2.077 Lecture Notes
Finite deformation Mises theory of rate-independent plasticity

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1 Summary of the large deformation rate-independent Mises theory

The complete set of constitutive equations for a large deformation rate-independent Mises theory with isotropic hardening consists of (cf., Gurtin et al., 2010, Section 99.4) :

(i) the Kröner decomposition

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \qquad \text{with} \qquad \det \mathbf{F}^p = 1, \tag{1.1}$$

in which $\mathbf{F}$ is the deformation gradient, while $\mathbf{F}^e$ and $\mathbf{F}^p$ are the elastic and plastic distortions.

(ii) The elastic strain is defined as

$$\mathbf{E}^e \stackrel{\text{def}}{=} \tfrac{1}{2}(\mathbf{C}^e - \mathbf{1}), \qquad \mathbf{C}^e \stackrel{\text{def}}{=} \mathbf{F}^{e\top}\mathbf{F}^e\,. \tag{1.2}$$

(iii) The free-energy per unit volume of the intermediate space is taken as

$$\psi_{\mathrm{I}} = G|\mathbf{E}^e_0|^2 + \tfrac{1}{2}K(\operatorname{tr}\mathbf{E}^e)^2, \tag{1.3}$$

where

$$G > 0 \quad \text{and} \quad K > 0, \tag{1.4}$$

are the shear modulus and the bulk modulus respectively.

(iv) The elastic second Piola stress is

$$\mathbf{T}^e = \mathbb{C}\mathbf{E}^e \qquad \text{with} \qquad \mathbb{C} \stackrel{\text{def}}{=} 2G\,(\mathbb{I}^{\text{sym}} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}) + K\,\mathbf{1}\otimes\mathbf{1},. \tag{1.5}$$

Remark.

In the general theory of large deformation isotropic plasticity the constitutive equations for plastic flow are formulated in terms of the Mandel stress (cf., Gurtin et al., 2010, Section 99.4)

$$\mathbf{M}^e \stackrel{\text{def}}{=} \mathbf{C}^e\mathbf{T}^e\,. \tag{1.6}$$

We will restrict our attention to materials (like metals) for which the elastic stretches are small ($\mathbf{C}^e \approx \mathbf{1}$), and make the approximation that Mandel stress may be approximated by the elastic second Piola stress,

$$\mathbf{M}^e \approx \mathbf{T}^e\,. \tag{1.7}$$

Henceforth, we will work with the elastic second Piola stress $\mathbf{T}^e$ instead of the Mandel stress $\mathbf{M}^e$ to formulate the plastic part of the theory.

$\square$

(v) Then, with

$$\bar{\sigma} \overset{\text{def}}{=} \sqrt{3/2}|\mathbf{T}^e_0| \tag{1.8}$$

denoting the equivalent tensile stress and $Y = Y(\bar{\epsilon}^p) > 0$ representing a flow resistance, we introduce a yield function

$$f(\bar{\sigma}, Y) = \bar{\sigma} - Y, \tag{1.9}$$

and an elastic range

$$\mathcal{E}(Y)\{\mathbf{T}^e \in \text{Sym}|f(\bar{\sigma}, Y) \leq 0\} \tag{1.10}$$

which limits the admissible stress $\mathbf{T}^e$ at fixed values of the internal variable $\bar{\epsilon}^p$.

(vi) Evolution equation for $\mathbf{F}^p$ is given by,

$$\begin{aligned} &\dot{\mathbf{F}}^p = \mathbf{D}^p\mathbf{F}^p, \\ &\mathbf{D}^p = \sqrt{3/2}\,\dot{\bar{\epsilon}}^p\mathbf{N}^p, \qquad \mathbf{N}^p = \sqrt{3/2}\left(\frac{\mathbf{T}^e_0}{\bar{\sigma}}\right) \end{aligned} \tag{1.11}$$

where

$$\dot{\bar{\epsilon}}^p \overset{\text{def}}{=} \sqrt{2/3}|\mathbf{D}^p| \tag{1.12}$$

is the equivalent tensile plastic strain rate and

$$\bar{\epsilon}^p = \int_0^t \dot{\bar{\epsilon}}^p(\zeta)d\zeta, \tag{1.13}$$

is the equivalent tensile plastic strain.

