

Mesh Parameterization

Acknowledgement: Olga Sorkine-Hornung
CSC 305 - Introduction to Computer Graphics - Teseo Schneider

Projections

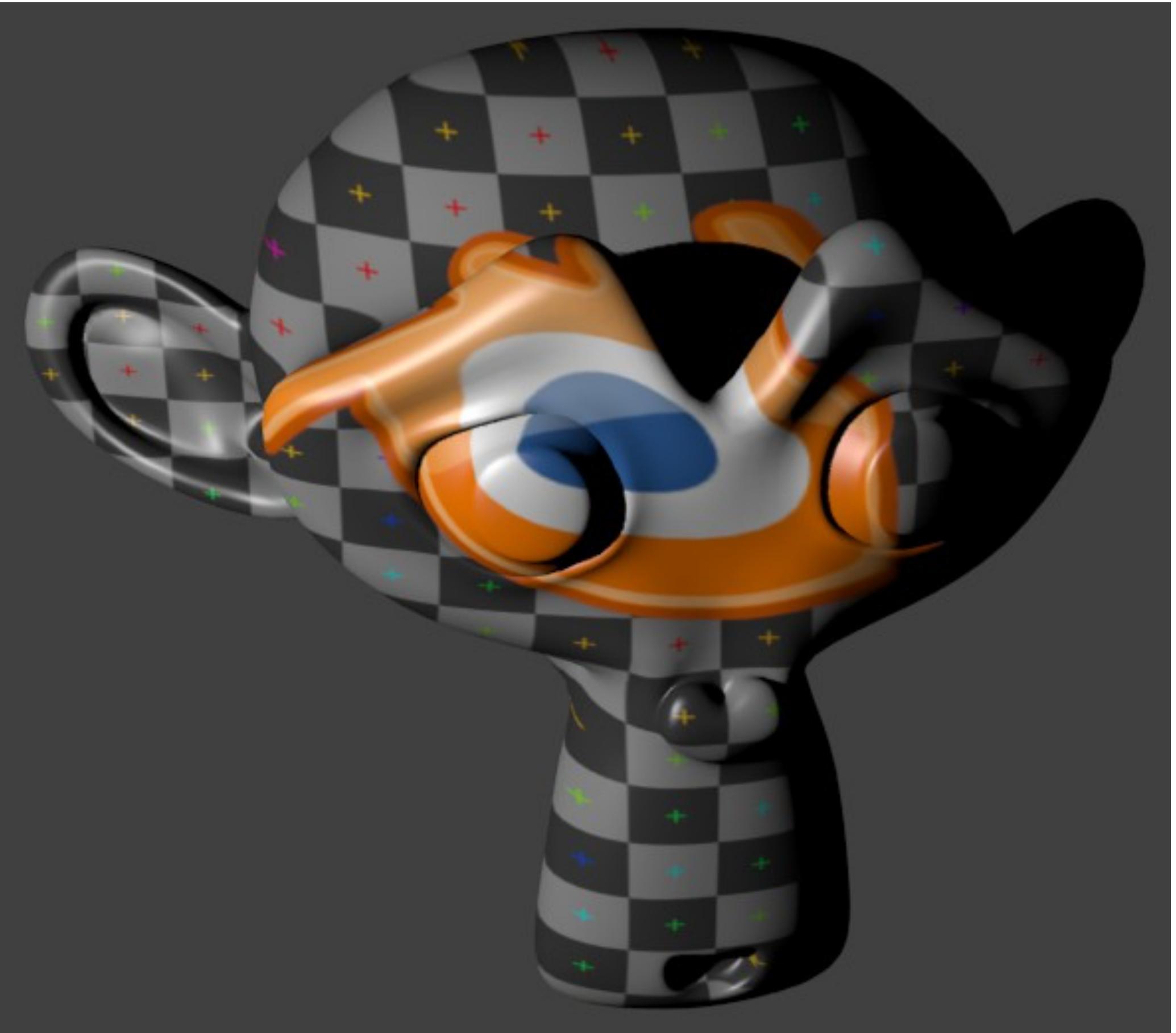
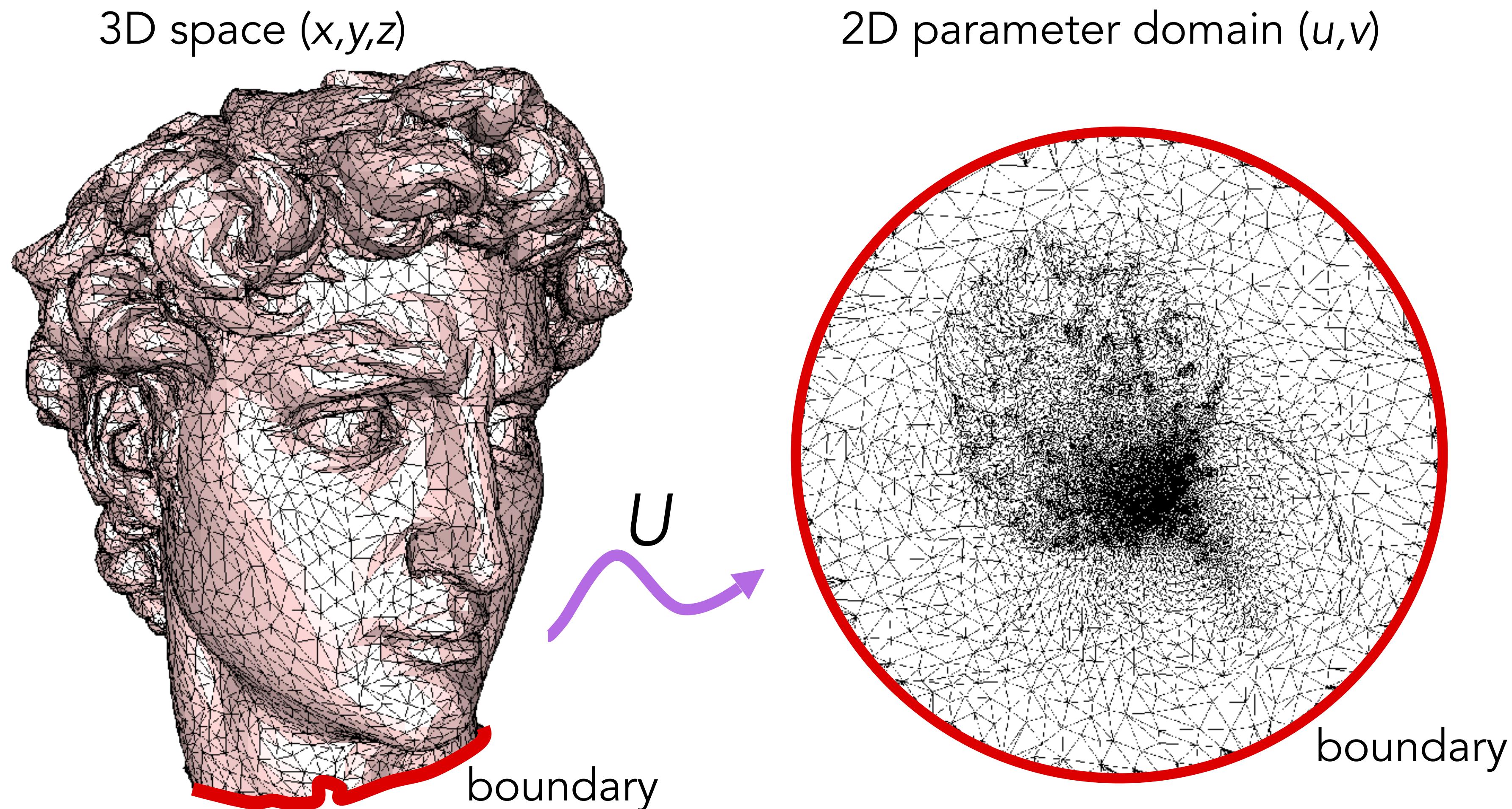


Image Courtesy of Blender

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Surface Parameterization

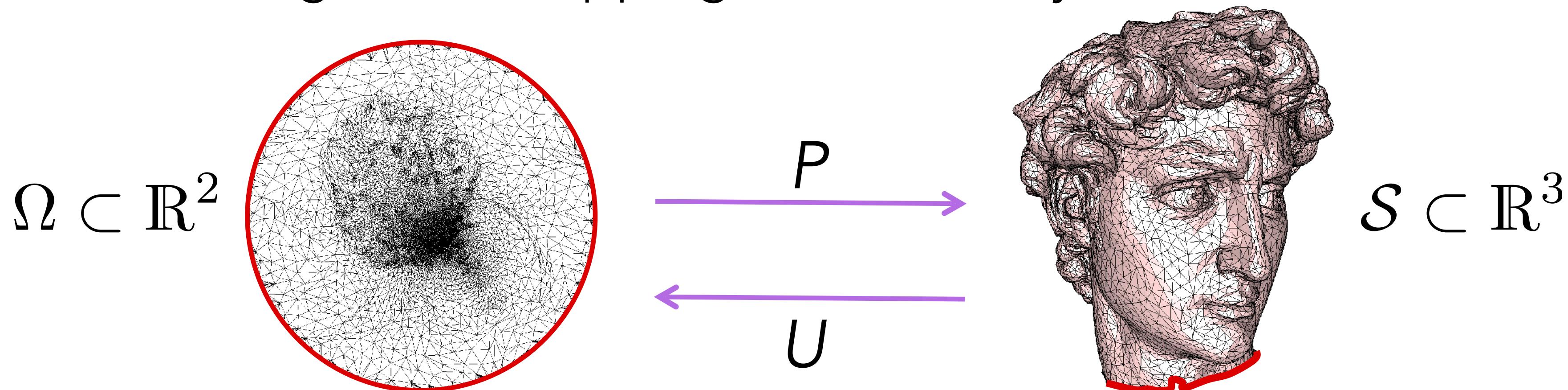


Parameterization – Definition

- Mapping P between a 2D domain Ω and the mesh S embedded in 3D (the inverse = flattening)
- Each mesh vertex has a corresponding 2D position:

$$U(\mathbf{v}_i) = (u_i, v_i)$$

- Inside each triangle, the mapping is affine (barycentric coordinates)

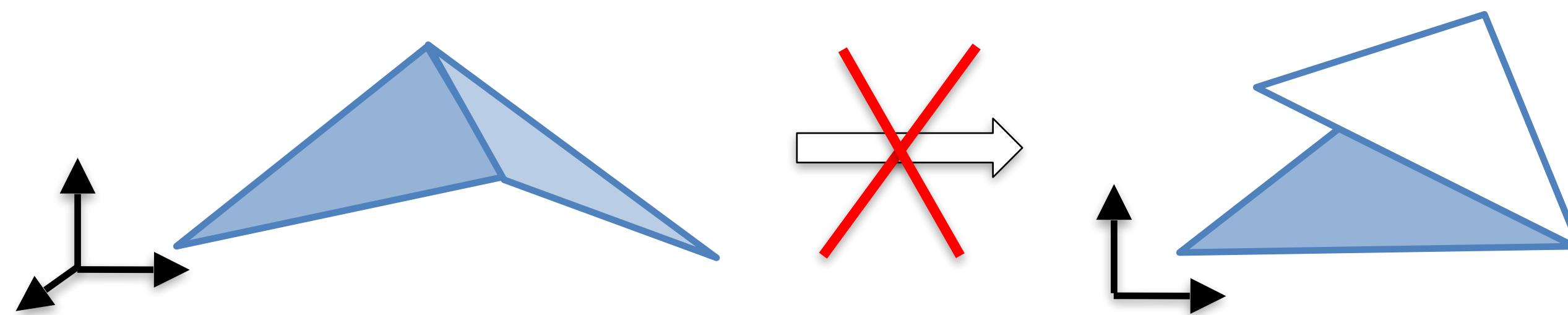


What is a good parametrization?

- It depends on the application, but usually:
 - Bijectivity
 - Number of cuts and charts
 - Geometric distortion

Bijection

- Locally bijective (1-1 and onto): No triangles fold over.



