hat{z} = Y\lambda^* + z^0, \tag{61}$$

where λ^* is an optimal solution of the linear program in (59). Note that here we have

$$c^* = \sum \lambda_i^* > 1,$$

since otherwise the cone K would be removed from further consideration. Geometrically, $\hat{z}$ is a point of the set P which stands farthest from the hyperplane containing $\bar{z}^1, \ldots, \bar{z}^{n+1}$.

If $\hat{z} \in D$, then we simply set $\bar{P} = P$. Otherwise, we compute the point w where the line segment $[z^0, \hat{z}]$ meets the boundary ∂D and set

$$\bar{P} = P \cap \{z : l(z) = (z - w)\xi \le 0\}, \tag{62}$$

where ξ is a subgradient at the point w of the convex function defining the convex set D.

It is known from the outer approximation concept (cf. e.g. Ref. 58), that we have $\hat{z} \notin \bar{P}$ and $\bar{P} \supseteq D$, which imply that $P \supseteq \bar{P} \subseteq D$.

Convergence. Algorithm 5.2 belongs to the class of branch-and-bound algorithms in which a nonempty set of feasible points is determined at each iteration. The following result is a general condition for the convergence of this class; see e.g. Refs. 2 and 54.

Theorem 5.2. For any infinite nested subsequence $\{K^q\} \subset \{K^k\}$ such that $K^{q+1} \subset K^q$ for all q, if the condition

$$\lim_{q \to \infty} (\gamma_q - \mu_q) = 0, \tag{63}$$

is fulfilled, then

$$\mu = \lim_{k \to \infty} \mu_k = \lim_{k \to \infty} \varphi(z^k) = \lim_{k \to \infty} \gamma_k = \gamma, \tag{64}$$

and every accumulation point z^* of the sequence $\{z^k\}$ is an optimal solution of problem (55).

Proof. Let z^* be an accumulation point of $\{z^k\}$. Then, it is easy to see that an infinite nested subsequence $\{K^q\}$ of $\{K^k\}$ exists such that $\lim_{q \to \infty} z^q = z^*$. From the continuity of the function φ, it follows that

$$\lim_{q \to \infty} \varphi(z^q) = \varphi(z^*).$$

Let μ and γ be the limits of $\{\mu_k\}$ and $\{\gamma_k\}$, respectively. Then, μ and γ exist, since $\{\mu_k\}$ is monotonically increasing and bounded by the optimal value φ^* of problem (55) and $\{\gamma_k\}$ is monotonically decreasing and bounded by φ^*. Obviously, we have

$$\mu \le \varphi^* \le \lim_{k \to \infty} \varphi(z^k) = \gamma.$$

Therefore, condition (63) implies (64), and z^* is an optimal solution to problem (55). □

We now establish a sufficient condition for (63). For this purpose, we need some additional notions. For each $q \ge 1$, we denote by $z^{q,i}$, $\bar{z}^{q,i}$, $\hat{z}^{q,i}$, $\hat{z}^q$ the points z^i, $\bar{z}^i$, $\hat{z}^i$, $\hat{z}$, respectively, constructed according to the cone K^q as in previous subsections. Further, we denote by z^q and $\bar{z}^q$ the points where the line segment $[z^0, \hat{z}^q]$ intersects the simplex $[z^{q,1}, \ldots, z^{q,n+1}]$ and $[\bar{z}^{q,1}, \ldots, \bar{z}^{q,n+1}]$, respectively.

Proposition 5.2. Condition (63) is fulfilled for every subsequence $\{K^q\} \subset \{K^k\}$ such that $K^{q+1} \subset K^q$, $\forall q$, if it holds that

$$\lim_{q \to \infty} \|\bar{z}^q - \hat{z}^q\| = 0. \tag{65}$$

Proof. Since the hyperplanes containing $[\bar{z}^{q,1}, \ldots, \bar{z}^{q,n+1}]$ and $[\hat{z}^{q,1}, \ldots, \hat{z}^{q,n+1}]$, respectively, are parallel to each other, we have, for each $i = 1, \ldots, n+1$,

$$\|\bar{z}^{q,i} - \hat{z}^{q,i}\| / \|\bar{z}^{q,i} - z^0\| = \|\bar{z}^q - \hat{z}^q\| / \|\bar{z}^q - z^0\|.$$

But $\|\bar{z}^{q,i} - z^0\|$ and $\|\bar{z}^q - z^0\|$ are all bounded; therefore, it follows from the continuity of the function φ that

$$\gamma_q - \mu_q \leq \varphi(\bar{z}^{q,i}) - \varphi(\hat{z}^{q,i}) \to 0, \qquad \text{as } q \to \infty,$$

if $\|\bar{z}^{q,i} - \hat{z}^{q,i}\| \to 0$ as $q \to \infty$, i.e., if (65) holds. □

Clearly, condition (65) depends upon the conical division process performed throughout the algorithm. As the readers may presume, a sufficient condition for (65) is the exhaustiveness of the conical division process.

