So far, we studied how to derive differential equations governing systems' dynamics.

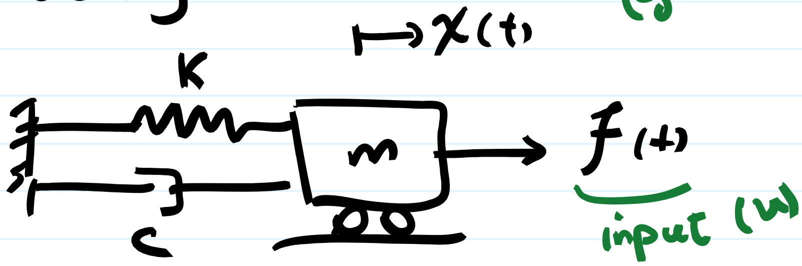
In general, for continuous-time linear dynamic systems, we have two standard representations

1- State-space (time-domain)

2- Transfer function (s-domain, freq domain)

* State-space modeling

Example:



The governing ODE of the system is:

$$m\ddot{x} + c\dot{x} + kx = f$$

Goal of SS modeling is to convert the n^{th}-order ODE to n 1^{st} order ODE and put it in the matrix form.

$$\underbrace{\dot{x}}_{n\times 1} = \underbrace{A}_{n\times n}\ \underbrace{x}_{n\times 1} + \underbrace{B}_{n\times m}\ \underbrace{u}_{m\times 1}$$

State vector

input

$$\underbrace{y}_{p\times 1} = \underbrace{C}_{p\times n}\ \underbrace{x}_{n\times 1} + \underbrace{D}_{p\times m}\ \underbrace{u}_{m\times 1}$$

output

Let's choose:

$x_1(t) = x(t)$ position of the mass

$x_2(t) = \dot{x}(t)$ velocity of the mass

$$\Rightarrow \quad \dot{x}_1(t) = \dot{x}(t) = x_2(t) \qquad (1)$$

$$\dot{x}_2(t) = \ddot{x}(t) = \frac{1}{m}\Big(\underbrace{f(t)}_{u} - c\underbrace{\dot{x}(t)}_{x_2} - k\underbrace{x(t)}_{x_1}\Big)$$

$$\Rightarrow \quad \dot{x}_2 = -\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t) + \frac{1}{m}u(t) \qquad (2)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{\text{State Matrix } A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\text{Input Matrix } B} u$$

If we choose only position x as output:

$$y = x = x_1$$

$$\Rightarrow \quad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\text{Output Mat, } C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\text{input-output Mat. } D} u$$

$$m\ddot{x} + c\dot{x} + kx = f$$

$$\Downarrow$$

$$\begin{cases} \dot{\underline{x}} = A\underline{x} + Bu \\ y = C\underline{x} + Du \end{cases}$$

Let's choose both position & velocity as output:

$$\left.\begin{matrix} y_1 = x \rightarrow x_1 \\ y_2 = \dot{x} \rightarrow x_2 \end{matrix}\right. \Rightarrow \underline{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} u$$

Let's choose $x, \dot{x}, \ddot{x}$ as outputs

$$y_1 = x = x_1$$

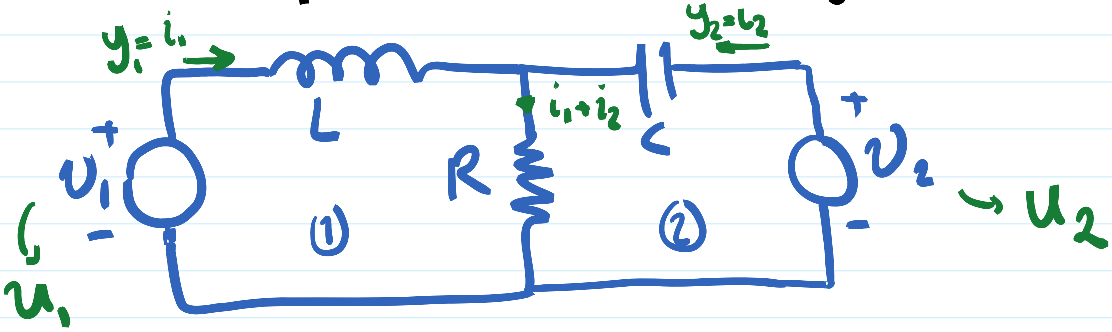
$$y_2 = \dot{x} = x_2$$

$$y_3 = \ddot{x} = \frac{1}{m}(u - cx_2 - kx_1)$$

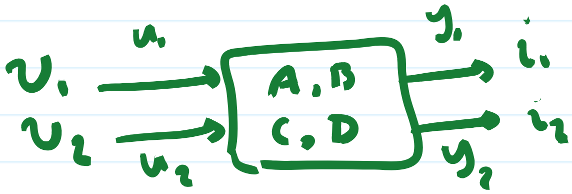
$\rightarrow$ output eq.

$$\underline{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} \end{bmatrix}}_{D \text{ (nonzero)}} u$$

+ Example 2: Electrical Systems



find:



KVL:
$$\begin{cases} +v_1 - L\dfrac{di_1}{dt} - R(\overbrace{i_1 + i_2}^{\text{KCL}}) = 0 & ① \\ +v_2 - \dfrac{1}{C}\displaystyle\int i_2\,dt - R(i_1 + i_2) = 0 & ② \end{cases}$$

Usually for electrical systems we choose "inductor current" and "capacitor charge" as state variables.

$$\begin{cases} x_1 = i_1 \\ x_2 = \int i_2\,dt \end{cases} \xRightarrow{d/dt} \begin{cases} \dot{x}_1 = \dfrac{di_1}{dt} = \overbrace{\dfrac{1}{L}(v_1 - R(i_1 + i_2))}^{\text{from ①}} \\ \dot{x}_2 = i_2 = \underbrace{\dfrac{1}{R}\left(v_2 - \dfrac{1}{C}\int i_2\,dt - Ri_1\right)}_{\text{from ②}} \end{cases}$$

$$\Rightarrow \dot{x}_2 = \frac{1}{R}\left(v_2 - \frac{1}{C}x_2 - Rx_1\right)$$

$$\dot{x}_1 = \frac{1}{L}\left(v_1 - Rx_1 - R\underbrace{\left(\frac{1}{R}\left(v_2 - \frac{1}{C}x_2 - Rx_1\right)\right)}_{i_2 = x_2}\right)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{LC} \\ -1 & -\frac{1}{RC} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ 0 & \frac{1}{R} \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$