- Globally bijective:
locally bijective +
no “distant” areas
overlap

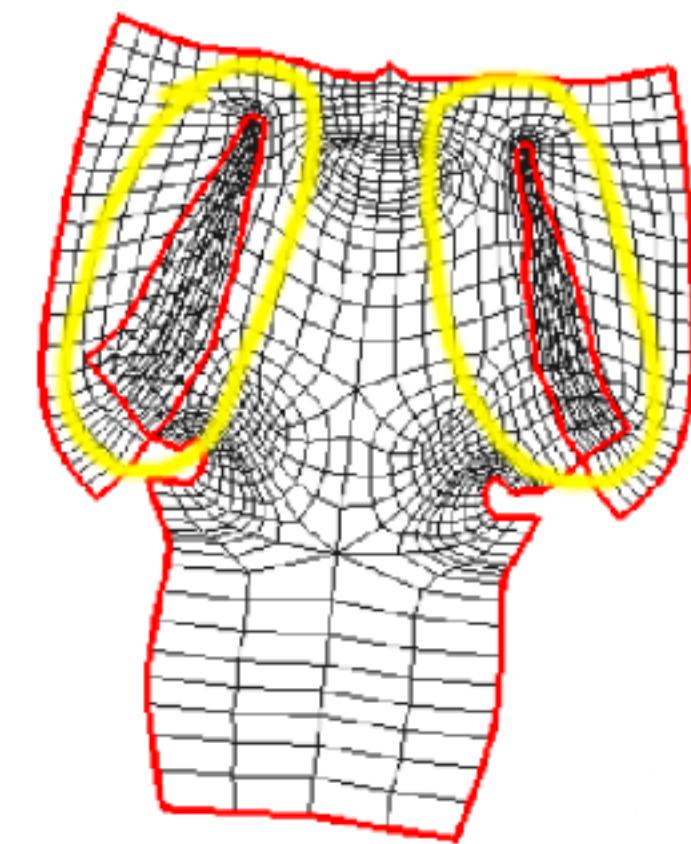
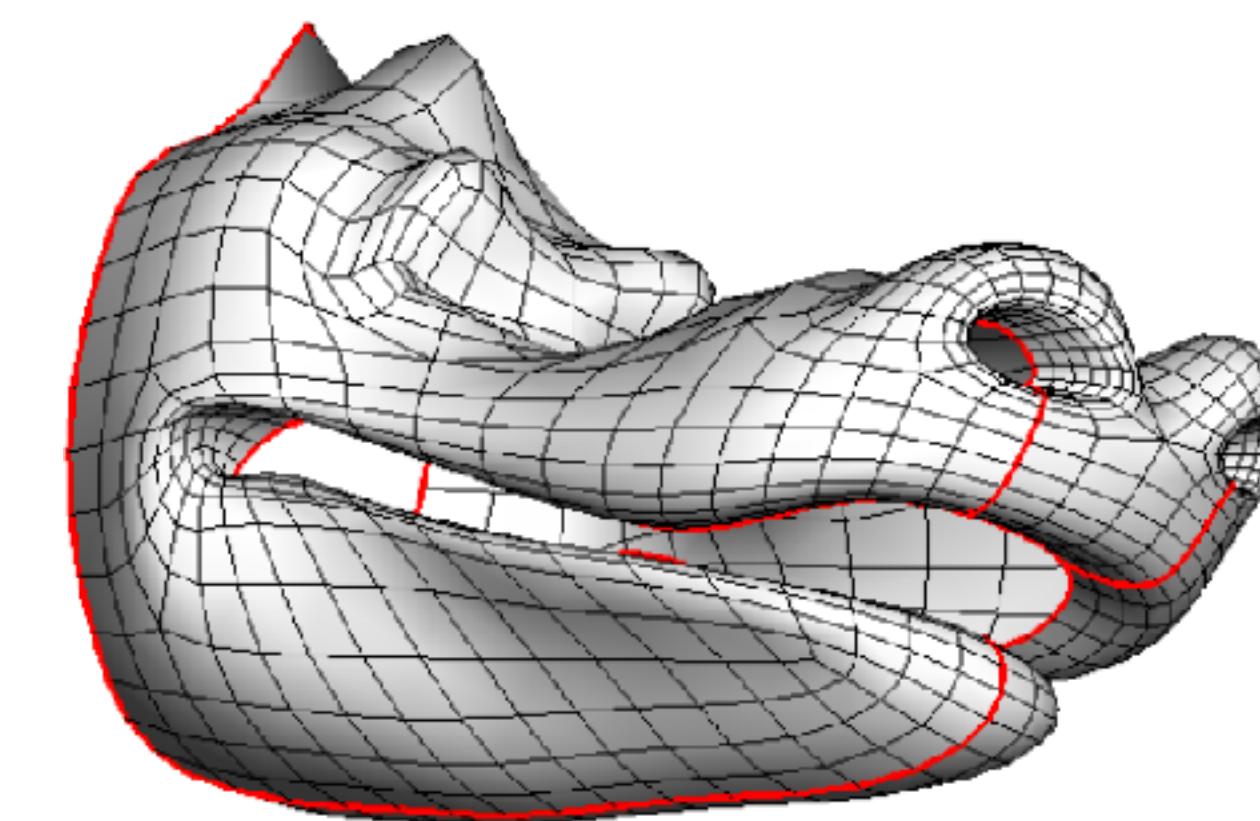
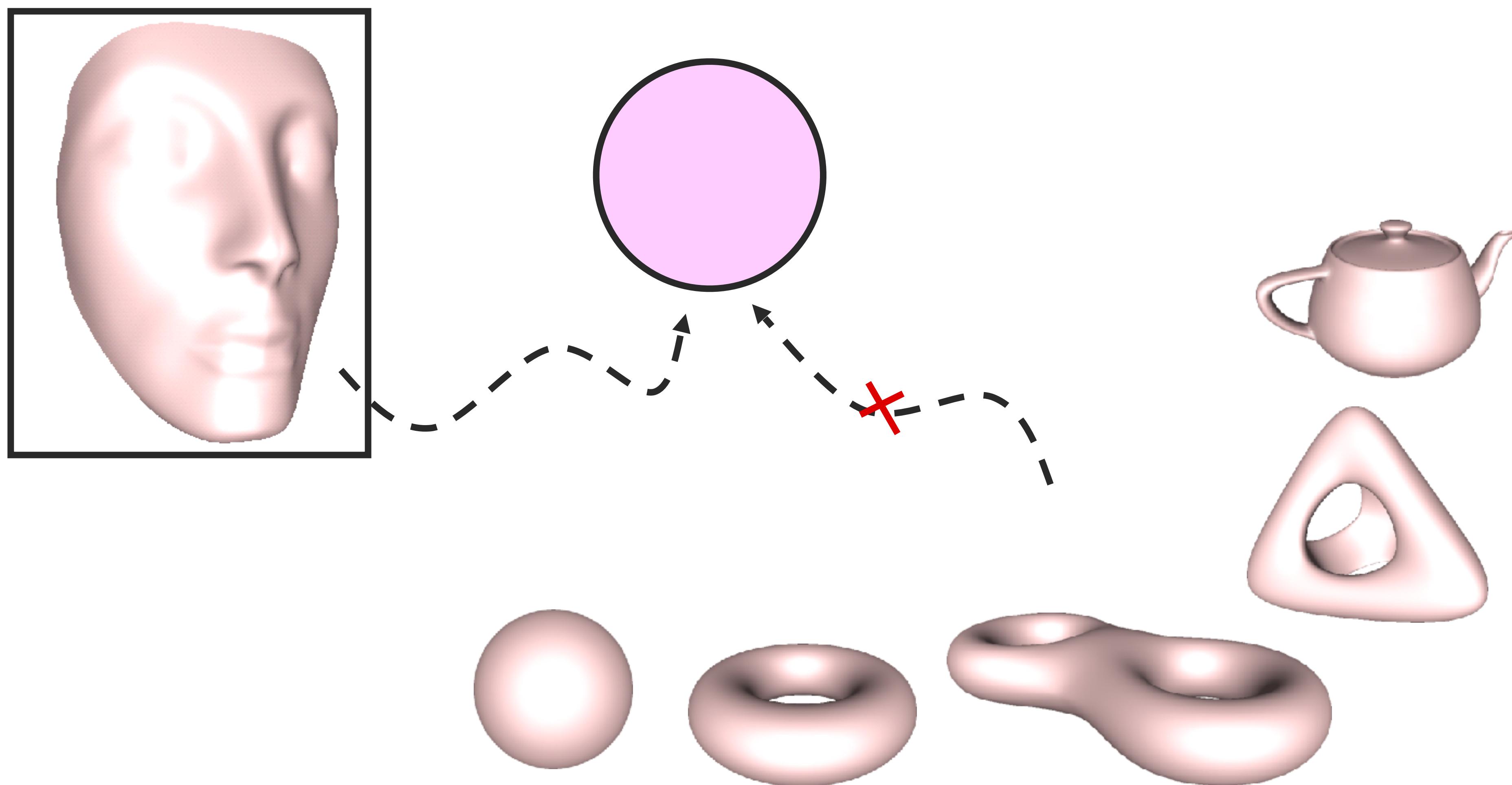
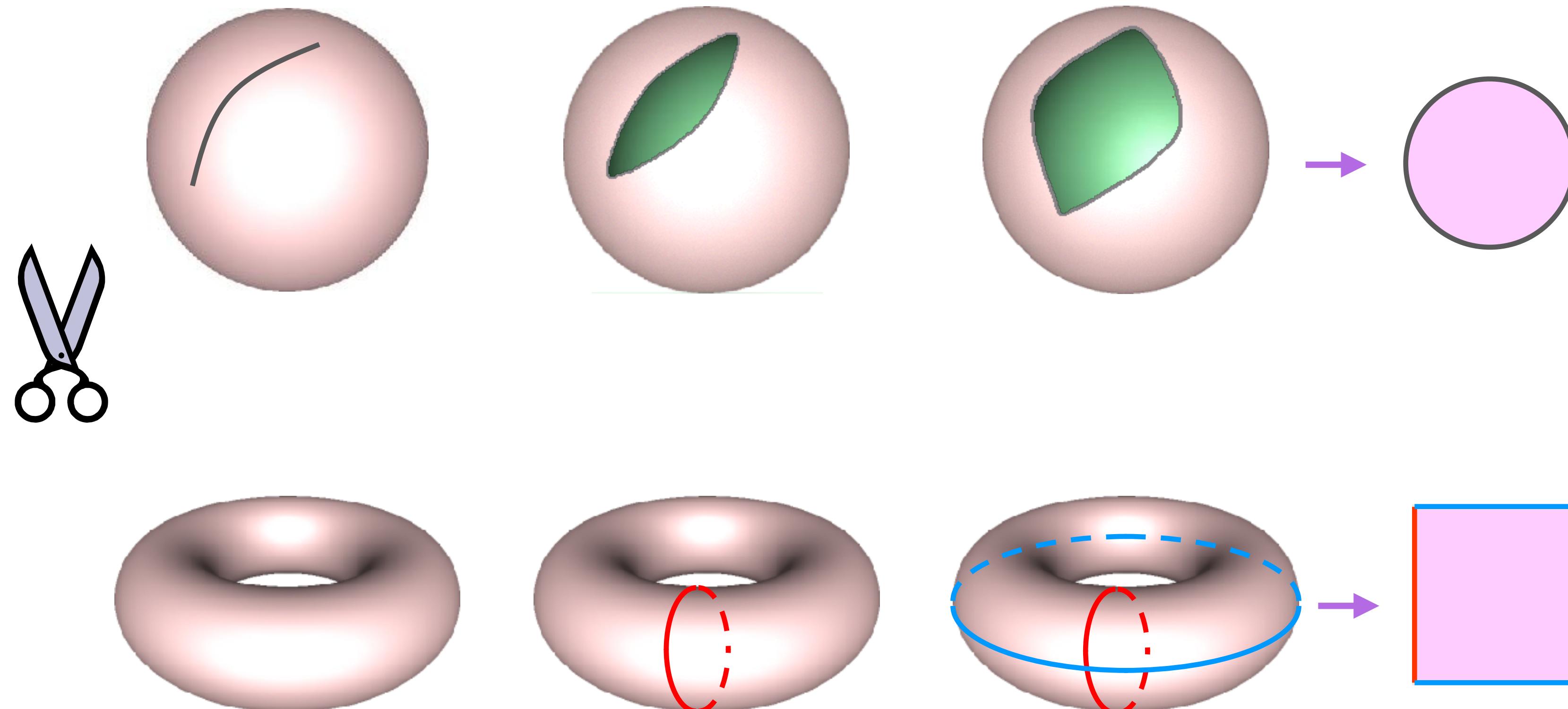