Proposition 5.3. Condition (65) is fulfilled for every exhaustive subdivision process.

Proof. Let $\{K^q\}$ be an exhaustive nested subsequence, and let Γ be the ray such that $\bigcap_q K^q = \Gamma$. Then, the point $\hat{z}^q$ approaches a point $z^* \in \Gamma$. From an outer approximation procedure (Refs. 54 and 58), it follows that $z^* \in \partial D$, i.e., z^* is the intersection point of Γ and ∂D. On the other hand, all the points $z^{q,i}$, $i = 1, \ldots, n+1$, and z^q approach the point z^* as well. Therefore, it follows that

$$\|\bar{z}^q - \hat{z}^q\| \leq \|z^q - \hat{z}^q\| \to 0, \qquad \text{as } q \to \infty.$$ □

In general, the radial division process is not exhaustive. However, the following proposition shows a case where condition (65) is fulfilled for a radial divison.

Proposition 5.4. Assume that, at each iteration $k \geq 1$, a radial division of the cone K^k is performed by using the point $u^k = \hat{z}^k$. Then, condition (65) is fulfilled if, for any subsequence $\{K^q\}$ such that each K^q is generated by a division of K^{q-1}, we have

$$1/\|eY_q^{-1}\| \geq a > 0, \qquad \forall q, \tag{66}$$

where Y_q is the matrix with columns $\bar{z}^{q,1} - z^0, \ldots, \bar{z}^{q,n+1} - z^0$ and $e = (1, \ldots, 1) \in \mathbb{R}^{n+1}$.

Proof. For each q, let us denote by H^q the hyperplane containing $\bar{z}^{q,1}, \ldots, \bar{z}^{q,n+1}$ and by $d(z, H^q)$ the distance from a point z to H^q; i.e.,

$$d(z, H^q) = \min\{\|z - z'\| : z' \in H^q\}.$$

First, we show that

$$\lim_{q \to \infty} d(\bar{z}^q, H^{q+1}) = 0. \tag{67}$$

Since $\{\bar{z}^q\}$ is bounded, we can assume, by passing to subsequence if necessary, that $\bar{z}^q \to \bar{z}^*$. But $\bar{z}^{q+1} \in H^{q+1}$, $\forall q$; therefore, we have

$$d(\bar{z}^q, H^{q+1}) \leq \|\bar{z}^q - \bar{z}^{q+1}\| \to 0, \qquad \text{as } q \to \infty.$$

Since K^{q+1} is generated from K^q by a radial division, we can fix an index j such that $z^{q+1,j}$ is the intersection point of $[z^0, \hat{z}^q]$ and ∂D. As in the proof of Proposition 2.3, if follows from an outer approximation concept that $\hat{z}^q$ and $z^{q+1,j}$ approach a unique point $\hat{z}^* \in \partial D$. On the other hand, we have

$$\begin{aligned} \|\bar{z}^q - z^{q+1,j}\| &\leq \|\bar{z}^q - \bar{z}^{q+1,j}\| \\ &= [d(\bar{z}^q, H^{q+1})/d(z^0, H^{q+1})], \|\bar{z}^{q+1,j}\|, \qquad \forall q. \end{aligned} \tag{68}$$

Since $\|\bar{z}^{q+1,j}\|$ is bounded, it follows from (66) and (67) that

$$\|\bar{z}^q - z^{q+1,j}\| \to 0, \qquad \text{as } q \to \infty,$$

which implies that $\bar{z}^q$ approaches $\hat{z}^*$, and hence (65). □

ϵ-Optimal Solutions and Heuristic Subdivision Rules. In order to determine an ϵ-optimal solution (x^*, t^*) in the sense that

$$\varphi(x^*, t^*) - \epsilon|\varphi(x^*, t^*)| \leq \varphi(x, t), \quad \forall (x, t) \in D, \quad \text{if } \varphi(x^*, t^*) \neq 0,$$

or

$$\varphi(x^*, t^*) - \epsilon \leq \varphi(x, t), \qquad \forall (x, t) \in D, \qquad \text{if } \varphi(x^*, t^*) = 0,$$

with a prescribed number $\epsilon > 0$, Algorithm 5.2 can be modified as discussed in Algorithm 5.1.