(vii) Loading-unloading conditions are expressed in the Kuhn-Tucker complementarity form

$$\dot{\bar{\epsilon}}^p \geq 0, \qquad f(\bar{\sigma}, Y) \leq 0, \qquad \dot{\bar{\epsilon}}^p\, f(\bar{\sigma}, Y) = 0. \tag{1.14}$$

(vii) Consistency condition

$$\dot{\bar{\epsilon}}^p\, \dot{\overline{f(\bar{\sigma}, Y)}} = 0 \qquad \text{when} \qquad f(\bar{\sigma}, Y) = 0. \tag{1.15}$$

(viii) Typical initial conditions presume that at time $t = 0$,

$$\mathbf{F}(\mathbf{X}, 0) = \mathbf{F}^p(\mathbf{X}, 0) = \mathbf{1}, \qquad \bar{\epsilon}^p(\mathbf{X}, 0) = 0, (= \text{constant}), \tag{1.16}$$

together with $\mathbf{F}^e(\mathbf{X}, 0) = \mathbf{1}$.

To complete the constitutive model for a particular material the constitutive parameter/functions that need to be specified are the elastic moduli (G, K), and the flow resistance function $Y = Y(\bar{\epsilon}^p)$. The latter is often given as the solution to an evolution equation

$$\dot{Y} = H(Y)\dot{\bar{\epsilon}}^p \qquad \text{with initial value} \qquad Y(\mathbf{X}, 0) = Y_0\,. \tag{1.17}$$

A particular useful form for the strain hardening function $H(Y)$ is

$$H(Y) = \begin{cases} H_0\left(1 - \dfrac{Y}{Y_\mathrm{s}}\right) & \text{for } Y_0 \leq Y \leq Y_\mathrm{s}, \\ 0 & \text{for } Y \geq Y_\mathrm{s}, \end{cases} \tag{1.18}$$

where H_0 and $Y_\mathrm{s} > Y_0$ are constant moduli. For later use we note that the integrated form of the evolution equation for Y in terms of the equivalent tensile plastic strain $\bar{\epsilon}^p$ is

$$Y = Y_\mathrm{s} - (Y_\mathrm{s} - Y_0)\exp\left(-\frac{H_0}{Y_s}\bar{\epsilon}^p\right)\,. \tag{1.19}$$

2 Time-integration procedure for the finite-deformation Mises theory

In this section we briefly consider the major aspects of numerical implementation of the large deformation rate-independent isotropic plasticity theory in the finite element program FEniCS. Let $[0, T]$ be the time interval of interest. We assume that at time $t_n \in [0, T]$ we are given

$$\{\mathbf{F}^p_n, \bar{\epsilon}^p_n\} \quad \text{as well as } \mathbf{F}_{\mathrm{n+1}} \text{ at time } t_{\mathrm{n+1}} = t_n + \Delta t. \tag{2.1}$$

The time-integration procedure in this deformation-driven problem is then to calculate

$$\left\{\mathbf{F}^p_{\mathrm{n+1}}, \bar{\epsilon}^p_{n+1}, \mathbf{T}_{\mathrm{Rn+1}}\right\}, \tag{2.2}$$

where $\mathbf{T}_{\mathrm{Rn+1}}$ is the Piola stress at the end of the time-step, and march forward in time.

