

Scalability of Struct SVM approaches

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<https://github.com/SoliElvis/structuredPredictionProject>

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1 Introduction

Structured prediction in machine learning is tasked with learning predictors where the labels on the data-points are more than simple tags but have inherent structure which implies the following:

- The number of potential labels for a given feature vector can grow exponentially with the input which makes traditional classification procedures intractable
- A certain intelligibility of the structure; hence a hope to leverage it to improve tractability

It is often quite hard to know in advance whether we can tackle a structured prediction problem with known approaches.

In this paper we discuss two different approaches to solving structured prediction problems, both focusing on a *large-margin approach* which translates into a support-vector machine - like problem formulation.

Let us introduce some notation which we will use throughout the paper. Define the dataset to be

$$S = \{x^{(i)}, y^{(i)}\}_{i=1}^n \in (\mathcal{X} \times \mathcal{Y})^n \quad (1)$$

If one were to frame the machine learning goal in the concisest and simplest way possible we could say that the goal is to *learn* a (parametrized) *predictor* function:

$$\begin{aligned} h_w &: \mathcal{X} \rightarrow \mathcal{Y} \\ &: \hat{x} \mapsto y \quad y \in \mathcal{Y}(\hat{x}) \end{aligned}$$

where \hat{x} is just some arbitrary sampled x and we abuse notation and mean the set valued mapping which outputs the feasible label set for a given x by $\mathcal{Y}(x)$

The traditional probabilistic approach is to compute h or its parameter by calculating the most likely parameter given conditioned on the observed data. However this is more often than not untractable in the structured prediction context as summed up in Taskar, Lacoste-Julien, and Jordan (2006)

In a large margin approach we wish to compute the predictor function the following way :

$$h_w(x) = \arg \max_{y \in \mathcal{Y}} \langle w, \phi(x, y) \rangle \quad (2)$$

where ϕ is just the feature map for the dataset.

This is a constrained optimization problem and the structure of \mathcal{Y} clearly has a big influence on how well we can solve the problem as well as which method should work well.

The dual extragradient approach presented in Taskar, Lacoste-Julien, and Jordan (ibid.) leverages a saddle-point approach to the problem to tackle the problem with a first order method and moreover “this approach allows us to exploit the structure of W and Z separately, allowing for efficient solutions for a wider range of parameterizations and structures.”(ibid.)

Despite its improvements at the time the dual extragradient approach suffers from two main computational draw-backs:

1. It is not seperable/'stochasizable' which is a severe problem as machine learning has seen much of its success in recent years by optimizing through first-order sampled method, i.e. first order method which use a probabilistic approximation of the gradient through sampling instead of computing the actual gradient (which might be unfeasible).
2. The algorithm requires a projection step which can be expensive depending on the structure of \mathcal{Y} and often involves a quadratic problem which doesn't scale too well.

This leads us to consider the Block Seperable Frank Wolfe algorithm presented in Lacoste-Julien et al. (2013)

2 Preliminaries

2.1 Structured SVM N-slack formulation

Let us recall the basic goal; to construct an accurate linear classifier¹

$$h_w(x) = \arg \max_{y \in \mathcal{Y}} \langle w, \phi(x, y) \rangle \quad (3)$$

In general finding the optimal separating hyperplane is an ill-defined problem. In the support vector machine (SVM) setting we want to find the separating hyperplane with the *largest margin*. The support vectors are the datapoints which lie on the the defined margins. It is not obvious that we should strive to completely separate the sets so we can include some slack variables. Indeed, some set of points might not be linearly separable but adding slack variables might allow one to misclassify some point but get a higher proportion of correctly classified points.

We call the following way to pose problem the n -slack formulation ² of the problem :

$$\min_{\omega, \xi} \quad \frac{\lambda}{2} \|w\|^2 + \frac{1}{n} \sum_{i=1}^n \varepsilon_i \quad s.t. \quad \langle w, \psi_i(y) \rangle \geq L(y_i, y) - \varepsilon_i, \quad \forall i, \forall y \in \mathcal{Y}(x) = \mathcal{Y}_i \quad (4)$$

Define $\psi(y) \triangleq \phi(x^{(i)}, y^{(i)}) - \phi(x^{(i)}, y)$ and $L_i(y) \triangleq L(y^{(i)}, y)$

Then the following problem is called the *loss-augmented decoding problem*.

$$H_i = \max_{y \in \mathcal{Y}_i} \{L_i(y) - \langle w, \psi_i(y) \rangle\} \quad (5)$$

We assume it can be solved efficiently; usually through ILPs that have a natural convex relaxation with integral optimal solutions which will be discussed later. Note that the structured hinge-loss is a convex upper bound to the task loss; hence it is quite natural to see that learning w amounts to the unconstrained problem,

$$\min_w \quad \frac{\lambda}{2} \|w\|^2 + \frac{1}{n} \sum_{i=1}^n H_i(w) \quad (6)$$