From practical point of view, the simplicial λ-bisection and λ-radial subdivision discussed in Algorithm 5.1 can also be applied accordingly to construct the conical λ-bisection and λ-radial subdivision in Algorithm 5.2.

Remark 5.2 In the construction of a starting cone K^1, if we set in (58) $t^0 = +\infty$, then Algorithm 5.2 becomes the so-called prismatic algorithm proposed in Refs. 2 and 48.

5.3. Cutting Plane Algorithm. In this section, we present an algorithm for solving DC programming problems of the form (54), which is established based upon the optimality condition (29).

For the development of the algorithm, we also need the assumption that the convex set X is compact having a nonempty interior, so that it can be packed into an n-simplex $S^0 \subset \mathbb{R}^n$. Moreover, we also assume that a point $y^0 \in \operatorname{int} X$ is available. The main idea of the algorithm is to generate a sequence of feasible points $\{y^0, y^1, \ldots\} \subset X$, until obtaining a feasible point y^k, which fulfills condition (29). For each polytope P, we denote by $V(P)$ the set of its vertices.

Algorithm 5.3.

Initialization.

(i) Set $\omega^0 = g(y^0) - h(y^0)$, first upper bound of the optimal value ω^* of problem (54);
(ii) construct a polytope P^0 containing the set $\{(x, t): x \in X,\ t \in \mathbb{R}^n,\ g(x) - t - \omega^* = 0\}$, and compute $V(P^0)$;
(iii) set $k = 0$.

Iteration k.

(i) Compute an optimal solution (x^k, t^k) of the problem $\min\{-h(x) + t: (x, t) \in V(P^k)\}$; if $-h(x^k) + t^k = 0$, then stop: y^k is an optimal solution of problem (54) with optimal value ω^k (cf. Theorem 21 below); otherwise, compute $y^{k+1} \in X$ such that $\omega^{k+1} = g(y^{k+1}) - h(y^{k+1}) \le \omega^k$;
(ii) construct a cutting plane, i.e., an affine function $l^k(x, t)$ such that

$$l^k(x^k, t^k) > 0, \tag{69a}$$

$$l^k(x, t) \le 0, \quad \text{for } x \in X,\ g(x) - t - \omega^{k+1} \le 0; \tag{69b}$$

(iii) set: $P^{k+1} = P^k \cap \{(x, t): l^k(x, t) \le 0\}$, and compute $V(P^{k+1})$;
(iv) set $k = k + 1$, and return to (i).

For the implementation of Algorithm 5.3, we have to specify the following main tasks:

(a) construction of a first polytope P^0 and its vertex set $V(S^0)$;
(b) choice of the point y^k for $k \ge 1$ and the construction of an affine function l^k, $k \ge 0$, satisfying condition (69);
(c) computation of the set $V(P^k)$, $k \ge 1$.

For calculating the vertex set of a polytope defined as the intersection of a polytope with a halfspace [Task (c)], the methods discussed in Refs. 2 and 59–61 can be used. Tasks (a) and (b) are implemented as follows.

Construction of a First Polytope P^0 and Its Vertex Set. Let

$$V(S^0)=\{v^1,\ldots,v^{n+1}\}$$

be the set of $n+1$ vertices of the n-simplex S^0, and let s be a subgradient of the convex function g at the point y^0. Further, define an affine function $l(x)$ by

$$l(x)=(x-y^0)s+g(y^0). \tag{70}$$

Then, $l(x)$ is an underestimation of the convex function $g(x)$, and we have

$$\begin{aligned}&\{(x,t): x\in X, t\in\mathbb{R}^n, g(x)-t-\omega^*=0\}\\ \subset&\{(x,t): x\in X, t\in\mathbb{R}^n, g(x)-t-\omega^0\le 0\}\\ \subset&\{(x,t): x\in X, t\in\mathbb{R}^n, l(x)-t-\omega^0\le 0\}.\end{aligned}$$