2.1 Time-integration procedure

Consider first a time-integration procedure in which the evolution equation for $\dot{\mathbf{F}}^p = \mathbf{D}^p \mathbf{F}^p$ is integrated by means of an exponential map

$$\mathbf{F}^p_{\mathrm{n+1}} = \exp\Big(\Delta t\, \mathbf{D}^p_{\mathrm{n+1}}\Big)\mathbf{F}^p_n, \qquad \text{with} \qquad \mathbf{D}^p_{\mathrm{n+1}} = \hat{\mathbf{D}}^p_{\mathrm{n+1}}(\mathbf{T}^e_{\mathrm{n+1}}, \bar{\epsilon}^p_{n+1}). \tag{2.3}$$

So that the inverse of $\mathbf{F}^p_{\mathrm{n+1}}$ is given by

$$\mathbf{F}^{p-1}_{\mathrm{n+1}} = \mathbf{F}^{p-1}_n \exp\Big(-\Delta t\, \mathbf{D}^p_{\mathrm{n+1}}\Big), \tag{2.4}$$

Thus, using $\mathbf{F}^e_{\mathrm{n+1}} = \mathbf{F}_{\mathrm{n+1}}\, \mathbf{F}^{p-1}_{\mathrm{n+1}}$, the elastic deformation gradient at the the end of the step is given by

$$\mathbf{F}^e_{\mathrm{n+1}} = \mathbf{F}^e_{\mathrm{tr}} \exp\Big(-\Delta t \mathbf{D}^p_{\mathrm{n+1}}\Big), \tag{2.5}$$

where

$$\mathbf{F}^e_{\mathrm{tr}} \stackrel{\text{def}}{=} \mathbf{F}_{\mathrm{n+1}}\, \mathbf{F}^{p-1}_n \tag{2.6}$$

is a *trial* value of the elastic deformation gradient. Trial values correspond to the value of a quantity at the end of the step when plastic flow is frozen.

Henceforth

- *we limit our considerations to small time increments* Δt.

Under this approximation

$$\exp\Big(-\Delta t \mathbf{D}^p_{\mathrm{n+1}}\Big) \approx \mathbf{1} - \Delta t \mathbf{D}^p_{\mathrm{n+1}}\,, \tag{2.7}$$

and equation (2.5) becomes

$$\mathbf{F}^e_{\mathrm{n+1}} = \mathbf{F}^e_{\mathrm{tr}}\Big(\mathbf{1} - \Delta t \mathbf{D}^p_{\mathrm{n+1}}\Big)\,. \tag{2.8}$$

A calculation for the Green elastic strain $\mathbf{E}^e_{\mathrm{n+1}} = (1/2)\left(\mathbf{C}^e_{\mathrm{n+1}} - \mathbf{1}\right)$ (using the approximation of small elastic stretches, $\mathbf{C}^e \approx \mathbf{1}$) then gives

$$\mathbf{E}^e_{\mathrm{n+1}} \approx \mathbf{E}^e_{\mathrm{tr}} - \Delta t \mathbf{D}^p_{\mathrm{n+1}} \qquad \text{where} \quad \mathbf{E}^e_{\mathrm{tr}} = (1/2)\left(\mathbf{C}^e_{\mathrm{tr}} - \mathbf{1}\right)\,. \tag{2.9}$$

The stress strain relation (1.5), $\mathbf{T}^e = \mathbb{C}[\mathbf{E}^e]$, then gives the update for the elastic second Piola stress,

$$\mathbf{T}^e_{\mathrm{n+1}} = \mathbf{T}^e_{\mathrm{tr}} - \mathbb{C}[\Delta t \mathbf{D}^p_{\mathrm{n+1}}], \tag{2.10}$$

where

$$\mathbf{T}^e_{\mathrm{tr}} \stackrel{\text{def}}{=} \mathbb{C}[\mathbf{E}^e_{\mathrm{tr}}], \tag{2.11}$$

is the trial elastic stress. Finally using

$$\mathbb{C} = 2G\,(\mathbb{I}^{\rm sym} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1}) + K\,\mathbf{1}\otimes\mathbf{1},$$

and

$$\mathbf{D}^p_{\rm n+1} = \sqrt{3/2}\,\dot{\bar{\epsilon}}^p{}_{\rm n+1}\mathbf{N}^p_{\rm n+1},$$

we have

$$\mathbf{T}^e_{\rm n+1} = \mathbf{T}^e_{\rm tr} - \sqrt{6}\,G\,\Delta\bar{\epsilon}^p\,\mathbf{N}^p_{\rm n+1}\,. \tag{2.12}$$

where

$$\Delta\bar{\epsilon}^p \overset{\rm def}{=} \Delta t\,\dot{\bar{\epsilon}}^p{}_{\rm n+1} \tag{2.13}$$

is an equivalent tensile plastic strain increment.