This last formulation into a non-smooth unconstrained problem is the one we will focus throughout this paper. Indeed it allows us to leverage convex analysis tools which emphasize replacing constraint sets with appropriate additions to the objective value (which often results in non-smooth problems)

¹clearly we can easily extend to non-linear cases, with kernel maps for example

²see Moguerza and Muñoz (2006) for example

3 Extra Gradient Algorithm

3.1 Saddle Point Formulation and Intuition

Let us focus on the loss-augmented formulation

$$\min_w \frac{\lambda}{2} \|w\|^2 + \frac{1}{n} \sum_{i=1}^n H_i(w) \quad (7)$$

which is an unconstrained non-smooth (strongly) convex problem as it is the sum of a strongly convex function, and the average of the H_i s which are convex by construction. What is important to note is that the *objective function is defined through solutions of another smaller optimization (maximization) problem*. Clearly it would be naive to just solve one after the other in batch. There are multiple approaches to this problem. A decade or so ago one formulation one could find was a small QP formulation where the loss augmented decoding is dualized to obtain a more standard quadratic problem which can be plugged into commercial solvers (Taskar, Lacoste-Julien, and Jordan, 2006).

In Taskar, Lacoste-Julien, and Jordan (ibid.) it is proposed to instead use first-order algorithms designed specifically for (continuous) saddle point problems in order to leverage the intrinsic min max structure. Intuitively one could say that the harder the inference/decoding problem is, the closer we are to a saddle-point problem, whereas if the decoding problem is trivial we are back to straight minimization hence standard classification.

This enables us to consider a richer class of problems to tackle, especially for problems where dualizing yields a quadratic program (QP) that is not necessarily efficiently solvable (even though it scales linearly ibid.).

It is not hard to see that the objective function of the non-smooth formulation of struct-svm is equivalent to the following:

$$\min_{w \in W} \max_{z \in Z} \sum_i (w^T F_i z_i + c_i^T z_i - w^T \phi_i(y_i)) \quad (8)$$

where the z_i 's correspond to the relaxation of the binary labels y_i and satisfy the constraints of the structured problem through their restriction to some feasible set Z . The terms F_i correspond to the matrix with columns $\phi(x_i, y)$ over labels y_i . The c_i 's correspond to the costs of a z_i and can be identified with the loss l for a label y'_i .

In equation 8, the term that is optimized is defined as:

$$L(w, z) \triangleq \sum_i w^T F_i z_i + c_i^T z_i - w^T \phi_i(y_i) \quad (9)$$

The latter function is bilinear in both w and z . Once can summon the idea of a duel between two masters of a zero-sum game who are playing against one another. The players, named w and z respectively (i.e. the parameters and the labels) play a zero-sum game. They perform updates using gradients of the objective w.r.t. their parameters. They then project the result to the set of feasible points given by the constraints imposed on the structure. The most common projections are obviously euclidean ones which are well studied and have fast implementations. However some problems call for other types of projections, i.e. based on other distance functions, or other divergence functions which are a weaker notion. The most important class for us are Bregman divergences which are Bregman divergences or distances which are *based on strictly convex functions*.

We have the following operator that is used to perform the updates for both players at the same time.

$$\begin{pmatrix} \nabla_w L(w, z) \\ -\nabla_{z_1} L(w, z) \\ \vdots \\ -\nabla_{z_m} L(w, z) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & F_1 & \dots & F_m \\ -F_1^T & & & \\ \vdots & & 0 & \\ -F_m^T & & & \end{pmatrix}}_F \underbrace{\begin{pmatrix} w \\ z_1 \\ \vdots \\ z_m \end{pmatrix}}_u - \underbrace{\begin{pmatrix} \sum_i \phi_i(y_i) \\ c_1 \\ \vdots \\ c_m \end{pmatrix}}_a = Fu - a \quad (10)$$

TODO averaging to prevent oscillations and $\eta < \|F\|^2$

3.2 Duality and Gap function

We can measure the “goodness” of the parameters using the gap function G :

$$G(w, z) \triangleq \left[\max_{z' \in Z} L(w, z') - L^* \right] + \left[L^* - \min_{w' \in W} L(w', z) \right] \quad (11)$$

where L^* gives the result of the min-max of the objective L . When we have a non-optimal point (i.e. not a saddle point), the gap is strictly positive. At an optimal point, the gap is exactly equal to 0. Now the restricted gap is exactly the same but the min and max are computed over a set of parameters that are within a certain distance of the start point $(\hat{u}_w, \hat{u}_z) \in U$:

$$G_{D_w, D_z}(w, z) = \max_{z' \in Z} [L(w', z') : d(z, z') \leq D_z] - \left[\min_{w' \in W} L(w', z) : d(w, w') \leq D_w \right] \quad (12)$$

The motivation for using this restricted gap function is that if we start “close” to an optimal point, of course we will converge more rapidly to it. This can be seen in the convergence analysis of the method.