Next, let $\bar{\omega}$ and $\bar{t}$ be real numbers satisfying

$$\begin{aligned}&\bar{\omega}=\min\{l(x): x\in V(S^0)\}-\max\{h(x): x\in V(S^0),\\ &\bar{t}>\max\{g(x): x\in V(S^0)\}-\bar{\omega}.\end{aligned}$$

Then, obviously we have $\bar{\omega}<\omega^*$ and

$$\begin{aligned}&\{(x,t): x\in X, t\in\mathbb{R}^n, g(x)-t-\omega^*=0\}\\ \subset&\{(x,t): x\in X, t\in\mathbb{R}^n, g(x)-t-\bar{\omega}\ge 0\}\\ \subset&\{(x,t): x\in X, t\in\mathbb{R}^n, t\le\bar{t}\}.\end{aligned}$$

Thus, a first polytope P^0 containing the set

$$\{(x,t): x\in X, t\in\mathbb{R}^n, g(x)-t-\omega^*=0\}$$

can be defined by

$$P^0=\{(x,t): x\in S^0, t\le\bar{t}, l(x)-t-\omega^0\le 0\}. \tag{71}$$

The set $V(P^0)$ consists of the $2(n+1)$ points

$$(v^i,\bar{t}),\qquad i=1,\ldots,n+1, \tag{72a}$$

$$(v^i, l(v^i)-\omega^0),\qquad i=1,\ldots,n+1. \tag{72b}$$

Moreover, by construction, the following property holds:

$$g(y^0) - \bar{t} - \omega < 0, \qquad \forall \omega \geq \omega^*. \tag{73}$$

Choice of y^k and Construction of l^k. In what follows, we assume that the convex set X is given by

$$X = \{x \in \mathbb{R}^n : \alpha(x) \leq 0\}, \tag{74}$$

where α is a convex function defined on $\mathbb{R}^n$. Moreover, at Iteration k of the algorithm, we define a convex function $\beta^k: \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\beta^k(x, t) = \max\{\alpha(x); g(x) - t - \omega^k\}. \tag{75}$$

Since $y^0 \in \operatorname{int} X$, it follows from (75) that

$$\beta^k(y^0, \bar{t}) < 0, \qquad \forall k \geq 0. \tag{76}$$

At Iteration k, consider the point (x^k, t^k). Notice that, whenever the algorithm does not terminate at this iteration, we have $-h(x^k) + t^k < 0$.

If $x^k \in X$, then set

$$\omega^{k+1} = \min\{\omega^k; g(x^k) - h(x^k)\};$$

choose y^{k+1} such that

$$g(y^{k+1}) - h(y^{k+1}) = \omega^{k+1}.$$

An affine function $l^k(x, t)$ is defined by

$$\begin{aligned} l^k(x, t) &= (x - x^k)s^k + g(x^k) - \omega^{k+1} - t \\ &= s^k x - t - (s^k x^k - g(x^k) + \omega^{k+1}), \end{aligned} \tag{77}$$

where s^k is a subgradient of g at x^k. Obviously, $l^k(x, t)$ satisfies condition (69).

If $x^k \notin X$, then $\beta^k(x^k, t^k) > 0$. Compute the point (ζ^k, θ^k) in the line segment $[(x^k, t^k), (y^0, \bar{t})]$ satisfying $\beta(\zeta^k, \theta^k) = 0$. Set

$$\omega^{k+1} = \min\{\omega^k; g(\zeta^k) - h(\zeta^k)\};$$

choose y^{k+1} such that

$$g(y^{k+1}) - h(y^{k+1}) = \omega^{k+1}.$$

An affine function $l^k(x, t)$ is defined by

$$l^k(x, t) = ((x, t) - (\zeta^k, \theta^k))s^k + \beta^k(\zeta^k, \theta^k), \tag{78}$$

where s^k is a subgradient of β^k at (ζ^k, θ^k). Obviously, the affine function $l^k(x, t)$ defined by (78) satisfies

$$l^k(x^k, t^k) > 0,$$

$$l^k(x, t) \leq 0, \qquad \text{for } x \in X, g(x) - t - \omega^k \leq 0.$$

Therefore, it satisfies condition (69), since $\omega^{k+1} \leq \omega^k$.

Convergence. The convergence of Algorithm 5.3 is stated in the following sense.

Theorem 5.3.

(i) At iteration k, if the condition $-h(x^k) + t^k = 0$ is fulfilled, then y^k is an optimal solution of problem (54).

