Convex Analysis

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Introduction

Convexity (of sets and functions) plays a central role in various areas of mathematics, e.g. optimization, optimal control, calculus of variations, statistics or convex geometry. The conceptual analysis of convex sets and functions goes as far back as to the work of Minkowski. Convex analysis as an autonomous mathematical discipline was supposedly forged by Werner Fenchel in his famous lecture notes. Together with Jean-Jacques Moreau and Terry Rockafellar (who wrote the first, outstanding standard reference on convex analysis, see [5]) these researchers are the founding fathers of the field. Convex analysis builds a beautiful theoretical framework with subdifferential calculus and duality theory at its core. In this regard, it forms the basis for modern nonsmooth and non-convex variational analysis. Convexity has always been intimately tied to minimization and other variational problems; in fact, convexity is viewed by many people to play the role in variational analysis that linearity plays in classical analysis or as Rockafellar put it:

"The great watershed in optimization is not between linearity and nonlinearity, but convexity and nonconvexity",

see [6].

Recently, there has been a huge renaissance of convex analysis due to the numerous applications of convex optimization algorithms in machine learning and big data problems.

This course is loosely designed on basis of the standard references [3] by Hiriart-Urruty and Lemaréchal and [5] by Rockafellar, and the more recent book [1] by Bauschke and Combettes, which also applies to infinite dimensional Hilbert spaces. We will focus on the theoretical tools for convex sets (convex hulls, relative topology, separation theorems, conical approximations etc.) and convex functions (affine minorization, continuity/differentiability properties, subdifferential calculus, duality theory), but we will also highlight the connections to optimization.

1 Preliminaries

1.1 The Euclidean setting

In what follows, $\mathbb{E}$ will be an N-dimensional ($N \in \mathbb{N}$) Euclidean space, i.e. a finite-dimensional real vector space equipped with a *scalar product*, which we denote by $\langle \cdot, \cdot \rangle$. Recall that a scalar product on $\mathbb{E}$ is a mapping $\langle \cdot, \cdot \rangle : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{E}$ and $\lambda, \mu \in \mathbb{R}$ we have:

i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ *(linearity in first argument)*;

ii) $\langle x, y \rangle = \langle y, x \rangle$ (symmetry);

iii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$ *(positive definiteness)*.

Altogether $\langle \cdot, \cdot \rangle$ is a *positive definite, symmetric bilinear form* on $\mathbb{E}$.

By $\| \cdot \|$ we label the norm[1] on $\mathbb{E}$ induced by the scalar product, i.e.

$$\|x\| := \sqrt{\langle x, x \rangle} \quad (x \in \mathbb{E}),$$

Scalar product and induced norm obey they famous *Cauchy-Schwarz inequality*

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad (x, y \in \mathbb{E}),$$

where equality holds if and only if x and y are linearly dependent.

The open ball with radius $\varepsilon > 0$ centered around $x \in \mathbb{E}$ is denoted by $B_\varepsilon(x)$. In particular, we put $\mathbb{B} := B_1(0)$ for the open unit ball.

If $(\mathbb{E}_i, \langle \cdot, \cdot \rangle_i)$ $(i = 1, \dots, m)$ is a Euclidean space, then $\bigtimes_{i=1}^m \mathbb{E}_i$ is also a Euclidean space equipped with the canonical scalar product

$$\langle \cdot, \cdot \rangle : \bigtimes_{i=1}^m \mathbb{E}_i \to \mathbb{R}, \quad \langle (x_1, \dots, x_m), (y_1, \dots, y_m) \rangle := \sum_{i=1}^m \langle x_i, y_i \rangle_i .$$

[1] Recall that a norm on $\mathbb{E}$ is a mapping $\| \cdot \|_* : \mathbb{E} \to \mathbb{R}$ such that for all $x, y \in \mathbb{E}$ and $\lambda \in \mathbb{R}$:

i) $\|x\|_* = 0 \iff x = 0$ (definiteness)

ii) $\|\lambda x\|_* = |\lambda| \cdot \|x\|_*$ (absolute homogeneity)

iii) $\|x + y\|_* \leq \|x\|_* + \|y\|_*$ (triangle inequality)

For two Euclidean spaces $\mathbb{E}_1, \mathbb{E}_2$ a *set-valued mapping* is a function $S : \mathbb{E}_1 \to 2^{\mathbb{E}_2}$, i.e. S maps points in $\mathbb{E}_1$ to subsets of $\mathbb{E}_2$ and we will mainly write

$$S : \mathbb{E}_1 \rightrightarrows \mathbb{E}_2$$

in that situation. We call S *closed-, compact-* and *convex-valued*, respectively, if $S(x) \subset \mathbb{E}_2$ is closed, compact and convex, respectively (see Definition 2.1.1 below)

For $n = \dim \mathbb{E}_1, m = \dim \mathbb{E}_2 < \infty$, the set of all *linear (hence continuous) operators* is denoted by $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. Recall from Linear Algebra that $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ is isomorphic to $\mathbb{R}^{m\times n}$, the set of all real $m \times n$-matrices, and that $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ if and only if

1) $L(\lambda x) = \lambda L(x) \quad \forall x \in \mathbb{E}_1, \lambda \in \mathbb{R}$ (homogeneity);
2) $L(x + y) = L(x) + L(y) \quad \forall x, y \in \mathbb{E}_1$ (additivity).

It is known from Linear Algebra that (since we restrict ourselves to finite dimensional Euclidean spaces alone) for $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ there exists a mapping $L^* \in \mathcal{L}(\mathbb{E}_2, \mathbb{E}_1)$ such that

$$\langle Lx,\, y\rangle_{\mathbb{E}_2} = \langle x,\, L^*y\rangle_{\mathbb{E}_1} \quad \forall x, y \in \mathbb{E}_1.$$

The mapping L^* is called the *adjoint (mapping)* of L. If $\mathbb{E}_1 = \mathbb{E}_2$ and $L = L^*$, we call L *self-adjoint*.

With the well-known definitions for

$$\operatorname{im} L := \{L(x) \in \mathbb{E}_2 \mid x \in \mathbb{E}_1\} \quad \text{(image of } L\text{)}$$

and

$$\ker L := \{x \in \mathbb{E}_1 \mid L(x) = 0\} \quad \text{(kernel of } L\text{)}$$

for $L : \mathbb{E}_1 \to \mathbb{E}_2$ linear, the following important relations are standard knowledge from Linear Algebra.

At this, recall that, for some nonempty subset $S \in \mathbb{E}$ we define its *orthogonal complement* by

$$S^\perp := \{x \in \mathbb{E} \mid \langle s,\, x\rangle = 0 \quad \forall s \in S\}.$$

Theorem 1.1.1 (Fundamental subspaces) *Let $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. Then the following hold:*

a) $\ker L = (\operatorname{im} L^*)^\perp$ *and* $(\ker L)^\perp = \operatorname{im} L^*$*;*

b) $\ker L^* = (\operatorname{im} L)^\perp$ *and* $(\ker L^*)^\perp = \operatorname{im} L$*.*

Minkowski addition and multiplication

For two sets $A, B \subset \mathbb{E}$ we define their *Minkowski sum* by

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

If $B = \{b\}$ is a singleton, we put

$$A + B =: A + b.$$

Moreover, if $A = \emptyset$ (or $B = \emptyset$) then $A + B := \emptyset$.

In addition, for $\Lambda \subset \mathbb{R}$, we put

$$\Lambda \cdot A := \{\lambda a \mid a \in A, \lambda \in \Lambda\}.$$

The operation $(\Lambda, A) \subset 2^{\mathbb{R}} \times 2^{\mathbb{E}} \to 2^{\mathbb{E}}$ is called *(generalized) Minkowski multiplication.* For $\Lambda = \{\lambda\}$ we simply write

$$\lambda A := \{\lambda\} \cdot A.$$

Using the Minkowski sum notation, for instance, we have

$$B_\varepsilon(x) = x + \varepsilon\mathbb{B}$$

for all $x \in \mathbb{E}$ and $\varepsilon > 0$.

1.1.1 Euclidean matrix spaces

The most obvious realization of an N-dimensional Euclidean space is $\mathbb{R}^N$ equipped with the standard scalar product

$$\langle x, y\rangle := x^T y \quad (x, y \in \mathbb{R}^N).$$

From Linear Algebra we know that two real vector-spaces of the same *finite* dimension are isomorphic (even isometrically isomorphic), hence there would be no loss in generality to only consider $\mathbb{R}^N$ instead of an abstract Euclidean space $\mathbb{E}$ of finite-dimension. However, there are two upsides in doing so: First, many results are also going to be valid without the requirement that $\mathbb{E}$ be finite-dimensional and this becomes very obvious in the general notation. Secondly, we are strongly interested in (convex) functions on matrix spaces (see below), and a more general notation translates smoothly to this setting.

The space $\mathbb{R}^{m\times n}$ of all real $m \times n$ matrices is an mn-dimensional Euclidean space with the standard scalar product

$$\langle A, B\rangle := \mathrm{tr}\,(A^T B),$$

where $\mathrm{tr}\; : \mathbb{R}^{p\times p} \to \mathbb{R}, \mathrm{tr}\,(M) = \sum_{i=1}^p m_{ii}$ is the *trace* function .

Symmetric and orthogonal matrices

Recall that the set

$$\mathrm{O}(n) := \left\{ A \in \mathbb{R}^{n\times n} \;\middle|\; A^T A = I \right\}$$

is a group (in fact, a subgroup of the the invertible matrices in $\mathbb{R}^{n\times n}$) called the *orthogonal group* with its members being called *orthogonal matrices.*

An important subspace of $\mathbb{R}^{n\times n}$ is

$$\mathbb{S}^n = \left\{ A \in \mathbb{R}^{n\times n} \;\middle|\; A^T = A \right\},$$

the space of all $n \times n$ *symmetric matrices.*

In Linear Algebra it is shown that any symmetric matrix is orthogonally similar to a diagonal matrix with real entries (the eigenvalues), which is usually subsumed in the following theorem.

Theorem 1.1.2 (Spectral Theorem) *Let $A \in \mathbb{S}^n$. Then there exists $U \in \mathrm{O}(n)$ orthogonal such that*

$$A = U^T \mathrm{diag}\,(\lambda_1, \dots, \lambda_n) U$$

and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Some important subsets of $\mathbb{S}^n$ are:

- $\mathbb{S}^n_+ := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x \geq 0 \; \forall x \in \mathbb{R}^n \right\}$ *(positive semidefinite matrices),*
- $\mathbb{S}^n_{++} := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x > 0 \; \forall x \in \mathbb{R}^n \setminus \{0\} \right\}$ *(positive definite matrices),*
- $\mathbb{S}^n_- := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x \leq 0 \; \forall x \in \mathbb{R}^n \right\}$ *(negative semidefinite matrices),*
- $\mathbb{S}^n_{--} := \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x < 0 \; \forall x \in \mathbb{R}^n \setminus \{0\} \right\}$ *(negative semidefinite matrices).*

Note that

$$\mathbb{S}^n_- = -\mathbb{S}^n_+ \quad \text{and} \quad \mathbb{S}^n_{--} = -\mathbb{S}^n_{++}.$$

For $A, B \in \mathbb{S}^n$, we also make use of the convention

$$A \succeq B \quad :\iff \quad A - B \in \mathbb{S}^n_+$$

and

$$A \succ B \quad :\iff \quad A - B \in \mathbb{S}^n_{++}.$$

An immediate consequence of the spectral theorem is the existence of a square root of a positive semidefinite matrix.

Corollary 1.1.3 (Square root of a positive semidefinite matrix) *For $A \in \mathbb{S}^n_+$ there exists a unique matrix $B \in \mathbb{S}^n_+$ such that $B^2 = A$.*

In the scenario of Corollary 1.1.3, we put $\sqrt{A} := A^{1/2} := B$ and call it the square root of A. Moreover, we define $A^{-1/2} := \sqrt{A^{-1}} = (\sqrt{A})^{-1}$.

1.1.2 Differentiation in Euclidean space

In this paragraph we recall some basic differentiability notions for functions mapping from one Euclidean space to another.

Let $\Omega \subset \mathbb{E}_1$ be open. A function $f : \Omega \to \mathbb{E}_2$ is said to be *(Fréchet) differentiable at* $x \in \Omega$ if there exists $L_x \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ such that

$$f(x+h) = f(x) + L_x(h) + o(\|h\|), \tag{1.1}$$

i.e.

$$\lim_{h \to 0, h \neq 0} \frac{f(x+h) - f(x) - L_x(h)}{\|h\|} = 0.$$

If such a linear operator L_x exists, we call it the *(Fréchet) derivative* of f at x and we will denote it by $f'(x)$ in the sequel. We call f *differentiable on* Ω if it is differentiable at every point in Ω, and for $\Omega = \mathbb{E}_1$, we simply say that f is differentiable. If the *derivative mapping*

$$f' : x \in \Omega \mapsto f'(x) \in L(\mathbb{E}_1, \mathbb{E}_2)$$

is continuous, we say that f is *continuously differentiable* or *smooth* on Ω.

Recall that if f is differentiable at x, $f'(x) \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ is uniquely determined through the limit

$$f'(x)(h) = \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t} \quad \forall h \in \mathbb{E}_1.$$

Thus far, we have not even used the fact that our spaces are equipped with a scalar product; in fact, the above concepts read the same in an arbitrary normed (possibly infinite dimensional) space.

Exploiting the fact that we have a scalar product, in the scalar-valued case, $f : \Omega \subset \mathbb{E} \to \mathbb{R}$, if f is differentiable at $x \in \Omega$, we know from Exercise 1.2. that there exists a unique vector $\nabla f(x) \in \mathbb{E}$ such that

$$f'(x)(h) = \langle \nabla f(x),\, h \rangle \quad \forall h \in \mathbb{E}.$$

We call $\nabla f(x)$ the *gradient* of f at x. Note, however, that $\nabla f(x)$ heavily depends on the chosen scalar product.

If $f : \Omega \to \mathbb{R}$ is differentiable on the open set Ω and the derivative mapping

$$f' : x \in \Omega \mapsto f'(x) \in \mathcal{L}(\mathbb{E}, \mathbb{R})$$

itself is differentiable on Ω (in the sense of (1.1) with $\mathbb{E}_2 = \mathcal{L}(\mathbb{E}, \mathbb{R})$) this induces a mapping

$$x \in \Omega \mapsto (f')'(x) =: f''(x) \in \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{R})).$$

In this case we say that f is *twice differentiable at* x, and we call $f''(x)$ the *second-order derivative* of f at x. Since, by a simple dimension argument, $\mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{R}))$ is isomorphic the the space

$$\mathcal{L}^2(\mathbb{E}) := \{\beta : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \mid \beta \text{ bilinear}\}$$

of all bilinear forms on $\mathbb{E}$, we obtain a bilinear form

$$(h, d) \in \mathbb{E} \times \mathbb{E} \mapsto [(f''(x)(h)](d) =: f''(x)[h, d].$$

Exploiting the existence of a scalar product $\langle \cdot, \cdot \rangle$, from Linear Algebra we know that we can represent the bilinear form $f''(x)$ by a linear operator $\nabla^2 f(x) \in \mathcal{L}(\mathbb{E}, \mathbb{E})$ in the sense that

$$f''(x)[h, d] = \left\langle \nabla^2 f(x)h,\, d \right\rangle \quad \forall h, d \in \mathbb{E}$$

We call $\nabla^2 f(x)$ the *Hessian* of f at x, and, just like the gradient, it clearly depends on the chosen scalar product.

It can be seen (as we know from the case $\mathbb{E} = \mathbb{R}^n$) that $f''(x)[\cdot, \cdot]$ is a symmetric bilinear form, hence $\nabla^2 f(x)$ is a self-adjoint linear operator, i.e.

$$\left\langle \nabla^2 f(x)h,\, d \right\rangle = \left\langle h,\, \nabla^2 f(x)d \right\rangle \quad \forall h, d \in \mathbb{E}.$$

If the scalar-valued function $f : \Omega \to \mathbb{R}$ is twice differentiable at $x \in \Omega$, it it admits the following *second-order approximiation* at x:

$$f(x + h) = f(x) + \langle \nabla f(x),\, h \rangle + \frac{1}{2} \langle \nabla f(x)h,\, h \rangle + o(\|h\|^2). \tag{1.2}$$

We want to illustrate the differentiability concepts from above with some examples that will accompany us throughout our study.

Example 1.1.4 (Quadratic functions) For $L \in \mathcal{L}(\mathbb{E}, \mathbb{E})$, $b \in \mathbb{E}$ and $\gamma \in \mathbb{R}$ we define the *quadratic function*

$$q : \mathbb{E} \to \mathbb{R}, \quad q(x) := \frac{1}{2} \langle L(x),\, x \rangle + \langle b,\, x \rangle + \gamma.$$

We claim that f is twice (in fact arbitraly often) differentiable with

$$\nabla q(x) = \frac{1}{2}(L + L^*)(x) + b \quad \text{and} \quad \nabla^2 q(x) = \frac{1}{2}(L + L^*) \quad \forall x \in \mathbb{E}.$$

To show the expression for the gradient we compute that

$$\begin{aligned} q(x + h) - q(x) - \left\langle \frac{1}{2}(L + L^*)(x) + b,\, h \right\rangle &= \frac{1}{2} \langle L(h),\, h \rangle \\ &= o(\|h\|). \end{aligned}$$

Furthermore, it holds that

$$\nabla q(x + h) - \nabla q(x) - \frac{1}{2}(L + L^*)(h) = 0,$$

which gives the assertion for the Hessian.

Computing derivatives can be a highly nontrivial task.

Example 1.1.5 (Log-determinant function) Consider the function

$$f : \mathbb{S}^n_{++} \to \mathbb{R}, \quad f(X) = \log(\det X).$$

Note that $\mathbb{S}^n_{++} \subset \mathbb{S}^n$ is open (see Exercise 1.1.). We claim that f is differentiable on $\mathbb{S}^n_{++}$ with $\nabla f(X) = X^{-1}$ for all $X \succ 0$.

We first show that $\nabla f(I) = I$: For these purposes, for $\Delta \in \mathbb{S}^n$, close enough to 0 such that $I + \Delta \succ 0$, we realize that

$$\det(I + \Delta) = \prod_{i=1}^n (1 + \lambda_i),$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are the eigenvalues of Δ.

Hence, we obtain

$$\begin{aligned} f(X + \Delta) - f(X) - \langle I, \Delta\rangle &= \log\left(\prod_{i=1}^n (1 + \lambda_i)\right) - 0 - \operatorname{tr}(\Delta) \\ &= \sum_{i=1}^n \left[\log(1 + \lambda_i) - \lambda_i\right]. \end{aligned}$$

Since $\log(1 + t) - t = o(|t|)$ (use, e.g. l'Hôpital's rule), this implies that $\nabla f(I) = I$.

Now, let $X \succ 0$ and $\Delta \in \mathbb{S}^n$ such that $X + \Delta \succ 0$. Then we have

$$\begin{aligned} & f(X + \Delta) - f(X) - \left\langle X^{-1}, \Delta\right\rangle \\ = \;& \log\left(\det\left(X^{1/2}(I + X^{-1/2}\Delta X^{-1/2})X^{1/2}\right)\right) - \log(\det X) - \operatorname{tr}(X^{-1}\Delta) \\ = \;& \log(\det X) + \log(\det(I + X^{-1/2}\Delta X^{-1/2})) - \log(\det X) - \operatorname{tr}(X^{-1/2}\Delta X^{-1/2}) \\ = \;& \log(\det(I + X^{-1/2}\Delta X^{-1/2})) - \operatorname{tr}(X^{-1/2}\Delta X^{-1/2}) \\ = \;& o(\|\Delta\|), \end{aligned}$$

as $\nabla f(I) = I$. This shows that, in fact, $\nabla f(X) = X^{-1}$ for all $X \succ 0$.

1.2 Extended real-valued functions and optimization problems

1.2.1 Extended arithmetic and extended real-valued functions

Let $\overline{\mathbb{R}} := [-\infty, \infty]$ be the *extended real line*. The following conventions for an extended arithmetic have become standard in the optimization and convex analysis conmmunity: The uncritical ones are

$$\alpha + \infty = +\infty = \infty + \alpha \quad \text{and} \quad \alpha - \infty = -\infty + \alpha = -\infty \qquad (\alpha \in \mathbb{R}),$$

$$\alpha \cdot \infty = \operatorname{sgn}(\alpha) \cdot \infty = \infty \cdot \alpha \quad \text{and} \quad \alpha \cdot (-\infty) = -\operatorname{sgn}(\alpha) \cdot \infty = -\infty \cdot \alpha \quad (\alpha \in \mathbb{R} \setminus 0).$$

It is expedient for our purposes to also use the following:

$$\begin{aligned} 0 \cdot \infty = 0 = 0 \cdot (-\infty), \\ \infty - \infty = -\infty + \infty = \infty \quad \text{(inf-addition)} \end{aligned}$$

Every subset $S \subset \overline{\mathbb{R}}$ of the extended real line has a *supremum* (least upper bound) and an *infimum* (greatest lower bound), which could be infinite. We use the common convention

$$\inf \emptyset = +\infty \quad \text{and} \quad \sup \emptyset = -\infty.$$

Functions of the type $f : \mathbb{E} \to \overline{\mathbb{R}}$ occur naturally in various areas of mathematics, in particular in optimization (e.g. optimal value functions) or measure theory, as soon as suprema or infima are involved.
The *domain* of a function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is given by

$$\operatorname{dom} f := \{x \mid f(x) < \infty\}.$$

We call f *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{E}$.

A central object of study for extended real-valued functions is the *epigraph*, which for $f : \mathbb{E} \to \overline{\mathbb{R}}$ is defined by

$$\operatorname{epi} f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) \leq \alpha\}$$

The epigraph establishes a one-to-one correspondence of sets in $\mathbb{E} \times \mathbb{R}$ and functions $\mathbb{E} \to \overline{\mathbb{R}}$. Hence, all the important convex-analytical properties of an extended real-valued function (like lower semicontinuity, convexity, positive homogeneity or sublinearity) have there correspondence in the geometry and topology, respectively, of their epigraph.

Another useful tool are the *level sets* of f, which are defined by

$$\operatorname{lev}_{\leq \alpha} f := \{x \in \mathbb{E} \mid f(x) \leq \alpha\} \quad (\alpha \in \mathbb{R}).$$

We call f *level-bounded* if $\operatorname{lev}_{\leq \alpha} f$ is bounded for all $\alpha \in \mathbb{R}$. Level-boundedness is tremendously important in optimization, see Theorem 1.2.6.
The most prominent example of an extended real-valued function is as simple as it is important.

Definition 1.2.1 (Indicator function) *For a set $S \in \mathbb{E}$ the mapping $\delta_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ given by*

$$\delta_S(x) := \begin{cases} 0 & \text{if} \quad x \in S, \\ +\infty & \text{if} \quad x \notin S \end{cases}$$

is called the indicator (function) *of S.*

1.2.2 Lower semicontinuity

We now want to establish continuity notions for extended real-valued functions. To this end, for $f : \mathbb{E} \to \overline{\mathbb{R}}$ and $\bar{x} \in \mathbb{E}$ we define

$$\liminf_{x \to \bar{x}} f(x) := \inf \left\{ \alpha \in \overline{\mathbb{R}} \;\middle|\; \exists \{x^k\} \to \bar{x} : \; f(x^k) \to \alpha \right\} \tag{1.3}$$

as the *lower limit* of f at $\bar{x}$ and, analogously,

$$\limsup_{x \to \bar{x}} f(x) := \sup \left\{ \alpha \in \overline{\mathbb{R}} \;\middle|\; \exists \{x^k\} \to \bar{x} : \; f(x^k) \to \alpha \right\}$$

as the *upper limit* of f at $\bar{x}$.

Definition 1.2.2 (Continuity notions for extended real-valued functions) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ and $\bar{x} \in \mathbb{E}$. Then f is said to be* lower semicontinuous (lsc) *at $\bar{x}$ if*

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}),$$

We call f upper semicontinuous (usc) at $\bar{x}$ if

$$\limsup_{x \to \bar{x}} f(x) \leq f(\bar{x}).$$

If f is lower and upper semicontinuous at $\bar{x}$, it is said to be continuous *at $\bar{x}$, which holds true if and only if*

$$\lim_{x \to \bar{x}} f(x) = f(\bar{x}).$$

Example 1.2.3 ((Negative) log-determinant) Consider the function

$$f : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}, \quad f(x) := \begin{cases} -\log(\det X) & \text{if} \quad X \succ 0, \\ +\infty & \text{else} \end{cases} \tag{1.4}$$

which we call the *(negative) log-determinant* or the *(negative) logdet* function, for short. Then f is proper and continuous: The properness is clear (as $\operatorname{dom} f = S^n_{++} \neq \emptyset$). For the continuity, we only need to consider the critical cases of points on the boundary of the domain, i.e. in $\bar{X} \in \operatorname{bd}(\operatorname{dom} f) = \mathbb{S}^n_+ \setminus \mathbb{S}^n_{++}$ and sequences $\{X_k \in \mathbb{S}^n_{++}\} \to \bar{X}$. At this, recall that the determinant mapping $X \mapsto \det(X)$ is continuous (as it is a polynomial of the matrices' entries, cf. *Leibniz formula*). Hence, for $\bar{X} \in \operatorname{bd}(\operatorname{dom} f)$ and $\{X_k \in \mathbb{S}^n_{++}\} \to \bar{X}$ we have $\det(X_k) \to 0$, and thus, we obtain

$$\lim_{k \to \infty} f(X_k) = \lim_{k \to \infty} -\log(\det(X_k)) = +\infty = f(\bar{X}).$$

Lower semicontinuity plays a far more important role in our study then upper semicontinuity. In fact, we often find it useful to rectify the absence of lower semicontinuity of a function as follows:

We define the *lower semicontinuous hull* or *closure* of f to be the function $\mathrm{cl}\, f : \mathbb{E} \to \overline{\mathbb{R}}$,

$$(\mathrm{cl}\, f)(\bar{x}) := \liminf_{x \to \bar{x}} f(x).$$

As the constant sequence $\{x_k = \bar{x}\}$ is admitted in (1.3), we always have

$$\liminf_{x \to \bar{x}} f(x) \le f(\bar{x}) \quad (\bar{x} \in \mathbb{E}),$$

hence

$$\mathrm{cl}\, f \le f$$

for every function $f : \mathbb{E} \to \overline{\mathbb{R}}$. Moreover, we have

$$\mathrm{epi}\, (\mathrm{cl}\, f) = \mathrm{cl}\, (\mathrm{epi}\, f),$$

see Exercise 1.10. In particular, in view of the following result (which also clarifies why lsc functions are also called *closed*), this shows that $\mathrm{cl}\, f$ is always lsc.

Proposition 1.2.4 (Characterization of lower semicontinuity) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the following are equivalent:*

i) *f is lsc (on $\mathbb{E}$);*

ii) $\mathrm{epi}\, f$ *is closed;*

iii) $\mathrm{lev}_{\le \alpha} f$ *is closed for all* $\alpha \in \mathbb{R}$.

Proof: 'i)⇒ii': Let $\{(x_k, \alpha_k) \in \mathrm{epi}\, f\} \to (x, \alpha)$. By lower semicontinuity we have

$$f(x) \le \liminf_{k \to \infty} f(x^k) \le \lim_{k \to \infty} \alpha_k = \alpha,$$

hence $(x, \alpha) \in \mathrm{epi}\, f$. This shows that $\mathrm{epi}\, f$ is closed.

'ii)⇒ iii)': Fix $\alpha \in \mathbb{R}$ and let $\{x_k \in \mathrm{lev}_{\le \alpha} f\} \to x$. Then $\{(x_k, \alpha) \in \mathrm{epi}\, f\} \to (x, \alpha)$, and by closedness of $\mathrm{epi}\, f$, we have $(x, \alpha) \in \mathrm{epi}\, f$, i.e. $x \in \mathrm{lev}_{\le \alpha} f$. Thus, $\mathrm{lev}_{\le \alpha} f$ is closed, which proves the desired implication.

'iii)⇒ i):'(Contraposition) Suppose that f is *not* lsc. Then there exists $x \in \mathbb{E}$, $\{x_k\} \to x$ and such that

$$f(x_k) \to \alpha < f(x).$$

Now, pick $r \in]\alpha, f(x)[$. Then we have

$$f(x_k) \leq r < f(x) \quad \text{for all } k \text{ sufficiently large.}$$

But that means that $x \notin \mathrm{lev}_{\leq r} f$, although almost every member of the sequence $\{x_k\}$ lies in $\mathrm{lev}_{\leq r} f$. Hence, $\mathrm{lev}_{\leq r} f$ cannot be closed.

□

Due to the ubiquitiousness of the indicator function, we state a closedness result for it explicitly.

Corollary 1.2.5 (Lower semicontinuity of the indicator) *For a set $C \subset \mathbb{E}$ its indicator δ_C is proper and lsc if and only if C is nonempty and closed.*

Lower semicontinuity plays a central role in minimization problems, as the following paragraph illustrates.

1.2.3 Optimization problems

Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ and $C \subset \mathbb{E}$. We define

$$\inf_C f := \inf_{x \in C} f(x) := \inf \{f(x) \mid x \in C\}$$

and

$$\sup_C f := \sup_{x \in C} f(x) := \sup \{f(x) \mid x \in C\}.$$

In this scenario, $\inf_C f$ and $\sup_C f$ describe a(n extended) real number and an optimization problem at the same time. The set C is called the *constraint set*. If $C = \mathbb{E}$, the respective minimization/maximization problems are called *unconstrained* and otherwise *constrained*.

Using indicator functions, the constrained optimization problem $\inf_C f$ can be cast as an unconstrained optimization problem

$$\inf_C f = \inf_{\mathbb{E}} (f + \delta_C).$$

Note that we always have

$$\inf_C f = -\sup_C -f \quad \text{and} \quad \sup_C f = -\inf_C -f,$$

hence there is no big loss in generality if we primarily focus on minimization problems.

Furthermore, we define

$$\operatorname*{argmin}_C f := \operatorname*{argmin}_{x \in C} f(x) := \left\{ x \in C \;\middle|\; f(x) = \inf_C f \right\}$$

and analogously

$$\operatorname*{argmax}_C f := \operatorname*{argmax}_{x\in C} f(x) := \left\{ x \in C \;\middle|\; f(x) = \sup_C f \right\}.$$

We want to emphasize here that, when minimizing proper functions $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$, we always have the implication

$$\operatorname*{argmin}_{\mathbb{E}} f \neq \emptyset \quad \Rightarrow \quad \inf_{\mathbb{E}} f \in \mathbb{R}.$$

However, the converse implication does *not* hold in general: Consider for example the function

$$f : x \in \mathbb{R} \mapsto \begin{cases} \frac{1}{x} & \text{if} \quad x > 0, \\ +\infty & \text{else.} \end{cases}$$

Then clearly, $\inf_{\mathbb{R}} f = 0$ but $\operatorname{argmin}_f = \emptyset$.

The significance of lower semicontinuity for minimization problems is highlighted by the following famous result along the lines of the foregoing remark.

Theorem 1.2.6 (Existence of minima) *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be proper, lsc and level-bounded. Then*

$$\operatorname{argmin} f \neq \emptyset \quad \textit{and} \quad \inf_{\mathbb{E}} f \in \mathbb{R}.$$

Proof: Let $f^* := \inf_{\mathbb{E}} f < \infty$. There exists a sequence $\{x_k\}$ such that $f(x^k) \to f^*$. Then for all $\alpha \in (f^*, +\infty)$ we have $x^k \in \operatorname{lev}_\alpha f$ for all $k \in \mathbb{N}$ sufficiently large. Since f is lsc, $\operatorname{lev}_\alpha f$ is closed by Proposition 1.2.4, and since f is level-bounded by assumption, $\operatorname{lev}_\alpha$ is compact. Hence, by the Bolzano-Weierstrass Theorem, there exists $\bar{x} \in \operatorname{lev}_\alpha$ and an infinite subset $K \subset \mathbb{N}$ such that $x_k \to_K \bar{x}$.

Lower semicontinuity of f then implies

$$f(\bar{x}) \leq \liminf_{x \to \bar{x}} f(x) \leq \lim_{k \in K} f(x^k) = f^*,$$

hence $f(\bar{x}) = f^*$, which proves the assertion.

□

The above theorem is the blueprint for what is called the *direct method of the calculus of variations.*

Clearly, considering functions $f : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$, if we substitute lower for upper semicontinuity, inf for sup, and argmin for argmax, the analogous assertions in the above theorem are still valid.

1.3 Hull operators

Definition 1.3.1 *Let X be a nonempty set. A mapping* $\text{hull} : 2^X \to 2^X$ *is called a* hull operator *if the following holds for all $A, B \subset X$.*

i) $A \subset \text{hull } A \quad \text{(extensive)}$

ii) $A \subset B \quad \Rightarrow \quad \text{hull } A \subset \text{hull } B \quad \text{(monotonic)}$

iii) $\text{hull}(\text{hull } A) = \text{hull } A \quad \text{(idempotent)}$

There is a generic way of obtaining hull operators.

Proposition 1.3.2 *Let X be a nonempty set and let $\mathcal{S} \subset 2^X$ be a family of sets such that*

i) $X \in \mathcal{S}$*;*

ii) For any $\emptyset \neq \mathcal{A} \subset \mathcal{S}$, we have

$$\bigcap \{A \mid A \in \mathcal{A}\} \in \mathcal{S},$$

i.e. $\mathcal{S}$ is closed with respect to intersection.

Then the mapping $\mathcal{S}-\text{hull} : 2^X \to 2^X$

$$\mathcal{S}-\text{hull}(M) = \bigcap_{\substack{M \subset U, \\ \text{U} \in \mathcal{S}}} U.$$

is a hull operator, satisfying

$$\mathcal{S}-\text{hull}(S) = S \quad \forall S \in \mathcal{S},$$

i.e. $\mathcal{S}$ is invariant under $\mathcal{S}-\text{hull}$.

Proof: (Extensivity) Let $A, B \subset X$. As $X \in S$, we have

$$A \subset \bigcap_{\substack{A \subset U, \\ \text{U} \in \mathcal{S}}} U = \mathcal{S}-\text{hull}(A).$$

(Monotonicity) If $A \subset B$, we have

$$\mathcal{S}-\text{hull}(A) = \bigcap_{\substack{A \subset U, \\ \text{U} \in \mathcal{S}}} U \subset \bigcap_{\substack{B \subset U, \\ \text{U} \in \mathcal{S}}} U = \mathcal{S}-\text{hull}(B).$$

(Idempotence) Due to what was already proven, we have $\mathcal{S}-\text{hull}(A) \subset \mathcal{S}-\text{hull}(\mathcal{S}-\text{hull}(A))$. On the other hand, due to ii), $\mathcal{S}-\text{hull}(A) \in \mathcal{S}$. Hence,

$$\mathcal{S}-\text{hull}(\mathcal{S}-\text{hull}(A)) = \bigcap_{\substack{\mathcal{S}-\text{hull}(A)\subset U, \\ \text{U}\in\mathcal{S}}} U \subset \mathcal{S}-\text{hull}(A).$$

This shows that $\mathcal{S}-\text{hull}$ is a hull operator.

Moreover, for $S \in \mathcal{S}$ we have

$$\mathcal{S}-\text{hull}(S) = \bigcap_{\substack{S\subset U, \\ \text{U}\in\mathcal{S}}} U = S$$

□

A family of sets $\mathcal{S}$ which fulfills the requirements of i) and ii) is called a *hull system*. As we can see in Exercise 1.14., the converse direction of Propposition 1.3.2 is also valid, i.e. there is acutally a one-to-one correspondence of hull operators and systems.

Some well-known hull operators are listed below.

Example 1.3.3 (Hull operators)

a) For a K-vector space V, the *linear span operator*

$$\text{span} : 2^V \to 2^V, \quad \text{span}\, M = \bigcap_{\substack{M\subset U\subset V, \\ U \text{ subspace}}} U$$

is a hull operator with the subspaces of V as the corresponding hull system. We also should remember that for $M \neq \emptyset$, we have the intrinsic characterization

$$\text{span}\, M = \left\{ \sum_{i=1}^{r} \alpha_i v_i \mid r \in \mathbb{N},\ \alpha_i \in K, v_i \in M\ (i = 1, \dots, r) \right\}$$

of the linear span of M.

b) In a topological (e.g. metric, normed) vector space the *closure*

$$\text{cl} : 2^V \to 2^V, \quad \text{cl}\, M := \bigcap_{\substack{M\subset N\subset V, \\ N \text{ closed}}} U$$

is a hull operator with the closed sets as the corresponding hull system.

In the sequel we are going to encounter several other hull operators, including the *affine hull*, *convex hull* and the *(convex) conical hull*.

1.4 Affine sets and mappings

On our path to convex sets and functions, a central role is played by affine sets and functions, respectively, as we will soon see.

Definition 1.4.1 (Affine set) *A set $S \subset \mathbb{E}$ is said to be* affine *if*

$$\lambda x + (1-\lambda)y \in S \quad \forall x, y \in S,\ \lambda \in \mathbb{R}. \tag{1.5}$$

The geometric interpretation of S being affine is that, for any two points $x, y \in S$, the *line*

$$\mathbb{R}(y-x) + x = \{\alpha x + (1-\alpha y) \mid \alpha \in \mathbb{R}\}$$

through these two points lies in S. Extreme examples of affine sets in $\mathbb{E}$ are $\emptyset$ and $\mathbb{E}$ itself.

Theorem 1.4.2 (Characterization of subspaces) *An affine set $S \subset \mathbb{E}$ is a subspace if and only if $0 \in S$.*

Proof: Clearly, every subspace S contains 0 and is closed under addition and scalar mulitplication, hence is, in particular, affine.

Conversely, let S be affine with $0 \in S$. Then for $x \in S$ and $\lambda \in \mathbb{R}$ we have

$$\lambda x = \lambda x + (1-\lambda)0 \in S.$$

In addition, if $y \in S$, then

$$\frac{1}{2}(x+y) = \frac{1}{2}x + (1-\frac{1}{2})y \in S,$$

hence, by what was already shown,

$$x + y = 2 \cdot \frac{1}{2}(x+y) \in S.$$

All in all, S is a subspace. □

We call S *parallel to a subspace* $U \subset V$ if $S = U + x$ for some $x \in \mathbb{E}$. Necessarily, since $0 \in U$, we then have $x \in S$.

Corollary 1.4.3 *Let $S \subset \mathbb{E}$ be affine. Then there exists a unique subspace $U \subset \mathbb{E}$ such that S is parallel to U. That subspace is actually given by $U = S - S$.*

Proof: (Uniqueness) Suppose there are subspaces $U, W \subset \mathbb{E}$ and $x, y \in S$ such that $U + x = S = W + y$. Then we have $W = U + (x-y)$. As $0 \in W$, we get $y - x \in U$, hence $x - y \in U$, so $W = U + (x-y) = U$.

(Existence) Pick any $x \in S$. Then $U := S - x$ is an affine set and as $0 \in U$, by Theorem 1.4.2, U is a subspace. Since $U + x = S$, S is parallel to U.

Since x was arbitrarily chosen in S, we get that $U = S - S$ is *the* subspace parallel to U.
□

We can now easily characterize all affine sets of $\mathbb{E}$.

Corollary 1.4.4 *The nonempty affine sets in $\mathbb{E}$ are exactly the sets $x + U$ for some $x \in \mathbb{E}$ and a subspace $U \subset \mathbb{E}$.*

Corollary also 1.4.3 justifies the next definition.

Definition 1.4.5 (Affine dimension) *Let $S \subset \mathbb{E}$ be affine and U the subspace such that S is parallel to U. Then* $\dim S := \dim U$ *is called the* (affine) dimension *of S.*

Example 1.4.6 (Affine hyperplanes) Let $b \in \mathbb{E}\backslash\{0\}$ and $\gamma \in \mathbb{R}$. Then the *(affine) hyperplane*

$$H_{b,\gamma} := \{v \in \mathbb{E} \mid \langle b,\, v\rangle = \gamma\}$$

is an affine set of dimension $n - 1$, since we have the representation

$$H_{b,\gamma} = H_{b,0} + v_0 \quad (v_0 : \langle b,\, v_0\rangle = \gamma),$$

and $H_{b,0}$ is a subspace of dimension $n - 1$.

The next result shows that affineness is preserved under intersection of sets.

Proposition 1.4.7 *Let $\{S_i \subset \mathbb{E} \mid i \in I\}$ be a family of affine sets. Then*

$$S := \bigcap_{i \in I} S_i$$

is a an affine set.

Proof: Let $x, y \in S$ and $\lambda \in \mathbb{R}$. Then $x, y \in S_i$ for all $i \in I$, and since S_i is affine, we have $\lambda x + (1 - \lambda)y \in S_i$ for all $i \in I$, i.e. $\lambda x + (1 - \lambda)y \in S$. Hence, S is affine. □

In the spirit of Section 1.3 we now define the affine hull of sets in $\mathbb{E}$.

Definition 1.4.8 (Affine hull) *For a set $M \subset \mathbb{E}$ its* affine hull *is defined by*

$$\operatorname{aff} M := \bigcap_{\substack{M \subset S,\\ S \text{ affine}}} S,$$

i.e. (in view of Proposition 1.4.7) aff M *is the smallest affine set containing M.*

In view of Proposition 1.4.7, the affine subsets of $\mathbb{E}$ form a hull system, hence, invoking Proposition 1.3.2 and aff $: 2^{\mathbb{E}} \to 2^{\mathbb{E}}$ is a hull operator in the sense of Definition 1.3.1

For vectors $x_1, \dots, x_k \in \mathbb{E}$ and scalars $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ ($k \in \mathbb{N}$) such that

$$\sum_{i=1}^{k} \alpha_i = 1$$

the sum

$$\sum_{i=1}^{k} \alpha_i x_i$$

is called an *affine combination* of $x_1, \dots, x_k$. The set of all affine combinations of elements of nonempty set $M \subset \mathbb{E}$ is denoted by $A(M)$, i.e.

$$A(M) = \left\{ \sum_{i=1}^{r} \alpha_i v_i \;\middle|\; r \in \mathbb{N},\ v_i \in M,\ \alpha_i \in \mathbb{R}; (i = 1, \dots, r) : \sum_{i=1}^{r} \alpha_i = 1 \right\}.$$

Lemma 1.4.9 *Let $M \in \mathbb{E}$ be nonempty. Then the following hold:*

a) $A(M)$ is affine.

b) $A(M) \subset S$ for every affine set $S \supset M$.