The dual extragradient algorithm from Nesterov gives a convergence guarantee for the objective L .

This algorithm has a lookahead step (i.e. v) that serves to perform the actual gradient update u^t . The intuition behind the lookahead step is that given a function to optimize that is Lipschitz, Nesterov was able to show that we can upper bound $f_D(\bar{u}^n) = \max_y \{ \langle g(y), \bar{u}^n - y \rangle : d(\hat{u}, y) \leq D \}$, where \bar{u}^n is the weighted average over all the updates u^t up to iteration n . The function g corresponds to the objective L in our setting. When value of $f_D(\bar{u}^n)$ gets close to 0, we have that the value $g(y^*)$ for an optimal y^* is close to 0, which signifies that we have reached saddle point (i.e. what we wanted). Note that in the definition of f_D , we used a distance metric d . This corresponds to the Euclidean distance (or Bregman distance in non-Euclidean setting). The rojection operator Π_U in the algorithm simply projects a point back to the set U by finding the nearest point with respect to the distance metric used.

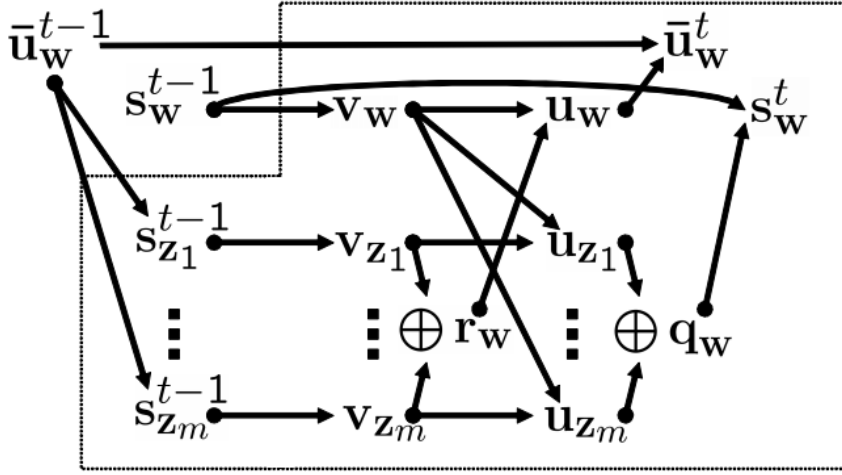


Figure 1: Memory efficient algorithm

3.2.1 Non-Euclidean setting

The main problem with the Euclidean projection operator is that for many problems, it is hard to compute the projection. Indeed for min-cut, we need to compute the partition function first, which is #P-complete. Thus, the authors of the paper introduced the Bregman operator, which computes the projection using the Bregman divergence. Using this operator has the great advantage of being easier to compute. We can see this for $L1$ regularization. Computing a projection using $L1$ distance is hard since it is not differentiable. Using the negative entropy as our function h (see 7), we get that the Bregman divergence is the KL divergence. This implies that we can differentiate the divergence to get the parameter that minimizes it. It is worth mentioning that for some problems, projections can still be hard to compute. This is why it may be wise to move to the FW algorithm especially if solving a linear problem over constraints the constraints is easy (or “easier”).

3.2.2 Memory-efficient tweak

In the dual extragradient algorithm, both a vector s^t and a vector \bar{u}^t are maintained. However, we can observe that the s_t ’s can be found using the running average \bar{u}^t since $s^t = -(t+1) \sum_{i=0}^t (F\bar{u}^i - a)$. We only have to store the vector \bar{u}^t . We can even do better when $|Z| \gg |W|$ since $\bar{u}^t = \{\bar{u}_w^t, \bar{u}_z^t\}$ and we only care about the part that corresponds to w . \bar{u}_z^t is maintained implicitly by storing a vector of size $|W|$ (although we now need to store s_w^t). It can be reconstructed using \bar{u}_w^t .

4 Frank Wolfe and Variants

4.1 Frank-Wolfe, the conditional gradient algorithm

Throughout this section we consider the problem of minimizing a continuously differentiable convex function - say f - over some convex set and compact set \mathcal{M} .