image from “Least Squares Conformal Maps”, Lévy et al., SIGGRAPH 2002

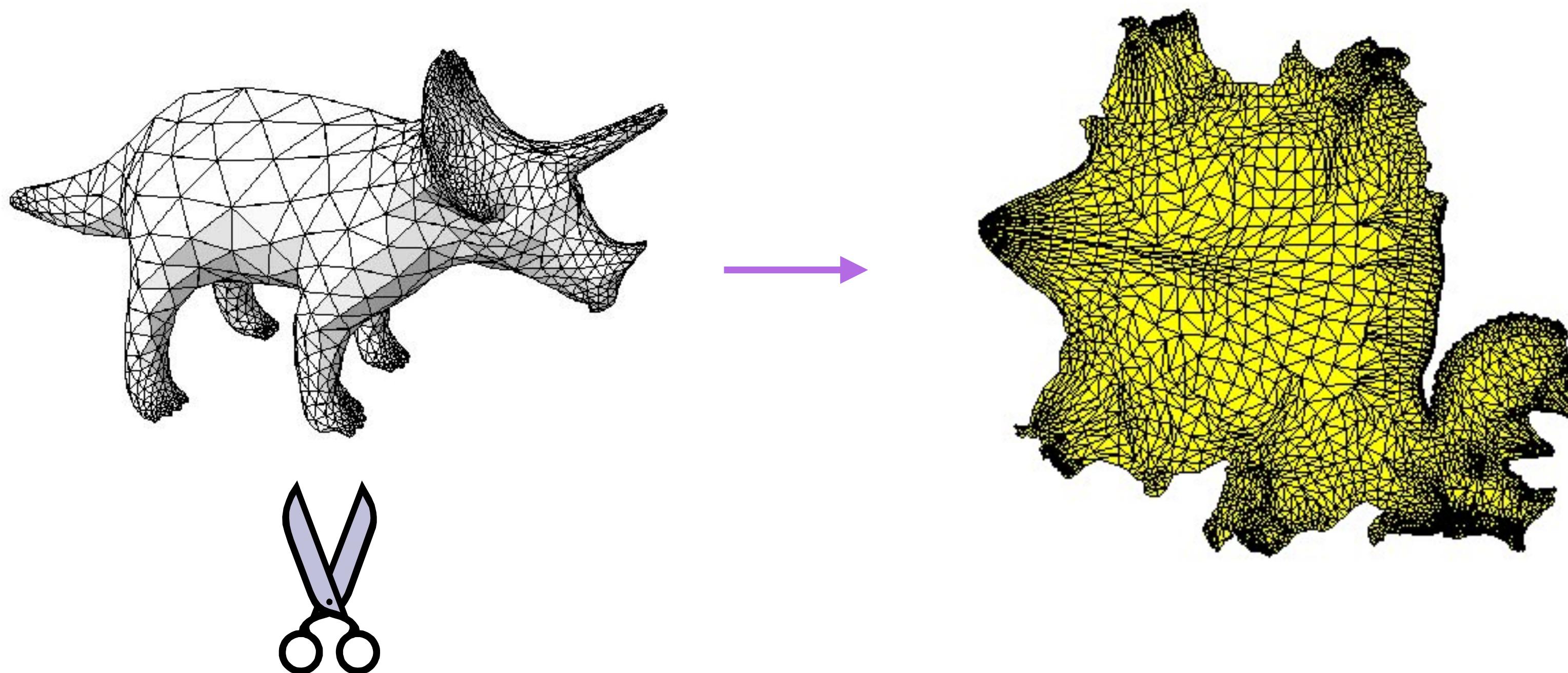
Bijectivity: Non-Disk Domains



Topological Cutting



Topological Cutting



A. Sheffer, J. Hart:

Seamster: Inconspicuous Low-Distortion Texture Seam Layout, IEEE Vis 2002

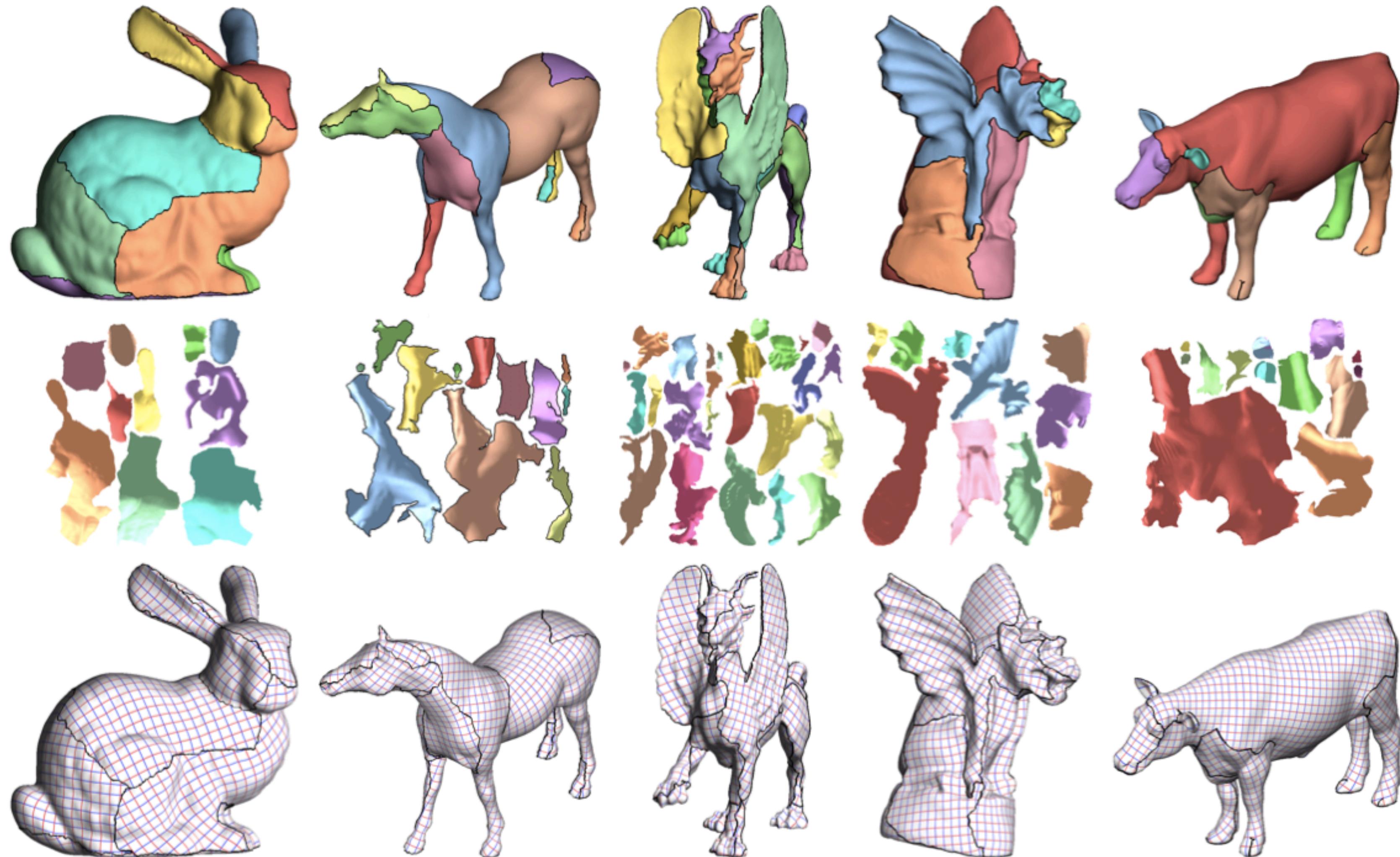
<http://www.cs.ubc.ca/~sheffa/papers/VIS02.pdf>



