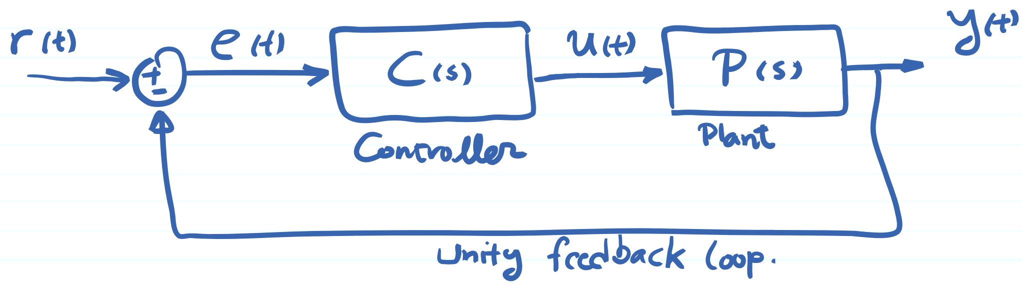
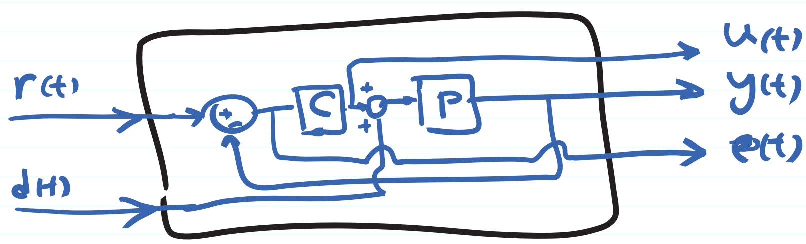


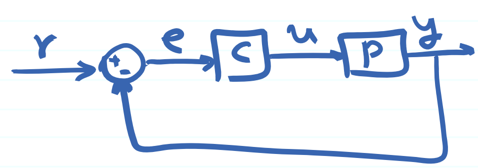
Control system TF:



Sensitivity transfer function.

Defined as the TF from reference to error

$$S(s) = \frac{E(s)}{R(s)}$$



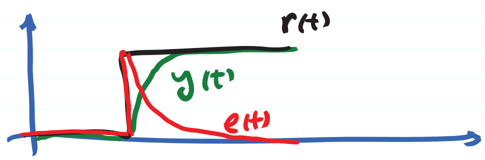
$$E(s) = R(s) - Y(s)$$
$$= R(s) - P(s)\,U(s)$$
$$= R(s) - P(s)\,C(s)\,E(s)$$
$$\Rightarrow \quad E(s)\left(1 + P(s)\,C(s)\right) = R(s)$$
$$\Rightarrow \boxed{S(s) = \frac{E(s)}{R(s)} = \frac{1}{1 + C(s)\,P(s)}}$$

Other transfer functions:

$$\frac{Y(s)}{R(s)} = \frac{CP}{1+CP} \qquad \frac{U(s)}{R(s)} = \frac{C}{1+CP}$$

In general: $$TF = \frac{\text{Feedforward}}{1 - \text{Feedback}}$$

Focusing the sensitivity TF



* Example: DC motor speed control:

1) Using a proportional Controller $C(s) = K_p$

Plant TF for a DC motor:

$$P(s) = \frac{W(s)}{V(s)} = \frac{K_t}{JR\,s + (Rb + K_t K_b)} = \frac{a}{b_1 s + b_0}$$

(Assuming coil inductance is negligible.)

$$S(s) = \frac{1}{1+CP} = \frac{1}{1 + K_p \dfrac{a}{b_1 s + b_0}} = \frac{s + b_0/b_1}{s + \dfrac{b_0 K_p a}{b_1}} = \frac{s+z}{s+p}$$

So the control design task is to choose K_p to provide desired closed-loop system response.

Zero of $S(s)$ is $s = -z = -b_0/b_1$

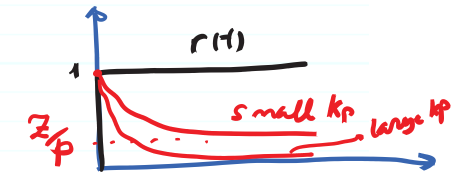
Pole of $S(s)$ is $s = -p = -\dfrac{b_0 K_p a}{b_1}$

Step response of the sensitivity TF:

$$E(s) = S(s)\underbrace{R(s)}_{1/s} = \frac{s+z}{s+p}\,\frac{1}{s} = \frac{\alpha}{s} + \frac{\beta}{s+p}$$