Definition 1. *A linear minimization oracle over a set \mathcal{M} is a map defined the following way:*

$$LMO_{\mathcal{M}}(d) \triangleq \arg \min_{s \in \mathcal{M}} \langle d, s \rangle \quad (13)$$

Intuitively the classical FW is a descent method where starting from some arbitrary feasible point we repeatedly take convex combinations with outputs of the oracle $s_t = LMO_{\mathcal{M}}(\nabla f(x_t))$. A reasonable step size can be chosen easily without having to do a line search as we are parametrizing the update by the convex combination "weight" (denoted γ below – as per convention.) Under convexity assumptions on the function and the feasible set we can tune this parameter between zero and one and be guaranteed to output a feasible iterate. This is a key feature of the algorithm as many other require to compute the gradient's lipschitz constant and/or the strong-convexity parameter, usually denoted μ . All in all parametrizing iterations through convex combinations is a strikingly simple and elegant to the feasibility problem. Moreover we also get sparsity for free, which is key in the application of its block-seperable version to the lagrangian dual of the loss augmented struct-svm.

Let $\alpha \in \mathcal{M}$

for $k = 0$ **to** K **do**

 Compute $s = \operatorname{argmin}_{s \in \mathcal{M}} \langle s, \nabla f(\alpha^k) \rangle$

 Let $\gamma = \frac{2}{k+2}$ (simple version) or optimize for line-search

 Update $\alpha^{k+1} = (1 - \gamma)\alpha^k + \gamma s$

end for

Algorithm 1: Classical Frank-Wolf

Sparsity is obviously a key feature for preventing the memory required to explode with the number of training samples. Moreover projection and proximal operations can usually involve solving a quadratic problem which is clearly harder to keep tractable. Many variants and extensions of the basic classical FW algorithm but Jaggi lists the main ones and the active research areas at the time of writing his paper in 2014. we mainly focus on the Block-Seperable variant applied to struct-svm problems as presented in Lacoste-Julien et al. (2013) While we can't necessarily compute the actual duality gap a linear surrogate can be used and has become widely standard in the literature, see for example Jaggi's review (Jaggi, 2013), it is defined as follows:

$$g(x) \triangleq \max_{s \in \mathcal{D}} \langle x - s, \nabla f(x) \rangle \quad (14)$$

By convexity we immediately have that $g(x) \geq f(x) - f(x^*)$ making exactly validating the tolerance of a solutionn very easy to compute.

4.2 Block Coordinate Frank Wolfe

Due to the exponential number of dual variables in the structured SVM setting, classical algorithms like projected gradient are intractable. Stochastic subgradient methods, on the other hand, achieve a sublinear convergence rate while only requiring a single call to the maximization oracle every step. They are nonetheless very sensitive to the sequence of stepsizes and it is unclear when to terminate the iterations.

In fact many algorithms with good theoretical guarantees introduce a dependency between the stepsize to be chosen and some constants which characterize the function, e.g. the lipschitz constant of the gradient (L) and/or the strong-convexity constant (μ) in order to provide some convergence guarantees. For example Nesterov (2003) in his analysis of dual extragradient the step size is bounded above by the gradient's Lipschitz constant which happens to correspond to the norm of the F matrix as defined in the previous section. Projections and proximal operators can involve quadratic programs but the natural convex combination approach ensures feasibility at no cost. Since the dual objective function for structured-svm is a quadratic problem we can compute analytically exactly the optimal line-search step size. The BCFW algorithm presented in Lacoste-Julien et al. (2013) also shows a couple equivalencies, the most important being that solving the loss-augmented problem is equivalent to a call to the linear minimization oracle in the dual.

Essentially BCFW can be seen as generalizing some subgradient and cutting plane methods. Tractability in the end, in the current literature, depends on how the inference problem (usually cast as an ILP since we are mostly dealing with underlying structures of combinatorial nature) and the task loss interact. It is shown in Taskar, Lacoste-Julien, and Jordan, 2006 that in many interesting cases such as when the weighted hamming distance is used as task loss for bipartite matching; the inference augmented loss inherits tractability and we have an LP with optimal integral solutions. It is a well known fact in combinatorial optimization that the incidence matrix of a bipartite graph is always totally unimodular which is a sufficient condition for optimal integral solution in its convex relaxation LP. The loss-augmented problems defined by the interaction of task loss and its convexification through the hinge loss “operator” are thus related in a very natural way to diverse combinatorial problems and the BCFW provides a theoretical pivot between those two problems.

5 Empirical

5.1 Experiments

In this section, we describe the implementation we performed for this project. The goal was to see how the Dual Extragradient algorithm compared to the Block-coordinate Frank-Wolfe algorithm. The task on which we performed the evaluation was word alignment in machine translation. We extracted the dataset from the Europarl dataset *Europarl Parallel Corpus* 2019. The data consisted of approximately 2 million sentence pairs in both english and french. Each the sentences in each pair were translations of one another. We extracted all sentences and performed a clean by splitting longer sentences into shorter ones. The goal of this step was to reduce the eventual number of matchings in training, which could take long to solve using a LP solver. Of course, each sentence was tokenized beforehand.

