Laplace transform transforms a time-domain function to an s-domain function through the following transformation:

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} f(t) e^{-st} dt$$

Example: Unit step function $1(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$

$$\mathcal{L}(1(t)) = \int_0^{\infty} 1(t) e^{-st} dt = \int_0^{\infty} 1 e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

Propertise of Laplace transform:

$$\mathcal{L}(\alpha f_1(t) + \beta f_2(t)) = \alpha F_1(s) + \beta F_2(s)$$ Superposition

$$\mathcal{L}(f(t-\lambda)) = e^{-\lambda s} F(s)$$ Time delay

$$\mathcal{L}\left(\frac{d}{dt} f(t)\right) = sF(s) - f(0)$$ Differentiation

$$\mathcal{L}\left(\frac{d^2}{dt^2} f(t)\right) = s^2 F(s) - s f(0) - \dot{f}(0)$$ Second time-der.

$$\mathcal{L}\left(\int f(t) dt\right) = \frac{F(s)}{s}$$ Integration

$$\mathcal{L}\left(\int_0^t f(t-\tau) g(\tau) d\tau\right) = F(s) G(s)$$ Convolution

* $$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$$ Final Value Theorem

$$f(t) = \mathcal{L}^{-1}(F(s))$$ Inverse Laplace Transform

* More properties are available on the Laplace Transform table!

* Transfer function:

1) Apply the Laplace transform to the differential eqs.
2) Set all the initial conditions to zero.
3) Determine the input-output relationship:

$$Y(s) = G(s)\,U(s) \quad \text{or} \quad G(s) = \frac{Y(s)}{U(s)}$$

$$\text{* Transfer function} = \frac{\text{Output}}{\text{Input}}$$

Example: $m\ddot{x} + c\dot{x} + kx = f$ find $G(s) = \dfrac{X(s)}{F(s)}$

Apply Laplace Transform:

$$m\left(s^2 X(s) - sX(0) - \dot{X}(0)\right) + c\left(sX(s) - X(0)\right) + KX(s) = F(s)$$

Set $x(0)$ and $\dot{x}(0)$ to zero: $(ms^2 + cs + k)X(s) = F(s)$

$$\Rightarrow G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + K}$$

Example: $\begin{cases} m\ddot{x} + c\dot{x} + kx = f(t) \\ \dot{f} + \sigma f = \alpha\, v(t) \end{cases}$

Get TF from $v \xrightarrow{G_1} f$ and from $v \xrightarrow{G_2} x$

$$G_1(s) = \frac{F(s)}{V(s)} \qquad G_2(s) = \frac{X(s)}{V(s)}$$

$$\left(sF(s) - \underbrace{f(0)}_{\downarrow 0}\right) + \sigma F(s) = \alpha\, V(s)$$

$$\Rightarrow F(s)(s + \sigma) = \alpha\, V(s) \Rightarrow G_1(s) = \frac{F(s)}{V(s)} = \frac{\alpha}{s + \sigma}$$

For $G_2(s)$: $ms^2 X(s) + csX(s) + kX(s) = F(s)$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + Cs + K}$$

$$G_2(s) = \frac{X(s)}{V(s)} = \frac{X(s)}{F(s)} \times \frac{F(s)}{V(s)} = \frac{1}{ms^2 + Cs + K} \times \frac{\alpha}{s + \sigma} = \frac{\alpha}{(ms^2 + Cs + K)(s + \sigma)}$$

+ Matlab Simulation Command

TFname = tf([Num], [Den])

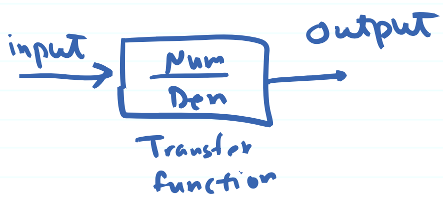
Num: Coefficients of the numerator polynomial

Den: " " " denominator "

Example: $$G(s) = \frac{2s^2 + 1}{s^3 + 3s + 5}$$

G = tf([2 0 1], [1 0 3 5])

Simulink:



* Multi-input Multi-output (MIMO)

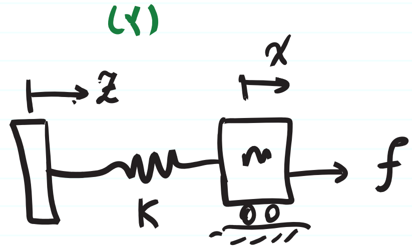
$$Y_1(s) = G_{11}(s)\,U_1(s) + G_{21}\,U_2(s) + \cdots$$

$$Y_2(s) = G_{12}\,U_1(s) + G_{22}\,U_2(s) + \cdots$$

$\vdots$

index of input (U) index of output (Y)

Example:



inputs: z, f (u_1, u_2)

outputs: $x, \dot{x}$ (y_1, y_2)

$$m\ddot{x} + Kx = f + Kz$$

Laplace Tr. with zero IC's: $ms^2X(s) + KX(s) = F(s) + KZ(s)$

$$\underbrace{X(s)}_{Y_1(s)} = \underbrace{\frac{K}{ms^2+K}}_{G_{11}(s)} \underbrace{Z(s)}_{U_1(s)} + \underbrace{\frac{1}{ms^2+K}}_{G_{21}(s)} \underbrace{F(s)}_{U_2(s)}$$

multiply by s:

$$\underbrace{s\,X(s)}_{\dot{x}(t) \to Y_2(s)} = \underbrace{\frac{ks}{ms^2+k}}_{G_{12}(s)} \underbrace{Z(s)}_{U_1} + \underbrace{\frac{s}{ms^2+k}}_{G_{22}} \underbrace{F(s)}_{U_2}$$

Final value Theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

Example: $\frac{F(s)}{U(s)} = \frac{\alpha}{s+\sigma}$ for unit step input calculate the steady-state response:

$$F(s) = \frac{\alpha}{s+\sigma} U(s) = \frac{\alpha}{s+\sigma} \underbrace{\frac{1}{s}}_{\text{Laplace transform of unit step}}$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s) = \lim_{s\to 0} s \frac{\alpha}{s+\sigma}\frac{1}{s} = \lim_{s\to 0} \frac{\alpha}{s+\sigma}$$

$$= \frac{\alpha}{\sigma}$$