University
of Victoria

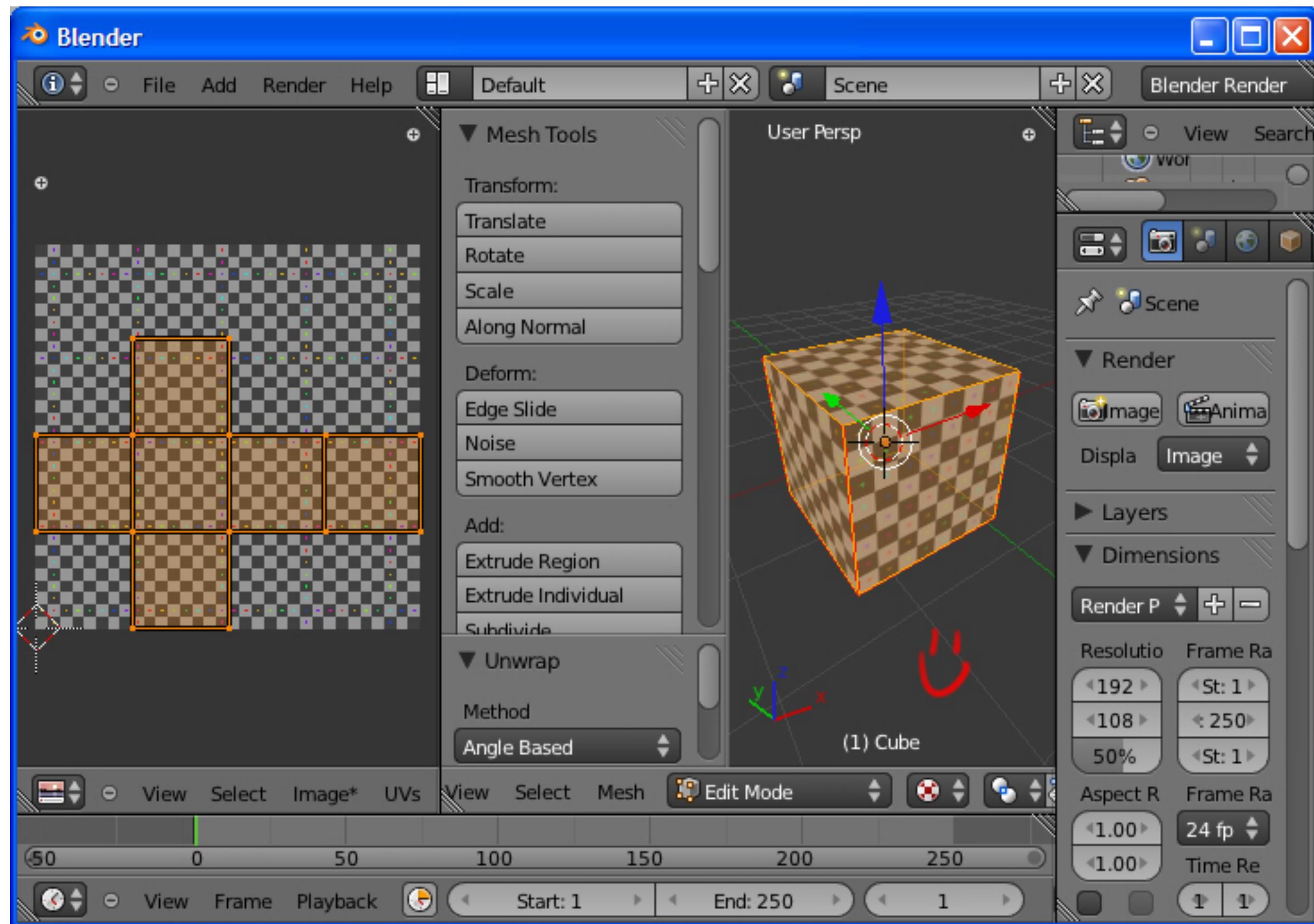
Computer Science

Segmentation



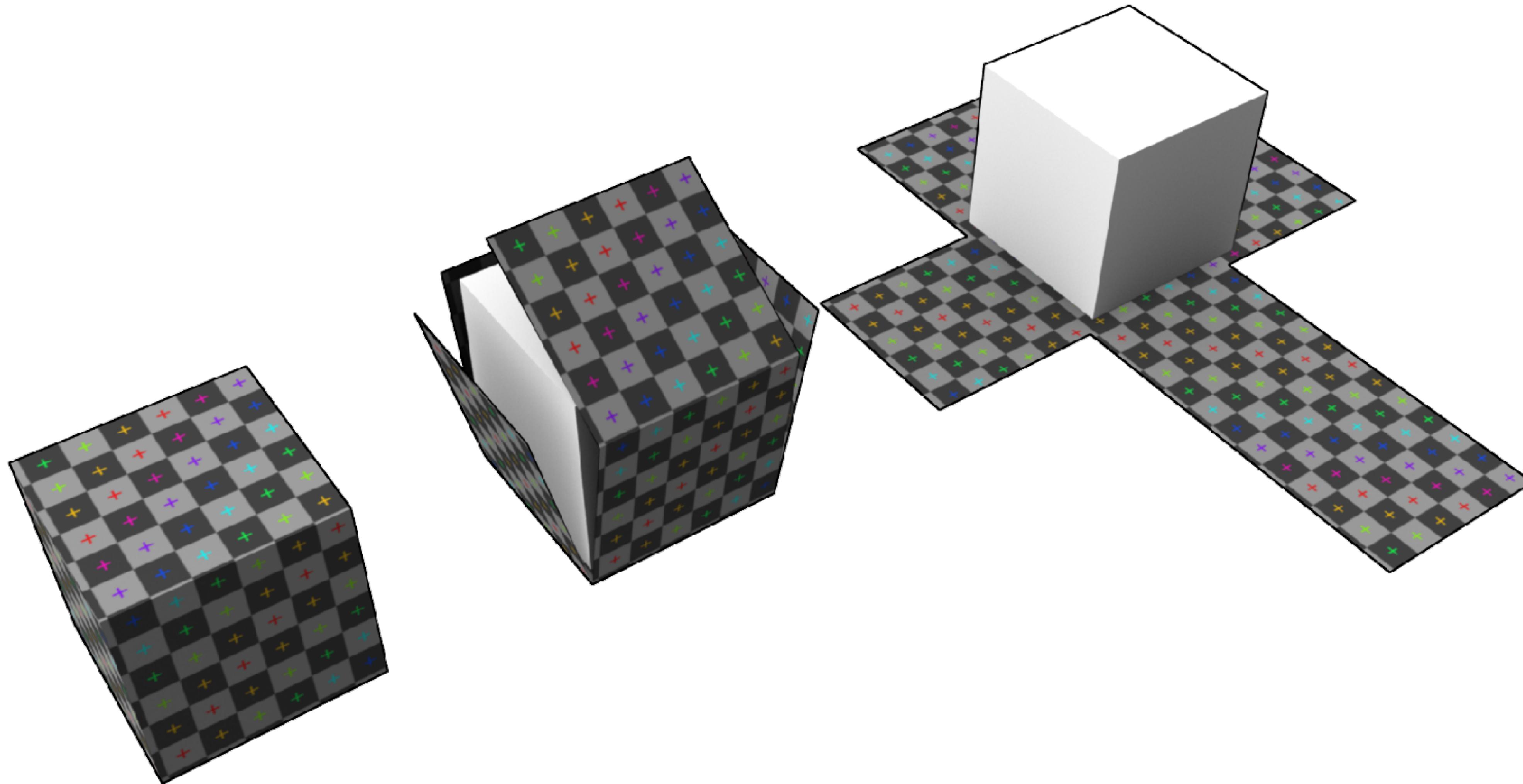
D-Charts: Quasi-Developable Mesh Segmentation,
D. Julius, V. Kraevoy, A. Sheffer, EUROGRAPHICS 2005

Segmentation



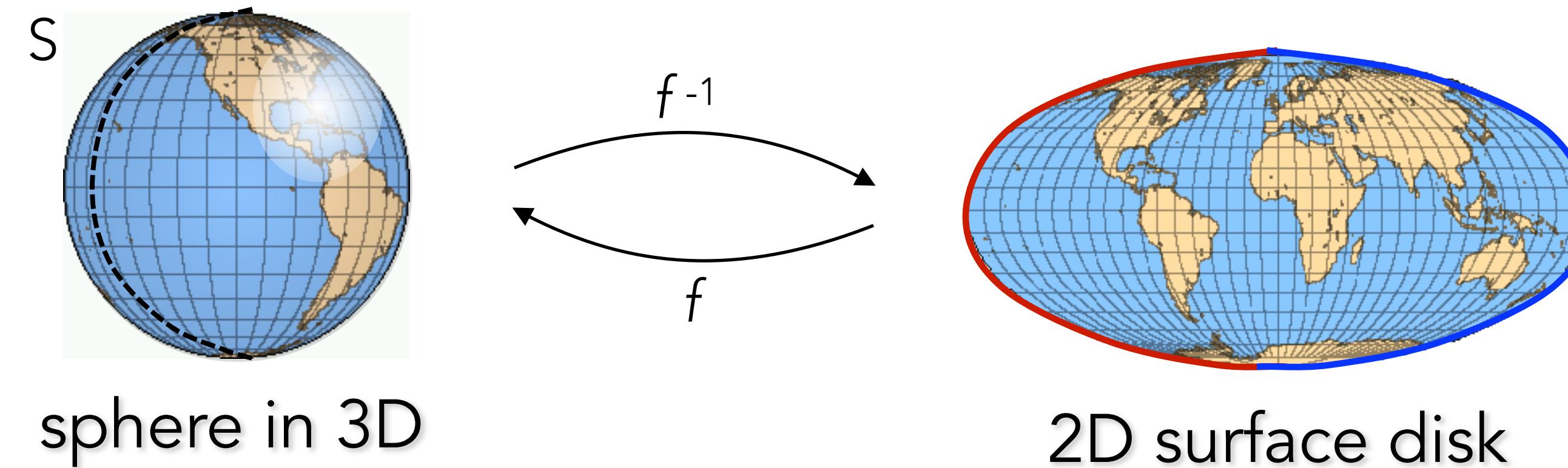
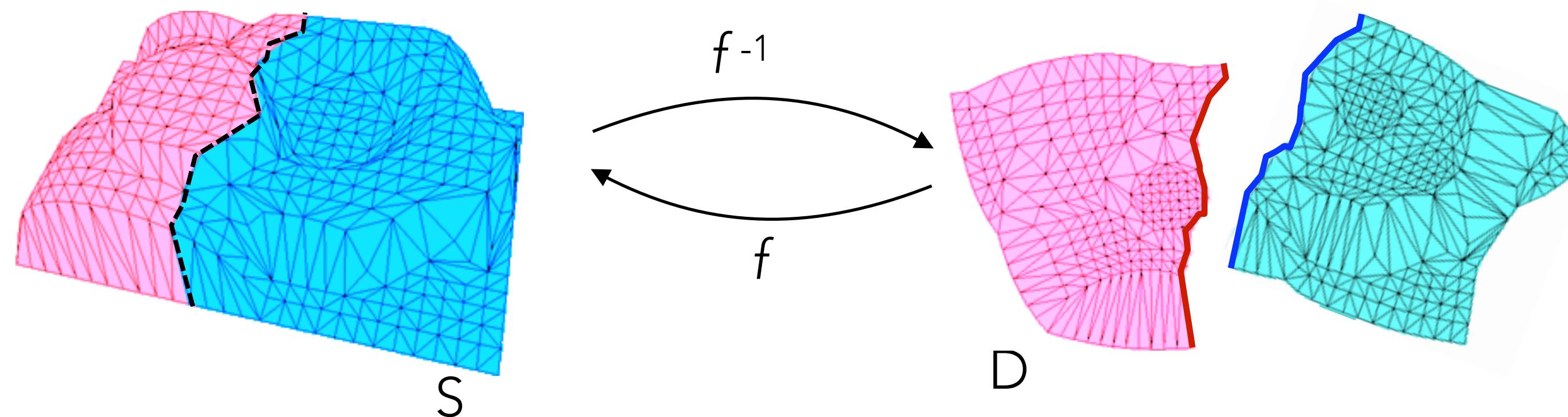
<http://sophiehoulden.com/tutorials/blender/unwrapTut.html>

Segmentation



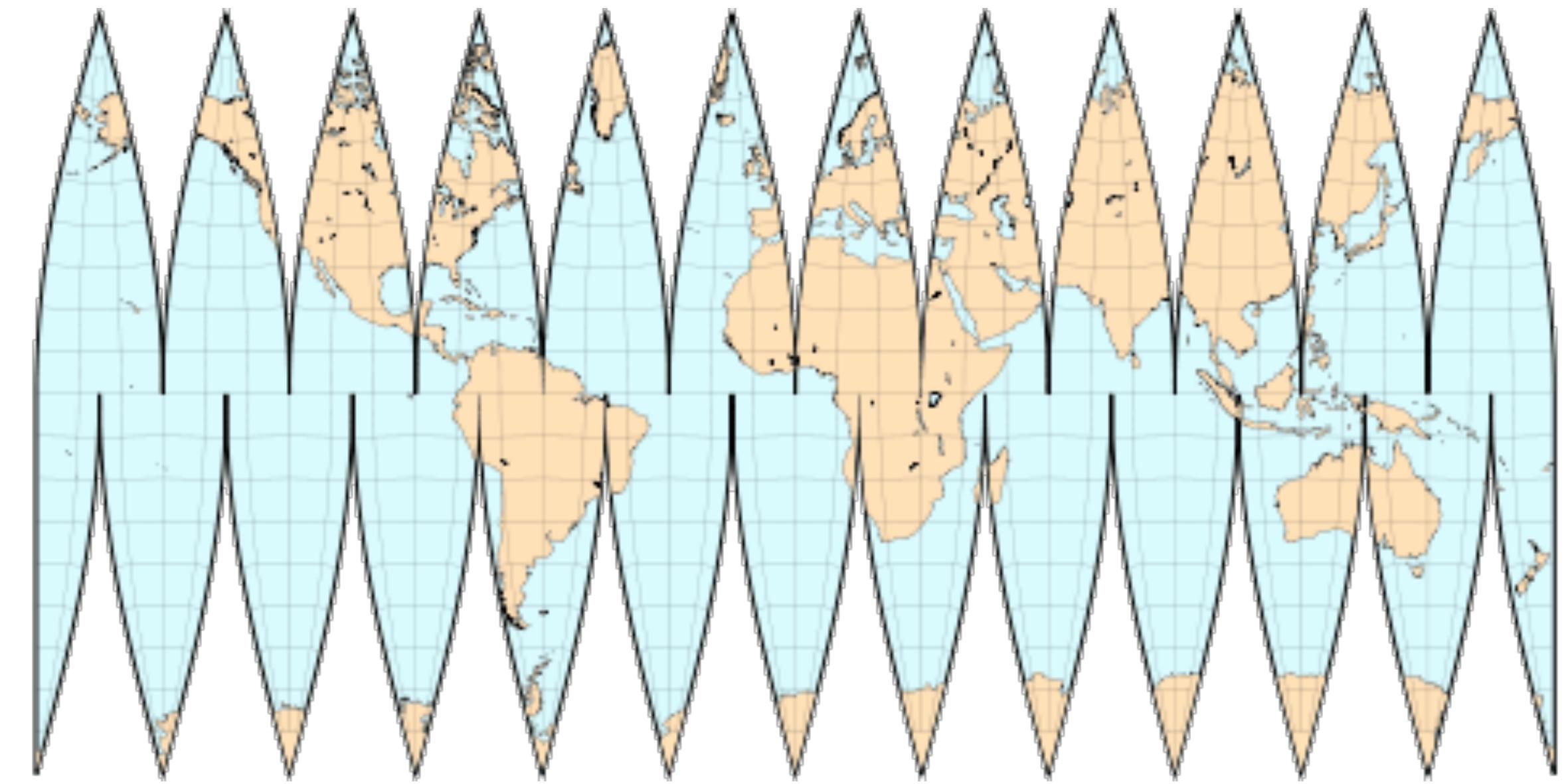
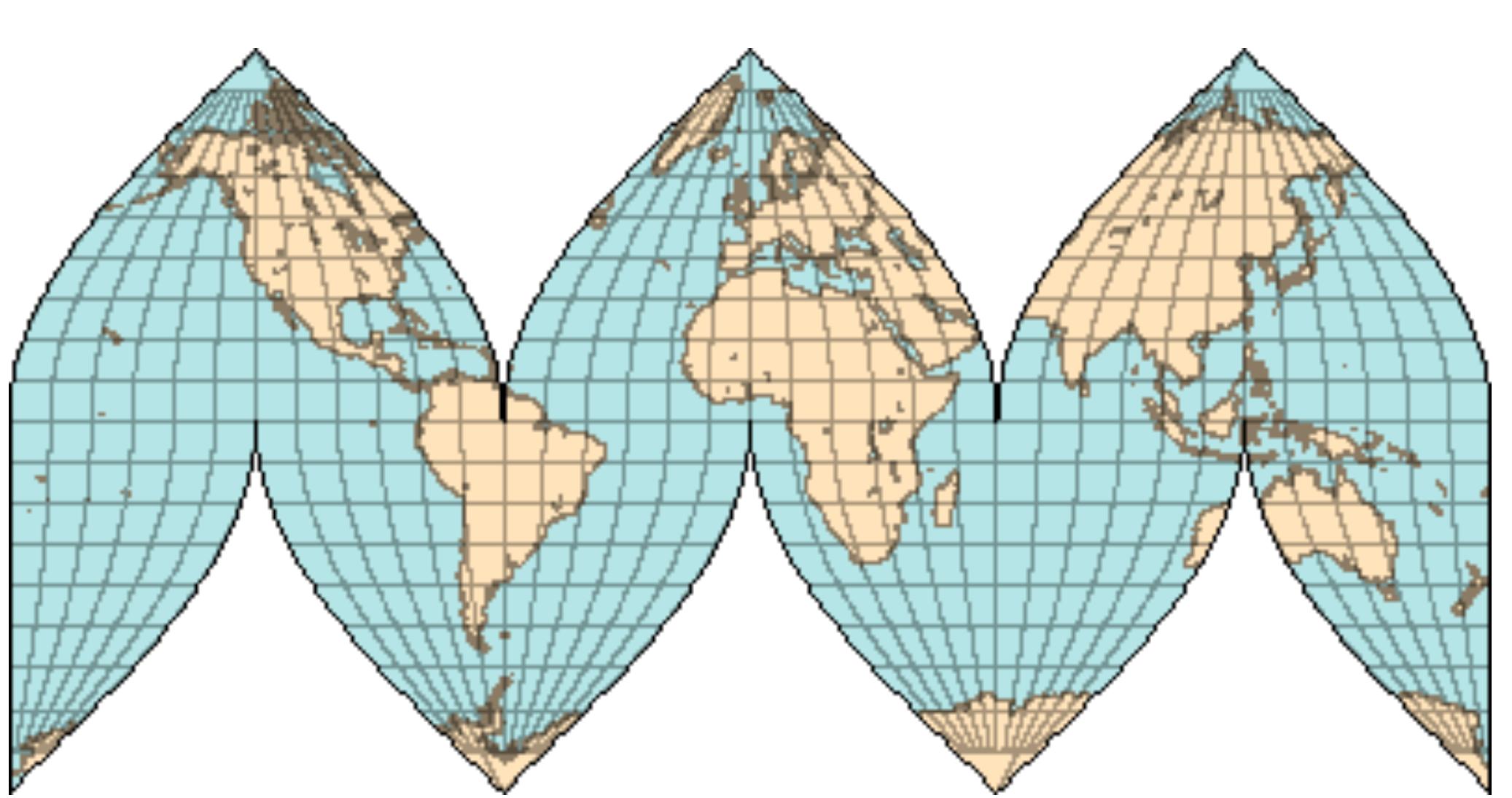
By Zephyris at en.wikipedia, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=7202834>