We then proceeded to implement the Dual Extragradient and BCFW algorithms using a SVM. We had to define a feature mapping for an input sentence pair. The features were extracted using the fastText library **fastText**. This library included a model that was previously trained to learn embeddings of words in both english and french. We later combined the embeddings in the two languages by applying a transformation found in Chojnacki and Kłopotek (2010). This transformation consisted in applying a matrix to each vector in each language (matrix W for english and Q for french). These matrices are in fact orthogonal (i.e. $Q^T W = I$). The idea behind such a transformation is that we sort of “put” or “align” both languages in the same vector space, a sort of “middle ground”. This way we can better compare the words “cat” and “chat” by getting their cosine similarity measure. We combined the cosine measure of each pair of words in the alignment by summing. The following blog post *Aligning Vector Representations* (2017) provides a good intuition using maps that are aligned. As an example, consider the following two sentences:

- This assignment was hard
- Ce travail etait ardu

We would compute the cosine distance for each word tuple (e.g. this/ce, this/travail, ..., hard/ardu). We were then able to get the highest match of each english word for french translation. This was how we extracted the “labels”. As the dataset was not annotated with alignments, we had to compute those according to the procedure mentioned.

For the features, we used the concatenated embeddings of each word pairs in the alignment. This gave us our edge score. Then, we simply performed a weighted combination of these vectors using the edge labels as weights. To clarify what we mean by edge labels, suppose that the edge linking “assignment” and “travail” has a value of 1. Then, we weight the vector extracted from these two words with 1. As another example, if the edge between “ce” and “ardu” has a value of 0 (i.e. no link between the words), we do not include the vector computed from the statistics of this word pair. This is exactly what was done in Taskar Taskar, Lacoste-Julien, and Jordan, n.d. modulo some other features.

In the implementation of BCFW, we used the solver from scipy with the simplex method. Since the constraint matrix given by the optimization of H_i in the algorithm is unimodular, when we relax the LP, we still get a solution to the ILP without relaxation. Thus, we take advantage of this fact and indeed use the LP solver. The loss that we used was the L_1 distance between the two labels, the proposal and the ground truth.

5.1.1 Results

Since running the experiments was computationally intensive and we did not dispose of a lot of computing power, we had to restrict the training set to a number a relatively small number of sentence pairs (100). This sanity check was simply a hint to the general applicability of the method as we scale up the number of training examples. We used the default λ value of 0.01 as our regularizer since, we did not want to train for too many iterations before getting decent results as per Theorem 3 found in Lacoste-Julien et al. (2013). We were able to get the following results for the BCFW algorithm:

To further motivate our intuition and to make sure the algorithm ran properly, we used the local scene dataset Müller and Behnke, 2014 from PyStruct to train a SVM using BCFW. It consisted of vectors of size 294 for the features of each training example (images). Each image was labeled according to the type of objects that were present in the scene.

These were:

$$beach \quad \cdot \quad sunset \quad \cdot \quad fall \quad \cdot \quad foliage \quad \cdot \quad field \quad \cdot \quad mountain \quad \cdot \quad urban \quad (15)$$

These 6 classes were not mutually exclusive so we had a total of $2^6 = 64$ possible labels. Training using all possible edges allowed to fully capture the complex relationships between them. The duality gap on the training data is given by the figure 2.

For the dual extragradient, we were only able to run the algorithm on a dataset of images Vemulapalli and Agarwala (2018). These were tuples of 3 images and a person hand-picked two images in each 3-tuple that resembled each other the most. Using the extragradient algorithm, we were able to get convergence but the results were not satisfying. We only obtained 46% accuracy on the prediction, which would compare with 33% if we had a random predictor (3 choose 2 gives 3 possibilities hence 33%). Thus, it motivated our change of dataset to obtain better results. We also moved on to the BCFW algorithm as we had not implemented it yet.