Proof:

a) Take $x, y \in A(M)$, i.e.

$$x = \sum_{i=1}^{r} \alpha_i x_i \quad \text{and} \quad y = \sum_{i=1}^{r} \beta_i y_i$$

where w.l.o.g. both sums have the same length (otherwise we add zeros). Then for $\lambda \in \mathbb{R}$ we have

$$\lambda x + (1 - \lambda) y = \sum_{i=1}^{r} \lambda \alpha_i x_i + (1 - \lambda) \beta_i y_i.$$

Putting

$$v_i := x_i \quad v_{r+i} := y_i \quad \text{and} \quad \gamma_i := \lambda \alpha_i, \quad \gamma_{r+i} := (1 - \lambda) \beta_i \quad (i = 1, \dots, r)$$

we see that $\sum_{i=1}^{2r} \gamma_i = 1$, hence

$$\lambda x + (1 - \lambda) y = \sum_{i=1}^{2r} \gamma_i v_i \in A(M),$$

which gives the assertion.

b) Let $x = \sum_{i=1}^r \alpha_i x_i \in A(M)$ (i.e. $\sum_{i=1}^r \alpha_i = 1,\ x_i \in M\ (i = 1, \dots, r)$) and $S \supset M$ be affine. Then $S = y + U$ with a (uniquely determined) subspace U, hence $x_i = y + u_i$ with $u_i \in U$. Thus,

$$x = \sum_{i=1}^r \alpha_i (y + u_i) = y + \sum_{i=1}^r \alpha_i u_i \in y + U = S.$$

□

Proposition 1.4.10 (Characterization of the affine hull) *Let $M \subset \mathbb{E}$ be nonempty. Then*

$$\operatorname{aff} M = A(M)$$

i.e. aff M *is the set of all affine combinations of points from* M*.*

Proof: Clearly, $M \subset A(M)$. Moreover, since $A(M)$ is affine by Lemma 1.4.9 a) and by the calculus for hull operators (cf. Proposition 1.3.2), we obtain

$$\operatorname{aff} M \subset \operatorname{aff} A(M) = A(M).$$

The converse inclusion follows from the fact that $A(M)$ is contained in every affine set that contains M, see Lemma 1.4.9 b).

□

The following result gives a representation of the affine hull of a finite set of vectors by means of the linear span.

Corollary 1.4.11 *Let $x_0, x_1, \dots, x_k \in \mathbb{E}$. Then*

$$x_0 + \operatorname{span}\{x_1 - x_0, \dots, x_k - x_0\} = \operatorname{aff}\{x_0, x_1, \dots, x_k\}$$

In particular, $\operatorname{aff}\{x_0, x_1, \dots, x_k\} = \operatorname{span}\{x_0, x_1, \dots, x_k\}$ *if* $\{x_0, x_1, \dots, x_k\}$ *contains the zero vector.*

Proof: Exercise 1.16.. □

The above corollary motivates the notion of *affine independence* as given below.

Definition 1.4.12 (Affine independence) *The vectors $x_0, x_1, \dots, x_p \in \mathbb{E}$ are said to be* affinely independent *if the vectors $x_1 - x_0, \dots, x_k - x_0$ are linearly independent.*

We have different ways of characterizing affine independence.

Proposition 1.4.13 (Characterization of affine independence) *Let $x_0, x_1, \dots, x_p \in \mathbb{E}$. Then the following are equivalent:*

i) $x_0, x_1, \dots, x_p \in \mathbb{E}$ *are affinely independent.*

ii) $(x_0, 1), (x_1, 1), \dots, (x_p, 1) \in \mathbb{E} \times \mathbb{R}$ *are linearly indpendent.*

iii) The system of equations

$$0 = \sum_{i=0}^{p} \alpha_i x_i, \quad 0 = \sum_{i=0}^{p} \alpha_i$$

has the unique solution $\alpha_0 = \alpha_1 = \dots = \alpha_p = 0$.

In this case, we have $\dim(\operatorname{aff}\{x_0, x_1, \dots, x_p\}) = p$ *and every* $x \in \operatorname{aff}\{x_0, x_1, \dots, x_p\}$ *has a unique representation*

$$x = \sum_{i=0}^{p} \alpha_i x_i, \quad \sum_{i=0}^{p} \alpha_i = 1.$$

Proof: 'i)⇒ii):' Let $\alpha_0, \alpha_1, \dots, \alpha_p \in \mathbb{R}$ such that $\sum_{i=0}^{p} \alpha_i (x_i, 1) = 0$. This can be written as

$$0 = \sum_{i=0}^{p} \alpha_i x_i \quad \text{and} \quad \sum_{i=0}^{p} \alpha_i = 0.$$

This yields

$$0 = -\sum_{i=1}^{p} \alpha_i x_0 + \sum_{i=1}^{p} \alpha_i x_i \quad \Longleftrightarrow \quad 0 = \sum_{i=1}^{n} \alpha_i (x_i - x_0).$$

As the vectors $x_0, x_1, \dots, x_p$ are affinely independent by assumption, this gives $\alpha_0 = \alpha_1 = \dots = \alpha_p = 0$, which proves ii).

'ii)⇒iii)': Let $\alpha_0, \alpha_1, \dots, \alpha_p \in \mathbb{R}$ such that

$$0 = \sum_{i=0}^{p} \alpha_i x_i, \quad 0 = \sum_{i=0}^{p} \alpha_i.$$

This means that $0 = \sum_{i=0}^{p} \alpha_i (x_i, 1)$, hence $\alpha_i = 0$ $(i = 0, 1, \dots, p)$ by ii).

'iii)⇒i)': Let $\alpha_1, \dots, \alpha_p \in \mathbb{R}$ such that

$$0 = \sum_{i=1}^{p} \alpha_i (x_i - x_0) = \sum_{i=1}^{p} \alpha_i x_i - \sum_{i=1}^{p} \alpha_i x_0.$$

Putting $\alpha_0 := -\sum_{i=1}^p \alpha_i$, we have

$$0 = \sum_{i=0}^{p} \alpha_i x_i, \quad 0 = \sum_{i=0}^{p} \alpha_i,$$

hence $\alpha_i = 0$ $(i = 0, 1, \dots, p)$ by assumption.

□

We now define the notion of an affine mapping.

Definition 1.4.14 (Affine mapping) *A mapping $F : \mathbb{E}_1 \to \mathbb{E}_2$ is said to be* affine *if*

$$F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y) \quad \forall x, y \in \mathbb{E}_1,\ \lambda \in \mathbb{R}.$$

Proposition 1.4.15 *Let $F : \mathbb{E}_1 \to \mathbb{E}_2$. Then the following are equivalent:*

i) F is affine.

ii) The mapping $T : \mathbb{E}_1 \mapsto \mathbb{E}_2$

$$T(x) := F(x) - F(0)$$

is linear.

iii) For every affine combination $\sum_{i=1}^k \lambda_i x_i \in \mathbb{E}_1$ we have

$$F\left(\sum_{i=1}^{k} \lambda_i x_i\right) = \sum_{i=1}^{k} \lambda_i F(x_i).$$

In particular, an affine mapping F is linear if and only if $F(0) = 0$.

Proof: i)⇒ii): Let $x \in \mathbb{E}_1$ and $\lambda \in \mathbb{R}$. Then we have

$$T(\lambda x) = F(\lambda x + (1-\lambda)0) - F(0) = \lambda F(x) + (1-\lambda)F(0) - F(0) = \lambda T(x),$$

therefore, T is homogeneous. Hence, for $x, y \in \mathbb{E}_1$, we have

$$T(x+y) = 2T\left(\frac{1}{2}x + \frac{1}{2}y\right) = 2\left(\frac{1}{2}F(x) + \frac{1}{2}F(y) - F(0)\right) = T(x) + T(y),$$

i.e. T is also additive thus, linear.

ii)⇒iii): For $i = 1, \dots, m$ let $x_1, \dots, x_m \in \mathbb{E}_1$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ with $\sum_{i=1}^m \alpha_i = 1$. Using the linear map T from ii), we get

$$F\left(\sum_{i=1}^{k} \alpha_i x_i\right) = T\left(\sum_{i=1}^{k} \alpha_i x_i\right) + F(0)$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \alpha_i T(x_i) + F(0) \\
&= \sum_{i=1}^{k} \alpha_i (F(x_i) - F(0)) + F(0) \\
&= \sum_{i=1}^{k} \alpha_i F(x_i).
\end{aligned}
$$

□

The following result is a trivial consequence of the preceeding one, but since we are going to make frequent use of it, we state it explicitly.

Corollary 1.4.16 *Let $F : \mathbb{E} \to \mathbb{E}'$ be affine. Then the following hold:*

a) For $\mathbb{E}' = \mathbb{R}$ there exist a unique $b \in \mathbb{E}$ and $\beta \in \mathbb{R}$ such that $F(x) = \langle b, x \rangle + \beta$ for all $x \in \mathbb{E}$.

b) F is continuous.

Proof: For a) combine Proposition 1.4.15 with Exercise 1.2. □

As another consequence we obtain that affine mappings preserve affineness of sets.

Corollary 1.4.17 *Let $F : \mathbb{E}_1 \to \mathbb{E}_2$ be affine and $S, M \subset \mathbb{E}$ such that S is affine. Then we have:*

a) $F(S)$ is affine.

b) $F(\mathrm{aff}\, M) = \mathrm{aff}\, F(M)$.

Proof: Part a) is obvious from the definition of an affine mapping and set, respectively. Part b) follows immediately from Proposition 1.4.15 iii).

□

The next theorem states that affine sets in a Euclidean space of the same dimension are one-to-one, affine images of each other.

Theorem 1.4.18 *Let $A, B \subset \mathbb{E}$ be affine sets of the same (affine) dimension $1 \leq k \leq N$. Then there exists an invertible affine mapping $F : \mathbb{E} \to \mathbb{E}$ with $F(A) = B$.*

Proof: Let $x_0, x_1, \dots, x_k$ be affinely independent such that aff $\{x^0, \dots, x^k\} = A$, and $y^0, y^1, \dots, y^k$ affinely independent such that aff $\{y_0, \dots, y_k\} = B$. By choice, the sets $\{x_1 - x^0, \dots, x_k - x^0\}$ and $\{y_1 - y_0, \dots, y_k - y_0\}$ are linearly independent and can be completed to bases of $\mathbb{E}$, e.g.

$$\{x_1 - x_0, \dots, x_k - x_0, x_{k+1}, \dots, x_N\} \quad \text{and} \quad \{y_1 - y_0, \dots, y_k - y_0, y_{k+1}, \dots, y_N\}.$$

We define a linear mapping $T : \mathbb{E} \to \mathbb{E}$ by

$$T(x_i - x_0) := y_i - y_0 \quad (i = 1, \dots, k)$$

and

$$T(x^i) = y^i \quad (1 = k+1, \dots, n),$$

which is invertible, as it maps one basis of $\mathbb{E}$ to another. Moreover, we define an affine mapping $F : \mathbb{E} \to \mathbb{E}$ through

$$F(x) = T(x) + y_0 - T(x_0).$$

As T is invertible, so is F and we have

$$F(x_i) = y_i \quad (i = 1, \dots, k),$$

thus,

$$F(A) = F(\text{aff}\,\{x_0, x_1, \dots, x_k\}) \overset{\text{Cor. 1.4.17}}{=} \text{aff}\,F(\{x_0, x_1, \dots, x_k\}) = \text{aff}\,\{y_0, y_1, \dots, y^k\} = B,$$

which concludes the proof. $\square$

Exercises to Chapter 1

1.1. **(Openness of $\mathbb{S}^n_{++}$)** Argue that $\mathbb{S}^n_{++}$ is open in $\mathbb{S}^n$.

1.2. **(Riesz representation theorem - finite dimensional version)** Let $(\mathbb{E}, \langle \cdot,\, \cdot \rangle)$ be a (finite dimensional) Euclidean space and $L \in \mathcal{L}(\mathbb{E}, \mathbb{R})$. Show that there exists a unique $b \in \mathbb{E}$ such that

$$L(x) = \langle b,\, x \rangle \quad \forall x \in \mathbb{E}.$$

1.3. **(Orthogonal matrices)** Show that $\mathrm{O}(n)$ is a compact subset of $\mathbb{R}^{n \times n}$.

1.4. **(Logdet and trace inequality)** Let $A \in S^n_{++}$. Show that

$$\log(\det A) + n \le \mathrm{tr}\,(A).$$

1.5. **(Matrix-fractional function)** For $\Omega := \mathbb{R}^n \times \mathbb{S}^n_{++}$ consider the function

$$f : \Omega \to \mathbb{R}, \quad f(x, V) = \frac{1}{2} x^T V^{-1} x.$$

Prove that f is differentiable by showing that

$$\nabla f(x, V) = \left[V^{-1} x, -\frac{1}{2} V^{-1} x x^T V^{-1} \right] \quad (x, V \in \mathbb{R}^n \times \mathbb{S}^n)$$

(w.r.t the standard scalar product on $\mathbb{R}^n \times \mathbb{S}^n$).

Hint: Use the fact that for $T \in \mathbb{R}^{n \times n}$ with $\|T\| < 1$ we have

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \quad \textit{(Neumann series).}$$

1.6. **(Hessian of the log-determinant)** Show that the function from Example 1.1.5 is twice differentiable by computing its Hessian.

1.7. **(Topology of Minkowski sum)** Let $A, B \in \mathbb{E}$ nonempty. Prove:

a) $A + B$ is open if A or B is open.

b) $A + B$ is closed if both A and B are closed and at least one of them is bounded. Illustrate by a counterexample that the boundedness assumption cannot be omitted in general.

1.8. **(Minimizing a quadratic function)** For $A \in \mathbb{S}^n$ and $b \in \mathbb{R}^n$ consider the quadratic function

$$q : \mathbb{R}^n \to \mathbb{R}, \quad q(x) = \frac{1}{2} x^T A x + b^T x.$$

Prove (without using first-order optimality conditions) that the following are equivalent:

i) $\inf_{\mathbb{R}^n} q > -\infty$;

ii) $A \succeq 0$ and $b \in \operatorname{im} A$;

iii) $\operatorname{argmin}_{\mathbb{R}^n} q \neq \emptyset$.

1.9. **(Closures and interiors of epigraphs)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the following hold:

a) $(\bar{x}, \bar{\alpha}) \in \operatorname{cl}(\operatorname{epi} f) \quad \Longleftrightarrow \quad \bar{\alpha} \geq \liminf_{x \to \bar{x}} f(x)$.

b) $(\bar{x}, \bar{\alpha}) \in \operatorname{int}(\operatorname{epi} f) \quad \Longleftrightarrow \quad \bar{\alpha} > \limsup_{x \to \bar{x}} f(x)$.

1.10. **(Lower semicontinuous hull)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show that

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f),$$

i.e. $\operatorname{cl} f$ is the function whose epigraph is the closure of $\operatorname{epi} f$.

1.11. **(Closure of a function)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show that

$$(\mathrm{cl}\, f)(x) = \inf \{ t \mid (x, t) \in \mathrm{epi}\, f \} .$$

1.12. **(Closedness of a positive combination)** For $p \in \mathbb{N}$ let $f_i : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc and $\alpha_i \geq 0$. Show that $f := \sum_{i=1}^{p} \alpha_i f_i$ is lsc.

1.13. **(Closedness preserving compositions)** Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc.

a) Show that $f \circ g$ is lsc if $g : \mathbb{E}' \to \mathbb{E}$ is continuous.

b) Show that $\phi \circ f$ is lsc if $\phi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and we use the convention $\phi(+\infty) = \sup_{t \in \mathbb{R}} \phi(t)$.

1.14. **(Hull operators and systems)** Let X be a nonempty set and $\mathrm{hull} : 2^X \to 2^X$ a hull operator. Show that there exists a hull system $\mathcal{S} \subset 2^X$.

1.15. **(Affine Hull of cartesian product)** Let $A \in \mathbb{E}_1$ and $B \in \mathbb{E}_2$. Then

$$\mathrm{aff}\,(A \times B) = \mathrm{aff}\, A \times \mathrm{aff}\, B.$$

1.16. **(Affine hull of a finite set)** Let $x_0, x_1, \ldots, x_k \in \mathbb{E}$. Show that

$$x_0 + \mathrm{span}\,\{x_1 - x_0, \ldots, x_k - x_0\} = \mathrm{aff}\,\{x_0, x_1, \ldots, x_k\}.$$

1.17. **(Affine sets, affine mappings, and hyperplanes)** Let S be a nonempty affine subset of $\mathbb{E}$ with $\dim S = r$. Moreover, let $\mathbb{E}'$ be another Euclidean space with $\dim \mathbb{E}' \geq N - r$. Prove:

a) There exists an affine mapping $F : \mathbb{E} \to \mathbb{E}'$ such that $S = \{x \mid F(x) = 0\}$.

b) S is the intersection of finitely many hyperplanes.

1.18. **(Affine hull of a polyhedron)** For $i = 1, \ldots, p$ ($p \in \mathbb{N}$) let $a_i \in \mathbb{E}$ and $b_i \in \mathbb{R}$. Then we define a *polyhedron*

$$P := \{x \in \mathbb{E} \mid \langle a_i,\, x\rangle \leq b_i \ (i = 1, \ldots, p)\}$$

along with

$$I := \{i \in \{1, \ldots, p\} \mid \forall x \in P : \ \langle a_i,\, x\rangle = b_i\} .$$

Show the following statements:

a) There exists $\hat{x} \in P$ such that $\langle a_j,\, \hat{x}\rangle < b_j$ for all $j \in \{1, \ldots, p\} \setminus I$.

b) $\mathrm{aff}\, P = \{x \in \mathbb{E} \mid \langle a_i,\, x\rangle \leq b_i \ (i \in I)\}$.

2 Convex Sets and Cones

In this chapter we study convex sets.

2.1 Elementary properties of convex sets

We start our study with the central definition of a convex set.

Definition 2.1.1 (Convex set) *A set* $C \subset \mathbb{E}$ *is said to be* convex *if*

$$\lambda x + (1-\lambda)y \in C \quad \forall x, y \in C, \forall \lambda \in [0,1]. \tag{2.1}$$

The geometric interpretation of (2.1) is that for two arbitrary points $x, y \in C$ also their *connecting line*

$$[x, y] := \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$$

lies entirely in C. Similarly, we define

$$[x, y) := \{\lambda x + (1-\lambda)y \mid \lambda \in (0,1]\}, \quad (x, y] := \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1)\}$$

and

$$(x, y) := \{\lambda x + (1-\lambda)y \mid \lambda \in (0,1)\}.$$

Clearly, every singleton $\{x\} \subset \mathbb{E}$ is convex. Moreover, the convex sets in $\mathbb{R}$ are exactly the (closed, open, half-open) intervals.
We now show that convexity is preserved under quite a number of important operations.

Proposition 2.1.2 (Convexity preserving operations)

a) (Intersection) Let $\{C_i \subset \mathbb{E} \mid i \in I\}$ *be a family of convex sets. Then*

$$C := \bigcap_{i \in I} C_i$$

is a convex set.

b) (Cartesian product) Let $C_i \in \mathbb{E}_i$ $(i = 1, \dots, m)$ *be convex. Then*

$$C := \bigtimes_{i=1}^{m} C_i \subset \bigtimes_{i=1}^{m} \mathbb{E}_i$$

is convex.

c) *(Minkowski sums/multiplications) For two convex sets $A, B \in \mathbb{E}$ and $\lambda, \mu \in \mathbb{R}$ the set*

$$\lambda A + \mu B = \{\lambda x + \mu y \mid x \in A, y \in B\}$$

is convex. In particular, translates of convex sets are convex.

d) *(Affine (pre-)images) For an affine (in particular for a linear) mapping $F : \mathbb{E}_1 \to \mathbb{E}_2$ and convex sets $A \in \mathbb{E}_1$ and $B \in \mathbb{E}_2$ the sets*

$$F(A) \quad \textit{and} \quad F^{-1}(B)$$

are convex.

Proof: Straightforward. □

Example 2.1.3 (Convex sets)

a) (Affine sets) Every affine set (hence every subspace) is convex.

b) (Half spaces) Let $H_{b,\gamma} := \{x \in \mathbb{E} \mid \langle b,\, x\rangle = \gamma\}$ be the (affine) hyperplane defined by $b \in \mathbb{E} \setminus \{0\}$ and $\gamma \in \mathbb{R}$. The sets

$$H^{\leq}_{b,\gamma} := \{v \in \mathbb{E} \mid \langle b,\, v\rangle \leq \gamma\}\,, \quad H^{\geq}_{b,\gamma} := \{v \in \mathbb{E} \mid \langle b,\, v\rangle \geq \gamma\}$$

and

$$H^{<}_{b,\gamma} := \{v \in \mathbb{E} \mid \langle b,\, v\rangle < \gamma\}\,, \quad H^{>}_{b\gamma} := \{v \in \mathbb{E} \mid \langle b,\, v\rangle > \gamma\}$$

are convex and called the *closed*, resp. *open half-spaces* associated with $H_{b,\gamma}$.

c) (Polyhedra) For $b_i \in \mathbb{E} \setminus \{0\}$ and $\gamma_i \in \mathbb{R}$ for $i = 1, \ldots, m$ the set

$$P := \{v \in \mathbb{E} \mid \langle b_i,\, v\rangle \leq \gamma_i\ (i = 1, \ldots, m)\}$$

is (see a) and Prop. 2.1.2) a convex set, called *Polyhedron.* If P is bounded (hence compact) it is called a *polytope.*

d) (Balls) For a given norm $\|\cdot\|_*$ on $\mathbb{E}$, $r > 0$ and $\bar{x} \in \mathbb{E}$ the ball $B^*_r(\bar{x}) := \{x \in \mathbb{E} \mid \|x - \bar{x}\|_* < r\}$ is convex due the triangle inequality and absolute homgeneity of the norm.

e) (Unit Simplices) For $k \in \mathbb{R}^k$ the set

$$\Delta_k := \left\{ x \in \mathbb{R}^k \;\middle|\; \sum_{i=1}^{k} x_i = 1,\ x_i \geq 0\ (i = 1, \ldots, k) \right\}$$

is called the *unit simplex* in $\mathbb{R}^k$ is convex, compact with $\operatorname{int} \Delta_k = \emptyset$.

Note that every affine hyperplane (see Example 1.4.6) can be written as the intersection of two closed half-spaces.

2.2 The convex hull

Analogous to the notion of the affine hull (or the linear hull), we now establish the convex hull of a set in the spirit of Proposition 1.3.2.

Definition 2.2.1 (Convex hull) *Let $M \subset \mathbb{E}$ be nonempty. The* convex hull *of M is the set*

$$\operatorname{conv} M := \bigcap_{\substack{M \subset C,\\ C \text{ convex}}} C,$$

i.e. (in view of Proposition 2.1.2 a)) the convex hull of M is the smallest convex set containing M.

Clearly, by definition and Proposition 2.1.2 a), $\operatorname{conv}\ : 2^{\mathbb{E}} \to 2^{\mathbb{E}}$ is a hull operator with the convex sets of $\mathbb{E}$ as a hull system.

In analogy to the affine (or linear) case, we can obtain an intrinsic characterization of the convex hull of a set by means of *convex combinations* of their elements. At this, for $x_1, \dots, x_p \in \mathbb{E}$ and $\lambda \in \Delta_p$ we call $\sum_{i=1}^p \lambda_i x_i$ a convex combination.

Proposition 2.2.2 (Characterization of the convex hull) *Let $M \subset \mathbb{E}$ be nonempty. The we have*

$$\operatorname{conv} M = \left\{ \sum_{i=1}^r \lambda_i x_i \mid r \in \mathbb{N},\ \lambda \in \Delta_r,\ x_i \in M\ (i = 1, \dots, r) \right\}$$

Proof: Exercise 2.1. □

Proposition 2.2.2 tells us that the convex hull of a set can be seen as the set of all convex combinations of elements from the set in question.

In an N-dimensional ($N \in \mathbb{N}$) space $\mathbb{E}$ this can be sharpened as follows.

Theorem 2.2.3 (Carathéodory's Theorem) *Let $M \subset \mathbb{E}$ be nonempty. Then we have*

$$\operatorname{conv} M = \left\{ \sum_{i=1}^{N+1} \lambda_i x_i \mid \lambda \in \Delta_{N+1},\ x_i \in M\ (i = 1, \dots, N+1) \right\},$$

i.e. every vector in $\operatorname{conv} M$ *can be written as a convex combination of at most $N+1$ elements from M.*

Proof: Let $x \in \operatorname{conv}\ M$. By Proposition 2.2.2 there exists $r \in \mathbb{N}$, $\lambda_1, \dots, \lambda_r > 0$ and $x_1, \dots, x_r \in M$ such that

$$\sum_{i=1}^r \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^r \lambda_i x_i = x. \tag{2.2}$$

If $r \leq N+1$ there is nothing to do.

Hence, let $r > N+1$. We are going to show that x can already be written as a convex combination of $r-1$ elements from M, which then (inductively) gives the assertion.

As $r > N+1$, the vectors

$$(x_1, 1), \dots, (x_r, 1) \in \mathbb{E} \times \mathbb{R}$$

are linearly dependent. Hence, there exist $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ such that

$$0 = \sum_{i=1}^{r} \alpha_i (x_i, 1) \tag{2.3}$$

and $\alpha_i \neq 0$ for at least one index i.

W.l.o.g. we can assume that

$$\left| \frac{\alpha_i}{\lambda_i} \right| \leq \left| \frac{\alpha_r}{\lambda_r} \right| \quad \forall i = 1, \dots, r-1, \tag{2.4}$$

(i.e. $r = \operatorname{argmax} \left\{ \left| \frac{\alpha_i}{\lambda_i} \right| \mid i = 1, \dots, r \right\}$) hence, in particular $\alpha_r \neq 0$. Putting $\beta_i := -\frac{\alpha_i}{\alpha_r}$ $(i = 1, \dots, r-1)$, equation (2.3) yields

$$x_r = \sum_{i=1}^{r-1} \beta_i x_i \quad \text{and} \quad \sum_{i=1}^{r-1} \beta_i = 1. \tag{2.5}$$

Then (2.2) and (2.5) imply

$$x = \sum_{i=1}^{r} \lambda_i x_i = \sum_{i=1}^{r-1} \lambda_i x_i + \lambda_r \sum_{i=1}^{r-1} \beta_i x_i = \sum_{i=1}^{r-1} (\lambda_i + \lambda_r \beta_i) x_i,$$

Setting $\tilde{\lambda}_i := \lambda_i + \lambda_r \beta_i$ $(i = 1, \dots, r-1)$ we thus obtain

$$x = \sum_{i=1}^{r-1} \tilde{\lambda}_i x_i,$$

and

$$\sum_{i=1}^{r-1} \tilde{\lambda}_i = \sum_{i=1}^{r-1} \lambda_i + \lambda_r \sum_{i=1}^{r-1} \beta_i = 1 - \lambda_r + \lambda_r = 1.$$

Due to (2.4) we also have

$$\tilde{\lambda}_i = \lambda_i - \lambda_r \frac{\alpha_i}{\alpha_r} > 0.$$

Thus, x is already a convex combination of the $r-1$ vectors $x_1, \dots, x_{r-1}$, which concludes the proof. $\square$

2.3 Topological properties of convex sets

We start by reviewing the fundamental concept of the (topological) closure, interior and boundary of an arbitray set $M \subset \mathbb{E}$. The *(topological) closure* $\operatorname{cl} M$ of M, which in terms of hull operations (cf. Example 1.3.3) is the intersection all closed sets containing M, can also be written as

$$\operatorname{cl} M = \{x \mid \forall \varepsilon > 0 \; \exists y \in B_\varepsilon(x) \cap M\},$$

in particular, $\operatorname{cl} M$ is the set of all cluster (or, equivalently, limit) points of sequences in M. It is easy to see that $\operatorname{cl} M$ can also be written as

$$\operatorname{cl} M = \bigcap_{\varepsilon > 0} M + \varepsilon \mathbb{B}. \tag{2.6}$$

The *interior* $\operatorname{int} M$ of M is defined by

$$\operatorname{int} M := \{x \in M \mid \exists \varepsilon > 0 : B_\varepsilon(x) \subset M\}.$$

The *boundary* $\operatorname{bd} M$ of M is given by

$$\operatorname{bd} M := \operatorname{cl} M \setminus \operatorname{int} M.$$

The first topological result shows that convexity of a set is inherited to its interior and its closure.

Proposition 2.3.1 (Closure and interior of a convex set) *Let $S \subset \mathbb{E}$ be convex. Then $\operatorname{cl} S$ and $\operatorname{int} S$ (which could well be empty, even if S is not) are convex, too.*

Proof: Since

$$\operatorname{cl} S = \bigcap_{\varepsilon > 0} S + \varepsilon \mathbb{B},$$

and $S + \varepsilon \mathbb{E}$ is convex (cf. Proposition 2.1.2 c)), the convexity of $\operatorname{cl} S$ follows from Proposition 2.1.2 a).

Now, let $x, y \in \operatorname{int} S$, hence there exist open neighborhoods $U_1, U_2 \subset S$ of x and y, respectively. It follows for $\lambda \in [0, 1]$ that

$$\lambda x + (1 - \lambda) y \subset \lambda U_1 + (1 - \lambda) U_2 \subset \lambda S + (1 - \lambda) S \subset S,$$

which proves the result as $\lambda U_1 + (1 - \lambda) U_2$ is open by Exercise 1.7. a). □

Boundedness, *closedness* and *compactness* are fundamental topological properties of subsets in $\mathbb{E}$. At this point, we want to study in how far they are preserved under the convex hull operation.

Recall that, in (the finite-dimensional space) $\mathbb{E}$, a subset is compact if and only if it is bounded and closed.

The following result shows that compactness is preserved under the convex hull operator.

Proposition 2.3.2 *Let $M \subset \mathbb{E}$ be compact. Then* $\operatorname{conv} M$ *is compact.*

Proof: Let $\{x_k \in \operatorname{conv} M\}$. By Theorem 2.2.3 (Carathéodory), for all $k \in \mathbb{N}$, there exist vectors $\alpha_k \in \Delta_{N+1}$ und $v_1^k, \dots, v_{N+1}^k \in M$ such that

$$x_k = \sum_{i=1}^{N+1} \alpha_i^k v_i^k \quad \forall k \in \mathbb{N}.$$

Since M and Δ_{N+1} are compact, there exists an infinite subset $K \subset \mathbb{N}$ and vectors $\alpha \in \Delta_{N+1}$ and $v_1, \dots, v_{N+1} \in M$ such that

$$\alpha_k \to_K \alpha \quad \text{and} \quad v_i^k \to_K v_i \quad (i = 1, \dots, N+1)$$

Hence,

$$x_k \to_K \sum_{i=1}^{n+1} \alpha_i v_i =: x.$$

As x is a convex combination of elements from M, we get (by Propostion 2.2.2) $x \in \operatorname{conv} M$, which proves the assertion. □

We point out that the proof of the above result relies on Carathéodory's Theorem and hence on the assumption that $\mathbb{E}$ be finite-dimensional.

As an immediate consequence, we obtain that boundedness, too, is preserved under the affine hull operator.

Corollary 2.3.3 *Let $M \subset \mathbb{E}$ be bounded. Then* $\operatorname{conv} M$ *is bounded.*

Proof: If M is bounded, $\operatorname{cl} M$ is compact. By Proposition 2.3.2 $\operatorname{conv}(\operatorname{cl} M)$ is hence compact, and, in particular, bounded. By the monotonicity of the convex hull we infer that

$$\operatorname{conv} M \subset \operatorname{conv}(\operatorname{cl} M),$$

hence, $\operatorname{conv} M$ is bounded. □

Unfortunately, closedness is, in general, not preserved under the convex hull operation, which we will illustrate by an example below. In the face of Proposition 2.3.2 the set to choose here, necessarily needs to be unbounded.

Example 2.3.4 Consider $M := \{\binom{0}{0}\} \cup \{\binom{a}{1} \mid a \geq 0\} \subset \mathbb{R}^2$. Then M is closed and

$$\begin{pmatrix} 1 \\ \frac{1}{k} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} k \\ 1 \end{pmatrix} + \left(1 - \frac{1}{k}\right)\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \operatorname{conv} M \quad (k \in \mathbb{N}).$$

On the other hand, as one can easily verify, $\lim_{k\to\infty} \binom{1}{\frac{1}{k}} = \binom{1}{0} \notin \operatorname{conv} M$, and hence $\operatorname{conv} M$ is not closed, see Figure 2.1.

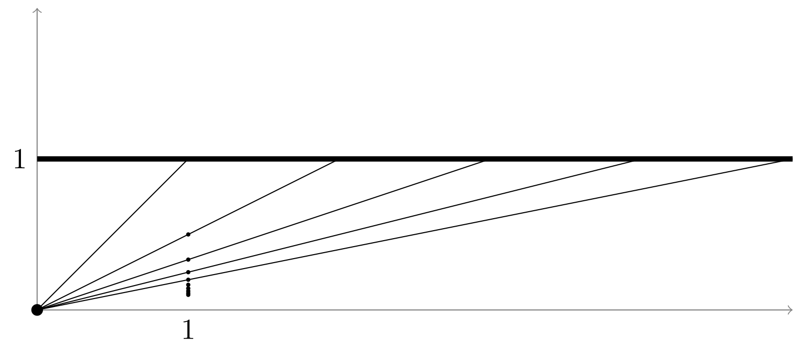

Figure 2.1: Closed set, non-closed convex hull

This justifies the following definition

Definition 2.3.5 (Closed convex hull) *Let $S \subset \mathbb{E}$ be nonempty. Then its* closed convex hull *is the intersection of all closed convex sets containing it, we denote it by* $\overline{\mathrm{conv}}\, S$.

It is not surprising that the closed convex hull equals the closure of the convex hull of a set.

Proposition 2.3.6 *Let $S \subset \mathbb{E}$ be nonempty. Then* $\overline{\mathrm{conv}}\, S = \mathrm{cl}\,(\mathrm{conv}\, S)$.

Proof: '$\subset$': $\mathrm{cl}\,(\mathrm{conv}\, S)$ is a closed convex set containing S, hence $\overline{\mathrm{conv}}\, S \subset \mathrm{cl}\,(\mathrm{conv}\, S)$, as $\overline{\mathrm{conv}}\, S$ is the smallest closed and convex set containing S.
'$\supset$': We have $S \subset \overline{\mathrm{conv}}\, S$, hence $\mathrm{conv}\, S \subset \overline{\mathrm{conv}}\, S$, thus $\mathrm{cl}\,(\mathrm{conv}\, S) \subset \overline{\mathrm{conv}}\, S$, as $\overline{\mathrm{conv}}\, S$ is closed and convex.

□

Exercise 2.2. complements Proposition 2.3.6, in that it shows that for a bounded set, the closed convex hull is also the convex hull of the closure of said set.

2.3.1 The relative interior

For convex sets the (topological) interior is too restrictive a concept as in many cases it is going to be empty. For instance, consider the line segment $C := [x, y]$ for some $x, y \in \mathbb{R}^n$. We have $\mathrm{int}\, C = \emptyset$, but on the other hand it would be nice to declare the set $(x, y)(:= C \setminus \{x, y\})$ as the 'interior of C in some sense'. Since $\mathrm{aff}\, C = \{\lambda x + (1 - \lambda) y \mid \lambda \in \mathbb{R}\}$, it is easily verified that

$$(x, y) = \{z \mid \exists \varepsilon > 0 : B_\varepsilon(z) \cap \mathrm{aff}\, C \subset C\},$$

i.e. (x, y) is the *interior* of C with respect to the *relative topology* induced by the affine hull of C.

We are going to make a general principle out of that.

C	$\operatorname{aff} C$	$\dim C$	$\operatorname{ri} C$
$\{x\}$	$\{x\}$	0	$\{x\}$
$[x, x']$	$\{\lambda x + (1-\lambda)x' \mid \lambda \in \mathbb{R}\}$	1	(x, x')
Δ_n	$\{a \mid e_n^T a = 1\}$	$n-1$	$\{a \in \Delta_n \mid a_i > 0\}$
$\overline{B}_\varepsilon(x)$	$\mathbb{E}$	N	$B_\varepsilon(x)$

Table 2.1: Examples for relative interiors

Definition 2.3.7 (Relative interior/boundary and convex dimension) *Let $C \subset \mathbb{E}$ be convex. Then the* relative interior $\operatorname{ri} C$ *of C is its interior with respect to the relative topology induced by* $\operatorname{aff} C$, *i.e.*

$$\operatorname{ri} C := \{x \in C \mid \exists \varepsilon > 0 : \ B_\varepsilon(x) \cap \operatorname{aff} C \subset C\}.$$

The relative boundary *of C is given by*

$$\operatorname{rbd} C := \operatorname{cl} C \setminus \operatorname{ri} C.$$

In addition, we define $\dim C := \dim(\operatorname{aff} C)$ *(cf. Definition 1.4.5) to be the* (convex) dimension *of C.*

By definition, for any convex set $C \subset \mathbb{E}$, we have

$$\operatorname{int} C \subset \operatorname{ri} C \subset C \subset \operatorname{cl} C. \tag{2.7}$$

One might ask the question why we did not define a relative closure along with the relative interior and boundary, respectively. The reason for this is that there is no difference between the closure in the standard topology and in the topology relative to the affine hull. This holds since, briefly speaking, $\operatorname{aff} C$ is closed and hence $\operatorname{cl} C \subset \operatorname{aff} C$.

Table 2.3.1 contains a list of examples of frequently used sets and their relative interiors. We urge the reader to verify them. Note that the first two show that the relative interior (as opposed to the interior) is non-monotonic in the sense that $C_1 \subset C_2$ does not necessarily imply $\operatorname{ri} C_1 \subset \operatorname{ri} C_2$.

Remark 2.3.8 The relative interior of a convex set $C \subset \mathbb{E}$ coincides with the interior if (and only if) C has full dimension, i.e. when $\operatorname{aff} C = \mathbb{E}$, in which case all topological questions reduce to the standard topology.

If, on the other hand, $\dim C = m < N (= \dim \mathbb{E})$, and we choose any m-dimensional subspace $U \subset \mathbb{E}$, Theorem 1.4.18 yields an invertible affine mapping $F : \mathbb{E} \to \mathbb{E}$ such that $F(\operatorname{aff} C) = U$, i.e. F maps $\operatorname{aff} C$ to U in a homeomorphic (even diffeomorphic) way. Thus, it holds that

$$\operatorname{aff} F(C) = F(\operatorname{aff} C) = U,$$

and therefore, it is often possible to reduce topological questions about arbitrary (lower dimensional) convex sets to the full dimensional case by just working with the affine, diffeomorphic image $F(A)$ as a full dimensional convex set in some (sub-)space U.

Remark 2.3.8 already comes into play in the proof of the next result.

Proposition 2.3.9 (Line segment principle) *Let $C \subset \mathbb{E}$ be convex as well as $x \in \operatorname{ri} C$ and $y \in \operatorname{cl} C$. Then we have $[x, y) \in \operatorname{ri} C$, i.e.*

$$(1-\lambda)x + \lambda y \in \operatorname{ri} C \quad (\lambda \in [0,1))$$

Proof: In view of Remark 2.3.8 we may assume w.l.o.g that $\dim C = N$, i.e. $\operatorname{ri} C = \operatorname{int} C$. Now, let $\lambda \in [0,1)$. Since $y \in \operatorname{cl} C$, we have (see (2.6))

$$y \in C + \varepsilon\mathbb{B} \quad \forall \varepsilon > 0. \tag{2.8}$$

Hence, using Minkowski addition, we get

$$\begin{aligned} B_\varepsilon((1-\lambda)x + \lambda y) &= (1-\lambda)x + \lambda y + \varepsilon\mathbb{B} \\ &\overset{(2.8)}{\subset} (1-\lambda)x + \lambda(C + \varepsilon\mathbb{B}) + \varepsilon\mathbb{B} \\ &= (1-\lambda)\Big[\underbrace{x + \frac{1+\lambda}{1-\lambda}\varepsilon\mathbb{B}}_{=B_{\varepsilon\frac{1+\lambda}{1-\lambda}}(x)}\Big] + \lambda C \end{aligned}$$

for all $\varepsilon > 0$. Since $x \in \operatorname{int} C$, we have $B_{\varepsilon\frac{1+\lambda}{1-\lambda}}(x) \subset C$ for all $\varepsilon > 0$ sufficiently small. Hence, for these $\varepsilon > 0$, we get

$$B_\varepsilon((1-\lambda)x + \lambda y) \subset (1-\lambda)C + \lambda C = C,$$

which shows that $(1-\lambda)x + \lambda y \in \operatorname{int} C$, and hence concludes the proof. $\square$

We encourage the reader to emulate the proof of Proposition 2.3.9 without the assumption that C has nonempty interior to see that nothing changes other than the necessity of intersecting with $\operatorname{aff} C$.

An immediate consequence of the line segment principle is the convexity of the relative interior of a convex set (just take $y \in \operatorname{ri} C$ in Proposition 2.3.9), which, in view of Remark 2.3.8, we also obtain from Proposition 2.3.1.

We now show that for $C \subset \mathbb{E}$ convex, the three convex sets $C, \operatorname{ri} C$ and $\operatorname{cl} C$ have the same affine hull, hence the same convex dimension.

Theorem 2.3.10 *Let $C \subset \mathbb{E}$ convex. Then $\operatorname{aff}(\operatorname{cl} C) = \operatorname{aff} C = \operatorname{aff}(\operatorname{ri} C)$. In particular, we have $\dim C = \dim \operatorname{cl} C = \dim \operatorname{ri} C$ and $\operatorname{ri} C \neq \emptyset$ if $C \neq \emptyset$.*

Proof: Since $\operatorname{aff} C$ is closed we have $\operatorname{cl} C \subset \operatorname{aff} C$. Using again the properties of the respective hull operators 'cl' and 'aff' we hence obtain

$$\operatorname{aff} C \subset \operatorname{aff}(\operatorname{cl} C) \subset \operatorname{aff}(\operatorname{aff} C) = \operatorname{aff} C,$$

therefore, in particular, we have $\operatorname{aff} C = \operatorname{aff}(\operatorname{cl} C)$.

We now show that $\operatorname{aff} C = \operatorname{aff}(\operatorname{ri} C)$: In view of Remark 2.3.8 we can assume that $\operatorname{aff} C = \mathbb{E}$, i.e. $\dim C = N$. Hence, it suffices to show that $\operatorname{int} C \neq \emptyset$ under this assumption, since then also $\operatorname{aff}(\operatorname{int} C) = \mathbb{E}$. For these purposes, let $x_0, x_1, \dots, x_N \in C$ be affinely independent and define

$$S := \operatorname{conv}\{x_0, x_1, \dots, x_N\} \subset C.$$

We show that

$$\bar{x} := \frac{1}{N+1} \sum_{i=0}^{N} x_i \in S$$

is an interior point of S hence also of C.