Good = “fewer cuts”?

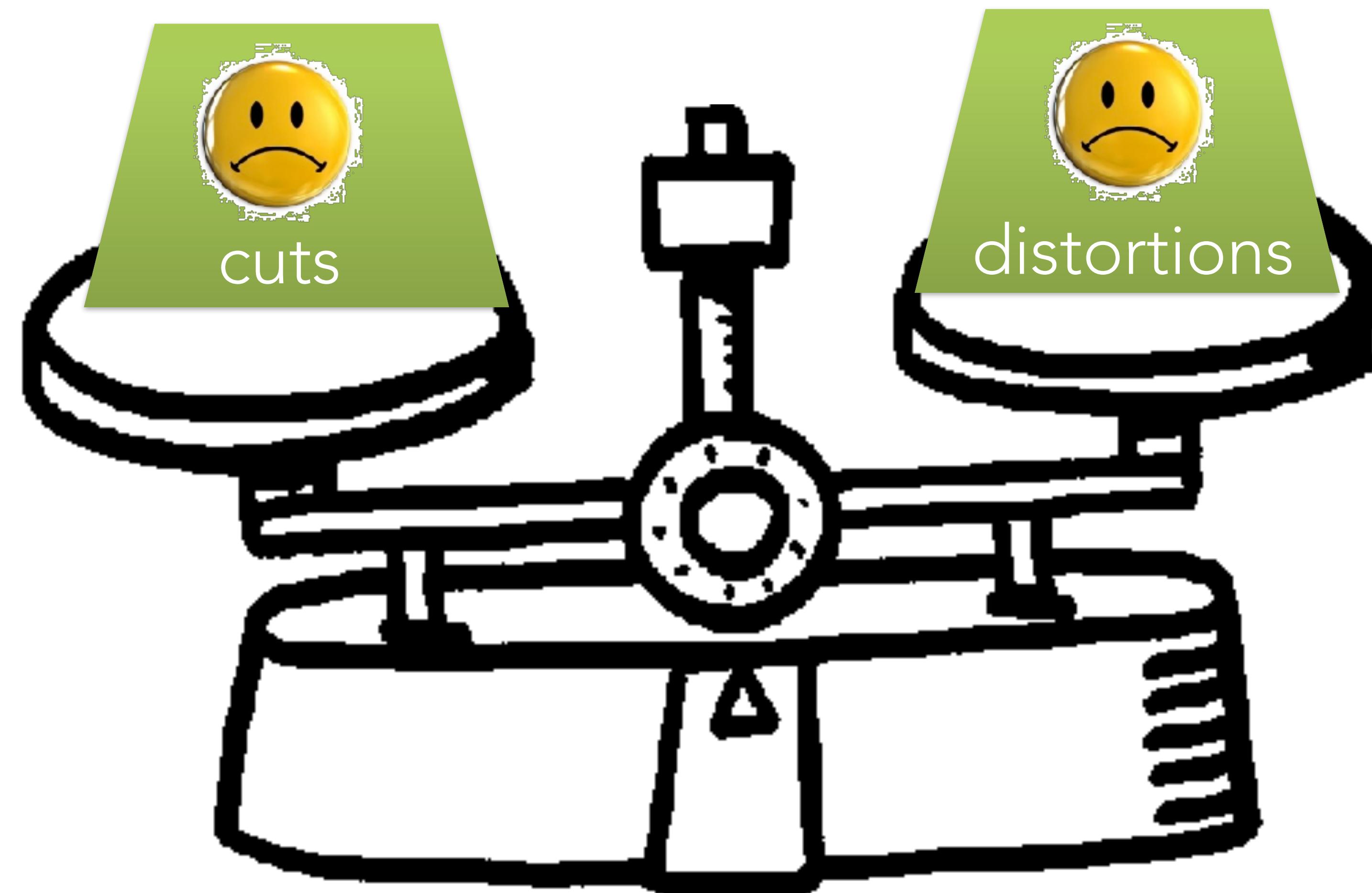


Good = “fewer cuts”?

but... more cuts => less distortion



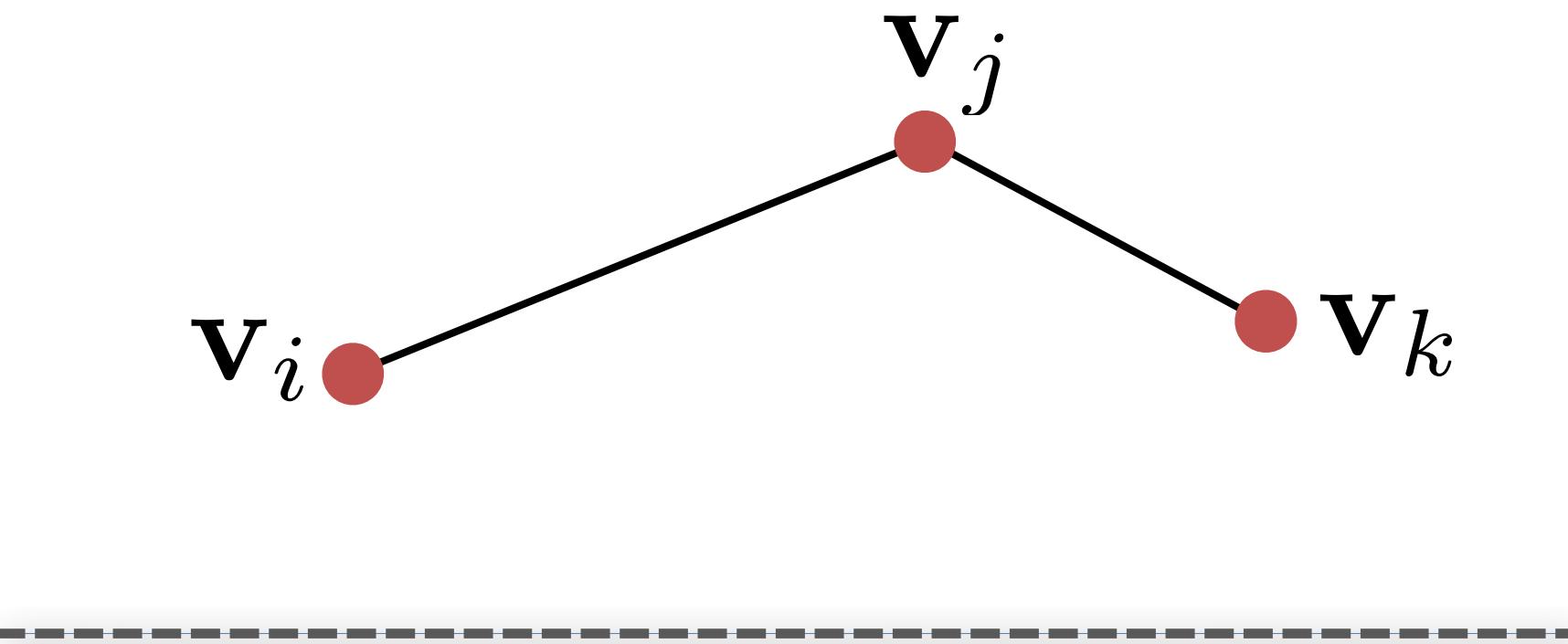
A difficult balance



Harmonic Mapping

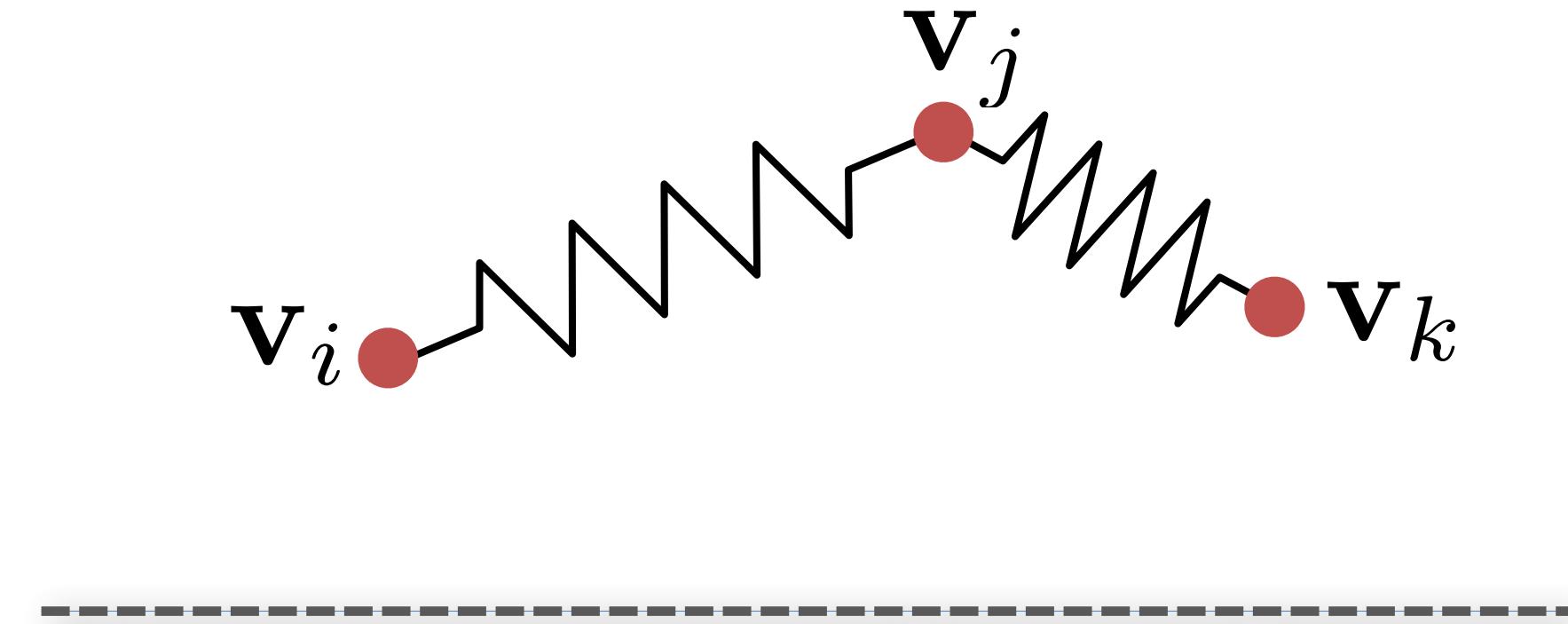
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