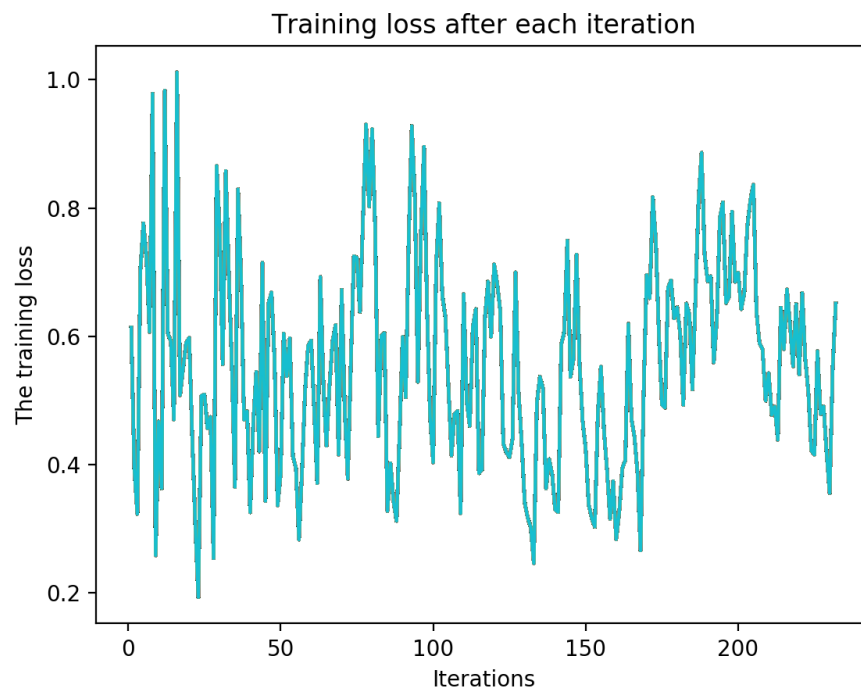


Figure 2: BCFW loss

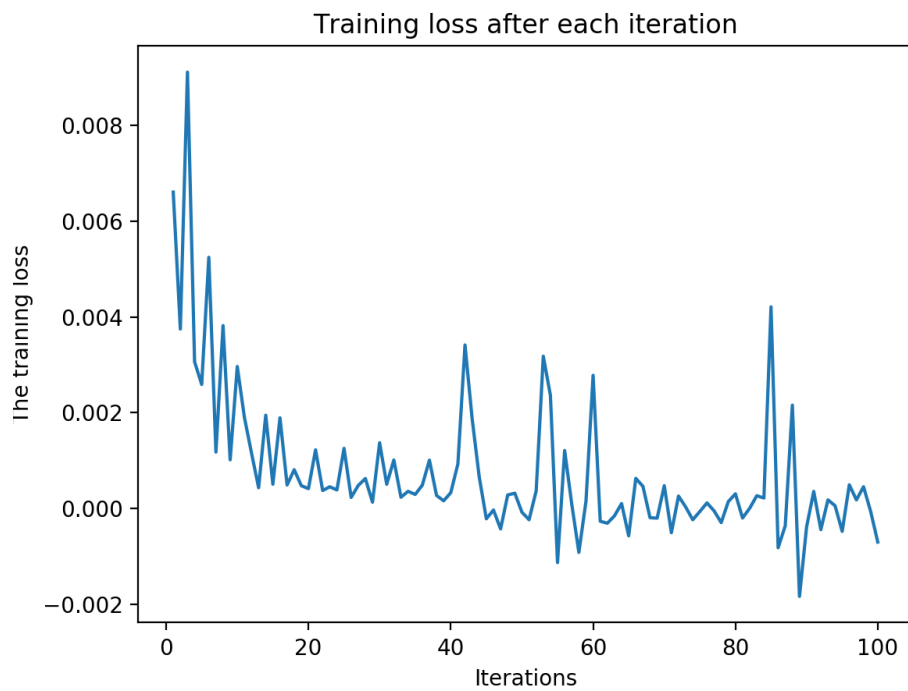


Figure 3: pystruct scene

6 Discussion

In this paper we have discussed mainly two methods which were first proposed in the papers by Taskar, Lacoste-Julien, and Jordan (2006) and Lacoste-Julien et al. (2013). Both of these papers contain insights that go beyond technicalities but try to cast problems in some new way and leverage strong guarantees with what seem to be, at the end of the day, quite strikingly simple solutions.

The essence of BCFW lies in the simplicity and elegance of convex structures but also shows how structure can be modular, in the abstraction of the LMO for computational costs and its equivalence to max oracles for example. In the ExtraGradient paper a key observation is that the tractability depends on the very interaction between the task loss, or its scoring scheme, on one hand and its structured surrogate loss on the other. As it turns out the combination of a structured hinge loss as a convex surrogate and a simple hamming distance results in a formulation of the *loss-augmented inference* problem for bipartite matchings as a tractable ILP. In other words if we have a tractable scoring scheme or task loss, we can map out how the interaction with specific loss surrogates affects the “transitivity” of tractability in the original problem. These kind of constructive frameworks allow us to build algorithms in a principled and elegant manner.

The dual extra gradient algorithm presented in Taskar, Lacoste-Julien, and Jordan (2006) casts some of the most common max margin estimation problems for structured output as a bilinear saddle point which opened the door for first order already known techniques and using different generalizations of the concept of projection could provide a new angle to attack those problems. All those approaches have in common that they leverage structures through specific black-boxable oracles to assemble algorithms with strong guarantees or tractability. It is obvious by how he opens his latest monograph on convex optimization that Nesterov (2018) wants to emphasize the need for a structural thinking pattern when trying to improve tractability of certain problems.