Notice that $\mathbb{E} = \operatorname{aff} S = \operatorname{aff}\{x_0, x_1, \dots, x_N\}$, and $x_0, x_1, \dots, x_N$ are affinely independent. Hence, for every $y \in \mathbb{E}$, we have unique scalars $\beta_0(y), \beta_1(y), \dots, \beta_N(y) \in \mathbb{R}$ with $\sum_{i=0}^N \beta_i(y) = 1$ such that

$$\sum_{i=0}^{N} \beta_i(y) x_i = \bar{x} + y = \frac{1}{N+1} \sum_{i=0}^{N} x_i + y.$$

Thus, putting $\alpha_i(y) := \beta_i(y) - \frac{1}{N+1}$ $(i = 0, 1, \dots, N)$, the vector $\alpha(y) \in \mathbb{R}^{N+1}$ is the unique solution of the linear system

$$y = \sum_{i=0}^{N} \alpha_i x_i, \quad 0 = \sum_{i=0}^{N} \alpha_i.$$

The thus induced mapping $y \in \mathbb{E} \mapsto \alpha(y)$ is linear, hence continuous. Thus, we can find $\delta > 0$ such that for all $y \in \delta\mathbb{B}$, we have $|\alpha_i(y)| \leq \frac{1}{N+1}$. But then $\beta_i(y) = \alpha_i(y) + \frac{1}{N+1} \geq 0$ and hence $\bar{x} + y = \sum_{i=0}^N \beta_i(y) x_i \in \operatorname{conv}\{x_0, \dots, x_N\} = S$ for all $y \in \delta\mathbb{E}$. Thus, $\bar{x} + \delta\mathbb{B} \subset S$, which gives the assertion.

□

We continue by providing a list of useful properties of the relative interior.

Proposition 2.3.11 *Let $C \subset \mathbb{E}$ convex. Then the following hold:*

a) $\operatorname{ri}(\operatorname{ri} C) = \operatorname{ri} C = \operatorname{ri}(\operatorname{cl} C)$*;*

b) $\operatorname{cl} C = \operatorname{cl}(\operatorname{ri} C)$*;*

c) $\operatorname{rbd} C = \operatorname{rbd}(\operatorname{ri} C) = \operatorname{rbd}(\operatorname{cl} C)$.

Proof: By Remark 2.3.8 we can again assume that $\operatorname{aff} C = \mathbb{E}$, i.e. the standard and the relative topology coincide. The statements then follow from the well-known facts for the interior, boundary and closure of sets with nonempty interior.

$\square$

The fact that two convex set with the same closure have the same relative interior is an immediate consequence, which we want to state explicitly because it is so frequently used.

Corollary 2.3.12 *Let $C_1, C_2 \subset \mathbb{E}$ such that $\operatorname{cl} C_1 = \operatorname{cl} C_2$. Then $\operatorname{ri} C_1 = \operatorname{ri} C_2$.*

Proof: From Proposition 2.3.11 we infer that $\operatorname{ri} C_1 = \operatorname{ri}(\operatorname{cl} C_1) = \operatorname{ri}(\operatorname{cl} C_2) = \operatorname{ri} C_2$. $\square$

We next present another useful principle for the relative interior of a convex set that we call the stretching principle.

Proposition 2.3.13 (Stretching principle) *Let $C \subset \mathbb{E}$ be a nonempty convex set . Then it holds that*

$$z \in \operatorname{ri} C \quad \Longleftrightarrow \quad \forall x \in C\ \exists \mu > 1 : \mu z + (1-\mu)x \in C.$$

Proof: First, let $z \in \operatorname{ri} C$. By definition, there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \cap \operatorname{aff} C \in C$. Moreover, for every $x \in C$ and $\mu \in \mathbb{R}$ we have $\mu z + (1-\mu)x \in \operatorname{aff} C$. For every μ sufficiently close to 1, $\mu z + (1-\mu)x \in B_\varepsilon(z)$. Hence, $\mu z + (1-\mu)x \in \operatorname{aff} C \cap B_\varepsilon(z) \subset C$.

Now, suppose that z satisfies the condition on the right-hand side of the equivalence: As $\operatorname{ri} C \neq \emptyset$ by Theorem 2.3.10, there exists $x \in \operatorname{ri} C$. By assumption we then have $y := \mu z + (1-\mu)x \in C$ for some $\mu > 1$. Then $z = \lambda y + (1-\lambda)x$, where $\lambda := \mu^{-1}$. By the line segment principle $z \in \operatorname{ri} C$. $\square$

Proposition 2.3.13 basically says that every line segment in C having $z \in \operatorname{ri} C$ as one endpoint can be, to some extent, *stretched* beyond z without leaving C.

In the following result we want to investigate how relative interiors and closures behave with regard to intersection of sets. For these purposes, recall that for an intersection of a family of sets $A_i \in \mathbb{E}$ $(i \in I)$, it always holds that

$$\operatorname{cl} \bigcap_{i \in I} A_i \subset \bigcap \operatorname{cl} A_i$$

due to the monotonicity and idempotence of the closure operator and the fact that an arbitrary intersection of closed sets is closed.

Proposition 2.3.14 *Let $C_i \subset \mathbb{E}$ be convex for $i \in I$ (an index set) such that $\bigcap_{i \in I} \operatorname{ri} C_i \neq \emptyset$. Then the following hold:*

a) $\mathrm{cl} \bigcap_{i\in I} C_i = \bigcap_{i\in I} \mathrm{cl}\, C_i$;

b) $\mathrm{ri} \bigcap_{i\in I} C_i = \bigcap_{i\in I} \mathrm{ri}\, C_i$ *if I is finite.*

Proof: Fix $x \in \bigcap_{i\in I} \mathrm{ri}\, C_i$. By the line segment principle, given any $y \in \bigcap_{i\in I} \mathrm{cl}\, C_i$, we have $(1-\lambda)x + \lambda y \in \bigcap_{i\in I} \mathrm{ri}\, C_i$ for all $\lambda \in [0,1)$. Moreover, we have $y = \lim_{\lambda\to 1}(1-\lambda)x + \lambda y$. As y was chosen arbitrarily in $\bigcap_{i\in I} \mathrm{cl}\, C_i$, we obtain

$$\bigcap_{i\in I} \mathrm{cl}\, C_i \subset \mathrm{cl} \bigcap_{i\in I} \mathrm{ri}\, C_i \subset \mathrm{cl} \bigcap_{i\in I} C_i \subset \bigcap_{i\in I} \mathrm{cl}\, C_i.$$

This proves a) and shows at the same time that $\bigcap_{i\in I} \mathrm{ri}\, C_i$ and $\bigcap_{i\in I} C_i$ have the same closure, hence, by Corollary 2.3.12, the same relative interior. Thus,

$$\mathrm{ri} \bigcap_{i\in I} C_i \subset \bigcap_{i\in I} \mathrm{ri}\, C_i.$$

In order to prove the opposite inclusion we assume that I is finite. Fix $z \in \bigcap_{i\in I} \mathrm{ri}\, C_i$. Then by the stretching principle (applied to every C_i), for every $x \in \bigcap C_i$ and every $i \in I$ there exists $\mu_i > 0$ such that $\mu_i z + (1-\mu_i) \in C_i$. Putting $\mu := \min_{i\in I} \mu_i > 1$ (I finite!), we see that $\mu z + (1-\mu)x \in \bigcap C_i$. Hence, again by the stretching principle, $z \in \mathrm{ri} \bigcap C_i$, which completes the proof.

□

Next, we want to show that affine mappings preserve relative interiors (as was already foreshadowed in Remark 2.3.8). To this end, recall that for a continuous function $f : \mathbb{E}_1 \to \mathbb{E}_2$ and $A \in \mathbb{E}_1$, we have

$$f(\mathrm{cl}\, A) \subset \mathrm{cl}\,(f(A)).$$

Proposition 2.3.15 (Relative interior under affine mappings) *Let $F : \mathbb{E}_1 \to \mathbb{E}_2$ be affine and $C \subset \mathbb{E}_1$ convex. Then*

$$\mathrm{ri}\, F(C) = F(\mathrm{ri}\, C).$$

Proof: First note that by Proposition 2.1.2 d) the set $F(C)$ is convex, so that we can even talk about its relative interior.

By the continuity of F, the monotonicity of the closure operation and Proposition 2.3.11 b) we obtain

$$F(C) \subset F(\mathrm{cl}\, C) = F(\mathrm{cl}\,(\mathrm{ri}\, C)) \subset \mathrm{cl}\, F(\mathrm{ri}\, C) \subset \mathrm{cl}\, F(C).$$

Hence, taking the (idempotent) closure again on both sides, we get $\mathrm{cl}\, F(C) = \mathrm{cl}\, F(\mathrm{ri}\, C)$. Therefore, see Corollary 2.3.12, $\mathrm{ri}\, F(C) = \mathrm{ri}\, F(\mathrm{ri}\, C) \subset F(\mathrm{ri}\, C)$.

In order to prove the converse inclusion take $z \in F(\operatorname{ri} C)$. Moreover, let $x \in F(C)$. In addition, choose $z' \in F^{-1}(z) \subset \operatorname{ri} C$ and $x' \in F^{-1}(x) \subset C$. By the stretching principle , there exists $\mu > 1$ such that $\mu z' + (1-\mu)x' \in C$, and thus,

$$F(\mu z' + (1-\mu)x') = \mu z + (1-\mu)x \in F(C).$$

As $x \in F(C)$ was chosen arbitrarily, we can apply the stretching principle (in the opposite direction then before) to z and infer that $z \in \operatorname{ri} F(C)$, which concludes the proof.

□

Proposition 2.3.15 has some nice consequences.

Corollary 2.3.16 *Let $C \subset \mathbb{E}$ be convex and $\lambda \in \mathbb{R}$. Then we have* $\operatorname{ri}(\lambda C) = \lambda \operatorname{ri} C$.

Proof: Apply Proposition 2.3.15 to $F : \mathbb{E} \to \mathbb{E}, F(x) = \lambda x$. □

Corollary 2.3.17 *Let $C_1, C_2 \in \mathbb{E}$ be convex. Then* $\operatorname{ri}(C_1 + C_2) = \operatorname{ri} C_1 + \operatorname{ri} C_2$.

Proof: By Exercise 2.6. $\operatorname{ri}(C_1 \times C_2) = \operatorname{ri} C_1 \times \operatorname{ri} C_2$. Now define the affine (in fact linear) mapping $F : \mathbb{E} \times \mathbb{E}, F(x_1, x_2) = (x_1 + x_2)$. Then by Proposition 2.3.15 we have

$$\operatorname{ri}(C_1 + C_2) = \operatorname{ri} F(C_1 + C_2) = F(\operatorname{ri}(C_1 \times C_2)) = F(\operatorname{ri} C_1 \times \operatorname{ri} C_2) = \operatorname{ri} C_1 + \operatorname{ri} C_2.$$

□

2.4 Cones and conical approximations of convex sets

2.4.1 Convex cones and conical hulls

Definition 2.4.1 (Cones) *A nonempty set $K \subset \mathbb{E}$ is said to be a* cone *if*

$$\lambda K \subset K \quad (\lambda \geq 0),$$

i.e. K is a cone if and only if it is closed under multiplication with nonnegative scalars. We call K pointed *if for any $p \in \mathbb{N}$ the implication*

$$x_1 + \cdots + x_p = 0 \quad \Rightarrow \quad x_i = 0 \ (i = 1, \ldots, p)$$

holds as soon as $x_i \in K$.

Note that, in our definition, a cone always contains the origin. In the literature (e.g. [3]) this is not necessarily the case.
For obvious reasons, *convex cones* are of particular interest to our study. We have the following handy characterization of convexity of a cone.

Proposition 2.4.2 (Convex cones) *Let $K \subset \mathbb{E}$ be a cone. Then K is convex if and only if $K + K \subset K$.*

Proof: Let K be convex. Then

$$x + y = 2 \cdot \underbrace{\frac{1}{2}(x + y)}_{\in K} \in K \quad (x, y \in K)$$

hence $K + K \subset K$.
If, in turn, $K + K \subset K$, then

$$\underbrace{\lambda x}_{\in K} + \underbrace{(1 - \lambda)y}_{\in K} \in K + K \subset K \quad \forall x, y \in K, \lambda \in [0, 1],$$

i.e. K is convex. $\square$

Pointedness of convex cones can be handily characterized.

Proposition 2.4.3 (Pointedness of convex cones) *Let $K \subset \mathbb{E}$ be a convex cone. Then K is pointed if and only if $K \cap (-K) = \{0\}$.*

Proof: Exercise 2.10. $\square$

We proceed with a list of prominent examples of cones.

Example 2.4.4 (Cones)

a) (Nonnegative Orthant) For all $n \in \mathbb{N}$, the nonnegative orthant $\mathbb{R}^n_+$ is a pointed, convex cone, which is also a polyhedron as

$$\mathbb{R}^n_+ = \left\{ x \in \mathbb{R}^n \;\middle|\; (-e_i)^T x \le 0 \ (i = 1, \dots, n) \right\}.$$

b) (Cone complementarity constraints) Let $K \subset \mathbb{E}$ be a cone. Then the set

$$\Lambda := \{(x, y) \in \mathbb{E} \times \mathbb{E} \mid x, y \in K, \quad \langle x, y \rangle = 0\}$$

is a cone. A prominent realization is $K = \mathbb{R}^n$, in which case Λ is called the complementarity constraint set.

c) (Positive semidefinite matrices) For $n \in \mathbb{N}$, the set $\mathbb{S}^n_+$ of positive semidefinite $n \times n$ matrices is a pointed, convex cone.

The next example is important enough to merit its own definition.

Definition 2.4.5 (Polar cone) *Let $K \in \mathbb{E}$ be a cone. Then the* polar (cone) *of K is defined by*

$$K^\circ := \{d \in \mathbb{E} \mid \langle d, x \rangle \leq 0 \; \forall x \in K\}.$$

respectively. Moreover, $K^{\circ\circ} := (K^\circ)^\circ$ is called the bipolar cone *of K.*

The cone $K^* := -K^\circ$ is sometimes referred to as the dual (cone) *of K, and K is called* self-dual *if $K^* = K$.*

In order to visualize the normal cone, we think of $\mathbb{E}$ as $\mathbb{R}^n$: Then the normal cone of the cone K is set of all vectors, which have an angle $\geq 90°$ to every vector in K.

Clearly, for an arbitrary cone $K \subset \mathbb{E}$, its polar K° is always a closed, convex cone. Hence, for K to be self-dual, it must necessarily be a closed, convex cone. Moreover, polarization is order-reversing, i.e. for $K_1 \subset K_2 \subset \mathbb{E}$, we have $K_2^\circ \subset K_1^\circ$.

We continue with some elementary examples of polar cones.

Example 2.4.6 (Polar cones)

a) It holds that $\{0\}^\circ = \mathbb{E}$ and $\mathbb{E}^\circ = \{0\}$, which is a special case of part b).

b) If S is a subspace, $S^\circ = S^\perp$ (cf. Exercise 2.13.).

c) For $0 \neq w \in \mathbb{E}$, the polar of the ray $\{tw \mid t \geq 0\}$ is the half-space $\{w \in \mathbb{E} \mid \langle w, x \rangle \leq 0\}$.

d) The negative orthant $\mathbb{R}^n_+$ and the positive semidefinite $n \times n$ matrices $\mathbb{S}^n_+$ are self-dual.

Since the intersection of convex cones is again a convex cone, using our usual routine, we can also build up *(convex) conical hulls* of arbitrary sets.

Definition 2.4.7 ((Convex) Conical hull) *Let $S \subset \mathbb{E}$ be nonempty. Then the* (convex) conical hull *of S is the set*

$$\operatorname{cone} S := \bigcap_{\substack{S \subset M, \\ M \text{ convex cone}}} M.$$

Moreover, we define the closed (convex) conical hull of S to be

$$\overline{\operatorname{cone}}\, S := \operatorname{cl}(\operatorname{cone} S).$$

We notice, without proof, that $\operatorname{cone} S$ is the intersection of all closed, convex cones that contain S.

Proposition 2.4.8 *Let $S \subset \mathbb{E}$ be nonempty. Then the following hold:*

a) $\operatorname{cone} S = \{\sum_{i=1}^r \lambda_i x_i \mid r \in \mathbb{N}, x_i \in S, \lambda_i \geq 0\ (i = 1, \dots, r)\} = \mathbb{R}_+(\operatorname{conv} S) = \operatorname{conv}(\mathbb{R}_+ S)$.

b) If S is compact with $0 \notin \operatorname{conv} S$ then $\overline{\operatorname{cone}} S = \operatorname{cone} S$.

Proof:

a) Exercise 2.12..

b) By Proposition 2.3.2, $\operatorname{conv} S$ is compact and, by assumption, does not contain the origin. We prove that $\operatorname{cone} S = \mathbb{R}_+(\operatorname{conv} S)$ is closed: To this end let $\{y_k \in \operatorname{cone} S\} \to x$, i.e. $y_k = t_k x_k$ with $t_k > 0$ and $x_k \in \operatorname{conv} S$ for all $k \in \mathbb{N}$. As $\operatorname{conv} S$ is compact, we may assume w.l.o.g. (otherwise we extract a subsequence) that $x_k \to x \in \operatorname{conv} S$, in particular, $x \neq 0$. Hence, $t_k = \frac{\|y_k\|}{\|x_k\|} \to \frac{\|y\|}{\|x\|} =: t \geq 0$. Hence, $y = tx \in \operatorname{cone} S$, which shows the closedness of $\operatorname{cone} S$.

□

2.4.2 Tangent and normal cone

Definition 2.4.9 (Tangent cone) *Let $S \subset \mathbb{E}$ and $x \in S$. Then the set*

$$T_S(\bar{x}) := \left\{ d \;\middle|\; \exists \{x_k \in S\} \to \bar{x}, \{t_k\} \downarrow 0 : \ \frac{x_k - \bar{x}}{t_k} \to d \right\}$$

is called the (Bouligand) tangent cone *of S at $\bar{x}$.*

The next result ensures that tangent cone is, in fact, a cone, and closed at that.

Proposition 2.4.10 *Let $S \subset \mathbb{E}$ and $x \in S$. Then $T_S(x)$ is a closed cone.*

Proof: Let $\{d_j \in T_S(\bar{x})\} \to d$. For every $j \in \mathbb{N}$ there exists $\{x_k^j \in S\} \to \bar{x}$, $\{t_k^j\} \downarrow 0$ such that

$$\frac{x_k^j - \bar{x}}{t_k^j} \to_{k \to \infty} d_j.$$

Given $j \in \mathbb{N}$, there exists $k(j) \in \mathbb{N}$ such that

$$\left\| \frac{x_{k(j)}^j - \bar{x}}{t_{k(j)}^j} - d_j \right\| \leq \frac{1}{j}, \quad \|x_{k(j)}^j - \bar{x}\| \leq \frac{1}{j}, \quad |t_{k(j)}^j| \leq \frac{1}{j} \quad (j \in \mathbb{N}).$$

Now put

$$x_j := x_{k(j)}^j \quad \text{and} \quad t_j := t_{k(j)}^j \quad (j \in \mathbb{N}).$$

Then $\{x_j \in S\} \to \bar{x}, \{t_j\} \downarrow 0$ and

$$\left\| \frac{x_j - \bar{x}}{t_j} - d \right\| \leq \frac{1}{j} + \|d_j - d\| \to_{j\to\infty} 0,$$

which shows that $d \in T_S(\bar{x})$. □

It should be noted that, for an arbitrary set $S \subset \mathbb{E}$, T_S is not convex-valued. For example, as a special case with $K = \mathbb{R}^2_+$, we get from Exercise 2.17. that for $K = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0,\ xy = 0\right\}$, we have

$$T_K(0) = K,$$

which is clearly not a convex set. On the other hand, for all other points $x \in K \setminus \{0\}$, one can easily compute that $T_K(x)$ is convex.

In fact, as we will see shortly, the tangent cone to a convex set is always convex-valued. To this end, we need the following preparatory result.

Lemma 2.4.11 *Let C be a nonempty convex set and $\bar{x} \in C$. Then*

$$C - \bar{x} \subset T_C(\bar{x}).$$

Proof: Let $x \in C$. We need to show that $x - \bar{x} \in T_C(\bar{x})$. To this end, choose $\{t_k \in [0, 1]\} \downarrow 0$, and put $x_k := \bar{x} + t_k(x - \bar{x}) = (1 - t_k)\bar{x} + t_k x \in C$, as C is convex. Moreover, $x_k \to \bar{x}$ and

$$\frac{x_k - \bar{x}}{t_k} = x - \bar{x} \to x - \bar{x}.$$

Hence, $x - \bar{x} \in T_C(\bar{x})$, which proves the assertion. □

Proposition 2.4.12 (Tangent cone of a convex set) *Let $C \subset \mathbb{E}$ be nonempty and convex and let $\bar{x} \in C$. Then*

$$T_C(\bar{x}) = \overline{\text{cone}}\,(C - \bar{x}) = \text{cl}\,(\mathbb{R}_+(C - \bar{x})).$$

In particular, $T_C(\bar{x})$ is (a closed and) convex (cone).

Proof: We show that $T_C(\bar{x}) = \text{cl}\,(\mathbb{R}_+(C - \bar{x}))$, the latter being equal to $\overline{\text{cone}}\,(C - \bar{x})$, since $C - \bar{x}$ is already convex (as a Minkowski sum of convex sets): By Lemma 2.4.11, we have $C - \bar{x} \subset T_C(\bar{x})$. Since $T_C(\bar{x})$ is a closed cone (cf. Proposition 2.4.10), this yields $\text{cl}\,(\mathbb{R}_+(C - \bar{x})) \subset T_C(\bar{x})$.

For the converse inclusion, let $d \in T_C(\bar{x})$, i.e. there exist $\{x_k \in C\} \to \bar{x}$ and $\{t_k\} \downarrow 0$ such that $\frac{x_k - \bar{x}}{t_k} \to d$. As $\frac{x_k - \bar{x}}{t_k} \in \mathbb{R}_+(C - \bar{x})$ we get $d \in \text{cl}\,(\mathbb{R}_+(C - \bar{x}))$, which completes the proof. □

Combining Proposition 2.4.12 with Exercise 2.6., we see that the tangent cone of a convex set at some point in the set is a subspace if and only if said point lies in the relative interior of the underlying set.

The polar of the tangent cone merits his own name.

Definition 2.4.13 (Normal cone) *Let $C \subset \mathbb{E}$ and $\bar{x} \in C$. Then the set*

$$N_C(\bar{x}) := (T_C(\bar{x}))^\circ$$

is called the (Fréchet) normal cone *of C at $\bar{x}$.*

For a convex set, the computation of the normal cone can be executed in a handy way without using the tangent cone.

Proposition 2.4.14 (Normal cone of a convex set) *Let $C \subset \mathbb{E}$ be nonempty and convex. Then*

$$N_C(\bar{x}) = \{v \in \mathbb{E} \mid \langle v, x - \bar{x}\rangle \le 0 \ \forall x \in C\}.$$

Proof: Follows immediately from Exercise 2.14.

□

Corollary 2.4.15 *Let $K \subset \mathbb{E}$ be nonempty, closed, convex cone. Then*

$$N_K(\bar{x}) = \{\bar{x}\}^\perp \cap K^\circ \quad (= \{v \in K^\circ \mid \langle v, \bar{x}\rangle = 0\}).$$

In particular, $N_K(0) = K^\circ$.

Proof: For $\bar{x} = 0$ the statement follows directly from the fact that $T_K(0) = K$, see Exercise 2.17.. Hence, let $\bar{x} \neq 0$: Also from Exercise 2.17. we know that $\operatorname{span} \bar{x} \subset T_K(\bar{x})$, hence $N_K(\bar{x}) \subset \{\bar{x}\}^\perp$ (cf. Example 2.4.6). Hence, by convexity, for $v \in N_K(\bar{x})$ we have

$$\langle v, x\rangle \le \langle v, \bar{x}\rangle = 0 \quad (x \in K)$$

see Proposition 2.4.14. This proves the assertion. □

Tangent and normal cones are frequently employed to state optimality conditions for minimization problems. For these purposes we first define/recall the notion of local minimizers.

Definition 2.4.16 (Minimizers) *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ and $C \in \mathbb{E}$. Then $\bar{x} \in \mathbb{E}$ called a* (global) minimizer of f over C *if $\bar{x} \in \operatorname{argmin}_C f$. It is called a* local minimizer of f over C *if there exists an $\varepsilon > 0$ such that $\bar{x} \in \operatorname{argmin}_{B_\varepsilon(\bar{x}) \cap C} f$.*

We now state first-order (local) optimality conditions using tangent and normal cones.

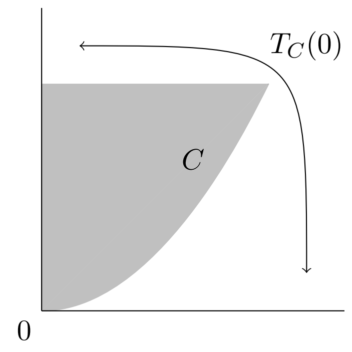

Figure 2.2: Tangent cone to a convex set at the origin

Theorem 2.4.17 (Basic first-order optimality conditions) *Let $f : \mathbb{E} \to \mathbb{R}$ be continuously differentiable and $C \subset \mathbb{E}$ nonempty. Suppose that $\bar{x}$ is a local minimizer of the optimization problem*

$$\min \; f(x) \quad \text{s.t.} \quad x \in C. \tag{2.9}$$

Then the following hold:

a) $\langle \nabla f(\bar{x}), \, d \rangle \geq 0 \quad \forall d \in T_C(\bar{x})$*;*

b) $0 \in \nabla f(\bar{x}) + N_C(\bar{x})$*.*

If C is convex, in addition, we have

$$\langle \nabla f(\bar{x}), \, x - \bar{x} \rangle \geq 0 \quad \forall x \in C.$$

Proof: We start by proving a) and b):

a) Let $d \in T_C(\bar{x})$. By definition of the tangent cone, there exists $\{x_k \in C\} \to \bar{x}$ and $\{t_k\} \downarrow 0$ sucht that $\frac{x_k - \bar{x}}{t_k} \to d$. The mean value theorem yields the existence of $\xi_k \in [x_k, \bar{x}]$ $(k \in \mathbb{N})$ such that

$$f(x_k) - f(\bar{x}) = \langle \nabla f(\xi_k), \, x_k - \bar{x} \rangle \, .$$

As $\bar{x}$ is a local minimizer and $x_k \in C$, we have $0 \leq f(x_k) - f(\bar{x})$ for all k sufficiently large. Hence, as ∇f is continuous, we obtain

$$0 \leq \lim_{k \to \infty} \left\langle \nabla f(\xi_k), \, \frac{x_k - \bar{x}}{t_k} \right\rangle = \nabla \langle f(\bar{x}), \, d \rangle \, ,$$

which proves the assertion.

b) From a) we infer that $\langle -\nabla f(\bar{x}), d\rangle \leq 0$ for all $d \in T_C(\bar{x})$, i.e. $-\nabla f(\bar{x}) \in (T_C(\bar{x}))^\circ = N_C(\bar{x})$. This is equivalent to $0 \in \nabla f(\bar{x}) + N_C(\bar{x})$, which is the assertion.

The convex case follows from Proposition 2.4.14.

□

Since $N_{\mathbb{E}}(x) = \{0\}$ for all $x \in \mathbb{E}$, we immediately obtain the well known first-order condition for unconstrained minimization of a smooth function.

Corollary 2.4.18 (Basic first-order optimality: unconstrained case) *Let $f : \mathbb{E} \to \mathbb{R}$ be continuously differentiable. If f is a local minimizer of f then $\nabla f(\bar{x}) = 0$.*

2.4.3 The horizon cone

The next conical approximation of a set is, loosely speaking, comprised of the *directions* in which one can go (starting at least one point in the set) without ever leaving the set, and thus takes account of the unboundedness of the set.

Definition 2.4.19 (Horizon cone) *For a nonempty set $C \subset \mathbb{E}$ the set*

$$S^\infty := \{v \in \mathbb{E} \mid \exists \{x_k \in C\}, \{t_k\} \downarrow 0 : \; t_k x_k \to v\}$$

is called the horizon cone *of S. We put $\emptyset^\infty := \{0\}$.*

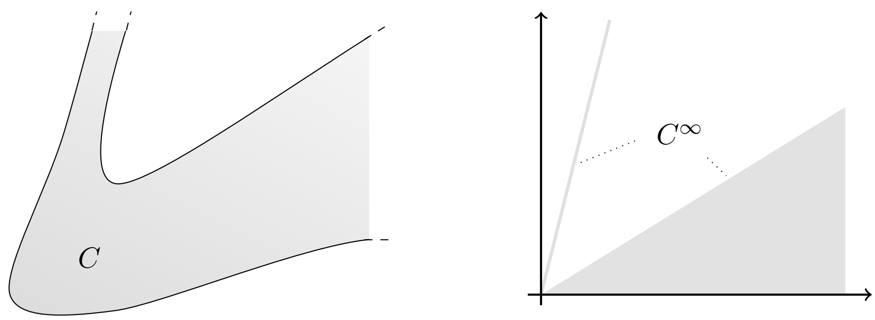

Figure 2.3: The horizon cone of an unbounded set

The following result shows that the horizon cone is indeed a closed cone.

Lemma 2.4.20 *The horizon cone of a set $C \subset \mathbb{E}$ is a closed cone.*

Proof: The fact that C^∞ is a cone is trivial. For the closedness of the horizon cone we can invoke analogous arguments as for the closedness of the tangent cone, see the proof of Proposition 2.4.10. □

The horizon cone can be used to very handily express boundedness of an arbitrary set in $\mathbb{E}$.

Proposition 2.4.21 (Horizon criterion for boundedness) *A set $S \subset \mathbb{E}$ is bounded if and only if $S^\infty = \{0\}$.*

Proof: If S is bounded, then for all sequences $\{x_k \in S\}, \{t_k\} \downarrow 0$, we have $t_k x_k = 0$, hence S^∞.

If, in turn, S is unbounded, then there exists $\{x_k \in S\}$ with $\|x_k\| \to \infty$. Putting $t_k = \frac{1}{\|x_k\|}$, we have $\{t_k\} \downarrow 0$ and $t_k x_k = \frac{x_k}{\|x_k\|}$ converges (at least on a subsequence) to some $v \neq 0$ and, hence $v \in S^\infty \supsetneq \{0\}$. □

Figure 2.3 nicely illustrates the foregoing result. It shows that the horizon cone of an unbounded nonconvex set, which is a non-trivial closed cone, which is not necessarily convex.

We now want to show that the horizon cone is always going to be convex if the said in question is convex itself. For these purposes, we introduce another conical approximation for convex sets.

Definition 2.4.22 (Recession cone) *For nonempty convex sets $C \subset \mathbb{E}$, the convex cone*

$$0^+(C) := \{v \mid \forall x \in C, \lambda \geq 0 : \; x + \lambda v \in C\},$$

is called the recession cone *of C,*

The recession cone is very closely related to the horizon cone as we will now see.

Proposition 2.4.23 (Horizon vs. recession cone) *Let $C \subset \mathbb{E}$ be nonempty and convex. Then*

$$C^\infty = 0^+(\operatorname{cl} C).$$

In particular, C^∞ is (a closed and) convex (cone) if C is convex.

Proof: Let $v \in C^\infty$. Then there exist $\{x_k \in C\}, \{t_k\} \downarrow 0$ such that $t_k x_k \to v$. Now, let $\lambda \geq 0$ and $x \in C$ be given. As C is convex, we have

$$(1 - \lambda t_k)x + \lambda t_k x_k \in C$$

for all $k \in \mathbb{N}$ sufficiently large. Hence,

$$x + \lambda v = \lim_{k \to \infty} (1 - \lambda t_k)x + \lambda t_k x_k \in \operatorname{cl}(C),$$

which shows that $v \in 0^+(\operatorname{cl} C)$.

Now let $v \in 0^+(\operatorname{cl} C)$. Fixing $x \in C$, we get $x + \lambda v \in \operatorname{cl} C$ for all $\lambda \geq 0$. Hence, we can find a sequence $\{x_k \in C\}$ such that $\|x + kv - x_k\| \leq \frac{1}{k}$ for all $k \in \mathbb{N}$. Putting $\lambda_k := \frac{1}{k}$ $(k \in \mathbb{N})$, we obtain

$$\|v - \lambda_k x_k\| \leq \frac{1}{k}(\|x\| + \|x + kv - x_k\|) \to 0,$$

hence $\lambda_k x_k \to v$, i.e. $v \in C^\infty$. $\square$

An obvious consequence of the foregoing proposition is the fact that horizon and recession cone coincide for nonempty, closed and convex sets.

Corollary 2.4.24 *Let $C \subset \mathbb{E}$ be nonempty, closed and convex. Then*

$$C^\infty = 0^+(C).$$

We now want to study how the horizon cone operation behaves as for union and intersection of sets.

Proposition 2.4.25 (Horizon cone of intersections and unions) *Let $C_i \subset \mathbb{E}$ $(i \in I)$ be a family of sets: Then the following hold:*

a) $\left[\bigcap_{i\in I} C_i\right]^\infty \subset \bigcap_{i\in I} C_i^\infty$*, where equality holds if C_i is closed and convex for all $i \in I$ and* $\bigcap_{i\in I} C_i \neq \emptyset$*.*

b) $\left[\bigcup_{i\in I} C_i\right]^\infty \supset \bigcup_{i\in I} C_i^\infty$*, where equality holds when I is finite.*

Proof:

a) Let $v \in \left[\bigcap_{i\in I} C_i\right]^\infty$, i.e. there exist sequences $\left\{x_k \in \bigcap_{i\in I} C_i\right\}$ and $\{\lambda_k\} \downarrow 0$ such that $\lambda_k x_k \to v$. But as $x_k \in C_i$ for all $k \in \mathbb{N}$ and $i \in I$, we have $v \in C_i^\infty$ for all $i \in I$, which gives the desired inclusion.

   Now, assume that C_i $(i \in I)$ is closed and convex and $\bigcap_{i\in I} C_i \neq \emptyset$. By Corollary 2.4.24 we hence have $\left[\bigcap_{i\in I} C_i\right]^\infty = 0^+(\bigcap_{i\in I} C_i)$ and $\bigcap_{i\in I} C_i^\infty = \bigcap_{i\in I} 0^+(C_i)$, hence it suffices to show that $\bigcap_{i\in I} 0^+(C_i) \subset 0^+(\bigcap_{i\in I} C_i)$ in this situation: To this end, let $v \in \bigcap_{i\in I} 0^+(C_i)$, i.e. $v \in 0^+(C_i)$ for all $i \in I$. Take $x \in \bigcap_{i\in I} C_i \neq \emptyset$ and $\lambda \geq 0$. As $v \in 0^+(C_i)$ $(i \in I)$, we have $\lambda v + x \in C_i$ for all $i \in I$, i.e. $v \in 0^+(\bigcap_{i\in I} C_i)$, which proves the assertion.

b) Let $v \in \bigcup_{i\in I} C_i^\infty$, i.e. there exists $j \in I$ such that $v \in C_j^\infty$. Hence, there exists $\{x_k \in C_j\}$ and $\{\lambda_k\} \downarrow 0$ such that $\lambda_k x_k \to v$. But as $x_k \in \bigcup_{i\in I} C_i$ for all $k \in \mathbb{N}$, we infer that $v \in \left[\bigcup_{i\in I} C_i\right]^\infty$, which proves the first inclusion.

   Now, assume that I is finite and take $v \in \left[\bigcup_{i\in I} C_i\right]^\infty$: Then there exists $\{x_k \in \bigcup_{i\in I} C_i\}$ and $\{\lambda_k\} \downarrow 0$ such that $\lambda_k x_k \to v$. As I is finite there exists $j \in I$ and an infinite subset $K \subset \mathbb{N}$ such that $x_k \in C_j$ for all $k \subset K$. This shows that $v \in C_j^\infty \subset \bigcup_{i\in I} C_i^\infty$, which concludes the proof.

□

We now want to study how horizon cones come into play in the preservation of closedness if certain operations are applied to closed sets.

A critical question that appears in many applications is when the linear image of a closed set is again closed. Using the horizon cone of the set in question, we can establish very handy sufficient conditions for this to hold.

Theorem 2.4.26 (Closedness of linear images of closed sets) *Let $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ and $C \subset \mathbb{E}_1$ be closed. If*

$$\ker L \cap C^\infty = \{0\} \tag{2.10}$$

then $L(C)$ is closed. In particular, this holds if L is injective or C is bounded (hence compact).

Proof: We first show that under the assumption (2.10), every sequence $\{L(x_k)\}$ is unbounded whenever $\{x_k \in C\}$ is unbounded: Suppose this were false, then there exists $\{x_k \in C\}$ that is unbounded, but $\{L(x_k)\}$ is bounded. In particular, we can assume that

$$\frac{x_k}{\|x_k\|} \to \bar{x} \in C^\infty \setminus \{0\}.$$

On the other hand, by linearity of L, we also have

$$L(\bar{x}) = \lim_{k\to\infty} \frac{1}{\|x_k\|} L(x_k) = 0,$$

since $\{L(x_k)\}$ was assumed to be bounded. Hence, $0 \neq \bar{x} \in \ker L \cap C^\infty$, which contradicts (2.10).

We now continue with the actual statement that $L(C)$ is closed: To this end, let $\{u_k \in L(C)\} \to u$, i.e. $u_k = L(x_k)$ with $x_k \in C$ for all $k \in \mathbb{N}$. As $\{L(x_k)\}$ converges it is bounded, hence, by what was just proven, $\{x_k\}$ is bounded, too. Hence, it has a cluster point $\bar{x} \in C$ (as C is closed), which by continuity fulfills $L(\bar{x}) = u$, hence $u \in L(C)$. This concludes the proof.

□

The following example shows that condition (2.10) is in fact crucial in that the conclusions of Theorem 2.4.26 can already fail without it in the simplest cases of a linear projection.

Example 2.4.27 Consider $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x) = (x_1, 0)$ and $C := \left\{ x \in \mathbb{R}^2 \mid x_1 x_2 = 1 \right\}$. Then we have $C^\infty = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$, hence $C^\infty \cap \ker T = \{0\} \times \mathbb{R}$, therefore (2.10) is violated. Moreover, $T(C) = \mathbb{R} \times \{0\} \setminus \{(0,0)\}$, which is not closed.

Proposition 2.4.28 (Horizon cones of cartesian products) *Let $C_1, \dots, C_p \subset \mathbb{E}$. Then*

$$\left(\mathop{\times}_{i=1}^{p} C_i \right)^\infty \subset \mathop{\times}_{i=1}^{p} C_i^\infty.$$

Equality holds under either of the following conditions:

i) C_i is nonempty and convex for all $i = 1, \dots, p$;

ii) There exists at most one $i_0 \in \{1, \dots, p\}$ such that C_{i_0} is unbounded.

Proof: Exercise 2.20. □

We close the section with a sufficient condition for the closedness of the Minkowski sum of closed sets, which siginifcantly generalizes Exercise1.7. b).

Corollary 2.4.29 (Minkowski sums and horizon cones) *Let $C_1, \dots, C_p \subset \mathbb{E}$ be closed. If the implication*

$$x_i \in C_i^\infty \ (i = 1, \dots, p) : \sum_{i=1}^p x_i = 0 \quad \Rightarrow \quad x_i = 0 \ (i = 1, \dots, p) \tag{2.11}$$

holds true then $\sum_{i=1}^p C_i$ is closed and

$$\left(\sum_{i=1}^n C_i \right)^\infty \subset \sum_{i=1}^p C_i^\infty.$$

This holds as an equation if the sets $C_1, \dots, C_p$ are nonempty and convex or if at most one of them is unbounded.

Proof: Consider $T \in \mathcal{L}(\mathbb{E}^p, \mathbb{E}), T(x_1, \dots, x_p) = \sum_{i=1}^p x_i$ applied to $\bigtimes_{i=1}^p C_i$ and use Proposition 2.4.28 and Theorem 2.4.26. □

2.5 Projection onto closed, convex sets

In this section we are concerned with the optimization problem

$$\inf_{v \in C} \frac{1}{2} \|v - x\|^2 \tag{2.12}$$

where $C \subset \mathbb{E}$ is nonempty and $x \in \mathbb{E}$.

Lemma 2.5.1 *Let $x \in \mathbb{E}$ and $C \subset \mathbb{E}$. The the following hold:*

a) If C is closed then (2.12) *has a solution.*

b) If C is convex, a solution of (2.12) *is unique.*

Proof: Put $f = \frac{1}{2} \|(\cdot) - x\|^2 + \delta_C$.

a) As f is lsc (as C is closed) and level-bounded, the assertion follows from Theorem 1.2.6.

b) Assume that $v_1, v_2 \in \operatorname{argmin}_C \frac{1}{2}\|(\cdot) - x\|^2$. Then we have $\bar{v} := \frac{v_1+v_2}{2} \in C$, by convexity of C. A short computation shows that

$$f(\bar{v}) = \frac{1}{2}\left[f(v_1) + f(v_2)\right] - \frac{1}{8}\|v_1 - v_2\|^2,$$

hence, necessarily, $v_1 = v_2$.

□

Lemma 2.5.1 justifies the following definition.

Definition 2.5.2 (Projection onto closed convex set) *Let $C \subset \mathbb{E}$ be nonempty, closed and convex. Then the mapping $P_C : \mathbb{E} \to C$ defined by*

$$P_C(x) := \operatorname*{argmin}_C \frac{1}{2}\|(\cdot) - x\|^2$$

is called the projection on C.

Needless to say that $x = P_C(x)$ if and only if $x \in C$. The following important theorem provides a variational characterization of the projection mapping.

Theorem 2.5.3 (Projection Theorem) *Let $C \subset \mathbb{E}$ nonempty, closed and convex and let $x \in \mathbb{E}$. Then $\bar{v} = P_C(x)$ if and only if*

$$\bar{v} \in C \quad \textit{and} \quad \langle \bar{v} - x,\, v - \bar{v} \rangle \geq 0 \quad (v \in C). \tag{2.13}$$

Proof: First, assume that $\bar{v} = P_C(x)$, i.e. $\bar{v}$ minimizes $g := \frac{1}{2}\|(\cdot) - x\|^2$ over C. Since, $\nabla g(\bar{v}) = \bar{v} - x$, Theorem 2.4.17 yields

$$\langle \bar{v} - x,\, v - \bar{v} \rangle \geq 0 \quad (v \in C).$$

In order to see the converse implication, let $\bar{v} \in \mathbb{E}$ such that (2.13) holds. For $v \in C$ we hence obtain

$$\begin{aligned} 0 &\geq \langle x - \bar{v},\, v - \bar{v} \rangle \\ &= \langle x - \bar{v},\, v - x + x - \bar{v} \rangle \\ &= \|x - \bar{v}\|^2 + \langle x - \bar{v},\, v - x \rangle \\ &\geq \|x - \bar{x}\|^2 - \|x - \bar{v}\| \cdot \|v - x\|, \end{aligned}$$

where the last inequality is due to the Cauchy-Schwarz inequality. As $v \in C$ was chosen arbitrarily, this yields

$$\|x - \bar{v}\| \leq \|x - v\| \quad (v \in C)$$

i.e. $\bar{v} = P_C(x)$. $\square$

The connection between the projection on a closed convex set and the normal cone operator associated with that set was already implictly used in the proof of the Projection Theorem. We make it even more apparent through the next result.