(ii) If the algorithm generates an infinite sequence $\{y^k\} \subset X$, then every accumulation point of this sequence is an optimal solution of problem (54).

Proof.

(i) From the property

$$-h(x^k) + t^k = 0,$$

it follows that

$$0 \leq \min\{-h(x) + t: x \in X, g(x) - t - \omega^k \leq 0\}.$$

Let

$$\bar{t}^k = h(y^k).$$

Since $y^k \in X$ and

$$g(y^k) - \bar{t}^k - \omega^k = g(y^k) - \bar{t}^k - (g(y^k) - h(y^k)) = h(y^k) - \bar{t}^k = 0,$$

it follows that the point $(y^k, \bar{t}^k)$ fulfills condition (29); therefore, y^k is an optimal solution of problem (54).

(ii) Let $\bar{y}$ be an accumulation point of $\{y^k\}$, and let $\{y^q\}$ be a subsequence of $\{y^k\}$ such that

$$\lim_{q \to \infty} y^q = \bar{y}.$$

Moreover, let

$$\bar{\omega} = \lim_{q \to \infty} \omega^q, \qquad (\bar{x}, \bar{t}) = \lim_{q \to \infty} (x^q, t^q).$$

By passing to a suitable subsequence, we can assume that

$$\text{either (a) } x^q \in X, \qquad \text{for all } q,$$

$$\text{or (b) } x^q \notin X, \qquad \text{for all } q.$$

From a well-known scheme of outer-approximation algorithms (see e.g. Ref. 58), it follows that we have

$$\text{in case (a),} \qquad \bar{x} \in X, g(\bar{x}) - \bar{t} - \bar{\omega} = 0,$$

$$\text{in case (b),} \qquad \lim_{q\to\infty} (\zeta^q, \theta^q) = (\bar{x}, \bar{t}), g(\bar{x}) - \bar{t} - \bar{\omega} = 0.$$

We have to show that

$$\lim_{q\to\infty} (-h(x^q) + t^q) = -h(\bar{x}) + \bar{t} = 0.$$

Suppose that

$$-h(\bar{x}) + \bar{t} < 0.$$

Then in case (a), it follows that

$$g(\bar{x}) - h(\bar{x}) < g(\bar{x}) - \bar{t} = \bar{\omega},$$

i.e., there is an index q_0 such that

$$q(x^{q_0}) - h(x^{q_0}) < \bar{\omega}.$$

Therefore,

$$\omega^{q_0+1} \le g(x^{q_0}) - h(x^{q_0}) < \bar{\omega},$$

which is a contradiction, because $\{\omega^q\}$ is a nonincreasing sequence converging to $\bar{\omega}$. In case (b), there is an index q_0 such that

$$-h(\zeta^q) + \theta^q < 0, \qquad \text{for all } q \ge q_0.$$

Since

$$g(\zeta^q) - \theta^q - \omega^q \le 0, \qquad \text{for all } q,$$

it follows that

$$g(\zeta^q) - h(\zeta^q) < \omega^q, \qquad \text{for all } q \ge q_0.$$

Therefore,

$$\omega^{q+1} = g(\zeta^q) - h(\zeta^q), \qquad \text{for all } q \ge q_0.$$

Thus,

$$-h(\zeta^q)+\theta^q=-(g(\zeta^q)-\theta^q-\omega^{q+1}), \qquad \text{for all } q\geq q_0;$$

therefore,

$$\begin{aligned}0>-h(\bar{x})+\bar{t}&=\lim_{q\to\infty}(-h(\zeta^q)+\theta^q)\\&=\lim_{q\to\infty}-(g(\zeta^q)-\theta^q-\omega^{q+1})\\&=g(\bar{x})-\bar{t}-\bar{\omega}\\&=0.\end{aligned}$$

This contradiction completes the proof. □

Remark 5.3. Algorithm 5.3 can be considered as a procedure for solving problem (30), which is a generalized canonical DC program derived from the DC programming problem (54). In fact, Algorithm 5.3 can be applied to every generalized canonical DC program satisfying the assumptions of Proposition 4.3.

6. Conclusions

DC programming plays an interesting and important part in the field of nonconvex optimization. This article surveys essential results on theory, applications, and solution methods of DC programming in the sense of global optimization. Some modifications and new results on optimality conditions and algorithms for DC programming problems are also presented.

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