In terms of structure the saddle-point formulation of Taskar, Lacoste-Julien, and Jordan (2006) to solve structured prediction problem feels very natural because saddle points are at the very heart of duality theory which is the main modern tool of convex analysis. As succinctly shown in Bot, 2010, the dualization of optimization problems, in its most general idea, is the creation of a related second problem to the first by taking the Fenchel conjugate of a given *perturbation function* which acts a link between the two topological spaces \mathcal{X} and \mathcal{Y} .

One should therefore expect their ubiquity and importance in optimization. More recently Gidel, Jebara, and Lacoste-Julien (2016) attacked convex-concave saddle point problems, i.e. models with the following structure:

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \mathcal{L}(x, y) \quad (16)$$

with a FW angle, i.e. by restricting themselves to only have access to a LMO black box. Instead of alternating projected gradient steps with some averaging scheme alternating FW steps are performed through convex combinations once again staying feasible and sparse. Moreover we still get decent step sizes in the general

case, but other situations where we get lucky as with the dual of struct-svm where the optimal step size ends up being computable analytically would not be too surprising. In the aforementioned paper problems considered challenging in optimization such as the *sparse struct-svm*, i.e. the minimization of the sum of loss-augmented decoding subproblems with an added l_1 regularization term. In their experiments it fares well when compared to state of the art stochastic subgradient methods.

All in all the LMO black box approach allow us to build optimization strategies around pieces of optimization problems which have become solid “technology” in the words of Boyd, i.e. optimization procedures which have been battle tested and extensively tuned with efficient code. The other key advantage of FW approaches is the sparsity of iterates can again often be the difference between impossible and tractable.

Finally seperable approaches are more and more relevant as datasets grow bigger and we want convergence in less expensive iteration steps. On a higher level, a fully stochastic treatment of optimization in machine learning contexts, meaning by treating quantities such as the gradient as random variables to be sampled from, seems to be a logical step and the seperability of an optimization algorithm would be hard to neglect as an important factor, even given comparable overall convergence rates. For instance in Lin et al. (2017) a stochastic primal dual extragradient method is proposed. Using proximal mappings instead of projections the author’s proposed algorithm, called Stochastic Primal-Dual Proximal ExtraGradient descent (SPDEG) was shown to converge in expectation with $\mathcal{O}(1/\sqrt{t})$ rate.

As for the numerical experiments we entertained in order to compare the two algorithms discussed the results provided with this paper clearly did not provide reliable benchmarks for other students of the field. However it was a good reminder that Structured optimization often involves structuring the data in the first place (in a physical and/or software sence.) Going through the process of trying to reproduce results underscored the danger of neglecting the importance of a framework for consistent, solid, and accurate benchmarks. We found ourselves biasing towards solutions that are easier to preprocess and load into memory -and this might be a harder bias to estimate than the common statistical one.

7 Citations and References

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Appendices

Proximal step operator

We define the proximal step operator as follows:

$$T_\eta(\mathbf{u}, \mathbf{s}) = \max_{\mathbf{u}' \in U} \left\{ \langle \mathbf{s}, \mathbf{u}' - \mathbf{u} \rangle - \frac{1}{\eta} d(\mathbf{u}, \mathbf{u}') \leq D \right\} \quad (17)$$

The operator is useful to compute projections since when we have a strongly convex function $h(\mathbf{u})$, we can find its convex conjugate $h^*(\mathbf{u}) = \max_{\mathbf{u} \in U} [\langle \mathbf{s}, \mathbf{u} \rangle - h(\mathbf{u})]$. From the definition of a strongly convex function, we have that:

$$h(\mathbf{u}') \geq h(\mathbf{u}) + \langle \nabla h(\mathbf{u}), \mathbf{u}' - \mathbf{u} \rangle + \frac{\sigma}{2} \|\mathbf{u}' - \mathbf{u}\|^2 \quad (18)$$

where σ is the strong convexity parameter. Rearranging, we can define an upper bound on the squared norm of $\mathbf{u}' - \mathbf{u}$. This comes out as:

$$d(\mathbf{u}', \mathbf{u}) \triangleq h(\mathbf{u}') - h(\mathbf{u}) - \langle \nabla h(\mathbf{u}), \mathbf{u}' - \mathbf{u} \rangle \geq \frac{\sigma}{2} \|\mathbf{u}' - \mathbf{u}\|^2 \quad (19)$$