Corollary 2.5.4 *Let $C \subset \mathbb{E}$ nonempty, closed and convex. Then*

$$P_C = (N_C + \mathrm{id}\,)^{-1}.$$

Proof: Let $x \in \mathbb{E}$. Then it holds that

$$\begin{aligned} \bar{v} = P_C(x) \quad &\Leftrightarrow \quad \bar{v} \in C \quad \text{and} \quad \langle x - \bar{v},\, v - \bar{v}\rangle \leq 0 \quad \forall v \in C \\ &\Leftrightarrow \quad \bar{v} \in C \quad \text{and} \quad x - \bar{v} \in N_C(\bar{v}). \end{aligned}$$

This proves the assertion. $\square$

The characterization of P_C from (2.13) takes a special form if C is a cone.

Proposition 2.5.5 (Projection on a closed, convex cone) *Let $K \subset \mathbb{E}$ be a closed, convex cone and $x \in K$. Then we have*

$$\bar{v} = P_K(x) \quad \Longleftrightarrow \quad \bar{v} \in K, \quad x - \bar{v} \in K^\circ \quad \textit{and} \quad \langle x - \bar{v},\, \bar{v}\rangle = 0.$$

Proof: '$\Rightarrow$': From Theorem 2.5.3 we know that $\bar{v} = P_K(x) \in K$ satisfies

$$\langle x - \bar{v},\, y - \bar{v}\rangle \leq 0 \quad (y \in K). \tag{2.14}$$

Taking $y = \alpha\bar{v}$ for some $\alpha \geq 0$, hence $y \in K$, this implies

$$(\alpha - 1)\,\langle x - \bar{v},\, \bar{v}\rangle \leq 0 \quad (\alpha \geq 0).$$

Since $\alpha - 1$ can take either sign, this yields $\langle x - \bar{v},\, \bar{v}\rangle = 0$. Thus, (2.14) becomes

$$\langle y,\, x - \bar{v}\rangle \leq 0 \quad (y \in K)$$

i.e. $x - \bar{v} \in K^\circ$. This shows one implication.

'$\Leftarrow$': Let $\bar{v}$ satisfy $\bar{v} \in K$, $x - \bar{v} \in K^\circ$ and $\langle x - \bar{v},\, \bar{v}\rangle = 0$. Then, for any $y \in K$, it tholds that

$$\begin{aligned} \frac{1}{2}\|x - y\|^2 \quad &= \quad \frac{1}{2}\|(x - \bar{v}) + (\bar{v} - y)\|^2 \\ &\geq \quad \frac{1}{2}\|x - \bar{v}\|^2 + \langle x - \bar{v},\, \bar{v} - y\rangle \end{aligned}$$

$$\overset{\langle x-\bar v,\bar v\rangle=0}{=} \quad \frac{1}{2}\|x-\bar v\|^2 - \langle x-\bar v,\, y\rangle$$
$$\overset{x-\bar v\in K^\circ}{\geq} \quad \frac{1}{2}\|x-\bar v\|^2,$$

i.e. $\bar v = P_K(x)$.

□

The following properties of the projection on closed, convex cones follow immediately from the characeterization in Proposition 2.5.5.

Corollary 2.5.6 *Let $K \subset \mathbb{E}$ be a closed, convex cone. Then the following hold:*

a) $P_K(x) = 0$ *if and only if* $x \in K^\circ$*;*

b) $P_K(\alpha x) = \alpha P_K(x) \quad (\alpha \geq 0)$ *(positive homogeneity);*

c) $P_K(-x) = -P_{-K}(x)$.

We close the section with the famous *Moreau decomposition Theorem* . It generalizes the familiar decomposition of $\mathbb{E}$ by means of the orthogonal complement operation: Recall that, for a subspace $U \subset \mathbb{E}$, we have

$$\mathbb{E} = U + U^\perp,$$

i.e. for $x \in \mathbb{E}$ there exists a (unique) decomposition $x = u + u'$ with $u \in U$ and $u' \in U^\perp$. It can be shown that $P_U(x) = u$, see Exercise 2.22.

Theorem 2.5.7 (Moreau decomposition) *Let $K \subset \mathbb{E}$ be a closed, convex cone and $x \in \mathbb{E}$. Then the following are equivalent:*

i) $x = u + v$ *with* $u \in K$, $v \in K^\circ$ *and* $\langle u,\, v\rangle = 0$*;*

ii) $u = P_K(x)$ *and* $v = P_{K^\circ}(x)$.

Proof: 'i)⇒ii)': Follows immediately from the characterization in Proposition 2.5.5.
'ii)⇒i)': Let $u = P_K(x)$ and $v = P_{K^\circ}(x)$. By Proposition 2.5.5, $u \in K$, $x - u \in K^\circ$ and $\langle x-u,\, u\rangle = 0$. Put $w := x - u$. We already know that $w \in K^\circ$. Moreover,

$$x - w = u \in K \subset K^{\circ\circ}.$$

In addition,

$$\langle x-w,\, w\rangle = \langle u,\, x-u\rangle = 0.$$

Hence, $w = P_{K^\circ}(x) = v$ (by Proposition 2.5.5), $u + v = x$ and $\langle u,\, v\rangle = 0$, which proves the desired implication. □

2.6 Separation of convex sets

We commence with a key observation, which is a simple consequence of the projection theorem.

Theorem 2.6.1 (Basic separation theorem) *Let $C \subset \mathbb{E}$ be nonempty, closed and convex, and let $x \notin C$. Then there exists $s \in \mathbb{E}$ with*

$$\langle s,\, x\rangle > \sup_{v\in C} \langle s,\, v\rangle\,.$$

Proof: Put $s := x - P_C(x) \neq 0$. Then the projection theorem yields

$$0 \geq \langle s,\, v - x + s\rangle = \langle s,\, v\rangle - \langle s,\, x\rangle + \|s\|^2 \quad \forall v \in C.$$

Thus,

$$\langle s,\, x\rangle - \|s\|^2 \geq \langle s,\, v\rangle \quad \forall v \in C,$$

hence, s fulfills the requirements of the theorem. □

We would like to note some technical things about the basic separation theorem, in order to refer to it later explicitly.

Remark 2.6.2

a) The vector s can always be substituted for $-s$ and thus, under the same assumptions, there exists $s \in \mathbb{E}$ such that $\langle s,\, x\rangle < \inf_{v\in C} \langle s,\, v\rangle$.

b) By positive homogeneity, we can assume w.l.o.g. that $\|s\| = 1$.

We still have not lost a word on why Theorem 2.6.1 is even called *separation* theorem: Let $s, x \in \mathbb{E}$ be the vectors and $C \subset \mathbb{E}$ be the closed, convex set from Theorem 2.6.1 and consider the hyperplanes $H_{s,\gamma}$ $(\gamma \in \mathbb{R})$. Putting

$$\bar{\gamma} := \frac{1}{2}\left(\langle s,\, x\rangle + \sup_{v\in C} \langle s,\, v\rangle\right)$$

we realize that

$$\langle s,\, x\rangle > \bar{\gamma} \quad \text{and} \quad \langle s,\, y\rangle < \bar{\gamma} \quad (y \in C),$$

i.e. $\{s\} \subset H^{>}_{s,\bar{\gamma}}$ and $C \subset H^{<}_{s,\bar{\gamma}}$. Geometrically, that means that the convex set $\{x\}$ and C are fully contained in opposite (open) half-spaces associated with $H_{s,\bar{\gamma}}$.

We are going to make a principle out of that.

Definition 2.6.3 (Separation of convex set) *Let $C_1, C_2 \subset \mathbb{E}$ nonempty and convex. A hyperplane $H := H_{b,\gamma}$ is said to* separate *C_1 and C_2*

- weakly *if $C_i \subset H^{\geq}$ and $C_j \subset H^{\leq}$ (for $i \neq j$);*
- properly *if H seperates them weakly and not both sets are actually contained in H itself.*
- strongly *if there exists an $\varepsilon > 0$ such that*

$$C_i + \varepsilon\mathbb{B} \subset H^{<} \quad \textit{and} \quad C_j + \varepsilon\mathbb{B} \subset H^{>} \quad (i \neq j).$$

Needless to say that stong separation implies proper separation which, in turn, implies weak separation.

Lemma 2.6.4 (Characterization of separation concepts) *Let $C_1, C_2 \subset \mathbb{E}$ be nonempty and convex. Then the following hold:*

a) There exists a hyperplane $H \subset \mathbb{E}$ that separates C_1 and C_2 properly if and only if there exists $s \in \mathbb{E}$ such that

$$\sup_{v \in C_1} \langle s,\, v\rangle \leq \inf_{v \in C_2} \langle s,\, v\rangle \quad \textit{and} \quad \inf_{v \in C_1} \langle s,\, v\rangle < \sup_{v \in C_2} \langle s,\, v\rangle \tag{2.15}$$

b) There exists a hyperplane $H \subset \mathbb{E}$ that separates C_1 and C_2 strongly if and only if there exists $s \in \mathbb{E}$ such that

$$\sup_{v \in C_1} \langle s,\, v\rangle < \inf_{v \in C_2} \langle s,\, v\rangle\,. \tag{2.16}$$

Proof:

a) Suppose s satisifies (2.15) and choose any $\gamma \in \left]\inf_{C_1} \langle s,\, \cdot\rangle, \sup_{C_2} \langle s,\, \cdot\rangle\right[$. Putting $H := H_{s,\gamma}$, we see that $C_1 \subset H^{\geq}$, while $C_2 \subset H^{\leq}$. The strict inequality in (2.15) implies that C_1 and C_2 are not actually both contained in H.
Conversely, if C_1 and C_2 can be separated properly by $H = H_{s,\gamma}$ in that $C_1 \subset H^{\geq}$ and $C_2 \subset H^{\leq}$ and not both are contained in H itself. Then clearly

$$\sup_{v \in C_2} \langle s,\, v\rangle \leq \gamma \leq \inf_{v \in C_1} \langle s,\, v\rangle\,,$$

and the strict inequality in (2.15) must hold since otherwise C_1 and C_2 were both contained in H.

b) If s satsifies the stronger condition (2.16), we can actually choose γ and $\delta > 0$ such that

$$\gamma + \delta \leq \langle s,\, v\rangle \quad (v \in C_1) \quad \text{and} \quad \gamma - \delta \geq \langle s,\, v\rangle \quad (v \in C_2).$$

Since the unit ball $\mathbb{B}$ is bounded (and $y \mapsto \langle s,\, y\rangle$ is continuous), $\varepsilon > 0$ can be chosen so small that $|\langle y,\, s\rangle| < \delta$ for all $y \in \varepsilon\mathbb{B}$. Thus,

$$C_1 + \varepsilon\mathbb{B} \subset H^{>}_{s,\gamma} \quad \text{and} \quad C_2 + \varepsilon\mathbb{B} \subset H^{<}_{s,\gamma},$$

i.e. $H_{s,\gamma}$ separates C_1 and C_2 strongly.
The converse inplication is straightforward.

□

Theorem 2.6.5 (Strong separation of convex sets) *Let $C_1, C_2 \subset \mathbb{E}$ be nonempty, closed and convex sets with $C_1 \cap C_2 = \emptyset$ and $C_1^\infty \cap C_2^\infty = \{0\}$. Then there exists $s \in \mathbb{E}$ such that*

$$\sup_{v \in C_1} \langle s,\, v\rangle < \inf_{v \in C_2} \langle s,\, v\rangle$$

i.e. C_1 and C_2 can be strongly separated (by a hyperplane).

Proof: First note that the assumption $C_1 \cap C_2 = \emptyset$ is equivalent to saying that $0 \notin C_1 - C_2$. The set $C_1 - C_2$ is convex, see Proposition 2.1.2 c). Moreover, as $C_1^\infty \cap C_2^\infty = \{0\}$, the set $C_1 - C_2$ is also closed, cf. Corollary 2.4.29. All in all, $C_1 - C_2$ is a closed, convex set with $0 \notin C_1 - C_2$. Hence, the basic separation theorem yields an $s \in \mathbb{E}$ such that

$$\begin{aligned} 0 &= \langle s,\, 0\rangle \\ &> \sup_{v \in C_1 - C_2} \langle s,\, v\rangle \\ &= \sup_{v_1 \in C_1, v_2 \in C_2} \{\langle s,\, v_1\rangle + \langle s,\, -v_2\rangle\} \\ &= \sup_{v_1 \in C_1} \langle s,\, v_1\rangle + \sup_{v_2 \in C_2} \langle s,\, -v_2\rangle \\ &= \sup_{v_1 \in C_1} \langle s,\, v_1\rangle - \inf_{v_2 \in C_2} \langle s,\, v_2\rangle\,, \end{aligned}$$

which concludes the proof. □

Corollary 2.6.6 *Let $C_1, C_2 \subset \mathbb{E}$ be nonempty, closed and convex sets with $C_1 \cap C_2 = \emptyset$. If C_2 is bounded, there exists $s \in \mathbb{E}$ such that*

$$\sup_{v \in C_1} \langle s,\, v\rangle < \min_{v \in C_2} \langle s,\, v\rangle\,.$$

In particular C_1 and C_2 can be strongly separated (by a hyperplane).

Proof: The assumptions yield that C_1 and C_2 are disjoint with no $C_1^\infty \cap C_2^\infty = \{0\}$. Hence, Theorem 2.6.5 yields the assertion.

□

We continue with the important concept of *supporting hyperplanes*.

Definition 2.6.7 (Supporting hyperplane) *An (affine) hyperplane H is said to* support *a set $S \subset \mathbb{E}$ if S is entirely contained in one of the two closed half-spaces associated with H.*

It is said to support S *at a point $x \in S$ if, in addition, $x \in H$.*

We say that $H = H_{b,\gamma}$ supports S nontrivially *(at x) if $b \notin V^\perp$ for V being the subspace parallel to* $\operatorname{aff} S$.

Note that a nontrivial supporting hyperplane H of C guarantees that $\operatorname{aff} C \subset H$ (which would be a somewhat meaningless achievement) does *not* hold, cf. Exercise 2.24..

The next result guarantees the existence of nontrivial supporting hyperplanes at every relative boundary point of a convex set.

Proposition 2.6.8 (Existence of nontrivial supporting hyperplanes) *Let $C \subset \mathbb{E}$ convex and $x \in \operatorname{rbd} C$ (and $C \neq \mathbb{E}$). Then there exists a nontrivial supporting hyperplane of C at x.*

Proof: Let V be the subspace parallel to $\operatorname{aff} C$, i.e. $\operatorname{aff} C = x + V$. Now define $D := C - x$. Then D is convex with $0 \in \operatorname{rbd} D$ and $D \subset \operatorname{aff} D \subset V$. Hence, there exists $\{x_k \in V \setminus \operatorname{cl} D\} \to 0$. For each $k \in \mathbb{N}$ we apply the basic separation theorem to x_k and $\operatorname{cl} D$ in the (sub)space V. Hence, for all $k \in \mathbb{N}$, there exists $s_k \in V$ with

$$\|s_k\| = 1 \quad \text{and} \quad \langle s_k,\, x_k - v\rangle > 0 \quad (v \in \operatorname{cl} D).$$

W.l.o.g. $s_k \to s$ with $s \neq 0$, thus passing to the limit above gives $\langle s,\, v\rangle \leq 0$ for all $v \in \operatorname{cl} D$. Hence, in particular,

$$\langle s,\, w\rangle \leq \langle s,\, x\rangle \quad (w \in C).$$

Putting $b := s$ and $\gamma := \langle b,\, x\rangle$, we find that $H_{b,\gamma}$ is a nontrivial supporting hyperplane that has the desired properties.

□

The existence of nontrivial supporting hyperplanes is the key for the our main theorem on proper separation.

Theorem 2.6.9 (Proper separation of convex sets) *Let $C_1, C_2 \in \mathbb{E}$ be nonempty and convex such that*

$$\operatorname{ri} C_1 \cap \operatorname{ri} C_2 = \emptyset. \tag{2.17}$$

Then there exists $s \in \mathbb{E}$ such that

$$\sup_{v \in C_1} \langle s,\, v\rangle \leq \inf_{v \in C_2} \langle s,\, v\rangle \quad \textit{and} \quad \inf_{v \in C_1} \langle s,\, v\rangle < \sup_{v \in C_2} \langle s,\, v\rangle\,,$$

i.e. C_1 and C_2 can be properly separated (by a hyperplane).

Proof: Consider the convex set $C = C_1 - C_2$. By Corollary 2.3.16 and 2.3.17, we have $\mathrm{ri}\, C = \mathrm{ri}\, C_1 - \mathrm{ri}\, C_2$, hence the assumption $\mathrm{ri}\, C_1 \cap \mathrm{ri}\, C_2 = \emptyset$ translates to $0 \notin \mathrm{ri}\, C$. If even $0 \notin \mathrm{cl}\, C$, we can stronlgy separate 0 from $\mathrm{cl}\, C$, hence from C (basic separation), which, by an analogous reasoning as in the proof of Theorem 2.6.5 gives strong separation of C_1 and C_2.

Otherwise, we have $0 \in \mathrm{rbd}\, C$ and by Proposition 2.6.8 there exists a nontrivial supporting hyperplane $H = H_{s,\gamma}$ $(s \neq 0)$ of C at 0, i.e.

$$\langle s, v \rangle \geq \gamma \quad (v \in C) \quad \text{and} \quad \langle s, 0 \rangle = \gamma$$

In particular, $\gamma = 0$. We infer that

$$\inf_{v \in C} \langle s, v \rangle \geq 0 \quad \text{and} \quad \sup_{v \in C} \langle s, v \rangle > 0,$$

where the strict inequality is due to the fact that H is nontrivial supporting hyperplane hence, $C \not\subset H$. As $C = C_1 - C_2$ this gives

$$\inf_{v \in C_1} \langle s, v \rangle \geq \sup_{v \in C_2} \langle s, v \rangle \quad \text{and} \quad \sup_{v \in C_1} \langle s, v \rangle > \inf_{v \in C_2} \langle s, v \rangle .$$

□

One of the most frequent versions of proper separation of a convex set and a point on its relative boundary is stated in the next result.

Proposition 2.6.10 *Let $C \subset \mathbb{E}$ be nonempty and convex and let $\bar{x} \in \mathrm{rbd}\, C$. Then the following are equivalent (and hence hold by Theorem 2.6.9):*

i) (Proper separation) There exists $s \neq 0$ such that

$$\langle s, \bar{x} \rangle \leq \inf_{v \in C} \langle s, v \rangle \quad \textit{and} \quad \langle s, \bar{x} \rangle < \sup_{v \in C} \langle s, v \rangle .$$

ii) There exists $s \neq 0$ such that

$$\langle s, \bar{x} \rangle < \langle s, v \rangle \quad (v \in \mathrm{ri}\, C).$$

Proof: The implication 'ii)⇒i)' is obvious, hence we only need to prove the converse implication. Suppose there exists $\bar{v} \in \mathrm{ri}\, C$ such that $\langle s, \bar{x} \rangle = \langle s, \bar{v} \rangle$. Since by assumption $\langle s, \bar{x} \rangle \leq \inf_{v \in C} \langle s, v \rangle$, this gives $\bar{v} \in \mathrm{argmin}_C \langle s, \cdot \rangle$. Then Exercise 2.8. implies that $\langle s, \cdot \rangle$ is constant on C, which is a contradiction to $\langle s, \bar{x} \rangle < \sup_{v \in C} \langle s, v \rangle$, hence such a $\bar{v}$ cannot exist.

□

2.7 First consequences of the separation theorems

2.7.1 Envelope representation of closed convex sets

Theorem 2.7.1 (Envelope representation of closed convex sets) *Let $C \subset \mathbb{E}$ be closed and convex. Then C is the intersection of all closed half-spaces containing it.*

Proof: We can assume that $\emptyset \neq C \neq \mathbb{E}$, since otherwise the statement is apparently true. Thus, there exists $x \notin C$, and by Theorem 2.6.1 (which is just a special case of Theorem 2.6.5) there exists a hyperplane H^x separating the closed convex sets $\{x\}$ and C strongly. In particular, one of the closed half-spaces associated with H^x contains C but not x. Since, $x \notin C$ was chosen arbitrarily, intersecting over all $x \notin C$ does not contain any other points than those of C. This gives the assertion, as the intersection over all closed half-spaces containing C contains C, but is potentially only smaller than the former intersection. $\square$

Since a closed half-space contains an arbitrary set if and only if it contains its closed convex hull the next result follows immediately.

Corollary 2.7.2 (Envelope representation of the closed convex hull) *Let $S \subset \mathbb{E}$. Then $\overline{\mathrm{conv}}\, S$ is the intersection of all closed half-spaces containing S.*

2.7.2 Farkas Lemma and Karush-Kuhn-Tucker conditions

Proposition 2.7.3 (Farkas Lemma) *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$. Then the following statemenst are equivalent:*

i) The polyhedron $P = \left\{x \in \mathbb{R}^m \;\middle|\; A^T x = b,\ x \geq 0\right\}$ is nonempty.

ii) We have $b^T d \geq 0$ for all $d \in \mathbb{R}^n$ with $Ad \geq 0$.

Proof: 'i)⇒ii)': Take $x \in P$ and d such that $Ad \geq 0$. Then

$$b^T d = (A^T x)^T d = x^T (Ad) \geq 0,$$

which proves the first implication.
'ii)⇒i)': (Contraposition) Suppose i) does not hold. Then $b \in \mathbb{R}^n$ is not an element of the closed, convex cone $K = \left\{A^T x \mid x \geq 0\right\} \subset \mathbb{R}^n$. Hence, by basic separation, we find $s \in \mathbb{R}^n$ such that

$$s^T b < s^T y \quad (y \in K).$$

From this we infer that

$$s^T b < 0 \leq s^T y \quad (y \in K). \tag{2.18}$$

Here, the first inequality is due to the fact that $0 \in K$, and the second one must hold since K is a cone, i.e. $\lambda y \in K$ for any $y \in K$ and $\lambda \geq 0$. Choosing $y = a_i = A^T e_i \in K \ (i = 1, \dots, m)$ (the columns of A^T) in (2.18), yields

$$s^T b < 0 \leq s^T a_i \quad (i = 1, \dots, m),$$

i.e. $As \geq 0$ but $s^T b < 0$, i.e. ii) does not hold, which concludes the proof. □

Consider the standard *nonlinear program (NLP)*

$$\min \ f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0, \tag{2.19}$$

for $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ continuously differentiable. We denote by X the feasible set of (2.19), i.e.

$$X = \{x \in \mathbb{R}^n \mid g(x) \leq 0, \ h(x) = 0\}.$$

Theorem 2.7.4 (Karush-Kuhn Tucker Theorem) *Let $\bar{x} \in \mathbb{R}^n$ be a local minimizer of* (2.19) *such that*

$$T_X(\bar{x}) = \left\{ d \in \mathbb{R}^n \;\middle|\; \nabla g_i(\bar{x})^T d \leq 0 \ (i : g_i(\bar{x}) = 0), \nabla h_j(\bar{x}) = 0 \ (j = 1, \dots, p) \right\}. \tag{2.20}$$

Then there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ such that

$$0 = \nabla f(\bar{x}) + \lambda^T g'(\bar{x}) + \mu^T h'(\bar{x}) \quad \textit{and} \quad \lambda^T g(\bar{x}) = 0.$$

Proof: Put $I := \{i \in \{1, \dots, m\} \mid g_i(x) = 0\}$ and $J := \{1, \dots, p\}$. Since $\bar{x}$ is a local minimizer of f over X, we know from Theorem 2.4.17 that

$$\nabla f(\bar{x})^T d \geq 0 \quad (d \in T_X(\bar{x})). \tag{2.21}$$

Putting

$$A := \begin{bmatrix} -\nabla g_i(\bar{x})^T & (i \in I) \\ -\nabla h_j(\bar{x})^T & (j \in J) \\ \nabla h_j(\bar{x})^T & (j \in J) \end{bmatrix} \in \mathbb{R}^{(|I|+2p) \times n}$$

we infer from (2.21) and the assumed representation of $T_X(\bar{x})$ that

$$\nabla f(\bar{x})^T d \geq 0 \quad \text{if} \quad Ad \geq 0.$$

The Farkas Lemma then yields a vector $\begin{pmatrix} \lambda \\ \mu_+ \\ \mu_- \end{pmatrix} \in \mathbb{R}^{|I|+2p}_+$ such that $A^T \begin{pmatrix} \lambda \\ \mu_+ \\ \mu_- \end{pmatrix} = \nabla f(\bar{x})$. Putting $\mu := \mu_+ - \mu_-$ and $\lambda_i = 0 \ (i \notin I(\bar{x}))$, we get the desired result. □

2.7.3 Minkowski's Theorem

Definition 2.7.5 *Let $C \subset \mathbb{E}$ be convex. A point $x \in C$ is said to be an* extreme point *if the following implication holds true for all $x_1, x_2 \in S$:*

$$\frac{1}{2}(x_1 + x_2) = x \quad \Rightarrow \quad x_1 = x_2.$$

The set of all extreme points of S is denoted by $\operatorname{ext} S$.

There are several ways of characterizing the fact that x is an extreme point of a set convex set $C \subset \mathbb{E}$, see Exercise 2.25.

These characterizations imply that every extreme point is, in fact, on the relative boundary of a convex set.

Example 2.7.6 (Extreme points)

a) Using the identity

$$\frac{1}{2}\|x + y\|^2 = \|x\|^2 + \|y\|^2 - \frac{1}{2}\|x - y\|^2, \quad (x, y \in \mathbb{E}) \tag{2.22}$$

we realize that $\operatorname{ext} \overline{\mathbb{B}} = \operatorname{bd} \mathbb{B}$.

b) If K is a convex cone, $\operatorname{ext} K = \{0\}$.

c) Nontrivial affine sets or half-spaces have no extrem points.

A given set does not need to have extreme points, but compactness is sufficent to guarantee their existence.

Proposition 2.7.7 *Let $S \subset \mathbb{E}$ be nonempty and compact. Then* $\operatorname{ext} S \neq \emptyset$.

Proof: Since S is compact there exists $\bar{x} \in \operatorname{argmax}_S \|\cdot\|^2$. We claim that $\bar{x} \in \operatorname{ext} S$: Indeed, suppose there are $x_1, x_2 \in S$ such that $\frac{1}{2}(x_1 + x_2) = \bar{x}$. Assuming that $x_1 \neq x_2$ we obtain (using (2.22))

$$\|\bar{x}\|^2 = \left\|\frac{1}{2}(x_1 + x_2)\right\|^2 < \frac{1}{2}\left(\|x_1\|^2 + \|x_2\|^2\right) \leq \|\bar{x}\|^2,$$

where the last inquality uses that $\bar{x}$ maximizes $\|\cdot\|^2$ over C. Altogether, this is a contradiction, hence $x_1 = x_2$, i.e. $\bar{x}$ is an extreme point of C.

□

Our ultimate goal in this section is to show that a nonempty convex and compact set is the convex hull of its extreme points. To this end, we first prove an auxiliary result.

Lemma 2.7.8 *Let H be a supporting hyperplane of the convex set $C \subset \mathbb{E}$. Then* $\mathrm{ext}\,(C \cap H) \subset \mathrm{ext}\, C$.

Proof: We can assume that $C \subset H^{\geq}$. Now, let $\bar{x} \in \mathrm{ext}\,(C \cap H)$ and asumme that $\bar{x} = \frac{1}{2}(x+y)$ with $x, y \in C \subset H^{\geq}$. If $x \in H^{>}$ or $y \in H^{>}$, we have $\bar{x} \in H^{>}$, which contradicts the choice of $\bar{x}$. Hence, $x, y \in H$, thus $x = y$, which proves the statement. $\square$

Theorem 2.7.9 (Minkowski's Theorem) *Let $C \subset \mathbb{E}$ be nonempty, convex and compact. Then* $C = \mathrm{conv}\,(\mathrm{ext}\, C)$.

Proof: We prove the statement by induction over $\dim C$: The result is certainly true if $\dim C = 0$, i.e. when C is a singleton. Now, assume that the statement holds true for all convex sets of dimension smaller than $\dim C$. Now, take $x \in \mathrm{rbd}\, C$. By Proposition 2.6.8 there exists a (nontrivial) hyperplane H supporting C at x. By the induction hypothesis and Lemma 2.7.8 we obtain

$$x \in H \cap C = \mathrm{conv}\,(\mathrm{ext}\,(C \cap H)) \subset \mathrm{conv}\,(\mathrm{ext}\, C).$$

Thus, we have shown that $\mathrm{rbd}\, C \subset \mathrm{conv}\,(\mathrm{ext}\, C)$, hence

$$\mathrm{conv}\,(\mathrm{rbd}\, C) \subset \mathrm{conv}\,(\mathrm{ext}\, C) \subset C.$$

But, since C is compact, we have $C = \mathrm{conv}\,(\mathrm{rbd}\, C)$ (cf. Exercise 2.7.), hence, in fact, $C = \mathrm{conv}\,(\mathrm{ext}\,(C \cap H))$.

$\square$

Exercises to Chapter 2

2.1. **(Convex hull)** Let $M \subset \mathbb{E}$ be nonempty. Show that

$$\mathrm{conv}\, M = \left\{ \sum_{i=1}^{r} \lambda_i x_i \mid r \in \mathbb{N},\ \lambda \in \Delta_r,\ x_i \in M\ (i = 1, \ldots, r) \right\}.$$

Hint: Proceed analogous to the proof of the characterization of the affine hull.

2.2. **(Closed convex hull of bounded sets)** Let $S \subset \mathbb{E}$ be nonempty and bounded. Show that $\overline{\mathrm{conv}}\, S = \mathrm{conv}\,(\mathrm{cl}\, S)$.

2.3. **(Convex hulls)** Let $F : \mathbb{E} \to \mathbb{E}'$ be affine and $A, C \subset \mathbb{E}$ and $B \subset \mathbb{E}'$ nonempty. Prove the following:

a) $\mathrm{conv}\, F(A) = F(\mathrm{conv}\, A)$;

b) $\operatorname{conv}(A \times B) = \operatorname{conv} A \times \operatorname{conv} B$;

c) $\operatorname{conv}(A + C) = \operatorname{conv} A + \operatorname{conv} C$.

2.4. **(Spectahedron)** For $n \in \mathbb{N}$ compute $\operatorname{conv}\left\{uu^T \in \mathbb{S}^n \mid u \in \mathbb{R}^n : \|u\| = 1\right\}$.

2.5. **(Characterization of relative interior points)** Let $C \subset \mathbb{E}$ be nonempty and convex and $x \in C$. Show that the following are equivalent:

i) $x \in \operatorname{ri} C$;

ii) $\mathbb{R}_+(C - x)$ is a subspace of $\mathbb{E}$.

2.6. **(Relative interiors and cartesian products)** For $i = 1, \dots, p$ let $C_i \subset \mathbb{E}_i$. Show that

$$\operatorname{ri}\left(\bigtimes_{i=1}^{p} C_i\right) = \bigtimes_{i=1}^{p} \operatorname{ri} C_i.$$

2.7. **(Conex hull and relative boundary)** Let $C \subset \mathbb{E}$ be nonempty, convex and compact. Show that $\operatorname{conv}(\operatorname{rbd} C) = C$.

2.8. **(Minimizers of a linear functional over a convex set)** Let $C \subset \mathbb{E}$ be nonempty and let $s \in \mathbb{E}$. Show that either

$$\operatorname*{argmin}_{C} \langle s, \cdot\rangle \subset \operatorname{rbd} C \quad \text{or} \quad \langle s, x\rangle = const. \quad \forall x \in \operatorname{aff} C.$$

2.9. **(Open mapping theorem - finite dimensional version)** Let $A \subset \mathbb{E}$ be open and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$ surjective. Then $L(A)$ is open.

2.10. **(Pointedness of convex cones)** Let $K \subset \mathbb{E}$ be a convex cone. Show that K is pointed if and only if $K \cap (-K) = \{0\}$.

2.11. **(Convex cones vs. subspaces)** Let $K \subset \mathbb{E}$ be a convex cone. Show that:

a) $K \cap (-K)$ is the largest subspace that is contained in K;

b) $K - K$ is the smallest subspace containing K, i.e. $K = K = \operatorname{span} K$.

c) K is a subspace if and only if $K = -K$.

2.12. **(Characterizing the conical hull)** Let $S \subset \mathbb{E}$ be nonempty. Prove that

$$\operatorname{cone} S = \left\{\sum_{i=1}^{r} \lambda_i x_i \mid r \in \mathbb{N}, x_i \in S, \lambda_i \geq 0\ (i = 1, \dots, r)\right\} = \mathbb{R}_+(\operatorname{conv} S) = \operatorname{conv}(\mathbb{R}_+ S).$$

2.13. **(Polar of subspace)** Let $U \subset \mathbb{E}$ be a subspace. Show that $U^\circ = U^\perp$.

2.14. **(Polar cone and normal vectors)** Let $C \subset \mathbb{E}$ be nonempty. Then it holds that

$$(\overline{\text{cone}}\, C)^\circ = \{w \in \mathbb{E} \mid \langle w,\, x\rangle \leq 0 \ (x \in C)\}\,.$$

2.15. **(Polar of a polyhedral cone)** For $p \in \mathbb{N}$ let $a_1, \dots, a_m \in \mathbb{E}$ and define

$$P := \{x \in \mathbb{E} \mid \langle a_i,\, x\rangle \leq 0 \ (i = 1, \dots, p)\}\,.$$

Show that

$$P^\circ = \overline{\text{cone}}\,\{a_1, \dots, a_m\}.$$

Hint: Exercise 2.23..

2.16. **(Second-order cone)** Let $K := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 \leq t\}$.

a) Show that K is a closed, convex cone which is self-dual.

b) For $x \in K$ compute $T_K(x)$ and $N_K(x)$.

Hint: In b) it can simplify things to use Exercise 2.23..

2.17. **(Tangent cone of closed cones)** Let $K \subset \mathbb{E}$ be a closed cone. Prove:

a) $T_K(0) = K$;

b) $\mathbb{R}\{\bar{x}\} \subset T_K(\bar{x})$ for all $\bar{x} \in K$.

2.18. **(Tangent and normal cone to smooth manifolds)** Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable and $\bar{x} \in C := F^{-1}(\{0\})$ such that $\operatorname{rank} F'(\bar{x}) = m$. Then the following hold:

a) $T_C(\bar{x}) = \ker F'(\bar{x})$;

b) $N_C(\bar{x}) = \operatorname{rge} F'(\bar{x})^T$.

2.19. **(Linear images of horizon cones)** Let $T \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ and let $C \subset \mathbb{E}$ be closed such that $\ker T \cap C^\infty = \{0\}$. Show that $T(C^\infty) = T(C)^\infty$.

2.20. **(Horizon cone of cartesian products)** Let $C_1, \dots, C_p \in \mathbb{E}$. Show that

$$\left(\bigtimes_{i=1}^{p} C_i\right)^\infty \subset \bigtimes_{i=1}^{p} C_i^\infty.$$

Moreover, prove that equality holds under either of the following conditions:

i) C_i is nonempty and convex for all $i = 1, \dots, p$;

ii) There exists at most one $i_0 \in \{1, \dots, p\}$ such that C_{i_0} is unbounded.

2.21. **(Necessary optimality conditions for convex constraints)** Let $f : \mathbb{E} \to \mathbb{R}$ be continuously differentiable and $C \subset \mathbb{E}$ closed and convex and let $L > 0$. Show that the following are equivalent:

i) $\langle \nabla f(\bar{x}),\, x - \bar{x} \rangle \geq 0 \quad (x \in C)$;

ii) $P_C(\bar{x} - \frac{1}{L}\nabla f(\bar{x})) = \bar{x}$.

2.22. **(Projection on subspaces)** Let $U \subset \mathbb{E}$ be a subspace. Then every $x \in \mathbb{E}$ has a unique decomposition $x = u + u'$ with $u \in U$ and $u' \in U^{\perp}$. Show that $P_U(x) = u$ and infer that P_U is linear.

2.23. **(Bipolar theorem for cones)** Let $K \subset \mathbb{E}$ be a cone. Then $K^{\circ\circ} = \overline{\operatorname{conv}}\, K$.

2.24. **(Nontrivial supporting hyperplanes)** Let $C \subset \mathbb{E}$ be a nonempty (convex) set and let H be a nontrivial supporting hyperplane of C at $x \in C$. Show that $\operatorname{aff} C \not\subset H$.

2.25. **(Extreme points)** Let $C \subset \mathbb{E}$ be convex. Show that for $x \in C$ the following are equivalent:

i) x is an extreme point of C;

ii) $x = \lambda x_1 + (1 - \lambda)x_2$ with $x_1, x_2 \in C$ and $\lambda \in [0, 1]$ implies $x = x_1 = x_2$;

iii) $x = \sum_{i=1}^{r} \lambda_i x_i$ for some $r \in \mathbb{N}$ and $x_i \in C$ $(i = 1, \dots, r)$, $\lambda \in \Delta_r$ implies $x_i = x$ for all $i = 1, \dots, r$;

iv) $C \setminus \{x\}$ is convex.

2.26. **(Linear functionals and extreme points)** Let C be convex and compact and $s \in \mathbb{E}$. Show that

$$\max_{C} \langle s,\, \cdot \rangle = \max_{\operatorname{ext} C} \langle s,\, \cdot \rangle \quad \text{and} \quad \operatorname*{argmax}_{C} \langle s,\, \cdot \rangle = \operatorname{conv}\left(\operatorname*{argmax}_{\operatorname{ext} C} \langle s,\, \cdot \rangle \right).$$

2.27. **(Converse Minkowski Theorem)** Let C be convex and compact and let $D \subset C$ such that $\overline{\operatorname{conv}}\, D = C$. Show that $\operatorname{ext} C \subset \operatorname{cl} D$.

3 Convex Functions

3.1 Convexity notions for functions and basic properties

We start the chapter with the basic definition of a convex function.

Definition 3.1.1 (Convex function) *A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is said to be* convex *if* $\mathrm{epi}\, f$ *is a convex set.*

Note that in the above definition we could have substitued the epigraph for the *strict epigraph* $\mathrm{epi}_{<} f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) < \alpha\}$ of f, see Exercise 3.3.. Moreover, note that convex functions have convex level sets, see Exercise 3.6.

Recall that the domain of a function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is defined by $\mathrm{dom}\, f := \{x \in \mathbb{E} \mid f(x) < \infty\}$. Using the linear mapping

$$L : (x, \alpha) \in \mathbb{E} \times \mathbb{R} \mapsto x \in \mathbb{E}, \tag{3.1}$$

we have $\mathrm{dom}\, f = L(\mathrm{epi}\, f)$, and hence Proposition 2.1.2 yields the following immediate but important result.

Proposition 3.1.2 (Domain of a convex function) *The domain of a convex function is convex.*

Recall that a (convex) function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is proper if $\mathrm{dom}\, f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{E}$.

Improper convex functions are somewhat pathological (cf. Exercise 3.4.), but they do occur; rather as by-products then as primary objects of study. For example the function

$$f : x \in \mathbb{R} \mapsto \begin{cases} -\infty & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| = 1, \\ +\infty & \text{if} \quad |x| > 1. \end{cases}$$

is improper and convex.
Convex functions have an important interpolation property, which we summarize in the next result for the case that f does not take the value $-\infty$.

Proposition 3.1.3 (Characterizing convexity) *A function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if for all $x, y \in \mathbb{E}$ we have*

$$f(\lambda x + (1 - \lambda)y) \leq (1 - \lambda)f(x) + \lambda f(y) \quad (\lambda \in [0, 1]). \tag{3.2}$$

Proof: First, let f be convex. Take $x, y \in \mathbb{E}$ and $\lambda \in [0,1]$. If $x \notin \operatorname{dom} f$ or $y \notin \operatorname{dom} f$ the inequality (3.2) holds trivially, since the right-hand side is going to be $+\infty$. If, on the other hand, $x, y \in \operatorname{dom} f$, then $(x, f(x)), (y, f(y)) \in \operatorname{epi} f$, hence by convexity

$$(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)) \in \operatorname{epi} f,$$

i.e. $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, which proves the first implication.

In turn, let (3.2) hold for all $x, y \in \mathbb{E}$. Now, take $(x, \alpha), (y, \beta) \in \operatorname{epi} f$ and let $\lambda \in [0,1]$. Due to (3.2) we obtain

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda\alpha + \lambda\beta,$$

i.e. $\lambda(x, \alpha) + (1-\lambda)(y, \beta) \in \operatorname{epi} f$, which shows the converse implication.

□

We move the analogous characterization of convexity for functions $\mathbb{E} \to \overline{\mathbb{R}}$ to Exercise 3.3., because these kinds of functions are not our primary object of study.

The next result is an extension of Proposition 3.1.3, which can be seen in various ways.

Corollary 3.1.4 (Jensen's Inequality) *A function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if*

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) \leq \sum_{i=1}^{p} \lambda_i f(x_i) \quad (x_i \in \mathbb{E}\ (i = 1, \dots, p),\ \lambda \in \Delta_p)$$

Proof: Exercise 3.5. □

It is sometimes expedient to consider convexity of a function restricted to a subset of its domain.

Definition 3.1.5 (Convexity on a set) *For a nonempty convex set $C \subset \operatorname{dom} f$, we call $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$* convex on C *if* (3.2) *holds for all $x, y \in C$.*

Corollary 3.1.6 *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the following are equivalent.*

i) f is convex.

ii) f is convex on its domain.

Proof: The implication' i)⇒ii)' is obvious from the characterization of convexity in Proposition 3.1.3

For the converse implication note that (3.2) always holds for any pair of points x, y if one of them is not in the domain.

This completes the proof.

□

Remark 3.1.7 As an immediate consequence of Corollary 3.1.6, we can make the following statement about proper, convex functions:
"The proper and convex functions $\mathbb{E} \to \mathbb{R}\cup\{+\infty\}$ are those for which there exists a nomempty, convex set $C \subset \mathbb{E}$ such that (3.2) *holds on C and f takes the value $+\infty$ outside of C."*

We are mainly interested in proper, convex (even lsc) functions $\mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Hence, we introduce the abbreviations

$$\Gamma := \Gamma(\mathbb{E}) := \{f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\} \mid f \text{ proper and convex}\}$$

and

$$\Gamma_0 := \Gamma_0(\mathbb{E}) := \{f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\} \mid f \text{ proper, lsc and convex}\}$$

which we will use frequently in the remainder.
Ever so often some stronger notions of convexity are needed, which we establish now.