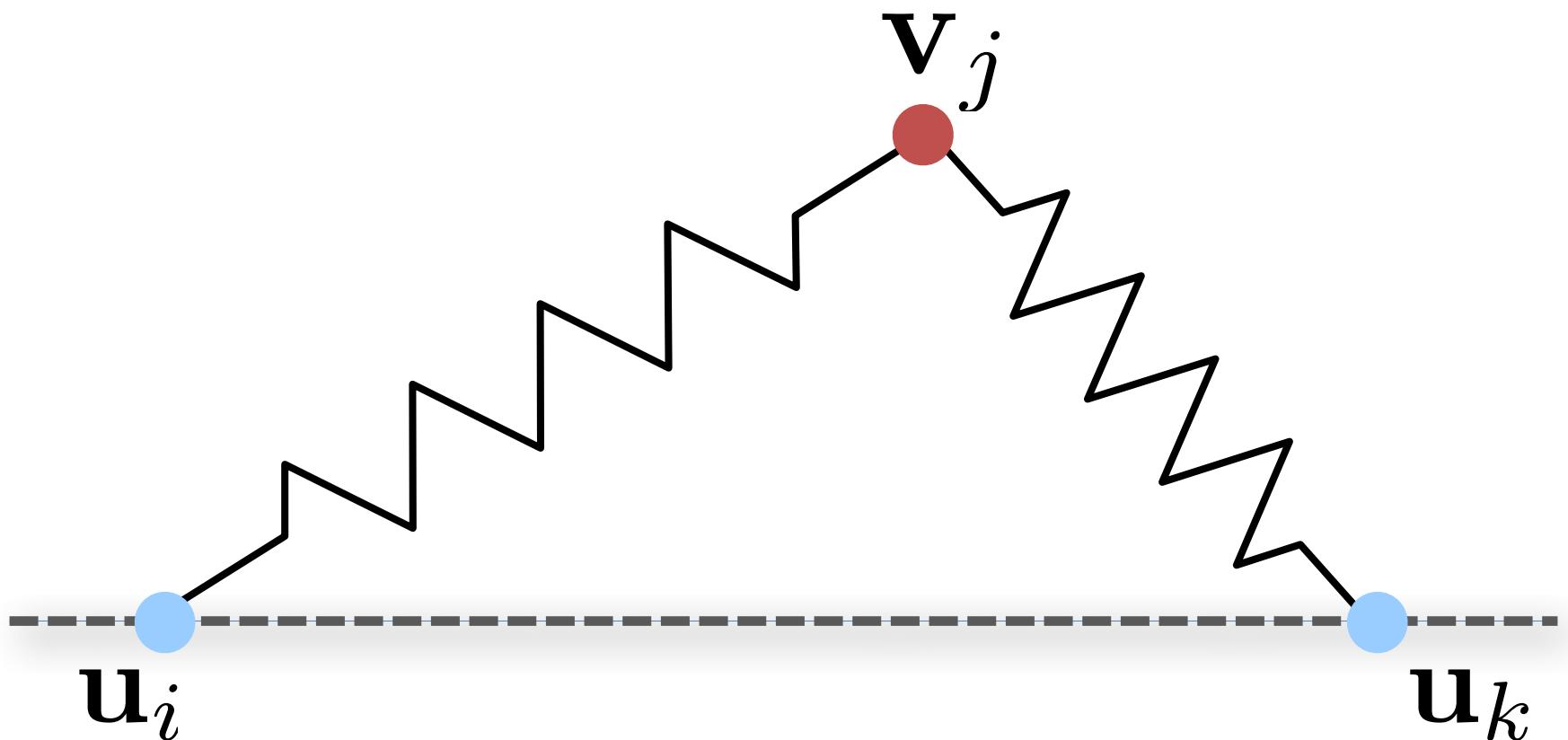
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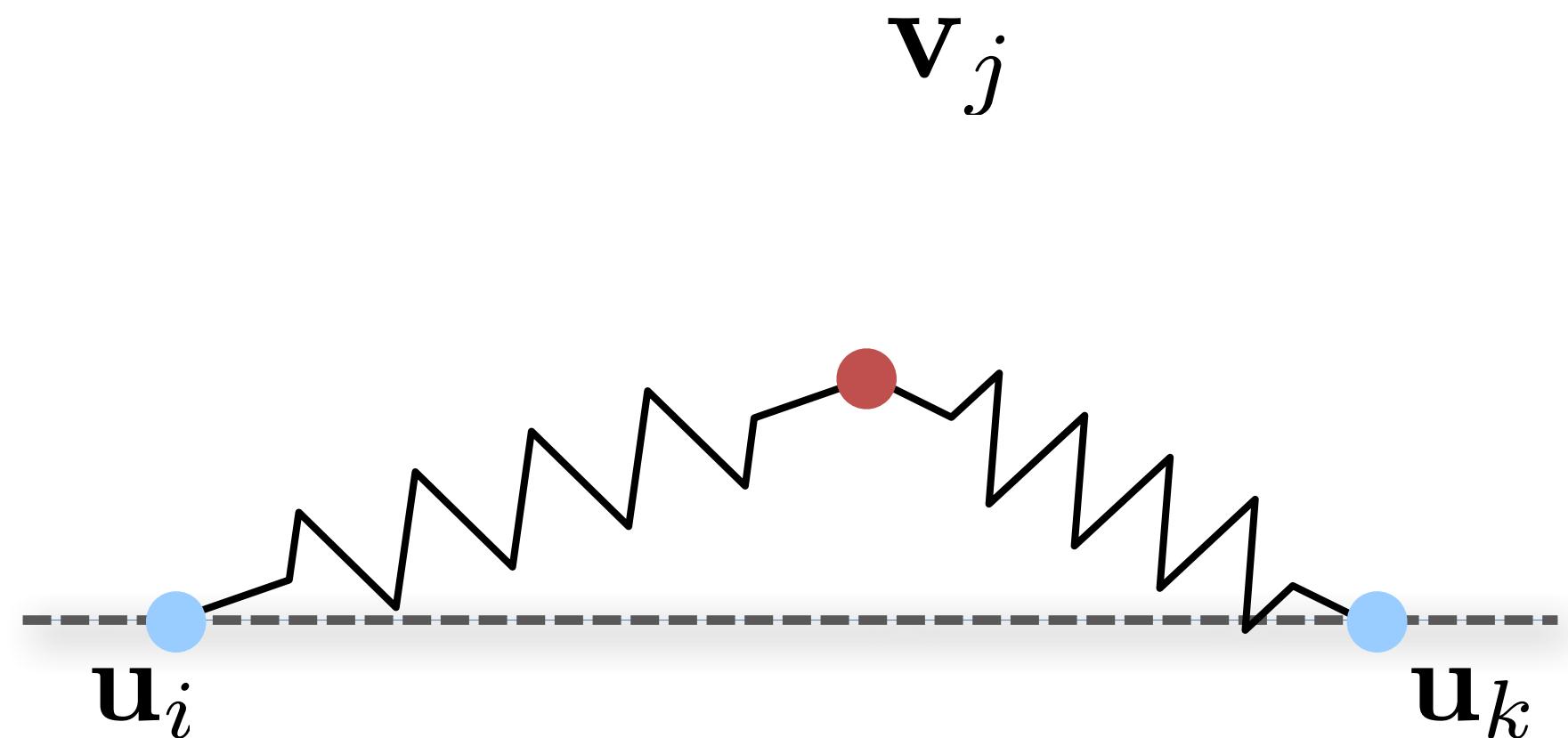
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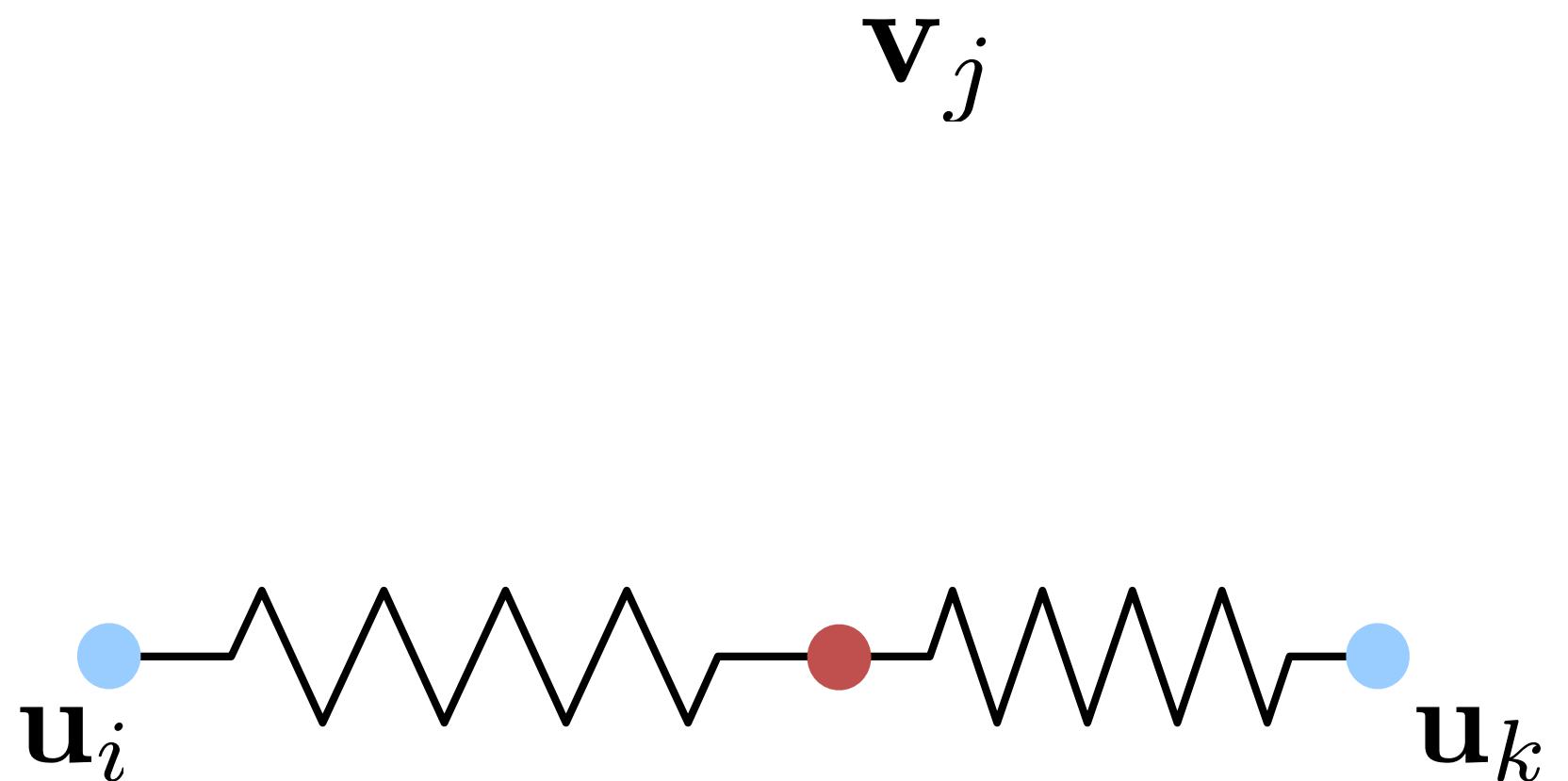
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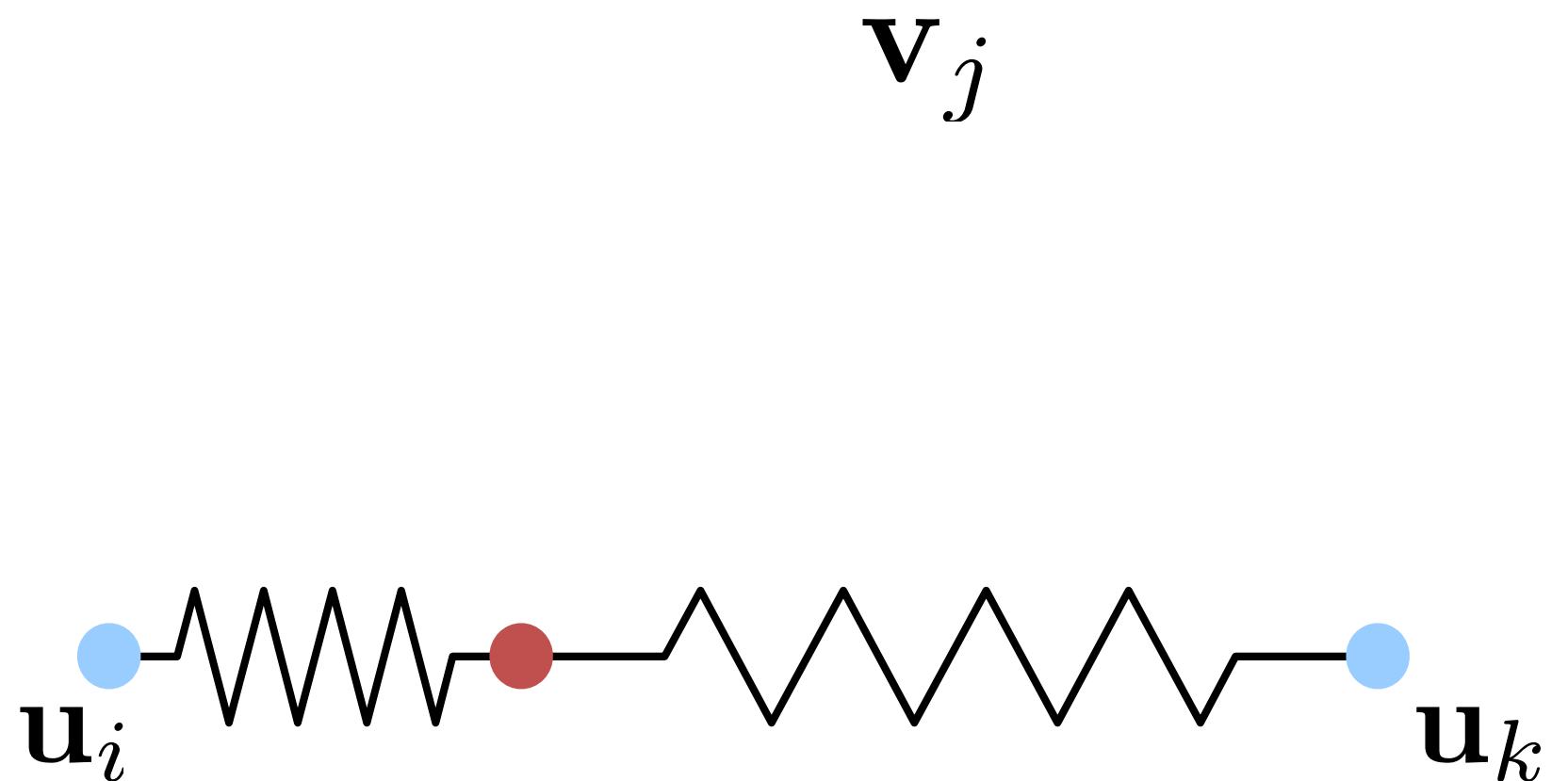
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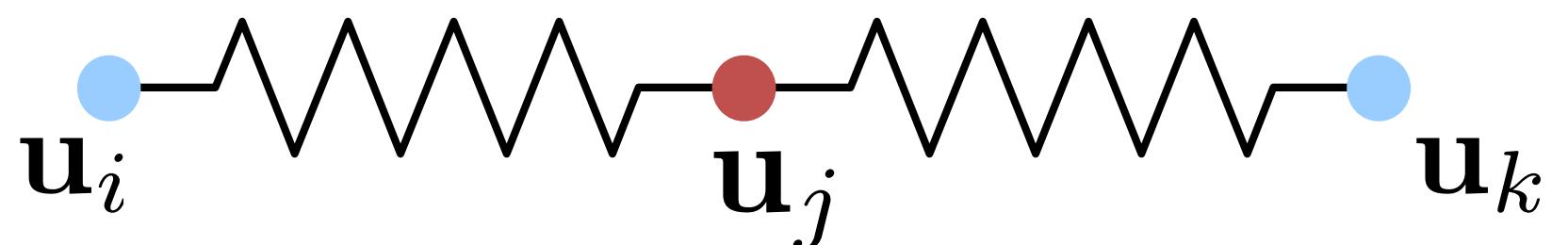
Harmonic Mapping