The distance metric d is called the Bregman divergence. The link between the Bregman divergence and the proximal step operator is that if we are given the function h inside the definition of the proximal step update, this induces the Bregman divergence, which in turn induces the update that is performed at each iteration of the extragradient algorithm. For example, if we have $h(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|_2^2$, the Bregman divergence becomes $d(\mathbf{u}', \mathbf{u}) = \frac{1}{2} \|\mathbf{u}' - \mathbf{u}\|_2^2$. We might wonder why we care about the Bregman divergence when the definition still includes the usual norm. After all, we still optimize the term $\langle \mathbf{s}, \mathbf{u}' - \mathbf{u} \rangle - \frac{1}{\eta} d(\mathbf{u}', \mathbf{u})$. This is because h^* is differentiable at every point of its domain by the strong convexity of h . Thus, it is easy to compute a projection in the usual fashion: we can compute the derivative of the term inside the projection operator and set it to 0. It is impossible to do for matchings for example as the distance is not even differentiable. We provide the steps to compute a projection:

$$\mathbf{s} - \nabla_{\mathbf{u}'} d(\mathbf{u}', \mathbf{u}) = \mathbf{s} - \frac{1}{\eta} \nabla_{\mathbf{u}'} d(\mathbf{u}, \mathbf{u}') = \mathbf{s} - \frac{1}{\eta} [\nabla h(\mathbf{u}') - \nabla h(\mathbf{u})] \quad (20)$$

By setting this equation to 0, it is possible to recover the optimal \mathbf{y}' when, let's say, $h(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}'\|^2$.

ExtraGradient

We present a simple formulation of the algorithm:

Initialize: Choose $\hat{u} \in U$, set $s^{-1} = 0$.

for $t = 0$ to $t = \tau$ **do**

$$v = \Pi_U(\hat{u} + \eta s^{t-1})$$

$$u^t = \Pi_U(v - \eta(Fv - a))$$

$$s^t = s^{t-1} - (Fu^t - a)$$

end for

return $\overline{u^\tau} = \frac{1}{1+\tau} \sum_{t=0}^{\tau} u^t$

Algorithm 2: Dual ExtraGradient

Convergence analysis

The restricted gap function \mathcal{G}_{D_w, D_z} is upper bounded by:

$$\mathcal{G}_{D_w, D_z}(\overline{w^\tau}, \overline{z^\tau}) \leq \frac{(D_w + D_z)L}{\tau + 1} \quad (21)$$

$$\mathcal{G}_{D_w, D_z}(w, z) = \begin{pmatrix} \sum_i F_i z_i^* \\ -F_1^T w^* \\ \vdots \\ -F_m^T w^* \end{pmatrix}^T \begin{pmatrix} w - w^* \\ z_1 - z_1^* \\ \vdots \\ z_m - z_m^* \end{pmatrix} - \begin{pmatrix} \sum_i f_i(y_i) \\ c_1 \\ \vdots \\ c_m \end{pmatrix}^T \begin{pmatrix} w - w^* \\ z_1 - z_1^* \\ \vdots \\ z_m - z_m^* \end{pmatrix}$$

We have the right tools for convergence of the algorithm. The Lipschitz constant of the operator $Fu -$ is given by L is equal to $\max_{u \in \mathcal{U}} \|F(u - u')\|_2 / \|u - u'\|_2 \leq \|F\|_2$. Of course, a point w, z that satisfies $\|w\|_2 \leq D_w$ and $\|z\|_2 \leq D_z$ also satisfies $\|(w, z)\|_2 \leq D$ when $D = \sqrt{D_w^2 + D_z^2}$ since $(w, 0) \perp (0, z)$. It is then easy to see that $f_D \geq \mathcal{G}_{D_w, D_z}$. Thus, the function \mathcal{G}_{D_w, D_z} is upper bounded by the right-hand side of equation 21.

Frank-Wolfe

Let $w^0 = w_i^0 = \bar{w}^0 = 0$, $l^0 = l_i^0 = 0$

for $k = 0 \dots K$ **do**

 Pick i at random in $\{1, \dots, n\}$

 Solve $y_i^* = \max_{y_i \in \mathcal{Y}_i} H_i(y, w^k)$

 Let $w_s = \frac{1}{n\lambda} \psi_i(y_i^*)$, and $l_s = \frac{1}{n} L_i(y_i^*)$

 Let $\gamma = \frac{\lambda(w_i^k - w_s)^T w^k - l_i^k + l_s}{\lambda \|w_i^k - w_s\|^2}$, and clip to $[0, 1]$

 Update $w_i^{k+1} = (1 - \gamma)w_i^k + \gamma w_s$, and $l_i^{k+1} = (1 - \gamma)l_i^k + \gamma l_s$

 Update $w^{k+1} = w^k + w_i^{k+1} - w_i^k$, and $l_i^{k+1} = (1 - \gamma)l_i^k + \gamma l_s$

end for

Algorithm 3: Block-Coordinate Frank-Wolfe – Lacoste-Julien et al. (2013)