Definition 3.1.8 (Strict and strong convexity) *Let f be proper and convex and $C \subset \operatorname{dom} f$ convex. Then f is said to be*

- strictly convex *on C if*

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y) \quad (x, y, \in C, x \neq y,\ \lambda \in (0,1)).$$

- strongly convex *on C if there exists $\sigma > 0$ such that*

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)-\frac{\sigma}{2}\lambda(1-\lambda)\|x-y\|^2 \quad (x, y, \in C,\ \lambda \in (0,1))$$

The scalar $\sigma > 0$ is called modulus of strong convexity *of f (on C).*

For $C = \operatorname{dom} f$ we simply call f strictly and strongly convex, respectively.

Proposition 3.1.9 (Characterization of strong convexity) *Let f be proper and convex and $C \subset \operatorname{dom} f$. Then f is strongly convex on C with modulus $\sigma > 0$ if and only if $f - \frac{\sigma}{2}\|\cdot\|^2$ is convex on C.*

Proof: First, let f be strongly convex on C with modulus $\sigma > 0$. Then for any $\lambda \in (0,1)$ and $x, y \in C$ we have

$$\begin{aligned}
&f(\lambda x + (1-\lambda)y) - \frac{\sigma}{2}\|\lambda x + (1-\lambda)y\|^2 \\
\leq\ & \lambda f(x) + (1-\lambda)f(y) - \frac{\sigma}{2}\left(\lambda(1-\lambda)\|x-y\|^2 + \|\lambda x + (1-\lambda)y\|^2\right) \\
\leq\ & \lambda\left(f(x) - \frac{\sigma}{2}\|x\|^2\right) + (1-\lambda)\left(f(y) - \frac{\sigma}{2}\|y\|^2\right),
\end{aligned}$$

i.e. $f - \frac{\sigma}{2}\|\cdot\|^2$ is convex on C.
If, in turn, $f - \frac{\sigma}{2}\|\cdot\|^2$ is convex on C we compute that

$$\begin{aligned} f(\lambda x + (1-\lambda y)) &\leq \lambda f(x) + (1-\lambda)f(y) + \frac{\sigma}{2}\left(\|\lambda x + (1-\lambda y)\|^2 - \lambda\|x\|^2 - (1-\lambda)\|y\|^2\right) \\ &= \lambda f(x) + (1-\lambda)f(y) - \frac{\sigma}{2}\lambda(1-\lambda)\|x-y\|^2 \end{aligned}$$

for all $x, y \in C$, i.e. f is strongly convex on C with modulus $\sigma > 0$. ☐

We stop our analysis for a short list of obvious convex functions. In Section 3.1.1 we learn how to build new convex functions from old ones.

Example 3.1.10 (Examples of convex functions)

a) (Affine functions) Every affine function $F : \mathbb{E} \to \mathbb{R}$ is convex.

b) (Indicator of convex sets) For a set $C \subset \mathbb{E}$ its indicator function δ_C is convex if and only if C is convex.

c) (Norms) Any Norm $\|\cdot\|_*$ on $\mathbb{E}$ is convex.

3.1.1 Functional operations preserving convexity

Proposition 3.1.11 (Positive combinations of convex functions) *For $p \in \mathbb{N}$ let $f_i : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be convex (and lsc) and $\alpha_i \geq 0$ for $i = 1, \dots, p$. Then*

$$\sum_{i=1}^{p} \alpha_i f_i$$

is convex (and lsc). If, in addtion, $\cap_{i=1}^{p} \mathrm{dom}\, f_i \neq \emptyset$, then f is also proper.

Proof: The convexity assertion is an immediate consequence of the characterization in (3.2). For the additional closedness see Exercise 1.12. The properness statement is obvious. ☐

Note that the latter result tells us that Γ and Γ_0 are convex cones.

Proposition 3.1.12 (Pointwise supremum of convex functions) *For an arbitrary index set I let f_i be convex (and lsc) for all $i \in I$. Then the function $f = \sup_{i \in I} f_i$, i.e.*

$$f(x) = \sup_{i \in I} f_i(x) \quad (x \in \mathbb{E})$$

is convex (and lsc).

Proof: It holds that

$$\operatorname{epi} f = \left\{(x,\alpha) \;\middle|\; \sup_{i\in I} f_i(x) \le \alpha\right\} = \{(x,\alpha) \mid \forall i \in I: f_i(x) \le \alpha\} = \bigcap_{i\in I} \operatorname{epi} f_i.$$

Since the intersection of (closed) convex sets it (closed) convex, this gives the assertion. $\square$

Proposition 3.1.13 (Pre-composition with and affine mapping) *Let $H : \mathbb{E}_1 \to \mathbb{E}_2$ be affine and $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ (lsc and) convex. Then the function $f := g \circ H$ is (lsc and) convex.*

Proof: Let $x, y \in \mathbb{E}_1$ and $\lambda \in (0,1)$. Then we have

$$f(\lambda x + (1-\lambda x)) = g(\lambda H(x) + (1-\lambda)y) \le \lambda g(H(x)) + (1-\lambda)g(H(y)) = \lambda f(x) + (1-\lambda) f(y),$$

which gives the convexity of f. The closedness of f, under the closedness of g, follows from the continuity (as a consequence of affineness) of H, cf. Exercise 1.13. $\square$

Proposition 3.1.14 (Post-composition with monotonically increasing, convex functions) *Let f be convex (and lsc) and let $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be convex (and lsc) and increasing. Under the convention $g(+\infty) := +\infty$ and $\lim_{x\to\infty} g(x) = +\infty$, the function $g \circ f$ is convex (and lsc). If in addition, there exists x_0 such that $f(x_0) \in \operatorname{dom} g$, then $g \circ f$ is proper.*

Proof: Exercise 3.8. $\square$

Proposition 3.1.15 (Convexity under epi-composition) *Let $f \in \Gamma$ and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$. Then the function $Lf : \mathbb{E}' \to \overline{\mathbb{R}}$ defined by*

$$(Lf)(y) := \inf\{f(x) \mid L(x) = y\}$$

is convex.

Proof: We first show that , with $T : (x,\alpha) \mapsto (Lx, \alpha)$, we have

$$\operatorname{epi}_< Lf = T(\operatorname{epi}_< f). \tag{3.3}$$

To this end, recall that

$$\operatorname{epi}_< Lf = \{(y,\alpha) \mid Lf(y) < \alpha\} \quad \text{and} \quad \operatorname{epi}_< f = \{(x,\alpha) \mid f(x) < \alpha\}.$$

First, let $(x, \alpha) \in \operatorname{epi}_< f$. Then $T(x, \alpha) = (L(x), \alpha)$ and

$$(Lf)(L(x)) = \inf_z \{f(z) \mid L(z) = L(x)\} \le f(x) < \alpha,$$

thus, $T(x, \alpha) \in \operatorname{epi}_< Lf$.

In turn, if $(y, \alpha) \in \operatorname{epi}_< Lf$, i.e. $\inf \{f(z) \mid L(z) = y\} < \alpha$, then $L^{-1}(y) \neq \emptyset$, hence, there exists $x \in L^{-1}(y)$ with $f(x) < \alpha$. Thus, we have $T(x, \alpha) = (y, \alpha)$ and $(x, \alpha) \in \operatorname{epi}_< f$. This proves (3.3).

Now, as f is convex, $\operatorname{epi}_< f$ is convex (see Exercise 3.3.). But, since T is linear, from (3.3) it follows that also $\operatorname{epi}_< Lf$ is convex, which proves the convexity of Lf.

□

3.1.2 Differentiable convex functions

We want to apply the notion of differentiability to extended-real valued functions. This only makes sense at points for which there exists a whole neighborhood on which the function in question is at least finitely valued, i.e. at points in the interior of the domain:

For $f : \mathbb{E} \to \overline{\mathbb{R}}$ we say that f is differentiable at $x \in \operatorname{int}(\operatorname{dom} f)$ if f restricted to $\operatorname{int}(\operatorname{dom} f)$ is differentiable at x. Stronger notions of differentiability are defined accordingly.

Convexity of differentiable functions can be handily characterized.

Theorem 3.1.16 (First-order characterizations) *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be differentiable on a convex, open set $C \subset \operatorname{int}(\operatorname{dom} f)$. Then the following hold:*

a) f is convex on C if and only if

$$f(x) \ge f(\bar{x}) + \langle \nabla f(\bar{x}),\, x - \bar{x} \rangle \quad (x, \bar{x} \in C). \tag{3.4}$$

b) f is strictly convex on C if and only if (3.4) *holds with strict inequality whenever $x \neq \bar{x}$.*

c) f is strongly convex with modulus $\sigma > 0$ on C if and only if

$$f(x) \ge f(\bar{x}) + \langle \nabla f(\bar{x}),\, x - \bar{x} \rangle + \frac{\sigma}{2}\|x - \bar{x}\|^2 \quad (x, \bar{x} \in C). \tag{3.5}$$

Proof:

a) First, let f be convex and take $x, \bar{x} \in C$ and $\lambda \in (0, 1)$. Then by convexity it holds that

$$f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) \le \lambda(f(x) - f(\bar{x})).$$

As f is differentiable on C, dividing by λ and letting $\lambda \to 0$ gives

$$\langle \nabla f(\bar{x}),\, x - \bar{x} \rangle \le f(x) - f(\bar{x}),$$

which establishes (3.4).

In turn, if (3.4) holds, we take $x_1, x_2 \in C$, $\lambda \in (0,1)$ and put $\bar{x} := \lambda x_1 + (1-\lambda)x_2 \in C)$. By (3.4) it follows that

$$f(x_i) \geq f(\bar{x}) + \langle \nabla f(\bar{x}),\, x_i - \bar{x} \rangle \quad (i = 1, 2).$$

Multiplying these two inequalities by λ and $(1-\lambda)$, respectively, summation of the resulting inequalities yields

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\bar{x}) + \langle \nabla f(\bar{x}),\, \lambda x_1 + (1-\lambda)x_2 - \bar{x} \rangle = f(\lambda x_1 + (1-\lambda)x_2).$$

As x_1, x_2 were taken arbitrarily from C, f is convex on C.

b) If f is strictly convex on C, for $x, \bar{x} \in C, x \neq \bar{x}$ and $\lambda \in (0,1)$, we have

$$f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) < \lambda(f(x) - f(\bar{x})).$$

In addition, since f is, in particular, convex, part a) implies that

$$\langle \nabla f(\bar{x}),\, \lambda(x - \bar{x}) \rangle \leq f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}).$$

Combining these inequalities gives the desired strict inequality.

The converse implications is proven analogously to the respective implication in a), starting from the strict inequality.

c) Using Proposition 3.1.9, applying part a) to $f - \frac{\sigma}{2}\|\cdot\|^2$ gives the assertion.

$\square$

Theorem 3.1.16 opens the door for another characterization of convexity of differentiable functions on open sets in terms of so-called *monotonicity properties* of the gradient mapping.

Before we prove it we would like to recall the reader of the *chain rule* for differentiable functions.

For $i = 1, 2$ let $\Omega_i \in \mathbb{E}_i$ be open. If $f : \Omega_1 \subset \mathbb{E}_1 \to \mathbb{E}_2$ is differentiable at $\bar{x} \in \Omega_1$ and $g : \Omega_2 \to \mathbb{E}_3$ is differentiable at $f(\bar{x}) \in \Omega_2$, then $g \circ f : \Omega_1 \to \mathbb{E}_3$ is differentiable at $\bar{x}$ with

$$(g \circ f)'(\bar{x}) = g'(f(\bar{x})) \circ f'(\bar{x}).$$

Corollary 3.1.17 (Monotonicity of gradient mappings) *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be differentiable on the open set $\Omega \subset \operatorname{int}(\operatorname{dom} f)$ and let $C \subset \Omega$ be convex. Then the following hold:*

a) f *is convex on* C *if and only if*

$$\langle \nabla f(x) - \nabla f(y),\, x - y\rangle \geq 0 \quad (x, y \in C). \tag{3.6}$$

b) f *is strictly convex on* C *if and only if* (3.6) *holds with a strict inequality whenever* $x \neq y$.

c) f *is strongly convex with modulus* $\sigma > 0$ *on* C *if and only if*

$$\langle \nabla f(x) - \nabla f(y),\, x - y\rangle \geq \sigma \|x - y\|^2 \quad (x, y \in C). \tag{3.7}$$

Proof: We are first going to show one direction in c) and a), respectively: To this end, first, let f be strongly convex with modulus $\sigma > 0$ on C. Hence, by Theorem 3.1.16 c), for $x, y \in C$, we obtain

$$f(x) \geq f(y) + \langle \nabla f(y),\, x - y\rangle + \frac{\sigma}{2}\|x - y\|^2$$

and

$$f(y) \geq f(x) + \langle \nabla f(x),\, y - x\rangle + \frac{\sigma}{2}\|x - y\|^2.$$

Adding these two inequalities yields (3.7), which shows one implication in c). Setting $\sigma = 0$ gives the same implication in a).

We now show the converse directions in a) and c): For these purposes, let $x, y \in C$ be given, and consider the function

$$\varphi : I \to \mathbb{R}, \quad \varphi(t) := f(x + t(y - x)).$$

with I an open interval containing $[0, 1]$. We put $x_t := x + t(y - x) \in C$ for all $t \in [0, 1]$ and realize that φ is differentiable on I with $\varphi'(t) = \langle \nabla f(x_t),\, y - x\rangle$ for all $t \in [0, 1]$ (chain rule). Hence, we obtain

$$\varphi'(t) - \varphi'(s) = \langle \nabla f(x_t) - \nabla f(x_s),\, y - x\rangle = \frac{1}{t - s}\langle \nabla f(x_t) - \nabla f(x_s),\, x_t - x_s\rangle \tag{3.8}$$

for all $0 \leq s < t \leq 1$.

If (3.6) holds, this implies that φ' is nondecreasing on $[0, 1]$, hence φ is convex on $(0, 1)$, cf. Exercise 3.2., i.e. f is convex on (x, y). Since $x, y \in C$ were chosen arbitrarily this implies that f is actually convex on C.

For the strong convexity, set $s := 0$ in (3.8) and use

$$\varphi'(t) - \varphi'(0) \geq \frac{\sigma}{t}\|x_t - x\|^2 = t\sigma\|y - x\|^2. \tag{3.9}$$

Integrating and exploiting the definition of φ then yields

$$f(y) - f(x) - \langle \nabla f(x),\, y - x\rangle \quad = \quad \varphi(1) - \varphi(0) - \varphi'(0)$$

$$
\begin{aligned}
&= \int_0^1 \varphi'(t) - \varphi'(0)\ dt \\
&\geq \int_0^1 t\sigma \|y - x\|^2\ dt \\
&= \frac{\sigma}{2}\|y - x\|^2,
\end{aligned}
$$

which gives (3.5) for x, y. As they were chosen arbitrarily in C, f is strongly convex on C by Theorem 3.1.16 c).

The same technique of prove gives part b), where (3.9) becomes a strict inequality with $\sigma = 0$, and remains strict after integration.

□

We now investigate convexity criteria for even twice differentiable functions.

Theorem 3.1.18 (Twice differentiable convex functions) *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be twice differentiable on the open convex set $\Omega \subset \mathrm{int}\,(\mathrm{dom}\, f)$. Then the following hold:*

a) f is convex on Ω if and only if $\nabla^2 f(x)$ is positive semidefinite for all $x \in \Omega$.

b) If $\nabla f(x)^2$ is positive definite for all $x \in \Omega$ then f is strictly convex on Ω.

c) f is strongly convex with modulus $\sigma > 0$ on Ω if and only if, for all $x \in \Omega$, the smallest eigenvalue of $\nabla^2 f(x)$ is bounded by σ from below.

Proof: Let $x \in \Omega, d \in \mathbb{E}$. Since Ω is open, the intervall $I := I(x, d) := \{t \in \mathbb{R} \mid x + td \in \Omega\}$ is open. We define

$$\varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi(t) := f(x + td). \tag{3.10}$$

Then, in particular, φ is twice differentiable on I with $\varphi''(t) = \left\langle \nabla^2 f(x + td)d,\, d \right\rangle$ for all $t \in I$.

a) First, assume that f is convex on Ω. Now, let $x \in \Omega$ and $d \in \mathbb{E} \setminus \{0\}$. Then φ from (3.10) is convex on I by Proposition 3.1.13. Using Exercise 3.2. it follows that

$$0 \leq \varphi''(t) = \left\langle \nabla^2 f(x + td)d,\, d \right\rangle,$$

which gives the first implication.

Conversely, take $x, y \in \Omega$ arbitrarily, put $d := y - x$ and assume that $\nabla^2 f(x + td)$ is positive semidefinite. Then for φ from (3.10) we have $\varphi''(t) \geq 0$ for all $t \in [0, 1] \subset I$. Therefore Exercise 3.2. tells us that φ is convex on $(0, 1)$, i.e. f is convex on (x, y). Since $x, y \in \Omega$ were chosen arbitrarily, f is convex on Ω.

b) Again take $x, y \in \Omega$ with $x \neq y$ and put $d := y - x$. Applying the mean-value theorem to the function φ', which is differentiable on $(0,1)$, yields some $\tau \in (0,1)$ such that

$$\langle \nabla f(y) - \nabla f(x),\, y - x\rangle = \varphi'(1) - \varphi' 0) = \varphi''(\tau) = \left\langle \nabla^2 f(x + \tau d)d,\, d\right\rangle > 0.$$

Corollary 3.1.17 then gives the assertion.

c) Using Proposition 3.1.9, we apply a) to the function $f - \frac{\sigma}{2}\|\cdot\|^2$, whose Hessian at $x \in \Omega$ is $\nabla^2 f(x) - \sigma I$ which has the eigenvalues $\lambda_i - \sigma$ with $\lambda_1, \dots, \lambda_N$ the eigenvalues of $\nabla^2 f(\bar{x})$. This gives the assertion, as a symmetric matrix is positive semidefinite if and only if all of its (real) eigenvalues are nonnegative.

□

Note that the condition in Theorem 3.1.18 b) is only sufficient for strict convexity. As an example that it is not necessary notice that $x \mapsto \frac{1}{4}x^4$ is strictly convex, but $f''(0) = 0$.

We continue with an example where we can succesfully apply a second-order criterion to detect convexity of an important function.

Example 3.1.19 (The log-determinant function) Consider the function

$$f : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}, \quad f(x) := \begin{cases} -\log(\det X) & \text{if} \quad X \succ 0, \\ +\infty & \text{else} \end{cases} \tag{3.11}$$

which we call the *(negative) log-determinant* or the *(negative) logdet* function, for short. Then f is proper, continuous and strictly convex, in particular, $f \in \Gamma_0(\mathbb{S}^n)$: The continuity of f is easily verified and as $\operatorname{dom} f = \mathbb{S}^n_{++}$, f is proper and twice differentiable on $\operatorname{dom} f$ with

$$\nabla f(X) = -X \quad \text{and} \quad \nabla^2 f(X) = X^{-1}(\cdot)X^{-1} \quad (X \succ 0),$$

see Example 1.1.5 and Exercise 1.6..

In particular, it holds for all $X \in \operatorname{dom} f$ and $H \in \mathbb{S}_n \setminus \{0\}$ that

$$\left\langle \nabla^2 f(X)(H),\, H\right\rangle = \operatorname{tr}\left(X^{-1}HX^{-1}H\right) = \operatorname{tr}\left((HX^{-1/2})^T X^{-1}(HX^{-1/2})\right) > 0,$$

as $X^{-1} \succ 0$ and hence also $HX^{-1/2} \neq 0$. Thus, by Theorem 3.1.18, f is strictly convex.

3.2 Minimization and convexity

We turn our attention to minimization problems of the form

$$\inf_{x \in C} f(x), \tag{3.12}$$

where $C \subset \mathbb{E}$ is nonempty and closed and $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ at least lsc. Note that (3.12) is equivalent to the problem

$$\inf_{x \in \mathbb{E}} f(x) + \delta_C(x),$$

a simple fact that we are going to exploit frequently. If $f \in \Gamma_0$ and C is convex, we call (3.12) a *convex minimization (optimization) problem.*

When talking about minimizers the questions for uniqueness and existence arise naturally. We start our study with some general existence results.

3.2.1 General existence results

Existence results tradionally employ *coercivity* properties of the objective function, and, more or less, do not depend too much on convexity.

Definition 3.2.1 (Coercivity and supercoercivity) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then f is called*

i) coercive *if*

$$\lim_{\|x\| \to +\infty} f(x) = +\infty;$$

ii) supercoercive *if*

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

The nomenclature for the above coercivitiy concepts is not unified in the literature. We use the same naming as in [1]. In [3], for instance, the authors use 0-*coercive* and 1-*coercive* for *coercive* and *supercoercive* instead.

In fact, we have already dealt with coercivity under a different moniker as the following result shows whose elementary proof is left to the reader as an exercise.

Lemma 3.2.2 (Level-boundedness = coercivity) *A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is coercive if and only if it is level-bounded.*

Proof: Exercise 3.7. □

In the lsc and convex case, coercivity is checked much easier.

Proposition 3.2.3 (Coercivity of convex functions) *Let $f \in \Gamma_0$. Then f is coercive if and only if there exists $\alpha \in \mathbb{R}$ such that $\mathrm{lev}_{\leq \alpha} f$ is nonempty and bounded.*

Proof: By Lemma 3.2.2, coercivity implies that all level sets are bounded and, as f is proper, there is a nonempty one, too.

In turn, assume that $\mathrm{lev}_{\leq \alpha} f$ is nonempty and bounded for some $\alpha \in \mathbb{R}$ and pick $x \in \mathrm{lev}_{\leq \alpha}$. Clearly, all level sets to levels smaller than α are bounded, too. Hence, we still need to show that $\mathrm{lev}_{\leq \gamma}$ is bounded for all $\gamma > \alpha$. To this end take $v \in (\mathrm{lev}_{\leq \gamma})^{\infty}$.

Since $\text{lev}_{\leq\gamma} f$ is closed and convex (as f is lsc and convex, cf. Proposition 1.2.4 and Exercise 3.6.), Corollary 2.4.24 yields

$$x + \lambda v \in \text{lev}_{\leq\gamma} f \quad (\lambda \geq 0). \tag{3.13}$$

Hence, for all $\lambda > 1$, it can be seen that

$$x + v = \left(1 - \frac{1}{\lambda}\right) x + \frac{1}{\lambda}(x + \lambda v),$$

and hence, by convexity and (3.13), we obtain

$$f(x+v) \leq \left(1 - \frac{1}{\lambda}\right) f(x) + \frac{1}{\lambda} f(x + \lambda v) \leq \left(1 - \frac{1}{\lambda}\right) f(x) + \frac{1}{\lambda}\gamma. \tag{3.14}$$

Letting $\lambda \to +\infty$ and recalling that $x \in \text{lev}_{\leq\alpha} f$, (3.14) gives

$$f(x+v) \leq f(x) \leq \alpha.$$

As $v \in (\text{lev}_\gamma f)^\infty$ was chosen arbitrarily, we infer that

$$x + (\text{lev}_{\leq\gamma} f)^\infty \subset \text{lev}_{\leq\alpha} f.$$

However, by the choice of α, $\text{lev}_{\leq\alpha} f$ is bounded, and hence, necessarily the cone $(\text{lev}_{\leq\gamma} f)^\infty$ is bounded, too. That leaves only $(\text{lev}_{\leq\gamma} f)^\infty = \{0\}$, and hence, by Proposition 2.4.21, $\text{lev}_{\leq\gamma} f$ is bounded, which completes the proof.

□

We now present the main existence result for minimization problems, which is, in fact, only a corollary to the existence result Theorem 1.2.6 using our new terminology and stating the constrained case explicitly.

Corollary 3.2.4 (Existence of minimizers) *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc and let $C \subset \mathbb{E}$ be closed such that* $\text{dom}\, f \cap C \neq \emptyset$ *and suppose that one of the following holds:*

i) f is coercive;

ii) C is bounded.

Then f has a minimizer over C.

Proof: Consider the function $g = f + \delta_C$. Then it holds that

$$\text{lev}_{\leq\alpha} g = C \cap \text{lev}_{\leq\alpha} f \quad (\alpha \in \mathbb{R}).$$

Hence, under either assumption i) (cf. Lemma 3.2.2) or ii), g has closed and bounded level-sets and is hence lsc and level-bounded. The assertion hence follows from Theorem 1.2.6. □

We now apply this result to the sum of functions:

Corollary 3.2.5 (Existence of minimizers II) *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *If f is coercive and g is bounded from below, then $f + g$ is coercive and has a minimizer (over $\mathbb{E}$).*

Proof: In view of Corollary 3.2.4 it suffices to show that $f + g$ is coercive, as it is already lsc, cf. Exercise 1.12. Putting $g^* := \inf_{\mathbb{E}} g > -\infty$, we see that

$$f(x) + g(x) \geq f(x) + g^* \to_{\|x\| \to \infty} +\infty,$$

which proves the result. □

3.2.2 Convex minimization

We now turn our attention to convex minimization problems: Recall the notion of global and local minimizers from Definition 2.4.16. It turns out that there is no distinction needed in the convex setting.

Proposition 3.2.6 *Let $f \in \Gamma$. Then every local minimizer of f (over $\mathbb{E}$) is a global minimizer.*

Proof: Let $\bar{x}$ be a local minimizer of f and suppose there exists $\hat{x}$ such that $f(\hat{x}) < f(\bar{x})$. Now let $\lambda \in (0, 1)$ and put $x_\lambda := \lambda\hat{x} + (1 - \lambda)\bar{x}$. By convexity, we have

$$f(x_\lambda) \leq \lambda f(\hat{x}) + (1 - \lambda)f(\bar{x}) < f(\bar{x}) \quad \forall \lambda \in (0, 1).$$

On the other hand, we see that $x_\lambda \to \bar{x}$ as $\lambda \downarrow 0$, which all in all contradicts the fact that $\bar{x}$ is a local minimizer of f. Hence, $\hat{x}$ cannot exist, which means that $\bar{x}$ is even a global minimizer of f.

□

Using our usual technique of casting a constrained optimization problem as an unconstrained problem by means of the indicator function of the constraint set, we immediately get the following result.

Corollary 3.2.7 (Minimizers in convex minimization) *Let $f \in \Gamma$ and $C \subset \mathbb{E}$ nonempty and convex. Then every local minimizer of f over C is a global minimizer of f over C.*

Proof: Apply Proposition 3.2.6 to the function $f + \delta_C$ which is convex by Proposition 3.1.11. □

The following results show that convex minimization problems have convex solution sets.

Proposition 3.2.8 *Let $f \in \Gamma$. Then* $\operatorname{argmin} f$ *is a convex set.*

Proof: If $f^* := \inf f \in \mathbb{R}$, we have that $\operatorname{argmin} f = \operatorname{lev}_{\leq f^*} f$. But as a convex function, f has convex level sets, cf. Exercise 3.6. □

We state the constrained case explicitly.

Corollary 3.2.9 *Let $f \in \Gamma$ and $C \subset \mathbb{E}$ convex. Then $\operatorname{argmin}_C f$ is convex.*

Proof: Apply Proposition 3.2.8 to $f + \delta_C \in \Gamma$. □

Uniqueness of minimizers of (convex) minimization problems comes into play with strict convexity, see Definition 3.1.8.

Proposition 3.2.10 (Uniqueness of minimizers) *Let $f \in \Gamma$ be strictly convex. Then f has at most one minimizer.*

Proof: Assume that $x, y \in \operatorname{argmin} f$, i.e. $\inf f = f(x) = f(y)$. If $x \neq y$, then strict convexity of f implies for all $\lambda \in (0,1)$ that

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y) = \inf f,$$

This is a contradiction, hence $x = y$. □

Corollary 3.2.11 (Minimizing the sum of convex functions) *Let $f, g \in \Gamma_0$ such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Suppose that one of the follwoing holds:*

i) f is supercoercive;

ii) f is coercive and g is bounded from below.

Then $f + g$ is coercive and has a minimizer (over $\mathbb{E}$). If f or g is strictly coercive, $f + g$ has exaclty one minimizer.

Proof: Since $f + g \in \Gamma_0$, $f + g$ is, in particular, lsc, hence for the first assertion, in view of Corollary 3.2.4, we only need to prove that $f + g$ is coercive in either of the cases i) or ii). If f is supercoercive, then $f + g$ is supercoercive by Exercise 3.9., hence in particular, coercive. In the second case, everything works also without convexity, see Corollary 3.2.5.

The uniqueness result follows immediately from Proposition 3.2.10, realizing that $f + g \in \Gamma_0$ is strictly convex if one of the summands is. □

We close out the section with a very powerful result on *optimal value functions* of parameter-dependent convex minimization problem.

Theorem 3.2.12 (Parametric minimization) *Let $h : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ be convex. Then the* optimal value function

$$\varphi : \mathbb{E}_1 \to \overline{\mathbb{R}},\ \varphi(x) := \inf_{y \in \mathbb{E}_2} h(x, y)$$

is convex. Moreover, the set-valued mapping

$$x \mapsto \operatorname*{argmin}_{y \in \mathbb{E}_2} h(x, y) \subset \mathbb{E}_2.$$

is convex-valued.

Proof: It can easily be shown that $\mathrm{epi}_{<}\varphi = L(\mathrm{epi}_{<}h)$ under the linear mapping $L : (x, y, \alpha) \mapsto (x, \alpha)$. This immediately gives the convexity of φ.

The remaining assertion follows immediately from Proposition 3.2.8, since $y \mapsto h(x, y)$ is convex for all $x \in \mathbb{E}_1$. □

3.3 Affine minorization of convex functions

In this section we will prove that every proper, convex function that does not take the value $-\infty$ is minorized by an affine mapping at every point of the relative interior of its domain, and this minorant can actually be chosen such that it coincides with the convex function at the point in question.

This result is a very useful tool for proofs involving convex functions and has tremendous consequences for subdifferential and duality theory of convex function as we will see later on.

For these purposes we need to study the relative interior of the epigraph of a convex function and how it is related to the relative interior of the domain of the function in question. Note that we can acutally speak of these relative interiors, since a convex function has (by definition) a convex epigraph and (see Proposition 3.1.2) also a convex domain.

Proposition 3.3.1 (Relative interior of epigraph) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be convex. Then*

$$\mathrm{ri}\,(\mathrm{epi}\, f) = \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid x \in \mathrm{ri}\,(\mathrm{dom}\, f), f(x) < \alpha\}.$$

Proof: Let L be the linear mapping from (3.1). By Proposition 2.3.15 we have

$$\mathrm{ri}\,(\mathrm{dom}\, f) = L(\mathrm{ri}\,(\mathrm{epi}\, f)). \tag{3.15}$$

Now, take $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$. For the subset of $\mathrm{ri}\,(\mathrm{epi}\, f)$ that is mapped to x under L, we compute

$$\begin{aligned} L^{-1}(\{x\}) \cap \mathrm{ri}\,(\mathrm{epi}\, f) &= (\{x\} \times \mathbb{R}) \cap \mathrm{ri}\,(\mathrm{epi}\, f) \\ &= \mathrm{ri}\,[(\{x\} \times \mathbb{R}) \cap \mathrm{epi}\, f] \end{aligned}$$

$$\begin{aligned} &= \text{ri}\,[\{x\} \times [f(x), +\infty)] \\ &= \{x\} \times (f(x), +\infty), \end{aligned}$$

where the third equality uses the fact that $\{x\} \times \mathbb{R}$ is relatively open and Proposition 2.3.14 b).

Thus, for $(x, \alpha) \in \text{ri}\,(\text{epi}\, f)$, we have $x \in \text{ri}\,(\text{dom}\, f)$ by (3.15), and hence $(x, \alpha) \in L^{-1}(\{x\}) \cap \text{ri}\,(\text{epi}\, f) = \{x\} \times (f(x), +\infty)$, in particular, $\alpha > f(x)$.

In turn, if $x \in \text{ri}\,(\text{dom}\, f)$ and $f(x) < \alpha$ then $(x, \alpha) \in \{x\} \times (f(x), +\infty) = L^{-1}(x) \cap \text{ri}\,(\text{epi}\, f)$, in particular, $(x, \alpha) \in \text{ri}\,(\text{epi}\, f)$.

□

Note that, by the description from (3.3.1), the relative interior of the epigraph of a given convex function f does not necessarily coincide with its *strict epigraph* $\{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) < \alpha\}$. We now come to the promised main theorem of this paragraph.

Theorem 3.3.2 (Affine minorization theorem) *Let $f \in \Gamma$ and $\bar{x} \in \text{ri}\,(\text{dom}\, f)$ ($\neq \emptyset$). Then there exists $g \in \mathbb{E}$ such that*

$$f(x) \geq f(\bar{x}) + \langle g,\, x - \bar{x} \rangle \quad (x \in \mathbb{E}). \tag{3.16}$$

In particular, there exists an affine mapping, namely

$$F : x \in \mathbb{E} \mapsto \langle g,\, x - \bar{x} \rangle + f(\bar{x}) \in \mathbb{R},$$

which minorizes f everywhere and coincides with f at $\bar{x}$.

Proof: By Proposition 3.3.1, we have $\text{ri}\,(\text{epi}\, f) = \{(x, \alpha) \mid x \in \text{ri}\,(\text{dom}\, f),\ f(x) < \alpha\}$. Hence, $(\bar{x}, f(\bar{x})) \in \text{rbd}\,(\text{epi}\, f)$. Thus, we can properly separate $(\bar{x}, f(\bar{x}))$ from epi f using Proposition 2.6.10, i.e. there exists $(s, \eta) \in (\mathbb{E} \times \mathbb{R}) \setminus \{0\}$ such that

$$\langle (s, \eta),\, (x, \alpha) \rangle \leq \langle (s, \eta),\, (\bar{x}, f(\bar{x})) \rangle \quad ((x, \alpha) \in \text{epi}\, f) \tag{3.17}$$

and

$$\langle (s, \eta),\, (x, \alpha) \rangle < \langle (s, \eta),\, (\bar{x}, f(\bar{x})) \rangle \quad ((x, \alpha) \in \text{ri}\,(\text{epi}\, f)).$$

For $\alpha > f(\bar{x})$ we have $(x, \alpha) := (\bar{x}, \alpha) \in \text{ri}\,(\text{epi}\, f)$, hence, the latter yields

$$\eta(\alpha - f(\bar{x})) < 0,$$

which immediately implies that $\eta < 0$.

Now put $g := \frac{s}{|\eta|}$. Dividing (3.17) by $|\eta|$ then yields

$$\langle (g, -1),\, (x, \alpha) \rangle \leq \langle (g, -1),\, (\bar{x}, f(\bar{x})) \rangle \quad ((x, \alpha) \in \text{epi}\, f)$$

or, equivalently,

$$\alpha \geq f(\bar{x}) + \langle g,\, x - \bar{x} \rangle \quad ((x, \alpha) \in \text{epi}\, f).$$

As f is proper, $(x, f(x)) \in \operatorname{epi} f$ for all $x \in \operatorname{dom} f$, thus

$$f(x) \geq f(\bar{x}) + \langle g, x - \bar{x} \rangle \quad (x \in \operatorname{dom} f).$$

For $x \notin \operatorname{dom} f$, this inequality holds trivially, hence the result is proven. $\square$

3.4 Infimal convolution of convex functions

Definition 3.4.1 (Infimal convolution) *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the function*

$$f\#g : \mathbb{E} \to \overline{\mathbb{R}}, \quad (f\#g)(x) := \inf_{u \in \mathbb{E}} \{f(u) + g(x - u)\}$$

is called the infimal convolution *of f and g. We call the infimal convolution $f\#g$* exact at $x \in \mathbb{E}$ *if*

$$\operatorname*{argmin}_{u \in \mathbb{E}} \{f(u) + g(x - u)\} \neq \emptyset.$$

We simply call $f\#g$ exact if it is a exact at every $x \in \operatorname{dom} f\#g$.

Observe that we have the representation

$$(f\#g)(x) = \inf_{u_1, u_2 : u_1 + u_2 = x} \{f(u_1) + g(u_2)\}. \tag{3.18}$$

This has some obvious, yet useful consequences.

Lemma 3.4.2 *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the following hold:*

a) $\operatorname{dom} f\#g = \operatorname{dom} f + \operatorname{dom} g$*;*

b) $f\#g = g\#f$.

Moreover, observe the trivial inequality

$$(f\#g)(x) \leq f(u) + g(x - u) \quad (u \in \mathbb{E}). \tag{3.19}$$

Infimal convolution preserves convexity, as can be seen in the next result.

Proposition 3.4.3 (Infimal convolution of convex functions) *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be convex. Then $f\#g$ is convex.*

Proof: Defining

$$h : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \cup \{+\infty\}, \quad h(x, y) := f(y) + g(x - y),$$

we see that h is convex (jointly in (x, y)) as a sum of the convex functions $(x, y) \mapsto f(y)$ and $(x, y) \mapsto g(x - y)$, the latter being convex by Proposition 3.1.13. By definition of the infimal convolution, we have

$$(f\#g)(x) = \inf_{y \in \mathbb{E}} h(x, y),$$

hence, Theorem 3.2.12 yields the assertion.

□

We continue with an important class of functions that can be constructed using infimal convolution, and that is intimately tied to projection mappings.

Example 3.4.4 (Distance functions) Let $C \subset \mathbb{E}$. Then the function $d_C := \delta_C \# \|\cdot\|$ is called the *distance (function)* to the set C. It holds that

$$\operatorname{dist}_C(x) = \inf_{u \in C} \|x - u\|.$$

Hence, from Lemma 2.5.1 it is clear that, if $C \subset \mathbb{E}$ is closed and convex, we have

$$\operatorname{dist}_C(x) = \|x - P_C(x)\|.$$

In order to preserve lower semicontinuity as well, it is not enough to simply assume that the the functions that are convoluted are lsc (and convex). Section 3.2.1, however, provides us with the necessary tools to deal with this issue.

Theorem 3.4.5 (Infimal convolution in $\mathbf{\Gamma_0}$) *Let $f, g \in \Gamma_0$ and suppose that one of the following conditions hold:*

i) f is supercoercive;

ii) f is coercive and g is bounded from below.

Then $f\#g \in \Gamma_0$ and is exact.

Proof: By Lemma 3.4.2, $\operatorname{dom} f\#g = \operatorname{dom} f + \operatorname{dom} g \neq \emptyset$. Now, take $x \in \operatorname{dom} f\#g$. Then, by the definition of $f\#g$, we have $\operatorname{dom} f \cap \operatorname{dom} g(x - (\cdot)) \neq \emptyset$. Hence, Corollary 3.2.11 implies that $f + g(x - (\cdot))$ has a minimizer. Thus, for all $x \in \operatorname{dom} f\#g$ there exists $u \in \mathbb{E}$ such that

$$(f\#g)(x) = f(u) + g(x - u) \in \mathbb{R}.$$

In particular, $f\#g$ is proper and exact. Since, by Proposition 3.4.3, $f\#g \in \Gamma$, it remains to be shown that $f\#g$ is lsc. For these purposes, let $\bar{x} \in \mathbb{E}$ and $\{x_k\} \to \bar{x}$ such that $(f\#g)(x_k) \to \alpha$.

We need to show that $\alpha \geq f(\bar{x})$, hence, w.l.o.g. we can assume that $\alpha < +\infty$ (since otherwise there is nothing prove), in particular, $x_k \in \mathrm{dom}\, f\#g$ for all $k \in \mathbb{N}$ (sufficiently large).

Then, by our recent findings, there exists $\{u_k \in \mathbb{E}\}$ such that

$$(f\#g)(x_k) = f(u_k) + g(x_k - u_k) \quad (k \in \mathbb{N}).$$

We claim that $\{u_k\}$ is bounded: Assume this were false, then (after passing to a subsequence if necessary) we have $0 \neq \|u_k\| \to +\infty$. We now show that under either of the assumptions i) and ii), respectively, this yields a contradiction:

i): By Theorem 3.3.2, there exists an affine minorant of g, say $x \mapsto \langle b,\, x\rangle + \gamma$. Using the Cauchy-Schwarz inequality , we have

$$\begin{array}{rcl} \|u_k\| \left(\dfrac{f(u_k)}{\|u_k\|} - \|b\| \right) + \langle b,\, x_k\rangle + \gamma & \leq & f(u_k) + \langle b,\, x_k - u_k\rangle + \gamma \\ & \leq & f(u_k) + g(x_k - u_k) \\ & = & (f\#g)(x_k) \\ & \to & \alpha \\ & < & +\infty. \end{array}$$

But, as f is supercoercive and we have (by assumption) that $\|u_k\| \to \infty$, the term on the left-hand side is unbounded from above, which is a contradiction, and hence, $\{u_k\}$ must be bounded.

ii): Since f is coercive, we have $f(u_k) \to +\infty$ under the assumption that $\|u_k\| \to +\infty$. But, since $f(u_k) + g(x_k - u_k) \to \alpha < +\infty$, we necessarily have $g(x_k - u_k) \to -\infty$, which is impossible if g is bounded from below, hence $\{u_k\}$ must be bounded.

All in all, we get in either case that $\{u_k\}$ is bounded and w.l.o.g. we can assume that $u_k \to u$. Relabeling the sequence $\{x_k\}$ if necessary, we obtain

$$\begin{array}{rcl} \alpha & = & \lim\limits_{k\to\infty} (f\#g)(x_k) \\ & = & \lim\limits_{k\to\infty} f(u_k) + g(x_k - u_k) \\ & \geq & \liminf\limits_{k\to\infty} f(u_k) + \liminf\limits_{k\to\infty} g(x_k - u_k) \\ & \geq & f(u) + g(x - u) \\ & \geq & (f\#g)(x). \end{array}$$

This concludes the proof.

$\square$

3.4.1 Moreau envelopes

One of the most important and frequently used instances of infimal convolutions is defined below.

Definition 3.4.6 (Moreau envelope and proximal mapping) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. The* Moreau envelope *(or Moreau-Yosida regularization) of f (to the parameter $\lambda > 0$) is the function $e_\lambda f : \mathbb{E} \to \overline{\mathbb{R}}$ defined by*

$$e_\lambda f(x) := \inf_{u \in \mathbb{E}} \left\{ f(u) + \frac{1}{2\lambda}\|x - u\|^2 \right\}.$$

The (possibly set-valued) mapping $P_\lambda f : \mathbb{E} \rightrightarrows \mathbb{E}$

$$P_\lambda f(x) := \operatorname*{argmin}_{u \in \mathbb{E}} \left\{ f(u) + \frac{1}{2\lambda}\|x - u\|^2 \right\}$$

is called the proximal mapping *or* prox-operator *to the parameter $\lambda > 0$ of f.*

Note that it is easily seen that

$$e_\lambda(\alpha f) = \alpha e_{\alpha\lambda} f \quad (\alpha, \lambda > 0). \tag{3.20}$$

From our findings in Section 3.2 and from above we can immediately state the following result.