Since

$$\mathbf{N}^p_{\rm n+1} = \sqrt{3/2}\left(\mathbf{T}^e_{0\,\rm n+1}/\bar{\sigma}_{\rm n+1}\right), \tag{2.14}$$

splitting (2.12) into its deviatoric and spherical parts we have

$$(\mathbf{T}^e_{\rm n+1})_0 = (\mathbf{T}^e_{\rm tr})_0 - \sqrt{6}\,G\,\Delta\bar{\epsilon}^p\,\mathbf{N}^p_{\rm n+1}, \tag{2.15}$$

$$\mathrm{tr}\,\mathbf{T}^e_{\rm n+1} = \mathrm{tr}\,\mathbf{T}^e_{\rm tr}. \tag{2.16}$$

Also, (2.14) and (2.15) gives

$$\left(\sqrt{2/3}\bar{\sigma}_{\rm n+1} + \sqrt{6}\,G\,\Delta\bar{\epsilon}^p\right)\mathbf{N}^p_{\rm n+1} = (\mathbf{T}^e_{\rm tr})_0. \tag{2.17}$$

Let

$$\bar{\sigma}_{\rm tr} \overset{\rm def}{=} \sqrt{3/2}|(\mathbf{T}^e_{\rm tr})_0| \tag{2.18}$$

$$\mathbf{N}^{p\,\rm tr}_{\rm n+1} \overset{\rm def}{=} \sqrt{3/2}\left((\mathbf{T}^e_{\rm tr})_0/\bar{\sigma}_{\rm tr}\right), \tag{2.19}$$

define the trial equivalent tensile stress and the the trial direction of plastic flow. Then (2.17) may be written as

$$\left(\sqrt{2/3}\bar{\sigma}_{\rm n+1} + \sqrt{6}\,G\,\Delta\bar{\epsilon}^p\right)\mathbf{N}^p_{\rm n+1} = \sqrt{2/3}\,\bar{\sigma}_{\rm tr}\mathbf{N}^{p\,\rm tr}_{\rm n+1}, \tag{2.20}$$

which immediately gives the following two important results:

1. *The direction of plastic flow at the end of the step is determined by the trial direction of plastic flow*:

$$\mathbf{N}^p_{\rm n+1} = \mathbf{N}^p_{\rm tr}. \tag{2.21}$$

2. The equivalent tensile stress at the end of the step and trial value of the equivalent tensile stress are related by

$$\bar{\sigma}_{\rm n+1} = \bar{\sigma}_{\rm tr} - 3\,\mathrm{G}\,\Delta\bar{\epsilon}^p. \tag{2.22}$$

Next, the *consistency condition* requires that

$$\bar{\sigma}_{\rm n+1} = Y(\bar{\epsilon}^p_{n+1}). \tag{2.23}$$

Hence (2.22) and (2.23) give the following *implicit equation for the equivalent tensile plastic strain increment* $\Delta\bar{\epsilon}^p$:

$$\Phi(\Delta\bar{\epsilon}^p) = \bar{\sigma}_{\rm tr} - 3\ \mathrm{G}\ \Delta\bar{\epsilon}^p - Y(\bar{\epsilon}^p_{n+1}) = 0. \tag{2.24}$$

We assume here that the strain-hardening rate is not very large, and that for small values of $\Delta\bar{\epsilon}^p$ we may approximate

$$Y(\bar{\epsilon}^p_{n+1}) = Y(\bar{\epsilon}^p_n) + H(\bar{\epsilon}^p_n)\,\Delta\bar{\epsilon}^p\,.$$

