Full-state Feedback Control

Goal: Stabilize unstable state-space system or improve response time of the system using complete state feedback:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Full-state control law:

$$\underset{m\times 1}{u} = -\underset{m\times n}{K}\,\underset{n\times 1}{x}$$

$\Rightarrow$ Closed-loop system:

$$\dot{x} = Ax + B(-Kx) = (A-BK)x$$

$$\Rightarrow \boxed{\dot{x} = (A-BK)x}$$

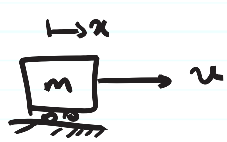
Control design task: Choose K such that the eigenvalues of $(A-BK)$ are on the negative side of the complex plane!

It is only possible to choose the eigenvalues arbitrarily if the pair $(A \& B)$ are "Controllable"

To find eigenvalues we solve for:

$$\det(A - \lambda I) = 0$$

* Example:



unstable (Assuming no friction)

$$m\ddot{x} = u \Rightarrow \begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = \frac{1}{m}u \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u$$

eigenvalues of A: $\det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2 - 0 = 0 \Rightarrow \lambda^2 = 0$$

$\lambda_1 = 0$; $\lambda_2 = 0$ $\quad$ (TF: $\frac{X(s)}{U(s)} = \frac{1}{ms^2} u$ $\quad$ Poles: 0, 0)

* Control Design:

Choose $u = -Kx = -[K_1 \;\; K_2]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}[K_1 \;\; K_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \frac{K_1}{m} & \frac{K_2}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_1}{m} & -\frac{K_2}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$ CL system: $\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{K_1}{m} & -\frac{K_2}{m} \end{pmatrix} x$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -\frac{K_1}{m} & -\frac{K_2}{m} - \lambda \end{bmatrix}\right) = \lambda^2 + \frac{K_2}{m}\lambda + \frac{K_1}{m} = 0$$

Let's Choose $\lambda_1 = -2$; $\lambda_2 = -2$

$(\lambda+2)(\lambda+2) = \lambda^2 + 4\lambda + 4$ $\quad$ (Desired)

$$\Rightarrow \begin{cases} \frac{K_2}{m} = 4 \\ \frac{K_1}{m} = 4 \end{cases} \Rightarrow \begin{matrix} K_2 = 4m \\ K_1 = 4m \end{matrix}$$