Proposition 3.4.7 *Let $f \in \Gamma_0$ and $\lambda > 0$. Then $e_\lambda f \in \Gamma_0$ and $P_\lambda f$ is single-valued (in particular nonempty).*

Proof: For the prox-operator everything follows from Corollary 3.2.11 since $\frac{1}{2\lambda}\|x - (\cdot)\|^2$ is strongly convex hence supercoercive and strictly convex, and it is continuous, hence lsc.

The fact that $e_\lambda f \in \Gamma_0$ follows from Theorem 3.4.5.

□

Note that by definition and the above result, for $f \in \Gamma_0$, $\lambda > 0$ and $x \in \mathbb{E}$, we have

$$e_\lambda f(x) = f(P_\lambda f(x)) + \frac{1}{2\lambda}\|x - P_\lambda f(x)\|^2 \le f(y) + \frac{1}{2\lambda}\|x - y\|^2 \quad (y \in \mathbb{E}). \tag{3.21}$$

The next results show that the prox-operator for closed, proper convex functions is in fact a generalization of the projection onto closed, convex sets.

Proposition 3.4.8 *Let $f \in \Gamma_0$ and let $x, p \in \mathbb{E}$. Then $p = P_1 f(x)$ if and only if*

$$\langle y - p,\, x - p \rangle + f(p) \le f(y) \quad (y \in \mathbb{E}). \tag{3.22}$$

Proof: First, assume that $p = P_1 f(x)$ and let $y \in \mathbb{E}$. Then put $p_\alpha := \alpha y + (1-\alpha)p \ (\alpha \in (0,1))$. Then, for every $\alpha \in (0,1)$, by convexity and (3.21) we have

$$\begin{array}{rcl} f(p) & \leq & f(p_\alpha) + \frac{1}{2}\|x - p_\alpha\|^2 - \frac{1}{2}\|x-p\|^2 \\ & \leq & \alpha f(y) + (1-\alpha) f(p) - \alpha \langle x - p,\, y - p \rangle + \frac{\alpha^2}{2}\|y-p\|^2, \end{array}$$

and hence

$$\langle y - p,\, x - p \rangle + f(p) \leq f(y) + \frac{\alpha}{2}\|y-p\|^2 \quad (y \in \mathbb{E}, \alpha \in (0,1)).$$

Letting $\alpha \downarrow 0$, we obtain (3.22).

Conversely, suppose that (3.22) holds. Then we deduce

$$\begin{array}{rcl} f(p) + \dfrac{1}{2}\|x-p\|^2 & \leq & f(y) + \dfrac{1}{2}\|x-p\|^2 + \langle x - p,\, p - y \rangle + \dfrac{1}{2}\|p-y\|^2 \\ & = & f(y) + \dfrac{1}{2}\|x-y\|^2 \end{array}$$

for all $y \in \mathbb{E}$. Thus, $p = P_1 f(x)$. □

Note that (3.22) is a generaliztion of (2.13), which can be seen by simply plugging in $f = \delta_C$ for some closed convex set C.

The next result shows that the prox-operator is globally Lipschitz continuous.

Proposition 3.4.9 (Lipschitz continuity of prox-operator) *Let $f \in \Gamma_0$. Then*

$$\|P_1 f(x) - P_1 f(y)\| \leq \|x - y\| \quad (x, y \in \mathbb{E}),$$

i.e. $P_1 f$ is globally Lipschitz continuous with Lipschitz modulus 1.

Proof: Let $x, y \in \mathbb{E}$ and put $p := P_1 f(x)$ and $q := P_1 f(y)$. Then Proposition 3.4.8 yields

$$\langle q - p,\, x - p \rangle + f(p) \leq f(q) \quad \text{and} \quad \langle p - q,\, y - q \rangle + f(q) \leq f(p).$$

Since $p, q \in \operatorname{dom} f$, adding thes two inequalities, we get

$$0 \leq \langle p - q,\, (x-y) - (p-q) \rangle = \langle p - q,\, x - y \rangle - \|p-q\|^2.$$

Using the Cauchy-Schwarz inequality we obtain the desired result. □

Can easily apply the results on the prox-operator of some $f \in \Gamma_0$ and the parameter $\lambda = 1$ to arbitrary parameters $\lambda > 0$ through the identity

$$P_\lambda f = P_1(\lambda f), \tag{3.23}$$

which is an immediate consequence of (3.20).

Theorem 3.4.10 (Differentiability of Moreau envelopes in Γ_0) *Let $f \in \Gamma_0$ and $\lambda > 0$. Then $e_\lambda f$ is differentiable with gradient*

$$\nabla(e_\lambda f) = \frac{1}{\lambda}(\mathrm{id} - P_\lambda f),$$

which is globally Lipschitz with modulus $\frac{1}{\lambda}$.

Proof: Assume that $x, y \in \mathbb{E}$ are distinct points and set $p := P_\lambda f(x)$ and $q := P_\lambda f(y)$. Using (3.21), (3.23) and Proposition 3.4.8, we obtain

$$\begin{aligned} e_\lambda f(y) - e_\lambda f(x) &= f(q) - f(p) + \frac{1}{2\lambda}\left(\|y-q\|^2 - \|x-p\|^2\right) \\ &= \frac{1}{2\lambda}\left(2[(\lambda f)(q) - (\lambda f)(p)] + \|y-q\|^2 - \|x-p\|^2\right) \\ &\geq \frac{1}{2\lambda}\left(2\langle q-p,\, x-p\rangle + \|y-q\|^2 - \|x-p\|^2\right) \\ &= \frac{1}{2\lambda}\left(\|y-q-x+p\|^2 + 2\langle y-x,\, x-p\rangle\right) \\ &\geq \frac{1}{\lambda}\langle y-x,\, x-p\rangle . \end{aligned}$$

Analogously, by simply changing the roles of x and y in the application of Proposition 3.4.8, we obtain

$$e_\lambda f(y) - e_\lambda f(x) \leq \frac{1}{\lambda}\langle y-x,\, y-q\rangle .$$

Using the last two inequalities and invoking Proposition 3.4.9, we obtain

$$\begin{aligned} 0 &\leq e_\lambda f(y) - e_\lambda f(x) - \frac{1}{\lambda}\langle y-x,\, x-p\rangle \\ &\leq \frac{1}{\lambda}\langle y-x,\, (y-x)-(p-q)\rangle \\ &\leq \frac{1}{\lambda}\left(\|x-y\|^2 - \|p-q\|^2\right) \\ &\leq \frac{1}{\lambda}\|y-x\|^2. \end{aligned}$$

Therefore,

$$\lim_{y\to x} \frac{e_\lambda f(y) - e_\lambda f(x) - \left\langle y-x,\, \frac{1}{\lambda}(x-p)\right\rangle}{\|x-y\|} = 0,$$

which proves the differentiabilty of $e_\lambda f$ and the gradient formula. The Lipschitz continuity of the gradient is then due to Proposition 3.4.9. □

As a nice application we can prove the differentiability of the squared Euclidean distance function.

Example 3.4.11 (Differentiability of squared distance function) Let $C \subset \mathbb{E}$ be nonempty, closed and convex. Then $\frac{1}{2}\mathrm{dist}_C^2 = e_1\delta_C$ is convex and differentiable with Lipschitz gradient

$$\nabla\left(\frac{1}{2}\mathrm{dist}_C^2\right) = \mathrm{id} - P_C.$$

We close out the section with a result that is tremendously interesting from an optimization perspective.

Proposition 3.4.12 (Minimizers of Moreau envelope) *Let $f \in \Gamma_0$, $\lambda > 0$ and $\bar{x} \in \mathbb{E}$. Then*

$$\operatorname*{argmin}_{\mathbb{E}} f = \operatorname*{argmin}_{\mathbb{E}} e_\lambda f \quad \textit{and} \quad \inf_{\mathbb{E}} f = \inf_{\mathbb{E}} e_\lambda f.$$

Proof: Let $\bar{x} \in \operatorname{argmin} f$. Then $\bar{x}$ also minimizes $f + \frac{1}{2\lambda}\|\bar{x} - (\cdot)\|^2$. But since the unique minimizer of the latter is $P_\lambda f(\bar{x})$, we must have $\bar{x} = P_\lambda f(\bar{x})$. Hence, by Theorem 3.4.10, we have $\nabla e_\lambda f(\bar{x}) = 0$. Thus, as $e_\lambda f$ is convex, $\bar{x} \in \operatorname{argmin}_{\mathbb{E}} e_\lambda f$, cf. Exercise 3.11.

In turn, if $\bar{x} \in \operatorname{argmin} e_\lambda f$, then $\nabla e_\lambda f(\bar{x}) = \frac{1}{\lambda}(\bar{x} - P_\lambda f(\bar{x})) = 0$, hence, $P_\lambda f(\bar{x}) = \bar{x}$ therefore

$$f(\bar{x}) = e_\lambda f(\bar{x}) \le e_\lambda f(y) \le f(y) \quad (y \in \mathbb{E}).$$

All in all we have proven the equality for the argmin sets and the identity for the infima if attained.

Since $e_\lambda f \le f$, we always have $\inf e_\lambda f \le \inf f$. Conversely, fix $x \in \mathbb{E}$. Then

$$\inf f \le f + \frac{1}{2\lambda}\|(\cdot) - x\|^2 = e_\lambda f(x).$$

Taking the infimum over $x \in \mathbb{E}$ gives the converse inequality.

□

Note that, implicitly, we proved above that $\bar{x} \in \mathbb{E}$ is a minimizer of f if and only if $P_\lambda f(\bar{x}) = \bar{x}$, i.e. the fixpoints of $P_\lambda f$ are exactly the minimizers of f.

3.5 Continuity properties of convex functions

In this section we want to study continuity properties of convex functions.

We start by defining continuity notions relative to a set.

Definition 3.5.1 (Continuity relative to a set) *Let $S \subset \mathbb{E}$. A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is said to be* continuous relative to *(or* on *)* S *if*

$$\lim_{k\to\infty} f(x_k) = f(x) \quad (x \in S,\ \{x_k \in S\} \to x).$$

In addition, we call f Lipschitz (continuous) relative to S *if there exists $L \ge 0$ such that*

$$|f(x) - f(y)| \le L\|x - y\| \quad (x, y \in S).$$

As a preparatory result, we compare a proper convex function with its closure.

Proposition 3.5.2 (Closure of convex functions) *Let* $f \in \Gamma$. *Then* $\mathrm{cl}\, f \in \Gamma_0$. *Moreover,* $\mathrm{cl}\, f$ *agrees with* f *except perhaps on* $\mathrm{rbd}\,(\mathrm{dom}\, f)$.

Proof: Since $\mathrm{epi}\,(\mathrm{cl}\, f) = \mathrm{cl}\,(\mathrm{epi}\, f)$, $\mathrm{cl}\, f$ is lsc and convex. Now let $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$. Since $f \in \Gamma$ there is an affine function $h \le f$ with $h(\bar{x}) = f(\bar{x})$, cf. Theorem 3.3.2. Since every affine function is continuous, in particular closed, we have $\mathrm{cl}\, f \ge \mathrm{cl}\, h = h$. Hence,

$$(\mathrm{cl}\, f)(\bar{x}) \le f(\bar{x}) = h(\bar{x}) \le (\mathrm{cl}\, f)(\bar{x}),$$

therefore $(\mathrm{cl}\, f)(\bar{x}) = f(\bar{x})$. This shows that f and $\mathrm{cl}\, f$ agree on $\mathrm{ri}\,(\mathrm{dom}\, f)$. In particular, we see that $\mathrm{cl}\, f$ is proper.

Now, let $x \notin \mathrm{cl}\,(\mathrm{dom}\, f)$. Clearly, all sequences $\{x_k\} \to x$ have that $x_k \notin \mathrm{dom}\, f$ for all k sufficiently large, i.e. $f(x_k) = +\infty$, hence $\liminf_{k\to\infty} f(x_k) = +\infty$, i.e $(\mathrm{cl}\, f)(x) = +\infty$. This proves the assertion.

□

Just like we argued in Remark 2.3.8 that for a single, given nonempty convex set, we can always assume w.l.o.g. that it has full dimension, we can do the same for the domain of a given proper convex function.

Remark 3.5.3 When dealing with a proper convex function $f : \mathbb{E} \to \overline{\mathbb{R}}$, we know that $\mathrm{dom}\, f$ is nonempty convex. Let U be the subspace parallel to $\mathrm{aff}\,(\mathrm{dom}\, f)$ (or any other subspace of $\mathbb{E}$ of the same dimenstion.). By Theorem 1.4.18, there exists an invertible affine mapping $F : \mathbb{E} \to \mathbb{E}$ such that $F(\mathrm{aff}\,(\mathrm{dom}\, f)) = U$. Defining $g : U \to \mathbb{E}$ by $g := f \circ F^{-1}$, we see that g is proper convex with $\mathrm{dom}\, g = F(\mathrm{dom}\, f)$, i.e. $\mathrm{aff}\,(\mathrm{dom}\, g) = \mathrm{aff}\,(F(\mathrm{dom}\, f)) = F(\mathrm{aff}\,(\mathrm{dom}\, f)) = U$. Hence, $\mathrm{dom}\, g$ has full dimension in U.

Clearly, Remark 3.5.3 does not apply if at least two functions are involved, whose domain does not generate the same affine hull.

We are now in a position to prove our first main result on continuity of convex functions.

We encourage the reader to recast the proof without using the assumption (justified through Remark 3.5.3) that f have a domain of full dimension.

Theorem 3.5.4 (Continuity of convex functions) *A convex function* $f : \mathbb{E} \to \overline{\mathbb{R}}$ *is continous relative to any relatively open convex subset of* $\mathrm{dom}\, f$. *In particular, it is continuous relative to* $\mathrm{ri}\,(\mathrm{dom}\, f)$.

Proof: Let $C \subset \mathrm{dom}\, f$ be relatively open and convex and consider $g := f + \delta_C$. Then $\mathrm{dom}\, g = C$ and g agrees with f on C. Hence, w.l.o.g. we may assume that $C = \mathrm{dom}\, f = \mathrm{ri}\,(\mathrm{dom}\, f)$; otherwise we substitute f for g. Moreover, in the face of Remark 3.5.3, we can assume that

C is N-dimensional, hence open instead of merely relatively open. If f is improper, we have by Exercise 3.4. that f is identically $-\infty$ on C and hence continuous on C. We can therefore assume that f is proper. Hence, Proposition 3.5.2 guarantees that $\operatorname{cl} f = f$ on C, i.e. f is lsc on C. To prove the result, it suffices to show that f is usc: By Proposition 3.3.1 and openness of $C = \operatorname{dom} f$, we have

$$\operatorname{int}(\operatorname{epi} f) = \{(x, \alpha) \mid f(x) < \alpha\}.$$

Therefore, for $\gamma \in \mathbb{R}$ and with $L : (x, \alpha) \to x$, we find that

$$\{x \mid f(x) < \gamma\} = L\left(\operatorname{int}(\operatorname{epi} f) \cap \{(x, \alpha) \mid \alpha < \gamma\}\right).$$

Since L is surjective and the intersection that it is applied to is open, the set $\{x \mid f(x) < \gamma\}$ is open, cf. Exercise 2.9. Thus, its complement, $\{x \mid f(x) \geq \gamma\}$ is closed. This is equivalent to saying that f is usc, which concludes the proof. $\square$

Since finite functions have the whole space as their domain, the next result follows trivially.

Corollary 3.5.5 (Continuity of finite convex functions) *A convex function $f : \mathbb{E} \to \mathbb{R}$ is continuous.*

We close out the section with our second main result of this section which is concerned with Lipschitz continuity of convex functions.

Theorem 3.5.6 *Let $f \in \Gamma$ and let $S \subset \operatorname{ri}(\operatorname{dom} f)$ be compact. Then f is Lipschitz relative to S.*

Proof: By Remark 3.5.3 we can assume w.l.o.g. that $\operatorname{dom} f$ is N-dimensional so that S actually lies in $\operatorname{int}(\operatorname{dom} f)$. By compactness of S, the sets $S + \varepsilon\overline{\mathbb{B}}$ are compact of all $\varepsilon > 0$, cf. Exercise 1.7. or Corollary 2.4.29. Clearly, for $\varepsilon > 0$ small enough, $S + \varepsilon\overline{\mathbb{B}} \subset \operatorname{int}(\operatorname{dom} f)$. Fix such an ε. By Theorem 3.5.4, f is continuous on $\operatorname{conv}(S + \varepsilon\overline{\mathbb{B}}) \subset \operatorname{int}(\operatorname{dom} f)$ hence, in particular, on $S + \varepsilon\overline{\mathbb{B}}$. As $S + \varepsilon\overline{\mathbb{B}}$ is compact, f is bounded on $S + \varepsilon\overline{\mathbb{B}}$, and let l and u be lower and upper bound, respectively. Now, take $x, y \in \mathbb{E}$ with $x \neq y$ and put

$$z := y + \frac{\varepsilon}{\|x - y\|}(y - x).$$

Then $z \in S + \varepsilon\overline{\mathbb{B}}$ and for $\lambda := \frac{\|x-y\|}{\varepsilon + \|x-y\|}$ we have $y = (1-\lambda)x + \lambda z$, and hence, by convexity of f, we see that

$$f(y) \leq (1-\lambda)f(x) + \lambda f(z) = f(x) + \lambda(f(z) - f(x)),$$

and consequently, for $L := \frac{u-l}{\varepsilon}$, we have

$$f(y) - f(x) \leq \lambda(u - l) \leq L\|x - y\|.$$

Interchanging the roles of x and y gives the desired inequality. $\square$

3.6 Conjugacy of convex functions

3.6.1 Affine approximation and convex hulls of functions

We have spent a significant amount of time studying affine functions. Affine mappings are tied utterly close to half-spaces. In fact, given an affine mapping $F : \mathbb{E} \to \mathbb{R}, F(x) = \langle b, x\rangle - \beta$ (cf. Exercise 1.2.), we have

$$\operatorname{epi} F = \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid \langle b, x\rangle - \beta \leq \alpha\} = H^{\leq}_{(b,-1),\beta} \subset \mathbb{E} \times \mathbb{R}.$$

Actually, it can be seen that every half-space in $\mathbb{E} \times \mathbb{R}$ has one of the following three forms:

1) $\{(x, \alpha) \mid \langle b, x\rangle \leq \beta\}$ (vertical),
2) $\{(x, \alpha) \mid \langle b, x\rangle - \alpha \leq \beta\}$ (upper),
3) $\{(x, \alpha) \mid \langle b, x\rangle - \alpha \geq \beta\}$ (lower),

for some $(b, \beta) \in \mathbb{E} \times \mathbb{R}$.

Theorem 3.6.1 (Envelope representation in Γ_0) *Let $f \in \Gamma_0$. Then f is the pointwise supremum of all affine functions minorizing f, i.e.*

$$f(x) = \sup\{h(x) \mid h \leq f,\ h \text{ affine}\}.$$

Proof: Since f is lsc and convex, epi f is a closed convex set in $\mathbb{E} \times \mathbb{R}$, and therefore, by Theorem 2.7.1, it is the intersection of all closed half-spaces in $\mathbb{E} \times \mathbb{R}$ containing it. No lower half-space can possibly contain epi f. Hence, only vertical and upper half-spaces can be involved in the intersection. We argue that not all of these half-spaces can be vertical: Since f is proper, there exists $x \in \operatorname{dom} f$. Then $(x, f(x) - \varepsilon)$ $(\varepsilon > 0)$ lies in every vertical half-space containing epi f, hence also in their intersection. On the other hand $(x, f(x) - \varepsilon)$ does not lie in epi f, hence not all half-spaces containing epi f can be vertical.

The upper half-spaces containing f, in turn, are simply the epigraphs of affine mappings $h \leq f$. The function that has the intersection of these epigraphs as its epigraphs is just the pointwise supremum of all these functions. Hence, to prove the theorem, we must show that the intersection of the upper half-spaces containing epi f equals the intersection of all upper and vertical half-spaces containing epi f, i.e. that the first intersection excludes every point that also the latter intersection excludes:

To this end, suppose that

$$V := \{(x, \alpha) \mid h_1(x) \leq 0\}, \quad h_1 : x \mapsto \langle b_1, x\rangle - \beta_1$$

is a vertical half-space containing epi f, and that $(x_0, \alpha_0) \notin V$. It suffices to show that there exists an upper half-space containing epi f that does not contain (x_0, α_0), i.e. we need to find an affine mapping $h : \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ such that $h \leq f$ and $h(x_0) > \alpha_0$. We already know that there exists at least on affine function $h_2 : x \mapsto \langle b_2, x\rangle - \beta_2$ such that $\mathrm{epi}\, f \subset \mathrm{epi}\, h_2$, i.e. $h_2 \le f$. For every $x \in \mathrm{dom}\, f$ we have $h_1(x) \le 0$ and $h_2(x) \le f(x)$, and hence

$$\lambda h_1(x) + h_2(x) \le f(x) \quad (\lambda \ge 0).$$

The above inequality holds trivially also for $x \notin \mathrm{dom}\, f$. Now, fix any $\lambda \ge 0$ and define $h_\lambda : \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ by

$$h_\lambda(x) := \lambda h_1(x) + h_2(x) = \langle \lambda b_1 + b_2, x\rangle - (\lambda\beta_1 + \beta_2).$$

Then, clearly, h_λ is affine with $h_\lambda \le f$. Since $h_1(x_0) > 0$, choosing $\bar\lambda > 0$ sufficiently large guarantees that $h_{\bar\lambda}(x_0) > \alpha_0$. Then $h := h_{\bar\lambda}$ has the desired properties, which concludes the proof. □

Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ and recall from Section 1.2.2 that the lower semicontinuous hull $\mathrm{cl}\, f$ of f is the largest lower semicontinuous function that minorizes f or, equivalently,

$$\mathrm{cl}\,(\mathrm{epi}\, f) = \mathrm{epi}\,(\mathrm{cl}\, f).$$

With the same approach, we can build the convex hull of f.

Definition 3.6.2 (Convex hull of a function) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then the pointwise supremum of all convex functions minorizing f, i.e.*

$$\mathrm{conv}\, f := \sup \left\{ h : \mathbb{E} \to \overline{\mathbb{R}} \mid h \le f,\ h \text{ convex} \right\}$$

is called the convex hull *of f.*

Moreover, we define the closed convex hull of f to be

$$\overline{\mathrm{conv}}\, f := \mathrm{cl}\,(\mathrm{conv}\, f),$$

i.e. $\overline{\mathrm{conv}}\, f$ is the largest lsc convex function that minorizes f.

Note that, for $f : \mathbb{E} \to \overline{\mathbb{R}}$, we have

$$\mathrm{epi}\,(\overline{\mathrm{conv}}\, f) = \overline{\mathrm{conv}}\,(\mathrm{epi}\, f), \tag{3.24}$$

cf. Exercise 3.12. An analogous statement does not hold for the convex hull, see the discussion in [3].

Corollary 3.6.3 (Envelope representation of closed, convex hull of proper functions) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ such that $\mathrm{conv}\, f$ is proper. Then $\overline{\mathrm{conv}}\, f$ is the pointwise supremum of all affine functions minorizing f.*

Proof: Since $\operatorname{conv} f$ is proper, so is $\overline{\operatorname{conv}}\, f$, by Exercise 3.4. d). Hence, $\overline{\operatorname{conv}}\, f \in \Gamma_0$ thus, by Theorem 2.7.1, $\overline{\operatorname{conv}}\, f$ is the pointwise supremum of all its affine minorants. Moreover, we have $\overline{\operatorname{conv}}\, f \leq f$. On the other, since all affine functions are lsc and convex, there cannot be an affine minorant of f, which is not a minorant of $\overline{\operatorname{conv}}\, f$. Hence, the affine minorants of $\overline{\operatorname{conv}}\, f$ and f coincide, which gives the desired result. $\square$

Note that the assumption that $\operatorname{conv} f$ be proper implies that f and $\operatorname{cl} f$ are proper, and is equivalent to demanding that f has an affine minorant, cf. Exercise 3.13.

3.6.2 The conjugate of a function

We start with the central definition of this section.

Definition 3.6.4 (Conjugate of a function) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Then its* conjugate *is the function $f^* : \mathbb{E} \to \overline{\mathbb{R}}$ defined by*

$$f^*(y) := \sup_{x \in \mathbb{E}} \{\langle x,\, y\rangle - f(x)\}.$$

The function $f^{**} := (f^*)^*$ is called the biconjugate *of f.*

Note that, clearly, we can restrict the supremum in the above definition of the conjugate to the domain of the underlying function f, i.e.

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \{\langle x,\, y\rangle - f(x)\}.$$

Moreover, by definition, we always have

$$f(x) + f^*(y) \geq \langle x,\, y\rangle \quad (x, y) \in \mathbb{E}, \tag{3.25}$$

which is known as the *Fenchel-Young inequality.*

The mapping $f \mapsto f^*$ from the space of extended real-valued functions to itself is called the *Legendre-Fenchel transform.*

We always have

$$f \leq g \quad \Longrightarrow \quad f^* \geq g^*,$$

i.e. the Legendre-Fenchel transform is order-reversing.

Before we start analyzing the conjugate function in-depth, we want to motivate why we would be interested in studying it: Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. We notice that

$$\operatorname{epi} f^* = \{(y, \beta) \mid \langle x,\, y\rangle - f(x) \leq \beta \quad (x \in \mathbb{E})\}. \tag{3.26}$$

This means that the conjugate of f is the function whose epigraph is the set of all (y, β) defining affine functions $x \mapsto \langle y,\, x\rangle - \beta$ that minorize f. In view of Corollary 3.6.3, if $\operatorname{conv} f$ is proper, the pointwise supremum of these affine mappings is the closed convex hull of f, i.e., through its epigraph, f^* encodes the family of affine minorants of $\overline{\text{conv}}\, f$, i.e. of f itself.

Since,

$$f^*(y) = \sup_{x\in\mathbb{E}}\{\langle x,\, y\rangle - f(x)\} = \sup_{(x,\alpha)\in\text{epi}\, f}\{\langle y,\, x\rangle - \alpha\} \quad (y \in \mathbb{E}), \tag{3.27}$$

we also have

$$\text{epi}\, f^* = \{(y,\beta) \mid \langle x,\, y\rangle - \alpha \le \beta \quad ((x,\alpha) \in \text{epi}\, f)\}$$

We use our recent findings to establish the first major result on conjugates and biconjugates:

Theorem 3.6.5 (Fenchel-Moreau Theorem) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ such that $\text{conv}\, f$ is proper (hence, so is f). Then the following hold:*

a) f^* and f^{**} are closed, proper and convex ;

b) $f^{**} = \overline{\text{conv}}\, f$;

c) $f^* = (\text{conv}\, f)^* = (\text{cl}\, f)^* = (\overline{\text{conv}}\, f)^*$.

Proof: First note that the assumption that $\text{conv}\, f$ is proper implies that both f and $\overline{\text{conv}}\, f$ are proper, cf. Exercise 3.13. and Exercise 3.4.

a) Applying Proposition 3.1.12 to (3.27), we see that f^* is lsc and convex. If f^* attained the value $-\infty$, f would be constantly $+\infty$, which is false. On the other hand, f^* is not identically $+\infty$, since that would imply that its epigraph, which, as $\text{conv}\, f$ is proper, encodes all minorizing affine mappings of $\overline{\text{conv}}\, f$, were empty, which is also false . Hence, f^* is proper.

   Applying the same arguments to $f^{**} = (f^*)^*$ gives that f^{**} is closed, proper and convex, too.

b) Applying (3.27) to f^{**}, for $x \in \mathbb{E}$, we have

$$f^{**}(x) = \sup_{(y,\beta)\in\text{epi}\, f^*}\{\langle y,\, x\rangle - \beta\}.$$

   Hence, in view of (3.26), f^{**} is the pointwise supremum of all affine minorants of f. Therefore, by Corollary 3.6.3, we see that $f^{**} = \overline{\text{conv}}\, f$.

c) Since the affine minorants of $f, \text{conv}\, f, \text{cl}\, f$ and $\overline{\text{conv}}\, f$ coincide their conjugates have the same epigraph and hence are equal.

□

Note that due to item b) from Theorem 3.6.5 we always have $f \ge f^{**}$ for a function $f : \mathbb{E} \to \overline{\mathbb{R}}$ with $\text{conv}\, f$ proper, and it holds that $f^{**} = f$ if and only if f is closed and convex. Thus, the

Legendre-Fenchel transform induces a one-to-one correspondence on Γ_0: For $f, g \in \Gamma_0$, f is conjugate to g if and only if g is conjugate to f and we write $f \overset{*}{\longleftrightarrow} g$ in this case. This is called the *conjugacy correspondence.* A property on one side is reflected by a dual property on the other.

A list of some elementary cases of conjugacy is given below.

Proposition 3.6.6 (Elementary cases of conjugacy) *Let $f \overset{*}{\longleftrightarrow} g$. Then the following hold:*

a) $f - \langle a, \cdot\rangle \quad \overset{*}{\longleftrightarrow} \quad g((\cdot) + a) \quad (a \in \mathbb{E})$*;*

b) $f + \gamma \quad \overset{*}{\longleftrightarrow} \quad g - \gamma \quad (\gamma \in \mathbb{R})$*;*

c) $\lambda f \quad \overset{*}{\longleftrightarrow} \quad \lambda g\left(\frac{(\cdot)}{\lambda}\right) \quad (\lambda > 0)$.

3.6.3 Special cases of conjugacy

Convex quadratic functions

For $Q \in \mathbb{S}^n, b \in \mathbb{R}^n$ we consider the *quadratic function* $q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$q(x) := \frac{1}{2}x^T Qx + b^T x. \tag{3.28}$$

From Theorem 3.1.18 we know that Q is (strongly) convex if and only if Q is positive (definite) semidefinite. Hence, for the remainder we assume that $Q \succeq 0$.

We are interested in computing the conjugate of q. This is easy if Q is postive definite. In the merely semidefinite case the following tool is very useful:

Theorem 3.6.7 (Moore-Penrose pseudoinverse) *Let $A \in \mathbb{S}^n_+$ with* $\operatorname{rank} A = r$ *and the spectral decomposition*

$$A = Q\Lambda Q^T \quad \text{with} \quad \Lambda = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad Q \in \mathrm{O}(n).$$

Then the matrix

$$A^\dagger := Q\Lambda^\dagger Q^T \quad \text{with} \quad \Lambda^\dagger := \begin{pmatrix} \lambda_1^{-1} & & & & \\ & \ddots & & & \\ & & \lambda_r^{-1} & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

called the (Moore-Penrose) pseudoinverse *of A, has the following properties:*

a) $AA^\dagger A = A$ *and* $A^\dagger AA^\dagger = A^\dagger$*;*

b) $(AA^\dagger)^T = AA^\dagger$ *and* $(A^\dagger A)^T = A^\dagger A$*;*

c) $(A^\dagger)^T = (A^T)^\dagger$;

d) If $A \succ 0$*, then* $A^\dagger = A^{-1}$*;*

e) $AA^\dagger = P_{\operatorname{im} A}$*, i.e.* $AA^\dagger$ *is the projection onto the image of* A*. In particular, if* $b \in \operatorname{rge} A$*, we have*

$$\{x \in \mathbb{R}^n \mid Ax = b\} = A^\dagger b + \ker A.$$

In fact, the Moore-Penrose pseudoinverse can be uniquely defined through properties a) and b) from above for any matrix $A \in \mathbb{C}^{m\times n}$, see, e.g. [4], but we confine ourselves with the positive semidefinite case.

We are now in a position to state the desired conjugacy result for convex quadratics.

Proposition 3.6.8 (Conjugate of convex quadratic functions) *For* q *from* (3.28) *with* $Q \in \mathbb{S}^n_+$ *we have*

$$q^*(y) = \begin{cases} \frac{1}{2}(y-b)^T Q^\dagger (y-b) & \text{if} \quad y-b \in \operatorname{rge} Q, \\ +\infty & \text{else.} \end{cases}$$

In particular, if $Q \succ 0$*, we have*

$$q^*(y) = \frac{1}{2}(y-b)^T Q^{-1}(y-b)$$

Proof: By definition, we have

$$q^*(y) = \sup_{x\in\mathbb{R}^n}\left\{x^Ty - \frac{1}{2}x^TQx - b^Tx\right\} = -\inf_{x\in\mathbb{E}}\left\{\frac{1}{2}x^TQx - (b-y)^Tx\right\}. \tag{3.29}$$

The necessary and sufficient optimality conditions of $\bar{x}$ to be a minimizer of the convex function $x \mapsto \frac{1}{2}x^TQx - (b-y)^Tx$ read

$$Q\bar{x} = y - b \tag{3.30}$$

cf. Exercise 3.11. Hence, if $y - b \notin \operatorname{im} Q$, from Exercise 1.8., we know that $\inf f = -\infty$, hence $q^*(y) = +\infty$ in that case.

Otherwise, we have $y - b \in \operatorname{im} Q$, hence, in view of Theorem 3.6.7, (3.30) is equivalent to

$$\bar{x} = Q^\dagger(y-b) + z, \quad z \in \ker A.$$

Inserting $\bar{x} = Q^\dagger(y-b)$ (we can choose $z = 0$) in (3.29) yields

$$q^*(y) \;=\; (Q^\dagger(y-b))^Ty - \frac{1}{2}(Q^\dagger(y-b))^TQQ^\dagger(y-b) - b^TQ^\dagger(y-b)$$

$$
\begin{aligned}
&= (y-b)Q^\dagger(y-b) - \frac{1}{2}(y-b)Q^\dagger Q Q^\dagger (y-b)\\
&= \frac{1}{2}(y-b)Q^\dagger(y-b),
\end{aligned}
$$

where we make use of Theorem 3.6.7 a) and c). Part d) of the latter result gives the remaining assertion.

□

We point out that, by the foregoing result, the function $f = \frac{1}{2}\|\cdot\|^2$ is self-conjugated in the sense that $f^* = f$. Exercise 3.14. shows that this is the only function on $\mathbb{R}^n$ that has this property. Clearly, by an isomorphy argument, the same holds for the respective function on an arbitrary Euclidean space.

Support functions

Definition 3.6.9 (Positive homogeneity, subadditivity, and sublinearity) *Let* $f : \mathbb{E} \to \overline{\mathbb{R}}$. *Then we call* f *with* $0 \in \mathrm{dom}\, f$

i) positively homogeneous *if*

$$f(\lambda x) = \lambda f(x) \quad (\lambda > 0, x \in \mathbb{E});$$

b) subadditive *if*

$$f(x+y) \le f(x) + f(y) \quad (x, y \in \mathbb{E});$$

c) sublinear *if*

$$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y) \quad (x, y \in \mathbb{E},\ \lambda, \mu > 0).$$

Note that in the definition of positive homogeneity we could have also just demanded an inequality, since $f(\lambda x) \le \lambda f(x)$ for all $\lambda > 0$ implies that

$$f(x) = f(\lambda^{-1}\lambda x) \le \frac{1}{\lambda} f(\lambda x).$$

We note that norms are sublinear.

Example 3.6.10 Every norm $\|\cdot\|$ is sublinear.

We next proivide a usful list of characerizations of positive homogeneneity and sublinearity, respectively.

Proposition 3.6.11 (Positive homogeneity, sublinearity and subadditivity) *Let* $f : \mathbb{E} \to \overline{\mathbb{R}}$. *Then the following hold:*

a) f *is positively homogeneous if and only if* $\operatorname{epi} f$ *is a cone. In this case* $f(0) \in \{0, -\infty\}$.

b) *If* f *is lsc and positively homogeneous with* $f(0) = 0$ *it must be proper.*

c) *The following are equivalent:*

   i) f *is sublinear;*

   ii) f *is positively homogeneous and convex;*

   iii) f *is positively homogeneous and subadditive;*

   iv) $\operatorname{epi} f$ *is a convex cone.*

Proof: Exercise 3.17. □

We continue with the prototype of a sublinear functions, so-called *support functions*, which will from now on occur ubiquitiously.

Definition 3.6.12 (Support functions) *Let* $C \in \mathbb{E}$ *nonempty. The* support function *of* C *is defined by*

$$\sigma_C : x \in \mathbb{E} \mapsto \sup_{s \in C} \langle s,\, x \rangle \,.$$

We start our investigation of support functions with a list of elementary properties.

Proposition 3.6.13 (Support functions) *Let* $C \subset \mathbb{E}$ *be nonempty. Then*

a) $\sigma_C = \sigma_{\operatorname{cl} C} = \sigma_{\operatorname{conv} C} = \sigma_{\overline{\operatorname{conv}}\, C}$.

b) σ_C *is proper, lsc and sublinear.*

c) $\delta_C^* = \sigma_C$ *and* $\sigma_C^* = \delta_{\overline{\operatorname{conv}}\, C}$.

d) *If* C *is closed and convex then* $\sigma_C \overset{*}{\longleftrightarrow} \delta_C$.

Proof:

a) Obviously, closures do not make a difference. On the other hand, we have

$$\left\langle \sum_{i=1}^{N+1} \lambda_i s_i,\, x \right\rangle = \sum_{i=1}^{N+1} \lambda_i \langle s_i,\, x \rangle \leq \max_{i=1,\ldots,r} \langle s_i,\, x \rangle$$

for all $s_i \in C, \lambda \in \Delta_{N+1}$, which shows that convex hulls also do not change anything.

b) By Proposition 3.1.12 σ_C is lsc and convex, and as $0 \in \operatorname{dom} \sigma_C$ and since $\lambda \sigma_C(x) = \sigma_C(\lambda x)$ for all $x \in \mathbb{E}$ and $\lambda > 0$ this shows properness and positive homogeneity, which gives the assertion in view of Proposition 3.6.11 c).

c) Clearly, $\delta_C^* = \sigma_C$. Hence, $\sigma_C^* = \delta_C^{**} = \overline{\text{conv}}\,\delta_C = \delta_{\overline{\text{conv}}\,C}$, since

$$\overline{\text{conv}}\,(\text{epi}\,\delta_C) = \overline{\text{conv}}\,(C \times \mathbb{R}_+) = \overline{\text{conv}}\,C \times \mathbb{R}_+ = \text{epi}\,(\delta_{\overline{\text{conv}}\,C}).$$

d) Follows immediately from c).

□

One of our main goals in this paragraph is to show that, in fact, part b) of Propostion 3.6.13 can be reversed in the sense that we will see that every proper, lsc and sublinear function is a support function. As a preparation we need the following result.

Proposition 3.6.14 *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be closed, proper and convex. Then the following are equivalent:*

i) f only takes the values 0 and $+\infty$;

ii f^* is positively homogeneous (i.e. sublinear, since convex).

Proof: 'i)⇒ii):' In this case $f = \delta_C$ for some closed convex set $C \subset \mathbb{E}$. Hence, $f^* = \sigma_C$, which is sublinear, cf. Proposition 3.6.13.

In turn, let f^* be positively homogeneous (hence sublinear). Then, for $\lambda > 0$ and $y \in \mathbb{E}$, we have

$$\begin{aligned} f^*(y) &= \lambda f^*(\lambda^{-1}y) \\ &= \lambda \sup_{x\in\mathbb{E}} \left\{ \left\langle x,\, \lambda^{-1}y \right\rangle - f(x) \right\} \\ &= \sup_{x\in\mathbb{E}} \left\{ \langle x,\, y\rangle - \lambda f(x) \right\} \\ &= (\lambda f)^*(y). \end{aligned}$$

Thus, $(\lambda f)^* = f^*$ for all $\lambda > 0$ and hence, by the Fenchel-Moreau Theorem, we have

$$\lambda f = (\lambda f)^{**} = f^{**} = f \quad (\lambda > 0).$$

But as f is proper, hence does not takte the value $-\infty$, this immediately implies that f only takes the values $+\infty$ and 0. □

Theorem 3.6.15 (Hörmander's Theorem) *A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is proper, lsc and sublinear if and only if it is a support function.*

Proof: By Proposition 3.6.13 b), every support function is proper, lsc and sublinear.

In turn, if f is proper, lsc and sublinear (hence $f = f^{**}$), by Proposition 3.6.14, f^* is the indicator of some set $C \subset \mathbb{E}$, which necessary needs to be nonempty, closed and convex, as $f^* \in \Gamma_0$. Hence, $f^{**} = \delta_C^* = \sigma_C$. □

We now want to give a slight refinement of Hörmander's Theorem, in that we describe the set that a proper, lsc sublinear function supports.

Corollary 3.6.16 *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and sublinear. Then $\mathrm{cl}\, f$ is the support function of the closed convex set*

$$\{s \in \mathbb{E} \mid \langle s,\, x\rangle \leq f(x)\ (x \in \mathbb{E})\}.$$

Proof: Since $\mathrm{cl}\, f$ is proper (cf. Exercise 3.4.) closed and sublinear it is a support function of a closed convex set C. Therefore, we have $\mathrm{cl}\, f = \delta_C^*$ and thus $f^* = (\mathrm{cl}\, f)^* = \delta_C$. Hence, $C = \{s \in \mathbb{E} \mid f^*(s) \leq 0\}$. But $f^*(s) \leq 0$ if and only $\langle s,\, x\rangle - f(x) \leq 0$ for all $x \in \mathbb{E}$. □

Gauges, polar sets and dual norms

We now present a class of functions that makes a connection between support functions and norms.

Definition 3.6.17 (Gauge function) *Let $C \subset \mathbb{E}$. The* gauge (function) *of C is defined by*

$$\gamma_C : x \in \mathbb{E} \mapsto \inf\{\lambda \geq 0 \mid x \in \lambda C\}.$$

For a closed convex set that contains the origin, its gauge has very desirable convex-analytical properties.