Using this in equation (2.24) gives the following equation for the equivalent tensile plastic strain increment $\Delta\bar{\epsilon}^p$,

$$\bar{\sigma}^{\rm tr} - 3\ {\rm G}\ \Delta\bar{\epsilon}^p = Y(\bar{\epsilon}^p_n) + H(\bar{\epsilon}^p_n)\,\Delta\bar{\epsilon}^p\,,$$

which gives

$$\Delta\bar{\epsilon}^p = \frac{\bar{\sigma}^{\rm tr} - Y(\bar{\epsilon}^p_n)}{3{\rm G} + H(\bar{\epsilon}^p_n)}. \tag{2.25}$$

The update for $\mathbf{F}^p$ is then obtained from

$$\mathbf{F}^p_{\rm n+1} = \left(\mathbf{1} + \Delta t\mathbf{D}^p_{\rm n+1}\right)\ \mathbf{F}^p_n, \qquad \text{with} \qquad \Delta t\mathbf{D}^p_{\rm n+1} = \sqrt{3/2}\,\Delta\bar{\epsilon}^p\,\mathbf{N}^p_{\rm tr}. \tag{2.26}$$

Then the following updates are easily calculated:

$$\begin{aligned}
\mathbf{F}^e_{\rm n+1} &= \mathbf{F}_{\rm n+1}\,(\mathbf{F}^p_{\rm n+1})^{-1}\,,\\
\mathbf{E}^e_{\rm n+1} &= \frac{1}{2}\left((\mathbf{F}^e_{\rm n+1})^\top\mathbf{F}^e_{\rm n+1} - \mathbf{1}\right)\,,\\
\mathbf{T}^e_{\rm n+1} &= 2G(\mathbf{E}^e_{\rm n+1})_0 + K\ {\rm tr}(\mathbf{E}^e_{\rm n+1})\,\mathbf{1}\,,\,,\\
\mathbf{T}_{\rm n+1} &= (J^e_{\rm n+1})^{-1}(\mathbf{F}^e_{\rm n+1})\mathbf{T}^e(\mathbf{F}^e_{\rm n+1})^\top\\
\mathbf{T}_{\rm R\,n+1} &= J\,\mathbf{T}_{\rm n+1}\mathbf{F}^{-\top}_{\rm n+1}\,.
\end{aligned} \tag{2.27}$$

3 Summary of time-integration algorithm

Given: $\{\mathbf{F}^p_n, \bar{\epsilon}^p_n\}$ and $\mathbf{F}_{\mathrm{n+1}}$ at time t_n

Calculate: $\left\{\mathbf{F}^p_{n+1}, \bar{\epsilon}^p_{n+1}, \mathbf{T}_{\mathrm{R}n+1}, \right\}$ at time $t_{n+1} = t_n + \Delta t$.

Step 1. Calculate the trial elastic deformation gradient

$$\mathbf{F}^e_{\mathrm{tr}} \overset{\mathrm{def}}{=} \mathbf{F}_{\mathrm{n+1}}\, \mathbf{F}^{p-1}_n. \tag{3.1}$$

Step 2. Calculate the trial elastic second Piola stress stress and the associated quantities

$$\begin{aligned}
\mathbf{C}^e_{\mathrm{tr}} &= (\mathbf{F}^e_{\mathrm{tr}})^\top \mathbf{F}^e_{\mathrm{tr}},\\
\mathbf{E}^e_{\mathrm{tr}} &= (1/2)\,(\mathbf{C}^e_{\mathrm{tr}} - \mathbf{1})\ ,\\
\mathbf{T}^e_{\mathrm{tr}} &= \mathbb{C}[\mathbf{E}^e_{\mathrm{tr}}]\,,\\
(\mathbf{T}^e_{\mathrm{tr}})_0 &= \mathbf{T}^e_{\mathrm{tr}} - \tfrac{1}{3}\,(\mathrm{tr}\,\mathbf{T}^e_{\mathrm{tr}})\mathbf{1},\\
\bar{\sigma}_{\mathrm{tr}} &= \sqrt{3/2}|(\mathbf{T}^e_{\mathrm{tr}})_0|, \qquad \mathbf{N}^p_{\mathrm{tr}} = \sqrt{3/2}\,\frac{(\mathbf{T}^e_{\mathrm{tr}})_0}{\bar{\sigma}_{\mathrm{tr}}},\\
f^{\mathrm{tr}}_{\mathrm{n+1}} &= \bar{\sigma}_{\mathrm{tr}} - Y_n.
\end{aligned} \tag{3.2}$$