$\alpha = z/p$ and $\beta = \dfrac{p-z}{p}$

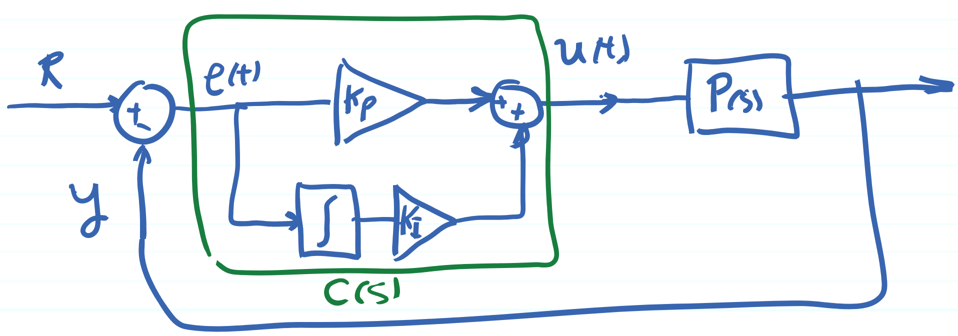
$$\Rightarrow e(t) = z/p + \frac{p-z}{p}\,e^{-pt}$$



$$p = \frac{b_0 a}{b_1} k_p$$

Making k_p large enough makes the steady-state error small!

* Proportional Integral Control:



$$u(t) = k_p e(t) + k_I \int_0^t e(\tau)\,d\tau$$

$$U(s) = k_p E(s) + k_I \frac{1}{s} E(s) \Rightarrow \frac{U(s)}{E(s)} = \underbrace{\frac{k_p s + k_I}{s}}_{C(s)}$$

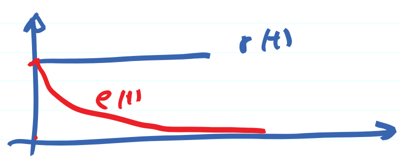
$$\Rightarrow S(s) = \frac{1}{1+CP} = \frac{1}{1+\dfrac{k_p s + k_I}{s}\,\dfrac{a}{b_1 s + b_0}} = \frac{b_1 s^2 + b_0 s}{b_1 s^2 + (b_0 + k_p a)s + k_I a}$$

$$\Rightarrow S(s) = \frac{s^2 + \frac{b_0}{b_1}s}{s^2 + \left(\frac{b_0 + K_p a}{b_1}\right)s + \frac{k_i}{b_1}a}$$

$$S(s) = \frac{E(s)}{R(s)} \Rightarrow E(s) = S(s)R(s) \quad \text{if } R = \frac{1}{s} \text{ (unit step)}$$

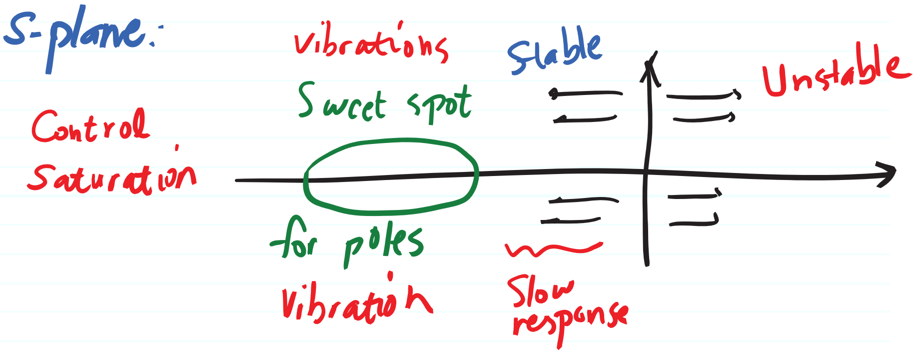
$$\Rightarrow E(s) = \frac{1}{s} S(s) = \frac{1}{s} \frac{s^2 + \frac{b_0}{b_1}s}{s^2 + (\dots)s + \frac{k_i a}{b_1}}$$

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s \frac{1}{s} S(s) = \lim_{s \to 0} S(s) = \frac{0}{k_i a / b_1} = 0$$



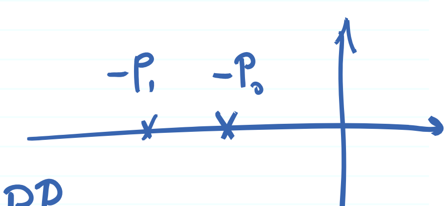
* Pole-Placement

place poles of the closed-loop system at desired locations in the s-plane:



Let's choose the poles of $S(s)$ to be $-P_0$, $-P_1$

Denominator polynomial or Characteristics polynomial



$$(s + P_0)(s + P_1) = s^2 + (P_0 + P_1)s + P_0 P_1$$

$$S(s) = \frac{s^2 + \frac{b_0}{b_1} s}{s^2 + \left(\frac{b_0 + k_p a}{b_1} \right) s + \frac{k_I a}{b_1}}$$

by comparison :
$$\begin{cases} \frac{b_0 + k_p a}{b_1} = P_0 + P_1 \\ \frac{k_I a}{b_1} = P_0 P_1 \end{cases}$$

$$\Rightarrow \begin{cases} k_p = \frac{(P_0 + P_1) b_1 - b_0}{a} \\ k_I = \frac{P_0 P_1 b_1}{a} \end{cases}$$