Proposition 3.6.18 *Let $C \subset \mathbb{E}$ be nonempty, closed and convex with $0 \in C$. Then γ_C is proper, lsc and sublinear.*

Proof: γ_C is obviously proper as $\gamma_C(0) = 0$. Moreover, for $t > 0$ and $x \in \mathbb{E}$, we have

$$\begin{aligned}
\gamma_C(tx) &= \inf\{\lambda \geq 0 \mid tx \in \lambda C\} \\
&= \inf\left\{\lambda \geq 0 \,\middle|\, x \in \frac{\lambda}{t} C\right\} \\
&= \inf\{t\mu \geq 0 \mid x \in \mu C\} \\
&= t \inf\{\mu \geq 0 \mid x \in \mu C\} \\
&= t\gamma_C(x),
\end{aligned}$$

i.e. γ_C is positively homogeneous (since also $0 \in \operatorname{dom} \gamma_C$). We now show that it is also subadditive, hence altogether, sublinear: To this end, take $x, y \in \operatorname{dom} \gamma_C$ (otherwise there is nothing to prove). Due to the identity

$$\frac{x+y}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}\frac{x}{\lambda} + \frac{\mu}{\lambda+\mu}\frac{y}{\mu} \quad (\lambda+\mu \neq 0),$$

we realize, by convexity of C, that $x + y \in (\lambda+\mu)C$ if $x \in \lambda C$ and $y \in \mu C$ for all $\lambda, \mu \geq 0$. This implies that $\gamma_C(x+y) \leq \gamma_C(x) + \gamma_C(y)$.

In order to prove lower semicontinuity of γ_C notice that (by Exercise 3.20. and positive homogeneity) we have $\operatorname{lev}_{\leq \alpha}\gamma_C = \alpha C$ for $\alpha > 0$, $\operatorname{lev}_{\leq \alpha}\gamma_C = \emptyset$ for $\alpha < 0$ and $\operatorname{lev}_{\leq 0}\gamma_C = C^\infty$ (again by Exercise 3.20.), hence all level sets of γ_C are closed, i.e. γ_C is lsc.

This concludes the proof.

$\square$

Note that in the proof of Proposition 3.6.18, we do not need the assumption that C contains the origin to prove sublinearity. We do need it, though, to get lower semicontinuity, cf. Exercise 3.20.

Since the gauge of a closed convex set that contains 0 is proper, lsc and sublinear we know, in view of Hörmander's Theorem (see Theorem 3.6.15), that it is the support function of some closed convex set. It can be described beautifully using the concept of *polar sets* which generalizes the notion of polar cones, cf. Definition 2.4.5.

Definition 3.6.19 (Polar sets) *Let $C \subset \mathbb{E}$. Then its* polar set *is defined by*

$$C^\circ := \{v \in \mathbb{E} \mid \langle v,\, x\rangle \leq 1 \ (x \in C)\}.$$

Moreover, we put $C^{\circ\circ} := (C^\circ)^\circ$ and call it the bipolar set *of C.*

Note that there is no ambiguity in notation, since the polar cone and the polar set of a cone coincide, see Exercise 3.19. Moreover, as an intersection of closed half-spaces, C° is a closed, convex set containing 0. In addition, like we would expect, we have

$$C \subset D \quad \Rightarrow D^\circ \subset C^\circ,$$

and

$$C \subset C^{\circ\circ}.$$

Before we continue to pursue our question for the support function representation of gauges, we provide the famous *bipolar theorem* which generalizes Exercise 2.23. Its proof is based once more on separation.

Theorem 3.6.20 (Bipolar Theorem) *Let $C \subset \mathbb{E}$. Then $C^{\circ\circ} = \overline{\operatorname{conv}}\,(C \cup \{0\})$.*

Proof: Since $C \cup \{0\} \subset C^{\circ\circ}$ and $C^{\circ\circ}$ is closed and convex, we clearly have $\overline{\mathrm{conv}}\,(C \cup \{0\}) \subset C^{\circ\circ}$. Now assume there were $\bar{x} \in C^{\circ\circ} \setminus \overline{\mathrm{conv}}\,(C \cup \{0\})$. By strong separation, there exists $s \in \mathbb{E} \setminus \{0\}$ such that

$$\langle s,\, \bar{x}\rangle > \sigma_{\overline{\mathrm{conv}}\,(C\cup\{0\})}(s) \geq \max\{\sigma_C(s), 0\}.$$

After rescaling s accordingly (cf. Remark 2.6.2) we can assume that

$$\langle s,\, \bar{x}\rangle > 1 \geq \sigma_C(s),$$

in particular, $s \in C^\circ$. On the other hand $\langle s,\, \bar{x}\rangle > 1$ and $\bar{x} \in C^{\circ\circ}$, which is a contradiction. $\square$

As a consequence of the bipolar theorem we see that every closed convex set $C \subset \mathbb{E}$ containing 0 satisfies $C = C^{\circ\circ}$. Hence, the mapping $C \mapsto C^\circ$ establishes a one-to-one correspondence on the closed convex sets that contain the origin. This is connected to conjugacy through gauge functions as is highlighted by the next result.

Proposition 3.6.21 *Let $C \subset \mathbb{E}$ be closed and convex with $0 \in C$. Then*

$$\gamma_C = \sigma_{C^\circ} \overset{*}{\longleftrightarrow} \delta_{C^\circ} \quad \textit{and} \quad \gamma_{C^\circ} = \sigma_C \overset{*}{\longleftrightarrow} \delta_C.$$

Proof: Since, by Proposition 3.6.18, γ_C is proper, lsc and sublinear we have

$$\gamma_C = \sigma_D, \quad D = \{v \in \mathbb{E} \mid \langle v,\, x\rangle \leq \gamma_C(x)\ (x \in \mathbb{E})\}$$

in view of Corollary 3.6.16. To prove that $\gamma_C = \sigma_{C^\circ}$, we need to show that $D = C^\circ$. Since $\gamma_C(x) \leq 1$ if (and only if; see Exercise 3.20.) $x \in C$, the inclusion $D \subset C^\circ$ is clear. In turn, let $v \in C^\circ$, i.e. $\langle v,\, x\rangle \leq 1$ for all $x \in C$. Now let $x \in \mathbb{E}$. By the definition of γ_C, there exists $\lambda_k \to \gamma_C(x)$ and $c_k \in C$ such that $x = \lambda_k c_k$ for all $k \in \mathbb{N}$. But then

$$\langle v,\, x\rangle = \lambda_k \langle v,\, c_k\rangle \leq \lambda_k \to \gamma_C(x),$$

hence $v \in D$, which proves $\gamma_C = \sigma_{C^\circ}$. Since $C^{\circ\circ} = C$, this implies $\gamma_{C^\circ} = \sigma_C$. The conjugacy relations are due to Proposition 3.6.13. $\square$

Exercise 3.20. tells us that the gauge of a symmetric, compact convex set with nonempty interior is a norm. This justifies the following definition.

Definition 3.6.22 (Dual norm) *Let $\|\cdot\|_*$ be a norm on $\mathbb{E}$ wih closed unit ball B_*. Then we call*

$$\|\cdot\|_*^\circ := \gamma_{B_*^\circ}$$

its dual norm.

Corollary 3.6.23 (Dual norms) *For any norm $\|\cdot\|_*$ with (closed) unit ball B its dual norm is σ_B, the support of its unit ball. In particular, we have $\|\cdot\|^\circ = \|\cdot\|$, i.e. the Euclidean norm is self-dual.*

3.6.4 Some dual operations

In this section, we give a list of conjugate functions obtained through convexity-preserving operations that we have studied earlier.

We start by conjugacy on product sets.

Proposition 3.6.24 (Conjugacy on product sets) *Let* $f_i : \mathbb{E}_i \to \overline{\mathbb{R}}$ $(i = 1, \dots, p)$ *put* $\mathbb{E} := \bigtimes_i^p \mathbb{E}_i$ *and define* $f : (x_1, \dots, x_p) \in \mathbb{E} \mapsto \sum_{i=1}^p f_i(x_i)$. *Then*

$$f^* : (y_1, \dots, y_p) \in \mathbb{E} \mapsto \sum_{i=1}^p f_i^*(y_i).$$

Proof: For $y = (y_1, \dots, y_p) \in \mathbb{E}$ we have

$$f^*(y) = \sup_{(x_1,\dots,x_p)\in\mathbb{E}} \left\{ \sum_{i=1}^p \langle x_i,\, y_i\rangle - \sum_{i=1}^p f(x_i) \right\} = \sum_{i=1}^p \sup_{x_i\in\mathbb{E}_i} \{\langle x_i,\, y_i\rangle - f_i(x_i)\} = \sum_{i=1}^p f_i^*(y_i).$$

□

One of the convexity-preserving operations that we have studied thoroughly is infimal convolution, which is, as we will now see, paired in duality with simple addition of functions.

Proposition 3.6.25 (Conjugacy of inf-convolution) *Let* $f, g : \mathbb{E} \to \overline{\mathbb{R}}$. *Then the following hold:*

a) $(f\#g)^* = f^* + g^*$;

b) If $f, g \in \Gamma_0$ *such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ *then* $(f + g)^* = \operatorname{cl}(f^*\#g^*)$.

Proof:

a) By definition, for all $y \in \mathbb{E}$, we have

$$\begin{aligned}
(f\#g)^*(y) &= \sup_x \Big\{ \langle x,\, y\rangle - \inf_u \{f(u) + g(x-u)\} \Big\} \\
&= \sup_{x,u} \{\langle x,\, y\rangle - f(u) - g(x-u)\} \\
&= \sup_{x,u} \{(\langle u,\, y\rangle - f(u)) + (\langle x-u,\, y\rangle - g(x-u))\} \\
&= f^*(y) + g^*(y).
\end{aligned}$$

b) From a) and the fact that f, g are closed, proper convex, we have

$$(f^*\#g^*)^* = f^{**} + g^{**} = f + g,$$

which is proper, as $\operatorname{dom} f$ meets $\operatorname{dom} g$, closed and convex. Thus,

$$\overline{\operatorname{conv}}\,(f^*\#g^*) = (f^*\#g^*)^{**} = (f+g)^*.$$

By Proposition 3.4.3 the convex hull on the left can be omitted, hence $\operatorname{cl}(f^*\#g^*) = (f+g)^*$.

$\square$

Note that it can be shown that the closure operation in Proposition 3.6.25 can be omitted under the qualifcation condition

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq 0. \tag{3.31}$$

This in fact is a prominent theorem which we now state and whose proof we postpone to the Appendix.

Theorem 3.6.26 (Attouch-Brézis) *Let $f, g \in \Gamma_0$ such that (3.31) holds. Then $(f+g)^* = f^*\#g^*$, and the infimal convolution is exact, i.e. the infimum in the infimal convolution is attained.*

Some very important cases of infimal convolutions that have occured in our study are considered below from a duality perspective.

Corollary 3.6.27 (Conjugacy for distance functions and Moreau envelopes) *Let $f \in \Gamma_0, \lambda > 0$ and C nonempty, closed and convex. Then the following hold:*

a) $\operatorname{dist}_C \overset{*}{\longleftrightarrow} \sigma_C + \delta_{\overline{\mathbb{B}}}$;

b) $e_\lambda f \overset{*}{\longleftrightarrow} f^* + \frac{\lambda}{2}\|\cdot\|^2$;

c) $e_\lambda f(x) + e_{\lambda^{-1}} f(\frac{x}{\lambda}) = \frac{1}{2\lambda}\|x\|^2 \quad (x \in \mathbb{E})$.

Proof:

a) Since $\operatorname{dist}_C = \delta_C \# \|\cdot\|$, $\delta_C^* = \sigma_C$ (see Proposition 3.6.13) and $\|\cdot\|^* = \sigma_{\overline{\mathbb{B}}}^* = \delta_{\overline{\mathbb{B}}}$, the assertion follows from Proposition 3.6.25 and Theorem 3.6.26.

b) Since $e_\lambda f = f\#(\frac{1}{2\lambda}\|\cdot\|^2)$ the assertion follows from Proposition 3.6.25 and Theorem 3.6.26 also using Proposition 3.6.6 c).

c) From b) we have for all $x \in \mathbb{E}$ that

$$\begin{aligned} e_\lambda f(x) &= \sup_y \left\{ \langle x, y \rangle - f^*(y) - \frac{\lambda}{2}\|y\|^2 \right\} \\ &= \frac{1}{2\lambda}\|x\|^2 - \inf_y \left\{ f^*(y) - \frac{\lambda}{2}\|y - \frac{1}{\lambda}x\|^2 \right\} \\ &= \frac{1}{2\lambda}\|x\|^2 - e_{\lambda^{-1}} f^* \left(\frac{x}{\lambda} \right). \end{aligned}$$

□

We continue with a conjugacy correspondence between pointwise infima and suprema.

Proposition 3.6.28 (Pointwise inf/sup) *Let $f_i : \mathbb{E} \to \overline{\mathbb{R}}$ $(i \in I)$. Then the following hold:*

a) $(\inf_{i \in I} f_i)^* = \sup_{i \in I} f_i^*$.

b) $(\sup_{i \in I} f_i)^* = \overline{\operatorname{conv}}\ (\inf_{i \in I} f_i^*)$ *for* $f_i \in \Gamma_0$ $(i \in I)$ *and* $\sup_{i \in I} f_i$ *proper.*

Proof:

a) For $y \in \mathbb{E}$ we have

$$\left(\inf_{i \in I} f_i\right)^*(y) = \sup_{x \in \mathbb{E}} \left\{ \langle x,\, y \rangle - \inf_{i \in I} f_i(x) \right\} = \sup_{i \in I} \sup_{x \in \mathbb{E}} \left\{ \langle x,\, y \rangle - f_i(x) \right\} = \sup_{i \in I} f_i^*(y).$$

b) Since $f_i = f_i^{**}$ $(i \in I)$, from a) we infer that $(\inf_{i \in I} f_i^*)^* = \sup_{i \in I} f_i$. Since the latter is lsc and convex (Proposition 3.1.12) and proper (by assumption), hence its convex hull is proper, then so is its conjugate, and thus, we have

$$\overline{\operatorname{conv}}\,(\inf_{i \in I} f_i^*) = (\inf_{i \in I} f_i^*)^{**} = (\sup_{i \in I} f_i)^*.$$

□

We proceed with a duality correspondence for parametric minimization.

Proposition 3.6.29 (Parametric minimization) *Let $f : \mathbb{E}_1 \times \mathbb{E}_2 \to \overline{\mathbb{R}}$. Then the following hold:*

a) For $p := \inf_{x \in \mathbb{E}_1} f(x, \cdot)$ *we have* $p^* = f^*(0, \cdot)$.

b) For $f \in \Gamma_0$, $\bar{u} \in \mathbb{E}_2$ *such that* $\varphi := f(\cdot, \bar{u})$ *is proper and* $q := \inf_{y \in \mathbb{E}} \{ f^*(\cdot, y) - \langle y,\, \bar{u} \rangle \}$, *we have* $\varphi^* = \operatorname{cl} q$.

Proof:

a) For $u \in \mathbb{E}_2$ we compute

$$p^*(u) = \sup_y \left\{ \langle y,\, u \rangle - \inf_x f(x, y) \right\} = \sup_{x,y} \left\{ \langle (x, y),\, ((0, u) \rangle - f(x, y) \right\} = f^*(0, u).$$

b) For $z \in \mathbb{E}_1$ we have

$$\begin{aligned} q^*(z) &= \sup_v \left\{ \langle v,\, z\rangle - \inf_y \{ f^*(v,y) - \langle y,\, \bar{u}\rangle\} \right\} \\ &= \sup_{v,y} \{ \langle (v,y),\, (z,\bar{u})\rangle - f^*(v,y)\} \\ &= f^{**}(z,\bar{u}) \\ &= f(z,\bar{u}) \\ &= \varphi(z). \end{aligned}$$

Here, the fourth equality is due to the fact that $f \in \Gamma_0$. Noticing that q is convex by Theorem 3.2.12, we obtain

$$\operatorname{cl} q = \overline{\operatorname{conv}}\, q = q^{**} = \varphi^*.$$

□

A sufficient condition such that the closure in Propositon 3.6.29 b) can be omitted is that $\bar{u}$ lies in the interior of $U := \{u \in \mathbb{E}_2 \mid \exists x \in \mathbb{E}_1 :\ f(x,u) < 0\}$, see, e.g., [7, Theorem 11.23 (c)]

We close this brief with a result on epi-composition, cf. Proposition 3.1.15.

Proposition 3.6.30 *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and $L \in \mathcal{L}(\mathbb{E},\mathbb{E}')$ and $T \in \mathcal{L}(\mathbb{E}',\mathbb{E})$. Then the following hold:*

a) $(Lf)^* = f^* \circ L^*$.

b) $(f \circ T)^* = \operatorname{cl}(T^* f^*)$ *if* $f \in \Gamma$.

Proof:

a) For $y \in \mathbb{E}'$ we have

$$\begin{aligned} (Lf)^*(y) &= \sup_{z\in\mathbb{E}'} \left\{ \langle z,\, y\rangle - \inf_{x:\ L(x)=z} f(x) \right\} \\ &= \sup_{z\in\mathbb{E}',\, x\in L^{-1}(\{z\})} \{ \langle z,\, y\rangle - f(x)\} \\ &= \sup_{x\in\mathbb{E}} \{ \langle x,\, L^*(y)\rangle - f(x)\} \\ &= f^*(L^*(y)). \end{aligned}$$

b) Follows from a) and the Fenchel-Moreau Theorem.

□

3.7 Fenchel-Rockafellar duality

In this section, we associate a very general (convex) minimization problem (the *primal* program) with a concave maximization problem (the *dual probolem*) that is built on conjugates of the functions occuring in the original problem.

Here the following notation is useful: For a function $h : \mathbb{E} \to \overline{\mathbb{R}}$ we define the function

$$h^{\vee} : x \in \mathbb{E} \mapsto h(-x) \in \overline{\mathbb{R}}.$$

We start our study with a basic duality result that goes back to Werner Fenchel, the founding father of convex analysis.

Theorem 3.7.1 (Fenchel Duality Theorem) *Let $f, g \in \Gamma_0$ such that* $\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset$. *Then*

$$\inf\,(f+g) = \max -(f^* + g^{*\vee}).$$

Proof: It is easily seen that $\inf f + g = -(f+g)^*(0)$. Using Theorem 3.6.26 (Attouch-Brézis), we infer that

$$\inf f + g = -(f^* \# g^*)(0) = -\min f^* + g^{*\vee},$$

which proves the statement. □

An interesting special case of the foregoing theorem is the following.

Corollary 3.7.2 *Let $f \in \Gamma_0$ and $K \subset \mathbb{E}$ be a closed, convex cone such that* $\mathrm{ri}\,(\mathrm{dom}\, f) \cap\ \mathrm{ri}\, K \neq \emptyset$. *Then*

$$\inf_K f = \max_{-K^\circ} -f^*.$$

Proof: Define $g = \delta_K$. Then by Exercise 3.21. we have $g^* = \delta_{K^\circ}$ and hence by Theorem 3.7.1 we have

$$\inf_K f = \inf(f+g) = \max -(f^* + g^{*\vee}) = \max_{-K^\circ} -f^*.$$

□

We now want to add a little more structure to the problem by introducing a linear operator.

Definition 3.7.3 (Fenchel-Rockafellar duality) *Let $f : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ be proper and $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. We call*

$$\inf_{\mathbb{E}_1} (f + g \circ L) \tag{3.32}$$

the primal problem *and*

$$\sup_{\mathbb{E}_2} -(f^* \circ L^* + g^{*\vee}).$$

the dual problem. *We call*

$$\Delta(f,g,L) = \inf_{\mathbb{E}_1}(f + g\circ L) - \sup_{\mathbb{E}_2} -(f^*\circ L^* + g^{*\vee})$$

the duality gap between primal and dual problem.

It is very easy to see that the duality gap in the sense of Definition 3.7.3 is always nonnegative, i.e. the dual optimal value is always a lower bound for the primal optimal value and vice versa.

Proposition 3.7.4 (Weak duality) *Let* $f : \mathbb{E}_1 \to \mathbb{R}\cup\{+\infty\}$ *and* $g : \mathbb{E}_2 \to \mathbb{R}\cup\{+\infty\}$ *be proper and* $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. *Then we have*

$$\inf_{\mathbb{E}_1}(f + g\circ L) \geq \sup_{\mathbb{E}_2} -(f^*\circ L^* + g^{*\vee}),$$

i.e. $\Delta(f,g,L) \geq 0$.

Proof: Let $x \in \mathbb{E}_1$ and $y \in \mathbb{E}_2$. Then by the Fenchel-Young inequality we have

$$f(x) + g(L(x)) \geq -f^*(L^*(y)) + \langle x,\, L^*(y)\rangle - g^*(-y) + \langle -y,\, L(x)\rangle = -(f^*\circ L^* + g^{*\vee})(y).$$

This already proves the statement. $\square$

The weak duality theorem tells us that the duality gap $\Delta(f,g,L)$ is always nonnegative. We now want to investigate under which assumptions it is, in fact, zero.

The following result is known as the *Fenchel-Rockafellar duality theorem*. In its proof, for $f : \mathbb{E}_1 \to \overline{\mathbb{R}}$ and $g : \mathbb{E}_2 \to \overline{\mathbb{R}}$, we use the convenient notation

$$f \oplus g : (x,y) \in \mathbb{E}_1 \times \mathbb{E}_2 \mapsto f(x) + g(y)$$

and call $f \oplus g$ the *separable sum* of f and g.

Theorem 3.7.5 (Strong duality) *Let* $f \in \Gamma_0(\mathbb{E}_1), g \in \Gamma_0(\mathbb{E}_2)$ *and* $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ *such that* $0 \in \operatorname{ri}(\operatorname{dom} g - L(\operatorname{dom} f))$. *Then* $\Delta(f,g,L) = 0$.

Proof: Define

$$C := \operatorname{dom} f \times \operatorname{dom} g - \operatorname{gph} L \subset \mathbb{E}_1 \times \mathbb{E}_2 \quad \text{and} \quad D := \operatorname{dom} g - L(\operatorname{dom} f) \in \mathbb{E}_2.$$

Using the calculus rules for affine hulls, see Corollary 1.4.17 and Exercise 1.15., and the fact that $\operatorname{gph} L$ is a subspace, we see that

$$\operatorname{aff} C = \operatorname{aff}(\operatorname{dom} f) \times \operatorname{aff}(\operatorname{dom} g) - \operatorname{gph} L \quad \text{and} \quad \operatorname{aff} D = \operatorname{aff}(\operatorname{dom} g) - L(\operatorname{aff}(\operatorname{dom} f)).$$

Now, let $(x, y) \in \operatorname{aff} C$, i.e. there exist $r \in \operatorname{aff}(\operatorname{dom} f)$, $s \in \operatorname{aff}(\operatorname{dom} g)$ and $u \in \mathbb{E}_1$ such that $(x, y) = (r, s) - (u, L(u))$. Hence,

$$y - L(x) = s - L(u) - L(r - u) = s - L(r) \in \operatorname{aff} D.$$

Since by assumption $0 \in \operatorname{ri} D$, there exists $t > 0$ such that $t(y - L(x)) \in D$. Hence, there exist $a \in \operatorname{dom} f$ and $b \in \operatorname{dom} g$ such that $y - L(x) = \frac{1}{t}(b - L(a))$. Putting $z := a - tx$, we have $x = \frac{a-z}{t}$ and $y = \frac{b-L(z)}{t}$, thus

$$(x, y) = \frac{1}{t}\left[(a, b) - (z, L(z))\right] \in \mathbb{R}_+ C.$$

Since $(x, y) \in \operatorname{aff} C$ were chosen arbitrarily, we thus have proven $\mathbb{R}_+ C = \operatorname{aff} C$. Since by assumption $0 \in D$, we have $0 \in C$, hence $\operatorname{aff} C = \operatorname{span} C$, i.e. $\mathbb{R}_+ C = \operatorname{span} C$. By Exercise 2.5., we have $0 \in \operatorname{ri} C$. Defining $\varphi := f \oplus g$, and $V := \operatorname{gph} L$ we hence have

$$\operatorname{ri}(\operatorname{dom} \varphi) \cap V = \operatorname{ri}(\operatorname{dom} f) \times \operatorname{ri}(\operatorname{dom} g) \cap \operatorname{gph} L \neq \emptyset.$$

Hence, using Corollary 3.7.2, we obtain

$$\inf_V \varphi = \max_{V^\perp} -\varphi^*.$$

By Proposition 3.6.24 we have $\varphi^* = f^* \oplus g^*$. Moreover, we easily compute that

$$V^\perp = \left\{(u, v) \in \mathbb{E}_1 \times \mathbb{E}_2 \mid u = -L^*(v)\right\}.$$

Therefore, we obtain

$$\begin{aligned}
\inf_{\mathbb{E}_1} (f + g \circ L) &= \inf_V \varphi \\
&= \max_{V^\perp} -\varphi^* \\
&= \max_{(u,v):\, u = -L^* v} -f^*(u) - g^*(v) \\
&= \max_{w \in \mathbb{E}_2} -f^*(L^*(w)) - g^*(-w) \\
&= \max_{\mathbb{E}_2} -(f^* \circ L^* + g^{*\vee}).
\end{aligned}$$

This proofs the assertion.

□

Corollary 3.7.6 *Let* $f \in \Gamma_0(\mathbb{E}_1), g \in \Gamma_0(\mathbb{E}_2)$ *and* $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ *such that* $0 \in \operatorname{ri}(\operatorname{dom} g - L(\operatorname{dom} f))$. *Then*

$$(f + g \circ L)^*(u) = \min_{v \in \mathbb{E}_2} \left\{f^*(u - L^*(v)) + g^*(v)\right\}.$$

Proof: For $u \in \mathbb{E}_1$ we have

$$\begin{aligned}(f + g \circ L)^*(u) &= \sup_{x\in\mathbb{E}_1} \{\langle x,\, u\rangle - f(x) - g(L(x))\} \\ &= -\inf_{x\in\mathbb{E}_1} \{f(x) - \langle x,\, u\rangle + g(L(x))\} \\ &= \min_{v\in\mathbb{E}_2} \{f^*(L^*(v) + u) + g^*(-v)\}\,.\end{aligned}$$

□

As one of the many applications of Fenchel-Rockafellar duality we would like to study linear programs in this regard.

Example 3.7.7 (Linear Programming duality) Let $A \in \mathbb{R}^{m\times n}, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The standard linear program reads

$$\inf\ c^T x \quad \text{s.t.} \quad Ax \geq b. \tag{3.33}$$

Using the functions

$$f : x \in \mathbb{R}^n \mapsto c^T x \quad \text{and} \quad g : y \in \mathbb{R}^m \mapsto \delta_{\mathbb{R}^m_+}(y - b)$$

we can write (3.33) as

$$\inf_{x\in\mathbb{R}^n} \{f(x) + g(Ax)\}.$$

Its dual program, in the sense of Definition 3.7.3, reads

$$\begin{aligned}\sup_{y\in\mathbb{R}^m} -f^*(A^T y) - g^*(-y) \quad &\Longleftrightarrow \quad \sup_{y\in\mathbb{R}^m} \delta_{\{c\}}(A^T y) - \delta_{\mathbb{R}^m_-}(-y) - b^T(-y) \\ &\Longleftrightarrow \quad \sup_{y\geq 0,\ A^T y = c} \quad b^T y.\end{aligned}$$

3.8 The convex subdifferential

In this section we would like to present a generalized notion of differentiability for (usually nondifferentiable) convex functions. The idea of so-called *subdifferentiabilty* is based on affine minorization properties of convex functions and deeply connected to conjugation.

3.8.1 Definition and basic properties

For $f \in \Gamma$, Theorem 3.3.2 tells us that at each point $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$ there exists $g \in \mathbb{E}$ such that

$$f(x) \geq f(\bar{x}) + \langle g,\, x - \bar{x}\rangle \quad (x \in \mathbb{E}). \tag{3.34}$$

We take this as a motivation for the following central concept.

Definition 3.8.1 (Subdifferential of a convex function) *Let $f:\mathbb{E}\to\overline{\mathbb{R}}$ be convex and $\bar{x}\in\mathbb{E}$. Then $g\in\mathbb{E}$ is called a* subgradient *of f at $\bar{x}$ if the* subgradient inequality (3.34) *holds at $\bar{x}$. The set*

$$\partial f(\bar{x}) := \{v\in\mathbb{E} \mid f(x)\geq f(\bar{x})+\langle v,\, x-\bar{x}\rangle \quad (x\in\mathbb{E})\}$$

of all subgradients is called the subdifferential *of f at $\bar{x}$. We denote the domain of the set valued-mapping $\partial f:\mathbb{E}\rightrightarrows\mathbb{E}$ by*

$$\operatorname{dom}\partial f := \{x\in\mathbb{E} \mid \partial f(x)\neq\emptyset\}\,.$$

Notice that, clearly, in the subgradient inequaltiy (3.34), we can restrict ourselves to points $x\in\operatorname{dom} f$, since the inequality holds trivially outside of $\operatorname{dom} f$.

We start our study of the subdifferential with some elementary properties.

Proposition 3.8.2 (Elementary properties of the subdifferential) *Let $f:\mathbb{E}\to\overline{\mathbb{R}}$ be convex and $\bar{x}\in\operatorname{dom} f$. Then the following holds:*

a) $\partial f(\bar{x})$ is closed and convex for all $\bar{x}\in\operatorname{dom} f$.

b) If f is proper then $\partial f(x)=\emptyset$ for $x\notin\operatorname{dom} f$.

c) If f is proper and $\bar{x}\in\operatorname{ri}(\operatorname{dom} f)$ then $\partial f(\bar{x})$ is nonempty.

d) We have $0\in\partial f(\bar{x})$ if and only if $\bar{x}$ minimizes f (over $\mathbb{E}$). (Generalized Fermat's rule)

e) $\partial f(\bar{x}) = \{v\in\mathbb{E} \mid (v,-1)\in N_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))\}$.

Proof:

a) We have

$$\partial f(\bar{x}) = \bigcap_{x\in\mathbb{E}}\{v \mid \langle x-\bar{x},\, v\rangle \leq f(\bar{x})-f(x)\}\,,$$

and intersection preserves closedness and convexity.

b) Obvious.

c) Follows immediately from Theorem 3.3.2.

d) By definition we have

$$0\in\partial f(\bar{x}) \quad\Longleftrightarrow\quad f(x)\geq f(\bar{x}) \quad (x\in\mathbb{E}).$$

e) Notice that

$$\begin{aligned} v\in\partial f(\bar{x}) \quad&\Longleftrightarrow\quad f(x)\geq f(\bar{x})+\langle v,\, x-\bar{x}\rangle \quad (x\in\operatorname{dom} f)\\ &\Longleftrightarrow\quad \alpha\geq f(\bar{x})+\langle v,\, x-\bar{x}\rangle \quad ((x,\alpha)\in\operatorname{epi} f)\\ &\Longleftrightarrow\quad 0\geq\langle (v,-1),\, (x-\bar{x},\alpha-f(\bar{x}))\rangle \quad ((x,\alpha)\in\operatorname{epi} f)\\ &\Longleftrightarrow\quad (v,-1)\in N_{\operatorname{epi} f}(\bar{x}, f(\bar{x})). \end{aligned}$$

□

Part b) and c) of the above Proposition imply that

$$\mathrm{ri}\,(\mathrm{dom}\, f) \subset \mathrm{dom}\,\partial f \subset \mathrm{dom}\, f \quad (f \in \Gamma).$$

The subdifferential of a convex function might well be empty, contain only a single point, be bounded (hence compact) or unbounded as the following examples illustrate.

Example 3.8.3

a) (Indicator function) Let $C \subset \mathbb{E}$ be convex and $\bar{x} \in C$. Then

$$\begin{aligned} g \in \partial\delta_C(\bar{x}) \quad &\Longleftrightarrow \quad \delta_C(x) \geq \delta_C(\bar{x}) + \langle g,\, x - \bar{x}\rangle \quad (x \in \mathbb{E}) \\ &\Longleftrightarrow \quad 0 \geq \langle g,\, x - \bar{x}\rangle \quad (x \in C), \end{aligned}$$

i.e. $\partial\delta_C(\bar{x}) = N_C(\bar{x})$.

b) (Euclidean norm) We have

$$\partial\|\cdot\|(\bar{x}) = \begin{cases} \frac{\bar{x}}{\|\bar{x}\|} & \text{if} \quad \bar{x} \neq 0, \\ \mathbb{B} & \text{if} \quad \bar{x} = 0 \end{cases}$$

as can be verified by elementary considerations.

c) (Empty subdifferential) Consider

$$f : x \mapsto \begin{cases} -(1 - |x|^2)^{1/2} & \text{if} \quad |x| \leq 1, \\ +\infty & \text{else}. \end{cases}$$

Then $\partial f(x) = \emptyset$ for $|x| \geq 1$.

There is a tight connection of subdifferentiation and conjugation of convex functions.

Theorem 3.8.4 (Subdifferential and conjugate function) *Let $f \in \Gamma_0$. Then the following are equivalent:*

i) $y \in \partial f(x)$;

ii) $x \in \mathrm{argmax}_z\{\langle z,\, y\rangle - f(z)\}$;

iii) $f(x) + f^*(y) = \langle x,\, y\rangle$;

iv) $x \in \partial f^*(y)$;

v) $y \in \mathrm{argmax}_w\{\langle x,\, w\rangle - f^*(w)\}$.

Proof: Notice that

$$\begin{aligned} y \in \partial f(x) &\iff f(z) \geq f(x) + \langle y,\, z - x\rangle \quad (z \in \mathbb{E}) \\ &\iff \langle y,\, x\rangle - f(x) \geq \sup_z \{\langle y,\, z\rangle - f(z)\} \\ &\iff f(x) + f^*(y) \leq \langle x,\, y\rangle \\ &\iff f(x) + f^*(y) = \langle x,\, y\rangle\,, \end{aligned}$$

where the last equality makes use of the Fenchel-Young inequality (3.25). This establishes the equivalences between i), ii) and iii). Applying the same reasoning to f^* and noticing that $f^{**} = f$ gives the missing equivalences. □

One consequence of Theorem 3.8.4 is that the set-valued mappings ∂f and ∂f^* are inverse to each other. We notice some other interesting implications of the latter theorem.

Corollary 3.8.5 *Let $C \subset \mathbb{E}$. Then the following hold:*

a) For $x \in \operatorname{dom} \sigma_C$, we have $\partial \sigma_C(x) = \operatorname{argmax}_C \langle \cdot,\, x\rangle$.

b) If C is a closed, convex cone the following are equivalent:

- *i) $y \in \partial \delta_C(x)$;*
- *ii) $x \in \partial \delta_{C^\circ}(y)$;*
- *iii) $x \in C, \quad y \in C^\circ \quad$ and $\quad \langle x,\, y\rangle = 0$.*

As another consequence we obtain the very desirable property that the subdifferential operator of a closed, proper and convex functions f has a closed graph

$$\operatorname{gph} \partial f := \{(x, y) \in \mathbb{E} \times \mathbb{E} \mid y \in \partial f(x)\}\,,$$

which is also referred to as *outer semicontinuity* of ∂f.

Corollary 3.8.6 (Outer semicontinuity of ∂f) *Let $f \in \Gamma_0$ and suppose $\{x_k\} \to x$ and $\{y_k \in \partial f(x_k)\} \to y$. Then $y \in \partial f(x)$, i.e. $\operatorname{gph} \partial f \in \mathbb{E} \times \mathbb{E}$ is closed.*

Proof: By Theorem 3.8.4 we have

$$f(x_k) + f^*(y_k) = \langle x_k,\, y_k\rangle \quad (k \in \mathbb{N}).$$

Using that f and f^* are lsc we obtain

$$f(x) + f^*(y) \leq \langle x,\, y\rangle\,,$$

which together with the Fenchel-Young inequality gives

$$f(x) + f^*(y) = \langle x,\, y\rangle\,.$$

But then, again, Theorem 3.8.4 implies that $y \in \partial f(x)$. $\square$

We close out the section with some useful boundedness properties of the subdifferential operator.

Theorem 3.8.7 (Boundedness properties ∂f) *Let $f \in \Gamma_0$ and $X \subset \operatorname{int}(\operatorname{dom} f)$ nonempty, open and convex. Then the following hold:*

a) f is Lipschitz continuous with modulus $L > 0$ on X if and only if $\|v\| \leq L$ for all $x \in X$ and $v \in \partial f(x)$.

b) ∂f maps bounded sets which are compactly contained in $\operatorname{int}(\operatorname{dom} f)$ to bounded sets.

Proof:

a) First, assume that f is Lipschitz on X with modulus $L > 0$. Now take $x \in X$ and $v \in \partial f(x)$. By the subgradient inequality we have

$$f(y) \geq f(x) + \langle v,\, y - x\rangle \quad (y \in \mathbb{E}). \tag{3.35}$$

Since X is open, there exists $r > 0$ such that $\overline{B}_r(x) \subset X$. Inserting the vector

$$y = x + \frac{r}{\|v\|} v \in \overline{B}_r(x).$$

in (3.35) yields

$$f\left(x + \frac{r}{\|v\|} v\right) \geq f(x) + r\|v\|.$$

Rearranging these terms gives

$$\|v\| \leq \frac{1}{r}\left| f\left(x + \frac{r}{\|v\|} v\right) - f(x)\right| \leq L,$$

which shows the first implication in a).

Conversely, assume that $\|v\| \leq L$ for all $x \in X$ and $v \in \partial f(x)$. For $x, y \in X$ and $v \in \partial f(x)$ we hence have

$$f(x) - f(y) \leq \langle v,\, x - y\rangle \leq \|v\| \cdot \|x - y\| \leq L\|x - y\|,$$

where we use the subgradient inequality and Cauchy-Schwarz. Interchanging the roles of x and y yields

$$f(y) - f(x) \leq L\|x - y\|,$$

which all in all gives

$$|f(x) - f(y)| \leq L\|x - y\|.$$

Therefore, f is Lipschitz continuous on X with modulus $L > 0$.

b) Let K be compactly contained in $\mathrm{int}\,(\mathrm{dom}\,f)$. Hence, we can assume w.l.o.g. that K is compact. Now, suppose there were a sequences $\{x_k \in K\}$ and $\{v_k \in \partial f(x_k)\}$ such that $\|v_k\| \to \infty$. Since K is compact, we can assume w.l.o.g. that $x_k \to x \in K$. Now take $r > 0$ such that $\overline{B}_r(x) \in X$. By Theorem 3.5.6, f is Lipschitz on $\overline{B}_r(x)$ with modulus, say, $L > 0$. In view of part a) we infer that $\|v\| \leq L$ for all $v \in \partial f(y)$ and $y \in \overline{B}_r(x)$. Since $x_k \in \overline{B}_r(x)$ for all k sufficiently large, we thus have $\|v_k\| \leq L$ for these k, which contradicts the assumption that $\{v_k\}$ were unbounded.

□

3.8.2 Connection to the directional derivative

The subdifferential of a convex function is intimately tied to its *directional derivative*, which we define now.

Definition 3.8.8 (Directional derivative) *Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ be proper. For $x \in \mathrm{dom}\, f$ we say that f is* directionally differentiable *at $\bar{x}$ in the direction $d \in \mathbb{E}$ if*

$$\lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

exists (in an extended real-valued sense). In this case we call

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

the directional derivative *of f at x in the direction of d.*

Proposition 3.8.9 (Directional derivative of a convex function) *Let $f \in \Gamma, x \in \mathrm{dom}\, f$ and $d \in \mathbb{E}$. Then the following hold:*

a) The difference quotient

$$t > 0 \mapsto q(t) := \frac{f(x + td) - f(x)}{t}$$

is nondecreasing.

b) $f'(x; d)$ exists (in $\overline{\mathbb{R}}$) with

$$f'(x; d) = \inf_{t > 0} q(t),$$

c) $f'(x; \cdot)$ is sublinear with $\mathrm{dom}\, f'(x; \cdot) = \mathbb{R}_+(\mathrm{dom}\, f - x)$.

d) $f'(x; \cdot)$ is proper and lsc for $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$.

Proof:

a) Fix $0 < s < t$ and put $\lambda := \frac{s}{t} \in (0,1)$ and $z := x + td$. If $f(z) = +\infty$, then $q(s) \leq q(t) = f(z) = +\infty$. Otherwise, by convexity of f, we have

$$f(x + sd) = f(\lambda z + (1-\lambda)x) \leq \lambda f(z) + (1-\lambda)f(x) = f(x) + \lambda(f(z) - f(x)),$$

hence, $q(s) \leq q(t)$ also in this case.

b) The infimum representation follows from a) since $q(t)$ decreases as $t \downarrow 0$. This also gives the existence statement, since an infimum always exists in the extended real-valued sense.

c) First notice that $0 \in \operatorname{dom} f$ as $f'(x;0) = 0$ and that $f'(x;\alpha d) = \alpha f'(x;d)$ for all $\alpha > 0$ and $d \in \mathbb{E}$, i.e. f is positively homogeneous. We now show that $f'(x;\cdot)$ is also convex, which then proves sublinearity: To this end, let $(d,\alpha), (h,\beta) \in \operatorname{epi}_< f'(x;\cdot)$. Then

$$\frac{f(x+td) - f(x)}{t} < \alpha \quad \text{and} \quad \frac{f(x+th) - f(x)}{t} < \beta$$

for all $t > 0$ sufficienty small. For such $t > 0$, by convexity of f, we compute

$$\begin{aligned} f(x + t(\lambda d + (1-\lambda)h)) - f(x) &= f(\lambda(x+td) + (1-\lambda)(x+th)) - f(x) \\ &\leq \lambda(f(x+td) - f(x)) + (1-\lambda)(f(x+th) - f(x)) \end{aligned}$$

for all $\lambda \in (0,1)$. This implies

$$\begin{aligned} &\frac{f(x + t(\lambda d + (1-\lambda)h)) - f(x)}{t} \\ \leq \;& \lambda \frac{f(x+td) - f(x)}{t} + (1-\lambda)\frac{(x+th) - f(x)}{t} \end{aligned}$$

for all $t > 0$ sufficiently small and $\lambda \in (0,1)$. Letting $t \downarrow 0$ gives

$$f'(x;\lambda d + (1-\lambda)h) \leq \lambda f'(x;d) + (1-\lambda)f'(x;h) < \lambda\alpha + (1-\lambda)\beta \quad (\lambda \in (0,1)),$$

which shows convexity of $\operatorname{epi}_< f'(x\,\cdot)$ and thus of $f'(x;\cdot)$. Hence, as $f'(x;\cdot)$ was proven to be positively homogeneous as well, it is sublinear, cf. Proposition 3.6.11.

The fact that $\operatorname{dom} f'(x;\cdot) = \mathbb{R}_+(\operatorname{dom} f - x)$ follows from b):

$$\begin{aligned} d \in \operatorname{dom} f'(x;\cdot) &\iff \exists t > 0: \ \frac{f(x+td) - f(x)}{t} < +\infty \\ &\iff \exists t > 0: \ f(x+td) - f(x) < +\infty \\ &\iff \exists t > 0: \ x + td \in \operatorname{dom} f \\ &\iff d \in \mathbb{R}_+(\operatorname{dom} f - x). \end{aligned}$$

d) From c) we know that $f'(x;\cdot)$ is, in particular, convex with $\mathrm{dom}\, f'(x;\cdot) = \mathbb{R}_+(\mathrm{dom}\, f - x)$ which is a subspace by Exercise 2.5. Since $f'(x;0) = 0$, by Exercise 3.4. we now see that $f'(x,\cdot)$ must be proper. Moreover, by Proposition 3.5.2 it follows that it agrees with its closure everywhere since its domain has no relative boundary. Thus, $f'(x;\cdot)$ is lsc.