Step 4 Determine if the step is elastic or plastic:

if ($f^{\mathrm{tr}}_{\mathrm{n+1}} \le 0$) then
    elastic step
else
    plastic step
end if

Step 5.

1. For an elastic step perform the update:

$$\begin{aligned}
\mathbf{T}_{\mathrm{n+1}} &= (J^e_{\mathrm{n+1}})^{-1}(\mathbf{F}^e_{\mathrm{n+1}})\mathbf{T}^e_{\mathrm{tr}}(\mathbf{F}^e_{\mathrm{n+1}})^\top\\
\mathbf{T}_{\mathrm{Rn+1}} &= J\,\mathbf{T}_{\mathrm{n+1}}\mathbf{F}^{-\top}_{\mathrm{n+1}}\\
\mathbf{F}^p_{\mathrm{n+1}} &= \mathbf{F}^p_n, \qquad \bar{\epsilon}^p{}_{\mathrm{n+1}} = \bar{\epsilon}^p{}_n\,.
\end{aligned} \tag{3.3}$$

2. For a plastic step:
   Calculate $\Delta\bar{\epsilon}^p$

$$\Delta\bar{\epsilon}^p = \frac{\bar{\sigma}^{\mathrm{tr}} - Y(\bar{\epsilon}^p_n)}{3\mathrm{G} + H(\bar{\epsilon}^p_n)}. \tag{3.4}$$

   and update $\bar{\epsilon}^p$,

$$\bar{\epsilon}^p_{n+1} = \bar{\epsilon}^p_n + \Delta\bar{\epsilon}^p\,. \tag{3.5}$$

   The update for $\mathbf{F}^p$ is then obtained from

$$\mathbf{F}^p_{\mathrm{n+1}} = \left(\mathbf{1} + \sqrt{3/2}\,\Delta\bar{\epsilon}^p\,\mathbf{N}^p_{\mathrm{tr}}\right)\mathbf{F}^p_n, \tag{3.6}$$

and the following updates are easily calculated:

$$
\begin{aligned}
\mathbf{F}^e_{\mathrm{n+1}} &= \mathbf{F}_{\mathrm{n+1}}\,(\mathbf{F}^p_{\mathrm{n+1}})^{-1}\,,\\
\mathbf{E}^e_{\mathrm{n+1}} &= \frac{1}{2}\left((\mathbf{F}^e_{\mathrm{n+1}})^{\top}\mathbf{F}^e_{\mathrm{n+1}} - \mathbf{1}\right)\,,\\
\mathbf{T}^e_{\mathrm{n+1}} &= 2G(\mathbf{E}^e_{\mathrm{n+1}})_0 + K\ \mathrm{tr}(\mathbf{E}^e_{\mathrm{n+1}})\,\mathbf{1}\,,\,,\\
\mathbf{T}_{\mathrm{n+1}} &= (J^e_{\mathrm{n+1}})^{-1}(\mathbf{F}^e_{\mathrm{n+1}})\mathbf{T}^e(\mathbf{F}^e_{\mathrm{n+1}})^{\top}\,,\\
\mathbf{T}_{\mathrm{R\,n+1}} &= J\,\mathbf{T}_{\mathrm{n+1}}\mathbf{F}^{-\top}_{\mathrm{n+1}}\,.
\end{aligned}
\tag{3.7}
$$

References

M. E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, Cambridge, 2010.