□

We now establish the connection between the subdifferential and the directional derivative of a proper convex functions. The first result in this regard characterizes subgradients using directionbal derivatives and shows that the latter is even proper on the domain of the subdifferential operator.

Proposition 3.8.10 *Let $f \in \Gamma$ and $x \in \mathrm{dom}\, \partial f$. Then we have:*

a) The following are equivalent.

i) $v \in \partial f(x)$;

ii) $f'(x;d) \geq \langle v,\, d\rangle \quad (d \in \mathbb{E})$.

b) $f'(x;\cdot)$ is proper and sublinear.

Proof:

a) We realize that the subgradient inequality for $v \in \mathbb{E}$ is equivalent to

$$\frac{f(x+\lambda d) - f(x)}{\lambda} \geq \langle d,\, v\rangle \quad (\lambda > 0,\ d \in \mathbb{E}). \tag{3.36}$$

As the left-hand side decreases to $f'(x;d)$ as $\lambda \downarrow 0$, this is equivalent to ii). This shows the equivalence of i) and ii).

b) Take $v \in \partial f(x)$. Then a) yields $f'(x;\cdot) \geq \langle \cdot,\, v\rangle$ and therefore, $f'(x;\cdot)$ does not take the value $-\infty$. Hence, in view of Proposition 3.8.9 c) and the fact that $f'(x;0) = 0$, $f'(x;\cdot)$ is proper and sublinear.

□

We continue with our main result of this section.

Theorem 3.8.11 (Directional derivative and subdifferential) *Let $f \in \Gamma$ and $x \in \mathrm{dom}\, \partial f$. Then*

$$\mathrm{cl}\,(f'(x,\cdot)) = \sigma_{\partial f(x)},$$

i.e. the lower semicontinuous hull of $f'(x;\cdot)$ is the support function of $\partial f(x)$.

Proof: Due to Proposition 3.8.9 we know that $f'(x;\cdot)$ is proper and sublinear. By Corollary 3.6.16, we thus see that $\mathrm{cl}\,(f'(x;\cdot)) = \sigma_C$ for $C = \{v \mid \langle v, d\rangle \le f'(x;d)\ (d\in\mathbb{E})\}$. But in view of Proposition 3.8.9 $C = \partial f(x)$, which concludes the proof. $\square$

Theorem 3.8.11 has a list of very important consequences.

Corollary 3.8.12 *Let $f\in\Gamma$ and $x\in\mathrm{ri}\,(\mathrm{dom}\, f)$. Then*

$$f'(x;\cdot) = \sigma_{\partial f(x)}.$$

Proof: Follows from Theorem 3.8.11 and Proposition 3.8.9 d).

$\square$

Corollary 3.8.13 *Let $f\in\Gamma$ and $x\in\mathrm{dom}\, f$. Then $\partial f(x)$ is nonempty and bounded if and only if $x\in\mathrm{int}\,(\mathrm{dom}\, f)$.*

Proof: If $x\in\mathrm{int}\,(\mathrm{dom}\, f)$, we know from Corollary 3.8.12 that $f'(x;\cdot) = \sigma_{\partial f(x)} > -\infty$. From Proposition 3.8.9 we know that $\mathrm{dom}\, f'(x;\cdot) = \mathbb{R}_+(\mathrm{dom}\, f - x)$. Since $x\in\mathrm{int}\,(\mathrm{dom}\, f)$, we have $\mathbb{R}_+(\mathrm{dom}\, f - x) = \mathbb{E}$, hence $f'(x;\cdot)$ is finite, i.e. $\sigma_{\partial f(x)}$ is finite, therefore (see Exercise 3.18.), $\partial f(x)$ is bounded (and nonempty).

In turn, if $\partial f(x)$ is bounded and nonempty then, by Theorem 3.8.11 and Exercise 3.18., $\mathrm{cl}\,(f'(x;\cdot)) = \sigma_{\partial f(x)}$ is finite. Hence, $f'(x;\cdot)$ must be finite, thus $\mathbb{R}_+(\mathrm{dom}\, f - x) = \mathrm{dom}\, f'(x;\cdot) = \mathbb{E}$. Using Exercise 2.5., this implies that $0\in\mathrm{int}\,(\mathrm{dom}\, f - x)$, i.e. $x\in\mathrm{int}\,(\mathrm{dom}\, f)$.

$\square$

Corollary 3.8.14 (Max formula) *Let $f\in\Gamma$ and $x\in\mathrm{int}\,(\mathrm{dom}\, f)$. Then*

$$f'(x;\cdot) = \max_{v\in\partial f(x)} \langle v, \cdot\rangle\,.$$

3.8.3 Subgradients of differentiable functions

In this section we want to study the subdifferential of convex functions at points of differentiablilty. We will ultimately proof that a convex function is differentiable, in fact continuously differentiable, at a point in the interior of its domain if and only if its subdifferentiable is a singleton, which then consists of the gradient only. Moreover, we will show that a differentiable convex functions is in fact continuously differentiable.

Theorem 3.8.15 *Let $f\in\Gamma$ and $x\in\mathrm{int}\,(\mathrm{dom}\, f)$. Then $\partial f(x)$ is a singleton if and only if f is differentiable at x. In this case we have $\partial f(x) = \{\nabla f(x)\}$.*

Proof: If f is differentiable at x then $f'(x;\cdot) = \langle \nabla f(x),\, \cdot\rangle$. Thus, by Proposition 3.8.10 a) the elements $v \in \partial f(x)$ are characterized through

$$\langle \nabla f(x),\, d\rangle \geq \langle v,\, d\rangle \quad (d \in \mathbb{E}),$$

which implies that $v = \nabla f(x)$, i.e. $\partial f(x) = \{\nabla f(x)\}$. This proves the first implication.

Conversely, assume that $\partial f(x) = \{v\}$. We have to show that

$$\lim_{d\to 0} \frac{f(x+d) - f(x) - \langle v,\, d\rangle}{\|d\|} = 0. \tag{3.37}$$

To this end, take $\{d_k\} \to 0$ arbitrarily and put

$$t_k := \|d_k\| \quad \text{and} \quad p_k := \frac{d_k}{\|d_k\|} = \frac{d_k}{t_k} \quad (k \in \mathbb{N}).$$

Then there exists $K \subset \mathbb{N}$ and $p \in \mathbb{E} \setminus \{0\}$ such that $p_k \to_K p$. Then we compute that

$$\begin{aligned}
\frac{f(x+d_k) - f(x) - \langle v,\, d_k\rangle}{\|d_k\|} &= \frac{f(x+t_kp_k) - f(x) - t_k\,\langle v,\, p_k\rangle}{t_k} \\
&= \frac{f(x+t_kp) - f(x)}{t_k} + \frac{f(x+t_kp_k) - f(x+t_kp)}{t_k} - \langle v,\, p_k\rangle
\end{aligned}$$

As we pass to the limit on K, the first summand tends to $f'(x;p)$, cf. Theorem 3.8.9. The second one goes to 0, since f is Lipschitz around $x \in \operatorname{int}(\operatorname{dom} f)$, cf. Theorem 3.5.6. Thus, we have

$$\begin{aligned}
\lim_{k\in K} \frac{f(x+d_k) - f(x) - \langle v,\, d_k\rangle}{\|d_k\|} &= f'(x;p) - \langle v,\, p\rangle \\
&= \max_{w\in\partial f(x)} \langle w,\, p\rangle - \langle v,\, p\rangle \\
&= 0,
\end{aligned}$$

where the second equality uses Corollary 3.8.14 and the last one exploits the fact that $\partial f(x) = \{v\}$. Since p was an arbitrary accumulation point of the bounded sequence $\{p_k\}$, we have that

$$\lim_{k\in\mathbb{N}} \frac{f(x+d_k) - f(x) - \langle v,\, d_k\rangle}{\|d_k\|} = 0.$$

Since $\{d_k\} \to 0$ was chosen arbitrarily this gives (3.37) and hence concludes the proof. □

Theorem 3.8.16 *Let $f \in \Gamma$ and $x \in \operatorname{int}(\operatorname{dom} f)$. Then f is continuously differentiable on* $\operatorname{int}(\operatorname{dom} f)$ *if and only if $\partial f(x)$ is a singleton for all $x \in \operatorname{int}(\operatorname{dom} f)$.*

Proof: If f is continuously differentiable on $\mathrm{int}\,(\mathrm{dom}\, f)$, Theorem 3.8.15 immediately implies that $\partial f(x) = \{\nabla f(x)\}$ for all $x \in \mathrm{int}\,(\mathrm{dom}\, f)$.

Conversely, let $\partial f(x)$ be a singleton for all $x \in \mathrm{int}\,(\mathrm{dom}\, f)$. By Theorem 3.8.15, f is differentiable at every point $x \in \mathrm{int}\,(\mathrm{dom}\, f)$. Now, fix $x \in \mathrm{int}\,(\mathrm{dom}\, f)$ and take $\{x_k \in \mathrm{int}\,(\mathrm{dom}\, f)\} \to x$. Then we have $\nabla f(x_k) \in \partial f(x_k)$ for all $k \in \mathbb{N}$. (In fact, we have $\partial f(x_k) = \{\nabla f(x_k)\}$, but that is unimportant to our reasoning.) Now choose $r > 0$ such that $\overline{B}_r(x) \subset \mathrm{int}\,(\mathrm{dom}\, f)$. Since $x_k \in \overline{B}_r(x)$ for all k sufficiently large, we also have $\nabla f(x_k) \in \partial f(\overline{B}_r(x))$, which is bounded due to Theorem 3.8.7 b). Hence, $\{\nabla f(x_k)\}$ has an accumulation point $g \in \mathbb{E}$ which, by Corollary 3.8.6 lies in $\partial f(x) = \{\nabla f(x)\}$. Hence, $\nabla f(x_k) \to g = \nabla f(x)$ on the respective subsequence. Since this holds true for any accumulation point, we acutally have $\nabla f(x_k) \to \nabla f(x)$ on the whole sequence. As $x_k \to x$ was chosen arbitrarily, this proves the statement. □

Corollary 3.8.17 (Differentiability of finite convex functions) *Let $f : \mathbb{E} \to \mathbb{R}$ convex. Then f is differentiable if and only if f is continuously differentiable.*

3.8.4 Subdifferential calculus

In this section we want to compute the subdifferential for various convex functions that come out of convexity-preserving operations.

We start with the subdifferential of the separable sum of convex functions.

Proposition 3.8.18 (Subdifferential of separable sum) *Let $f_i \in \Gamma(\mathbb{E}_i)$ $(i = 1, 2)$. Then*

$$\partial(f_1 \oplus f_2) = \partial f_1 \times \partial f_2.$$

Proof: For $(x_1, x_2) \in \mathbb{E}_1 \times \mathbb{E}_2$ we have

$$\begin{aligned}
&(v_1, v_2) \in \partial f_1(x) \times \partial f_2(y) \\
&\quad \Leftrightarrow \quad f_i(y_i) \geq f_i(x_i) + \langle v_i,\, y_i - x_i\rangle \quad (y_i \in \mathbb{E}_i,\ i = 1, 2) \\
&\quad \Leftrightarrow \quad f_1(y_1) + f_2(y_2) \geq f_1(x_1) + f_2(x_2) + \langle v_1,\, y_1 - x_1\rangle + \langle v_2,\, y_2 - x_2\rangle \quad ((y_1, y_2) \in \mathbb{E}_1 \times \mathbb{E}_2) \\
&\quad \Leftrightarrow \quad (f_1 \oplus f_2)(y_1, y_2) \geq (f_1 \oplus f_2)(x) + \langle (v_1, v_2),\, (y_1, y_2) - (x_1, x_2)\rangle \quad ((y_1, y_2) \in \mathbb{E}_1 \times \mathbb{E}_2) \\
&\quad \Leftrightarrow \quad (v_1, v_2) \in \partial(f_1 \oplus f_2)(x_1, x_2).
\end{aligned}$$

Here, the '$\Leftarrow$'-implication in the second equivalence follows from setting $x_j = y_j$ for $j \neq i$, $i = 1, 2$. □

Note that the above result, by induction, extends to artbitrary finite separable sums of convex functions, and without any more effort, can be proven for a separable some over much more general than only finite index sets, cf. [1, Proposition 16.8], but we only need that case of two functions in our study.

We continue with a subdifferential rule fo epi-compositions, cf. Proposition 3.1.15 and Proposition 3.6.30.

Proposition 3.8.19 (Subdifferential of epi-composition) *Let $f \in \Gamma_0(\mathbb{E})$ and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$. Then for $y \in \operatorname{dom} Lf$ and $x \in L^{-1}(\{y\})$ the following hold:*

a) If $(Lf)(y) = f(x)$ then $\partial(Lf)(y) = (L^*)^{-1}(\partial f(x))$.

b) If $(L^*)^{-1}(\partial f(x)) \neq \emptyset$ then $(Lf)(y) = f(x)$.

Proof: Let $v \in \mathbb{E}'$. From Proposition 3.6.30 a) and Theorem 3.8.4 we infer that

$$\begin{aligned} f(x) + (Lf)^*(v) = \langle y,\, v\rangle \quad &\Leftrightarrow \quad f(x) + f^*(L^*(v)) = \langle L(x),\, v\rangle \\ &\Leftrightarrow \quad f(x) + f^*(L^*(v)) = \langle x,\, L^*(v)\rangle \\ &\Leftrightarrow \quad L^*(v) \in \partial f(x) \\ &\Leftrightarrow \quad v \in (L^*)^{-1}(\partial f(x)). \end{aligned} \tag{3.38}$$

a) Theorem 3.8.4 and Proposition 3.6.30 a) imply that

$$\begin{aligned} v \in \partial(Lf)(y) \quad &\Leftrightarrow \quad (Lf)(y) + (Lf)^*(v) = \langle y,\, v\rangle \\ &\Leftrightarrow \quad f(x) + f^*(L^*v) = \langle L(x),\, v\rangle\,. \end{aligned} \tag{3.39}$$

Combining (3.38) and (3.39) gives a).

b) Suppose $v \in (L^*)^{-1}(\partial f(x))$. Then the Fenchel-Young inequality, the fact that $L(x) = y$ and (3.38) imply that

$$\langle y,\, v\rangle \leq (Lf)(y) + (Lf)^*(v) \leq f(x) + (Lf)^*(v) = \langle y,\, v\rangle\,,$$

hence $(Lf)(y) = f(x)$.

□

We exploit Proposition 3.8.19 to obtain an important subdifferential result for infimal convolutions.

Theorem 3.8.20 (Subdifferentiation of infimal convolutions) *Let $f, g \in \Gamma_0$ as well as $x \in \operatorname{dom}(f \# g) (= \operatorname{dom} f + \operatorname{dom} g)$. Then the following hold:*

a) We have

$$\partial(f\#g)(x) = \partial f(y) \cap \partial g(x-y) \quad (y \in \operatorname*{argmin}_{u\in\mathbb{E}} \{f(u) + g(x-u)\}).$$

b) If $\partial f(y) \cap \partial g(x-y) \neq \emptyset$ for some $y \in \mathbb{E}$ then $(f\#g)(x) = f(y) + g(x-y)$, i.e. $y \in \operatorname{argmin}_{u\in\mathbb{E}} \{f(u) + g(x-u)\}$.

Proof: Consider the linear mapping $L : (a, b) \in \mathbb{E} \times \mathbb{E} \mapsto a + b \in \mathbb{E}$. Then $L^* : z \in \mathbb{E} \mapsto (z, z) \in \mathbb{E} \times \mathbb{E}$. By definition of the respective operations we have $f \# g = L(f \oplus g)$, in particular, $\operatorname{dom} L(f \oplus g) = \operatorname{dom} f \# g$. Thus, $L(y, x - y) = x \in \operatorname{dom} L(f \oplus g)$.

a) Let $y \in \operatorname{argmin}_{u \in \mathbb{E}} \{f(u) + g(x - u)\}$. Since $(L(f \oplus g))(x) = (f \oplus g)(y, x - y)$, Proposition 3.8.19 a) and Proposition 3.8.18 imply that

$$\begin{aligned} \partial(f \# g)(x) &= \partial(L(f \oplus g))(x) \\ &= (L^*)^{-1}(\partial(f \oplus g)(y, x - y)) \\ &= (L^*)^{-1}(\partial f(x) \times \partial g(x - y)) \\ &= \partial f(x) \cap \partial g(x - y). \end{aligned}$$

b) By assumption we have

$$\emptyset \neq \partial f(x) \cap \partial g(x - y) = (L^*)^{-1}(\partial f(x) \times \partial g(x - y)) = (L^*)^{-1}(\partial(f \oplus g)(x, x - y)).$$

Thus, Proposition 3.8.19 b) implies

$$(f \# g)(x) = (L(f \oplus g))(x) = (f \oplus g)(x, x - y) = f(x) + g(x - y).$$

□

As a first application we obtain the subdifferential of the (Euclidean) distance function.

Example 3.8.21 (Subdifferential of Euclidean distance) Let $C \subset \mathbb{E}$ be nonempty, closed and convex. Then

$$\partial \operatorname{dist}_C(x) = \begin{cases} \left\{ \frac{x - P_C(x)}{\operatorname{dist}_C(x)} \right\} & \text{if} \quad x \notin C, \\ N_C(x) \cap \mathbb{B} & \text{if} \quad x \in \operatorname{bd} C, \\ \{0\} & \text{else}. \end{cases}$$

This can be seen using the Examples 3.4.4, 3.4.11 and 3.8.3 a) and Theorem 3.8.20.

Our next goal is to establish a subdifferential for the sum of convex functions as well as the composition of a convex function and a linear mapping.In fact, both of these problems will be answered by a general result for the subdifferentiation of the convex function

$$f + g \circ L \quad \text{where} \quad f \in \Gamma_0(\mathbb{E}_1),\ g \in \Gamma_0(\mathbb{E}_2),\ L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2).$$

To establish the subdifferential calculus for the latter function we need some preliminary results.

Lemma 3.8.22 *Let* $f \in \Gamma(\mathbb{E}_1), g \in \Gamma(\mathbb{E}_2)$ *and* $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. *Then*

$$\partial(f + g \circ L) \supset \partial f + L^* \circ (\partial g) \circ L.$$

Proof: Let $x \in \mathbb{E}$. A generic point in $\partial f(x) + (L^* \circ (\partial g) \circ L)(x)$ is of the form $u + L^* v$ with $u \in \partial f(x)$ and $v \in \partial g(L(x))$. The subdifferential inequality yields

$$f(y) \geq f(x) + \langle u,\, y - x\rangle \quad \text{and} \quad g(L(y)) \geq g(L(x)) + \langle v,\, L(y) - L(x)\rangle \quad (y \in \mathbb{E}_1).$$

Combining these two inequalities gives

$$f(y) + g(L(y)) \geq f(x) + g(L(x)) + \langle u + L^*(v),\, y - x\rangle \quad (y \in \mathbb{E}_1),$$

i.e. $u + L^* v \in \partial(f + g \circ L)(x)$. □

Proposition 3.8.23 *Let* $f \in \Gamma_0(\mathbb{E}_1), g \in \Gamma_0(\mathbb{E}_2)$ *and* $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ *such that* $(f + g \circ L)^* = \min_{v \in \mathbb{E}_2} \{f^*((\cdot) - L^*(v)) + g^*(v)\}$. *Then*

$$\partial(f + g \circ L) = \partial f + L^* \circ (\partial g) \circ L.$$

Proof: In view of Lemma 3.8.22 it remains to show that $\operatorname{gph} \partial(f + g \circ L) \subset \operatorname{gph} (\partial f + L^*(\partial g) \circ L)$: To this end, take $(x, u) \in \operatorname{gph} \partial(f + g \circ L)$. By Theorem 3.8.4, we have

$$(f + g \circ L)(x) + (f + g \circ L)^*(u) = \langle x,\, u\rangle\,. \tag{3.40}$$

On the other hand, by asumption, there exists $v \in \mathbb{E}_2$ such that

$$(f + g \circ L)^*(u) = f^*(u - L^*(v)) + g^*(v).$$

Combining this with (3.40), we obtain

$$[f(x) + f^*(u - L^*(v)) - \langle x,\, u - L^*(v)\rangle] + [g(L(x)) + g^*(v) - \langle x,\, L^*(v)\rangle] = 0.$$

By the Fenchel-Young inequality (3.25) we thus obtain

$$f(x) + f^*(u - L^*(v)) = \langle x,\, u - L^*(v)\rangle \quad \text{and} \quad g(L(x)) + g^*(v) - \langle x,\, L^*(v)\rangle = 0.$$

Invoking Theorem 3.8.4 again yields

$$u - L^*(v) \in \partial f(x) \quad \text{and} \quad v \in \partial g(L(x)),$$

hence, $u \in \partial f(x) + L^* \partial g(L(x))$ as desired.

□

We now come to the announced main result.

Theorem 3.8.24 (Generalized subdifferential sum rule) *Let* $f \in \Gamma_0(\mathbb{E}_1), g \in \Gamma_0(\mathbb{E}_2)$ *and* $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$

$$\partial(f + g \circ L) \supset \partial f + L^* \circ (\partial g) \circ L. \tag{3.41}$$

Under the qualification condition

$$L(\mathrm{ri}\,(\mathrm{dom}\, f)) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset \tag{3.42}$$

equality holds in (3.41).

Proof: By our qualification condition $\mathrm{ri}\,(\mathrm{dom}\, g) \cap L(\mathrm{ri}\,\mathrm{dom}\, f) \neq \emptyset$, we infer from Corollary 3.7.6 that $(f + g \circ L)^* = \min_{v \in \mathbb{E}_2} \{f^*((\cdot) - L^*(v)) + g^*(v)\}$. Hence, the assertion follows from Proposition 3.8.23. □

The generalized subdifferential sum rule has two many important consequences, two of which we present now.

Corollary 3.8.25 (Subdifferential sum rule) *Let* $f, g \in \Gamma$ *then*

$$\partial(f + g) \supset \partial f(x) + \partial g(x) \quad (x \in \mathbb{E}). \tag{3.43}$$

Under the qualification condition

$$\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset \tag{3.44}$$

equality holds in (3.43).

Corollary 3.8.26 (Subdifferential chain rule) *Let* $g \in \Gamma(\mathbb{E}_2)$ *and* $L \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. *Then*

$$\partial(g \circ L) \supset L^*(\partial g) \circ L. \tag{3.45}$$

Under the qualification condition

$$\mathrm{rge}\, L \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset \tag{3.46}$$

equality holds in (3.45).

We proceed with the subdifferential of the pointwise maximum of a finite collection of convex functions.

Theorem 3.8.27 (Subdifferential of maximum of convex functions) *For* $i \in I := \{1, \dots, m\}$ *let* $f_i \in \Gamma$ *and* $x \in \cap_{i \in I} \mathrm{int}\,(\mathrm{dom}\, f_i)$ *and set* $f := \max_{i \in I} f_i$ *and* $I(x) := \{i \in I \mid f_i(x) = f(x)\}$. *Then*

$$\partial f(x) = \overline{\mathrm{conv}} \bigcup_{i \in I(x)} \partial f_i(x).$$

Proof: Let $i \in I(x)$ and $u \in \partial f_i(x)$. Then, by the subdifferential inequality, we have

$$\langle u,\, y-x\rangle \le f_i(y) - f_i(x) \le f(y) - f(x) \quad (y \in \mathbb{E}),$$

i.e. $u \in \partial f(x)$. Using this and the fact that $\partial f(x)$ is closed and convex, cf. Proposition 3.8.2 a), we have

$$\partial f(x) \supset \overline{\mathrm{conv}} \bigcup_{i \in I(x)} \partial f_i(x).$$

Now assume the inclusion were strict, i.e. there exists

$$u \in \partial f(x) \setminus \overline{\mathrm{conv}} \bigcup_{i \in I(x)} \partial f_i(x). \tag{3.47}$$

By strong separation there hence exists $s \in \mathbb{E} \setminus \{0\}$ such that

$$\langle s,\, u\rangle > \max_{i \in I(x)} \sup_{z \in \partial f_i(x)} \langle s,\, z\rangle = \max_{i \in I(x)} f_i'(x; s), \tag{3.48}$$

where we use Corollary 3.8.14 for the second identity. In view of Remark 2.6.2 and the fact that $x \in \mathrm{int}\,(\mathrm{dom}\, f_i)$ for all $i \in I$, we realize that we can rescale s such that

$$x + s \in \bigcap_{i \in I} \mathrm{dom}\, f_i = \mathrm{dom}\, f. \tag{3.49}$$

Now let $\{\alpha_k \in (0, 1)\} \downarrow 0$. Since I is finite, we can assume w.l.o.g that there exists $j \in I$ such that

$$f_j(x + \alpha_k s) = f(x + \alpha_k s) \quad (k \in \mathbb{N}). \tag{3.50}$$

For $k \in \mathbb{N}$ we hence have $f_j(x + \alpha_k s) \le (1 - \alpha_k) f_j(x) + \alpha_k f_j(x + s)$ and thus

$$\begin{aligned}
(1 - \alpha_k) f_j(x) \quad &\ge \quad f_j(x + \alpha_k s) - \alpha_k f_j(x + s) \\
&\ge \quad f(x + \alpha_k s) - \alpha_k f(x + s) \\
&\ge \quad f(x) + \langle u,\, \alpha_k s\rangle - \alpha_k f(x + s) \\
&\ge \quad f_j(x) + \alpha_k \langle u,\, s\rangle - \alpha_k f(x + s).
\end{aligned}$$

Here, the second inequality uses (3.50) and the definition of f. The third one uses that $u \in \partial f(x)$ (see 3.47) and the last one is again due to the definition of f. Now letting $k \to \infty$ and using (3.49) yields

$$f_j(x) = f(x). \tag{3.51}$$

Finally, using (3.50), (3.51), (3.47) and (3.48), we obtain

$$f_j'(x; s) < \langle s,\, u\rangle \le \frac{f(x + \alpha_k s) - f(x)}{\alpha_k} = \frac{f_j(x + \alpha_k s) - f_j(x)}{\alpha_k} \to f_j(x; s),$$

which is the desired contradiction. $\square$

A frequently occuring special case of the foregoing result is the following.

Corollary 3.8.28 *For $i \in I := \{1, \dots, m\}$ let $f_i \in \Gamma$ be differentiable at $x \in \bigcap_{i \in I} \mathrm{int}\,(\mathrm{dom}\, f_i)$ and set $f := \max_{i \in I} f_i$ and $I(x) := \{i \in I \mid f_i(x) = f(x)\}$. Then*

$$\partial f(x) = \mathrm{conv}\,\{\nabla f_i(x) \mid i \in I(x)\}\,.$$

Proof: Combine Theorem 3.8.27 and Theorem 3.8.15. □

Exercises to Chapter 3

3.1. **(Domain of an lsc function)** Is the domain of an lsc function closed?

3.2. **(Univariate convex functions)** Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and $I \subset \mathrm{dom}\, f$ be an open intervall. Show the following (without using results from Section 3.1.2):

a) f is convex on I if and only if the *slope-function*

$$x \mapsto \frac{f(x) - f(x_0)}{x - x_0}$$

is nondecreasing on $I \setminus \{x_0\}$.

b) Let f is differentiable on I: Then f is convex on I if f' is nondecreasing on I, i.e.

$$f'(s) \le f'(t) \quad (s, t \in I : s \le t).$$

c) Let f is twice differentiable on I. Then f is convex on I if and only if $f''(x) \ge 0$ for all $x \in I$.

3.3. **(Characterization of convexity)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show the equivalance of:

i) f is convex;

ii) The *strict epigraph* $\mathrm{epi}_{<} f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) < \alpha\}$ of f is convex;

iii) For all $\lambda \in (0, 1)$ we have $f(\lambda x + (1 - \lambda) y) < \lambda \alpha + (1 - \lambda)\beta$ whenever $f(x) < \alpha$ and $f(y) < \beta$.

3.4. **(Properness and closedness of convex functions)** Prove the following:

a) An improper convex function $f : \mathbb{E} \to \overline{\mathbb{R}}$ must have $f(x) = -\infty$ for all $x \in \mathrm{ri}\,(\mathrm{dom}\, f)$.

b) An improper convex function which is lsc, can only have infinite values.

c) Let $f \in \Gamma$. Then $\mathrm{cl}\, f \in \Gamma_0$. Moreover, $\mathrm{cl}\, f$ agrees with f except perhaps on $\mathrm{rbd}\,(\mathrm{dom}\, f)$.

d) If f is convex then $\operatorname{cl} f$ is proper if and only f is proper.

3.5. **(Jensen's Inequality)** Show that $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if

$$f\left(\sum_{i=1}^{p} \lambda_i x_i\right) \le \sum_{i=1}^{p} \lambda_i f(x_i) \quad \forall x_i \in \mathbb{E}\ (i = 1, \dots, p),\ \lambda \in \Delta_p.$$

3.6. **(Quasiconvex functions)** A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is called *quasiconvex* if the level sets $\operatorname{lev}_{\le\alpha} f$ are convex for every $\alpha \in \mathbb{R}$. Show:

a) Every convex function is quasiconvex.

b) $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex if an only if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\} \quad (x, y \in \operatorname{dom} f,\ \lambda \in [0,1]).$$

c) If $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex then $\operatorname{argmin} f$ is a convex set.

3.7. **(Coercivity is level-boundedness)** Show that a function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is coercive if and only if it is level-bounded.

3.8. **(Post-composition with monotonically increasing, convex functions)** Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be convex (and lsc) and let $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be convex (and lsc) and nondecreasing. We put $g(+\infty) := +\infty$ and assume that $\lim_{x\to\infty} g(x) = +\infty$.

a) Show that $g \circ f$ is convex (and lsc);

b) Give a necessary and sufficient condition for $g \circ f$ to be proper.

3.9. **(Supercoercivity in sums)** Let $f \in \Gamma$ and $g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ supercoercive. Show that $f + g$ is supercoercive.

3.10. **(Convergence of prox-operator)** Let $f \in \Gamma_0$ and $\bar{x} \in \operatorname{dom} f$. Prove that

$$P_\lambda f(\bar{x}) \to \bar{x} \quad \text{and} \quad f(P_\lambda f(\bar{x})) \to f(\bar{x}) \quad (\lambda \downarrow 0).$$

3.11. **(Minimizing differentiable convex functions)** Let $f : \mathbb{E} \to \mathbb{R}$ be convex and differentiable and $C \subset \mathbb{E}$. Show that $\bar{x} \in C$ is a minimizer of f over C if and only if $-\nabla f(\bar{x}) \in N_C(\bar{x})$.

3.12. **(Convex hulls of functions)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show the following:

a) $\operatorname{epi}(\overline{\operatorname{conv}}\, f) = \overline{\operatorname{conv}}\,(\operatorname{epi} f)$;

b) $(\operatorname{conv} f)(x) = \inf\left\{\sum_{i=1}^{N+2} \lambda_i f(x_i) \;\middle|\; \lambda \in \Delta_{N+2},\ x = \sum_{i=1}^{N+2} \lambda_i x_i\right\} \quad (x \in \mathbb{E})$.

3.13. **(Properness of convex hull)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$.

a) Show that f is proper if $\operatorname{conv} f$ is. Does the reverse implication hold as well?

b) Show that $\operatorname{conv} f$ is proper if and only if f has an affine minorant.

3.14. **(Self-conjugacy)** Show that $\frac{1}{2}\|\cdot\|^2$ is the only function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $f^* = f$.

3.15. **(Conjugate of negative logdet)** Compute f^* and f^{**} for

$$f : X \in \mathbb{S}^n \mapsto \begin{cases} -\log(\det X) & \text{if} \quad X \succ 0, \\ +\infty & \text{else.} \end{cases}$$

3.16. **(Conjugacy and projections)** Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ and $V \subset \mathbb{E}$ be a subspace such that $\operatorname{dom} f \subset V$. Then $f^*_{|V} \circ P_V = f^* = f^* \circ P_V$.

3.17. **(Positive homogeneity, sublinearity and subadditivity)** Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. Show the following:

a) f is positively homogeneous if and only if $\operatorname{epi} f$ is a cone. In this case $f(0) \in \{0, -\infty\}$.

b) If f is lsc and positively homogeneous with $f(0) = 0$ it must be proper.

c) The following are equivalent:

i) f is sublinear;

ii) f is positively homogeneous and convex;

iii) f is positively homogeneous and subadditive;

iv) $\operatorname{epi} f$ is a convex cone.

3.18. **(Finiteness of support functions)** Let $S\mathbb{E}$ be nonempty. Then σ_S is finite if and only if S is bounded.

3.19. **(Polar sets)** Show the following:

a) If $C \in \mathbb{E}$ is a cone, we have $\{v \mid \langle v, x\rangle \leq 0 \ (x \in C)\} = \{v \mid \langle v, x\rangle \leq 1 \ (x \in C)\}$.

b) $C \subset \mathbb{E}$ is bounded if and only if $0 \in \operatorname{int} C^\circ$.

c) For any closed half-space H containing 0 we have $H^{\circ\circ} = H$.

3.20. **(Gauge functions)** Let $C \subset \mathbb{E}$ be nonempty, closed and convex with $0 \in C$. Prove:

a) $C = \operatorname{lev}_{\leq 1}\gamma_C, \quad C^\infty = \gamma_C^{-1}(\{0\}), \quad \operatorname{dom}\gamma_C = \mathbb{R}_+ C$

b) The following are equivalent:

i) γ_C is a norm (with C as its unit ball);

ii) C is bounded, symmetric ($C = -C$) with nonempty interior.

3.21. **(Cone polarity and conjugacy)** Let $K \subset \mathbb{E}$ be a convex cone. Then $\delta_K \overset{*}{\longleftrightarrow} \delta_{K^\circ}$.

3.22. **(Soft thresholding)** For $f : x \in \mathbb{R}^n \mapsto \|x\|_1$ compute ∂f and $e_\lambda f$ $(\lambda > 0)$.

4 Appendix

4.1 The Attouch-Brézis Theorem

In this section we are devoted to proving the Attouch-Brézis Theorem (see Theorem 3.6.26). To this end, we first present a preliminary version of said result.

Proposition 4.1.1 *Let* $f, g \in \Gamma_0$ *such that* $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$. *Then*

$$(f+g)^* = f^* \# g^*$$

and the infimal convolution is exact.

Proof: For $\eta \in \mathbb{R}$ and $\rho > 0$ define

$$S_{\eta,\rho} := \{(u,v) \in \mathbb{E} \times \mathbb{E} \mid f^*(u) + g^*(v) \leq \eta,\ \|u+v\| \leq \rho\}.$$

Since $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$, we have $\mathbb{R}_+(\operatorname{dom} f - \operatorname{dom} g) = \mathbb{E}$, hence, picking $a, b \in \mathbb{E}$ there exist $x \in \operatorname{dom} f, y \in \operatorname{dom} g$ and $t > 0$ such that $a - b = t(x - y)$. Define

$$\beta_{a,b} := \rho\|b - ty\| + t(f(x) + g(y) + \eta).$$

Then $\beta_{a,b} \in \mathbb{R}$ as $x \in \operatorname{dom} f, y \in \operatorname{dom} g$ and both functions are proper. Hence, for any $(u,v) \in S_{\eta,\rho}$ we compute

$$\begin{aligned}
\langle (a,b),\, (u,v)\rangle &= \langle a,\, u\rangle + \langle b,\, v\rangle \\
&= \langle b - ty,\, u+v\rangle + t(\langle x,\, u\rangle + \langle y,\, u\rangle) \\
&\leq \|b - ty\| \cdot \|u+v\| + t(f(x) + f^*(y) + g^*(y) + g^*(v)) \\
&\leq \beta_{a,b}.
\end{aligned}$$

Here the first inequality uses the Cauchy-Schwarz and the Fenchel-Young inequality. Since $(a,b) \in \mathbb{E} \times \mathbb{E}$ and $(u,v) \in S_{\eta,\rho}$ were chosen arbitrarily we thus obtain

$$\sup_{(u,v) \in S_{\eta,\rho}} |\langle (a,b),\, (u,v)\rangle| \leq \max\{\beta_{a,b}, \beta_{-a,-b}\} < +\infty \quad ((a,b) \in \mathbb{E} \times \mathbb{E}).$$

Exercise 4.1. then shows that $S_{\eta,\rho}$ is bounded.

By the lower semicontiuity of $f^* \oplus g^*$ and $(u,v) \mapsto \|u+v\|$, we have that $S_{\rho,\eta}$ is also closed, hence compact. Hence, the set

$$W_{\eta,\rho} = \{u+v \mid (u,v) \in S_{\eta,\rho}\}$$

is compact as well, in particular, closed. Exercise 4.2. then yields that also

$$W_\eta = \{u+v \mid (u,v) \in \mathbb{E}\times\mathbb{E} : \ f^*(u)+g^*(v) \leq \eta\} \quad (\eta \in \mathbb{R})$$

is closed (possibly empty). Thus, for every $\nu \in \mathbb{R}$, we have

$$\begin{aligned} \operatorname{lev}_{\leq \nu} f^* \# g^* &= \left\{ w \in \mathbb{E} \,\middle|\, \inf_u f^*(u) + g^*(w-u) \leq \nu \right\} \\ &= \bigcap_{\eta > \nu} \{w \in \mathbb{E} \mid \exists u \in \mathbb{E} : \ f^*(u) + g^*(w-u) \leq \eta\} \\ &= \bigcap_{\eta > \nu} W_\eta \end{aligned}$$

is closed. Hence, Proposition 1.2.4 implies that $f^*\#g^*$ is lsc, and, invoking 3.6.25 b) implies that $(f+g)^* = f^*\#g^*$. We still need to show exactness: For these purposes, fix $w \in \mathbb{E}$. If $w \notin \operatorname{dom}(f^*\#g^*)$ there is nothing to show. Hence, assume that $w \in \operatorname{dom}(f^*\#g^*)$. Now, define

$$F : (u,v) \in \mathbb{E}\times\mathbb{E} \mapsto f^*(u) + g^*(v) \in \mathbb{R}\cup\{+\infty\}$$

along with

$$C := \{(u,v) \in \mathbb{E}\times\mathbb{E} \mid u+v = w\} \quad \text{and} \quad D_\eta := C \cap \operatorname{lev}_{\leq \eta} F \quad (\eta \in ((f^*\#g^*)(w), +\infty)).$$

Then $F \in \Gamma_0(\mathbb{E}\times\mathbb{E})$, C is closed and convex and $\emptyset \neq D \subset S_{\eta,\|w\|}$ for all $\eta \in ((f^*\#g^*)(w), +\infty)$. Hence, by what was shown above, D_η is bounded and thus, cf. Corollary 3.2.4 and its proof, F achieves its minimum over C, which proves the exactness and hence concludes the proof. $\square$

We are now in a position to state the desired Attouch-Brézis Theorem which is, actually, a refinement of the foregoing result in that the assumptions can be weakened to get the same assertion.

Theorem 4.1.2 (Attouch-Brézis Theorem) *Let $f, g \in \Gamma_0$ such that*

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq 0. \tag{4.1}$$

holds. Then $(f+g)^* = f^*\#g^*$, and the infimal convolution is exact, i.e. the infimum in the infimal convolution is attained.

Proof: Since $f+g \in \Gamma_0$ (properness is due to the fact that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$) we infer from Theorem 3.6.5 that $(f+g)^* \in \Gamma_0$. Now fix $z \in \mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$ and define

$$\varphi : x \mapsto f(x+z) \quad \text{and} \quad \psi : y \mapsto g(y+z).$$

Then $\mathrm{dom}\,\varphi - \mathrm{dom}\,\psi = \mathrm{dom}\, f - \mathrm{dom}\, g$ and hence (4.1) implies that

$$K := \mathbb{R}_+(\mathrm{dom}\,\varphi - \mathrm{dom}\,\psi) = \mathrm{span}\,(\mathrm{dom}\,\varphi - \mathrm{dom}\,\psi).$$

In particular, we have

$$\mathrm{dom}\,\varphi \subset K \quad \text{and} \quad \mathrm{dom}\,\psi \in K.$$

We notice that the restrictions $\varphi_{|K}$ and $\psi_{|K}$ of φ and ψ, respectively, to the Euclidean space K fulfill $\varphi_{|K}, \psi_{|K} \in \Gamma_0(K)$. Moreover, the qualification condition (4.1) implies that

$$0 \in \mathrm{int}\,(\mathrm{dom}\,\varphi_{|K} - \psi_{|K}).$$

Now, set $h := \langle z, \cdot \rangle$ and let $u \in \mathbb{E}$. Then we compute

$$\begin{aligned}
(f+g)^*(u) - h(u) &= (\varphi+\psi)^*(u) \\
&= (\varphi_{|K} + \psi_{|K})^*(P_K(u)) \\
&= (\varphi^*_{|K} \# \psi^*_{|K})(P_K(u)) \\
&= \min_{v \in K} \left\{ \varphi^*_{|K}(v) + \psi^*_{|K}(P_K(u) - v)) \right\} \\
&= \min_{w \in \mathbb{E}} \left\{ \varphi^*_{|K}(_K(w)) + \psi^*_{|K}(P_K(u-w)) \right\} \\
&= \min_{w \in \mathbb{E}} \{ \varphi^*(w) + \psi^*(u-w) \} \\
&= (\varphi^* \# \psi^*)(u) \\
&= ((f^* - h) \# (g^* - h))(u) \\
&= (f^* \# g^*)(u) - h(u).
\end{aligned}$$

Here the first equation uses the definition of the conjugate and the functions employed. The second equation is due to Exercise 3.16. The third and fourth identity invoke Proposition 4.1.1. The fifth equation uses linearity and surjectivity of the projection. The sixth equation, again, employs Excercise 3.16.. the seventh identiy just the definition of the infimal convolution, where the eighth uses 3.6.6 a). The last equation is then again simply due to the definition of the infimal convolution.

All in all, we have shown that $(f+g)^* = f^* \# g^*$ and the infimal convolution is exact, which is the desired statement.

□

Exercises to Chapter 4

4.1. **(Banach-Steinhaus - finite dimensional version)** Let $T_i \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$ be a family of linear operators such that

$$\sup_{i\in I} \|T_i(x)\| < +\infty \quad (x \in \mathbb{E}_1).$$

Then it follows that $\sup_{i\in I} \|T_i\| < \infty$.

4.2. **(Closedness criterion)** Let $C \subset \mathbb{E}$ such that $C \cap \overline{B}_n(0)$ is closed for all $n \in \mathbb{N}$. Show that C is closed.

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