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Harmonic Mapping

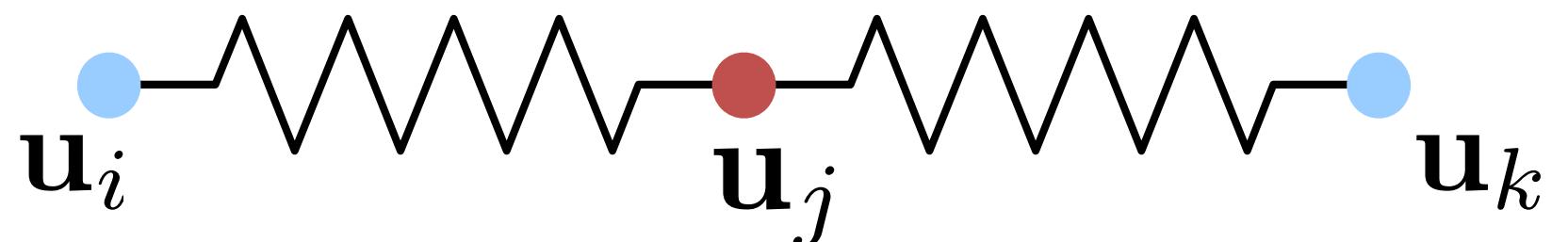
- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane
- Spring energy:

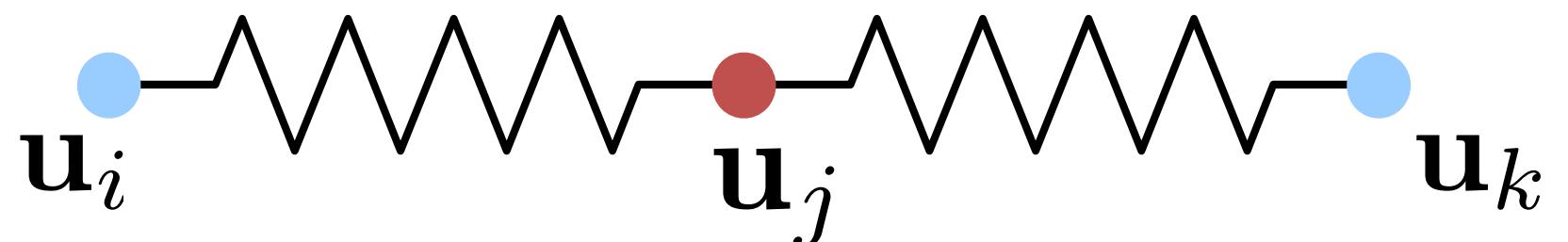
$$\frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$
$$\mathbf{u}_i, \mathbf{u}_j \in \mathbb{R}^2$$



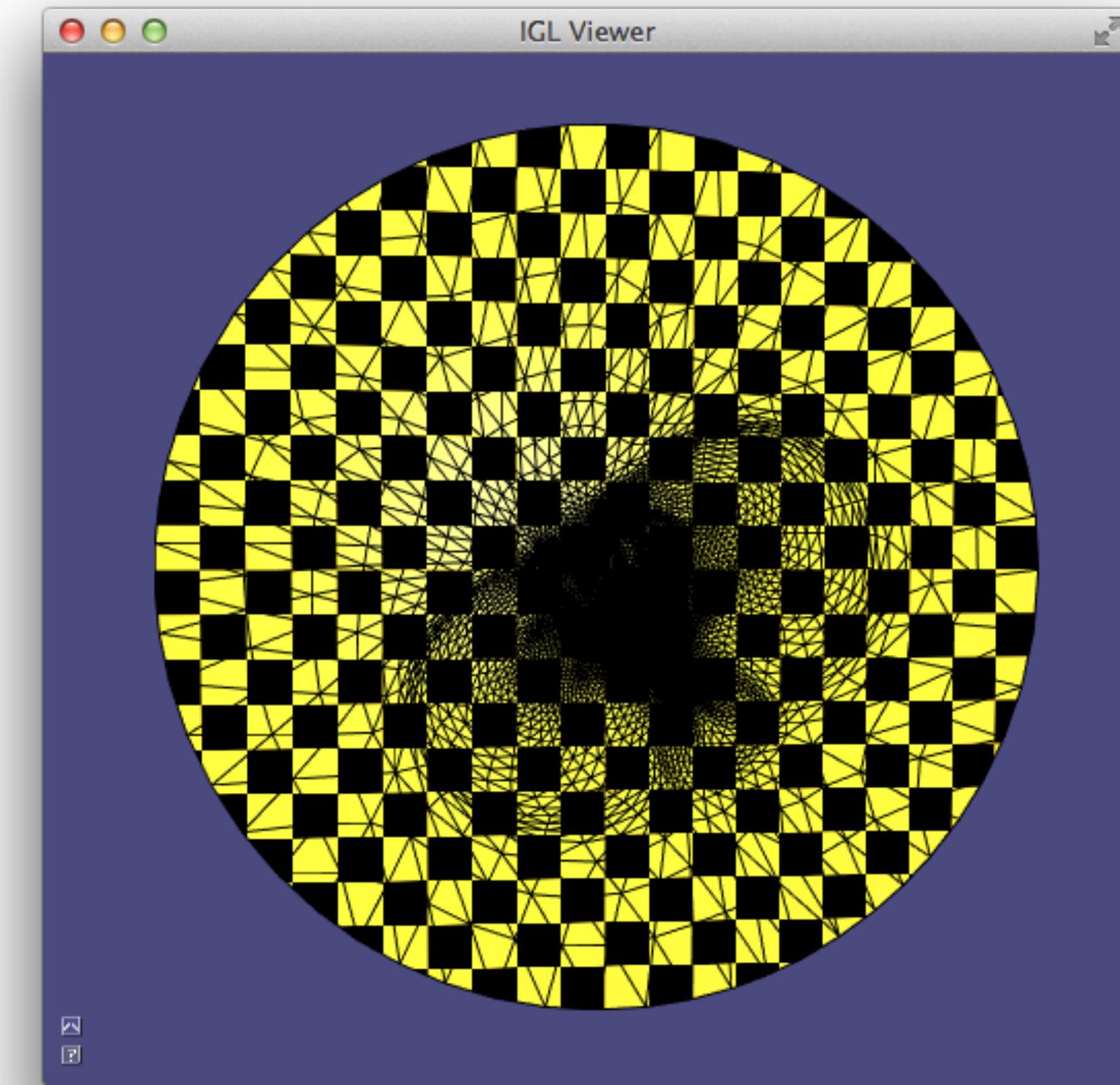
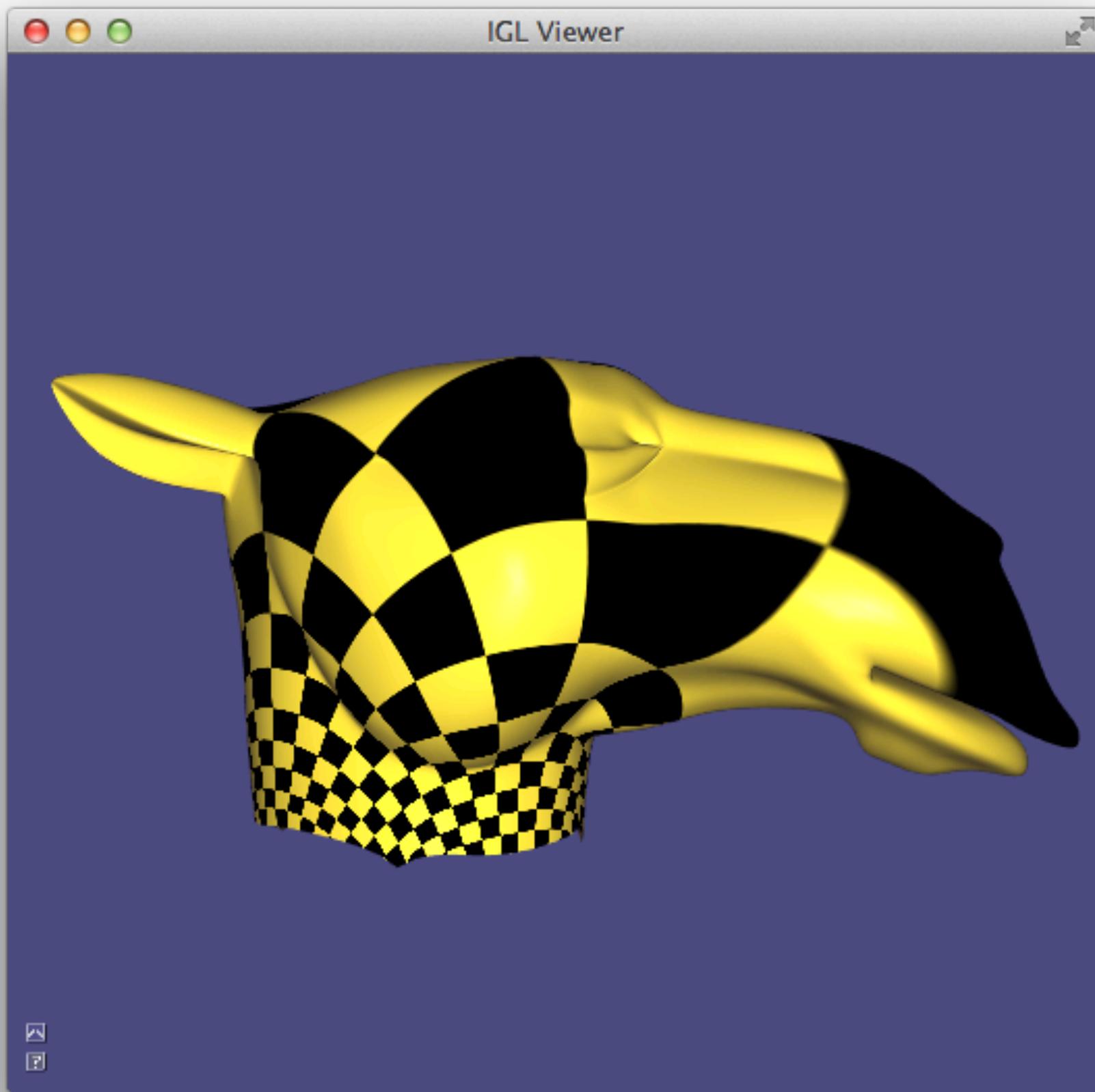
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane
- Total spring energy of the flattened mesh:

$$E(\mathbf{u}_1, \dots, \mathbf{u}_n) = \sum_{(i,j) \in \mathcal{E}} \frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$



Demo



<https://libigl.github.io/libigl/tutorial/tutorial.html#harmonicparametrization>

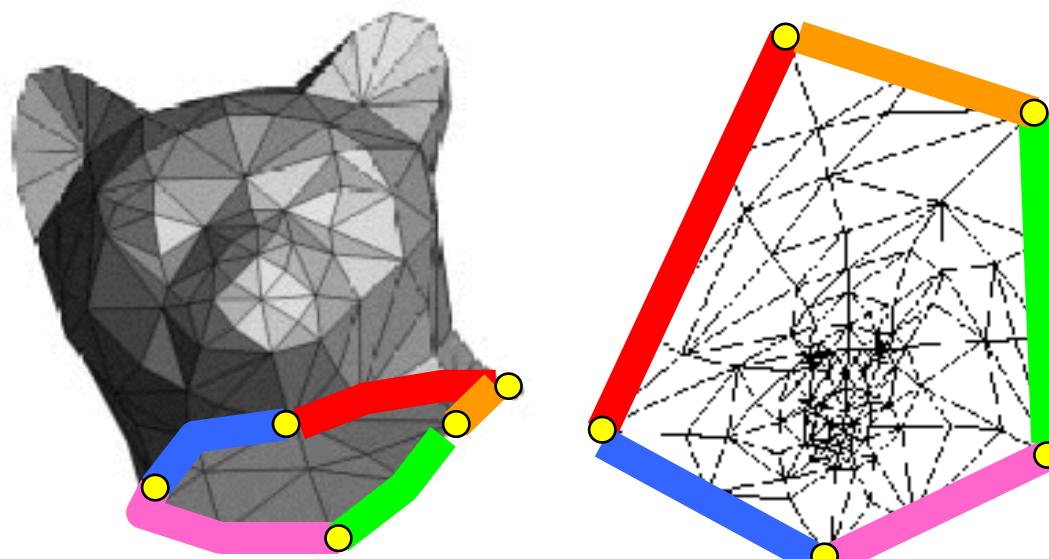
Minimizing Spring Energy

$$E(\mathbf{u}_1, \dots, \mathbf{u}_n) = \sum_{(i,j) \in \mathcal{E}} \frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$

$$\frac{\partial E(\mathbf{u}_1, \dots, \mathbf{u}_n)}{\partial \mathbf{u}_i} = \sum_{j \in \mathcal{N}(i)} k_{i,j} (\mathbf{u}_i - \mathbf{u}_j) = 0$$

$$\sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_i + \sum_{j \in \mathcal{N}(i) \setminus \mathcal{B}} k_{i,j} (\mathbf{u}_i - \mathbf{u}_j) = \sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_j$$

unknown
flat vertex
positions



$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ – inner vertices
 $\mathbf{v}_{n+1}, \dots, \mathbf{v}_N$ – boundary vertices

known fixed
boundary
positions

Minimizing Spring Energy

- Sparse linear system of n equations to solve!

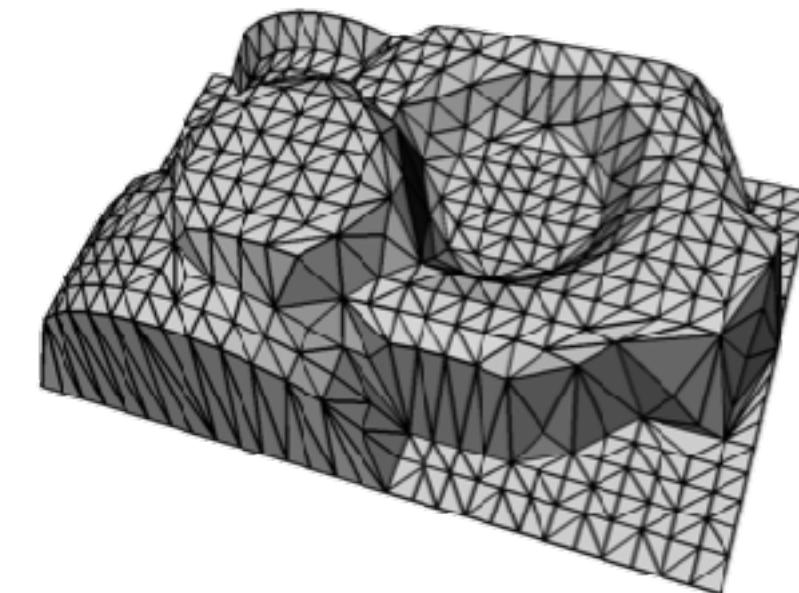
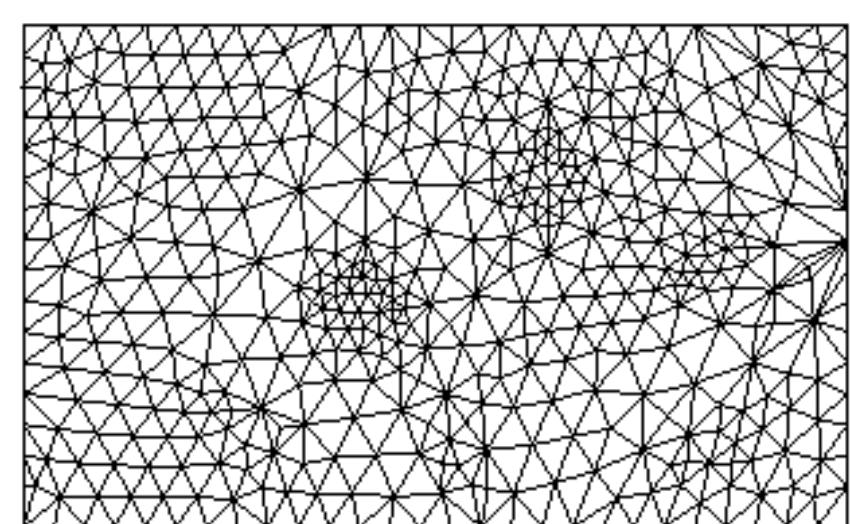
$$\sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_i + \sum_{j \in \mathcal{N}(i) \setminus \mathcal{B}} k_{i,j} (\mathbf{u}_i - \mathbf{u}_j) = \sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_j$$

$$\begin{pmatrix} \sum_j k_{i,j} & * & \cdots & -k_{i,j} \\ * & \sum_j k_{i,j} & * & \vdots \\ \vdots & * & \ddots & * \\ -k_{j,i} & \cdots & * & \sum_j k_{i,j} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{u}}_1 \\ \bar{\mathbf{u}}_2 \\ \vdots \\ \bar{\mathbf{u}}_n \end{pmatrix}$$

Choice of spring constants $k_{i,j}$

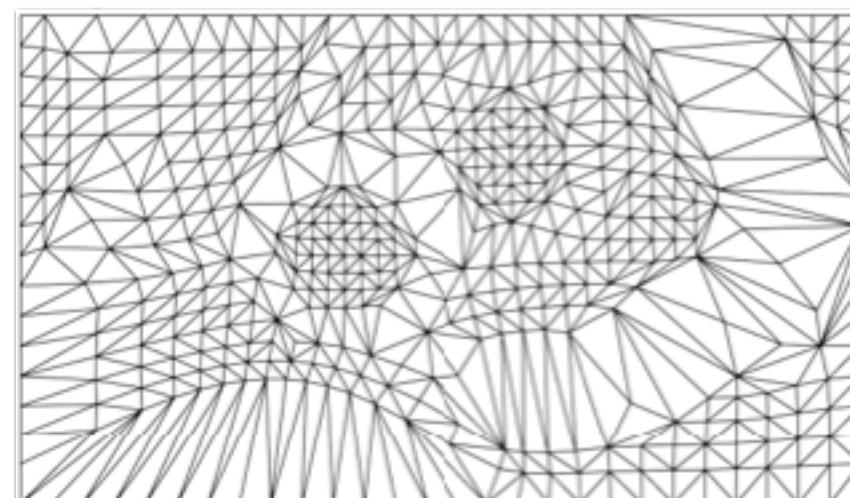
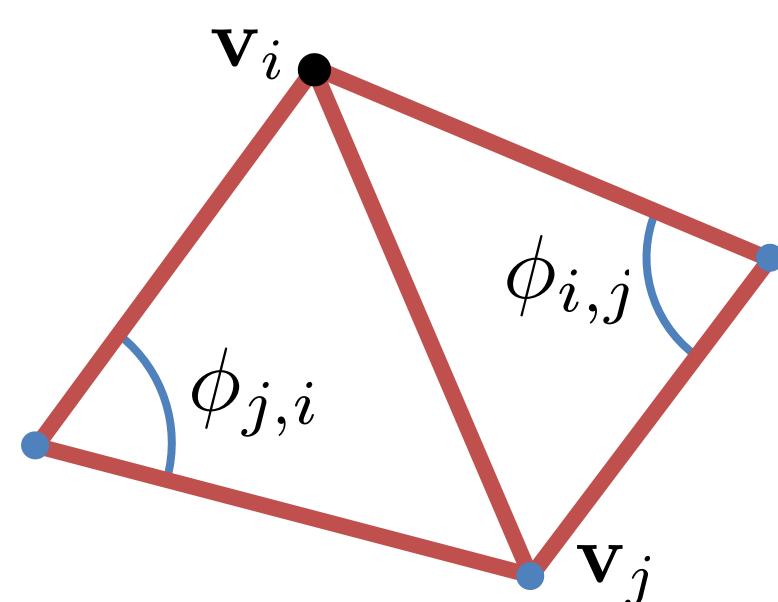
- Uniform

$$k_{i,j} = 1$$



- Cotangent

$$k_{i,j} = \cot \phi_{i,j} + \cot \phi_{j,i}$$

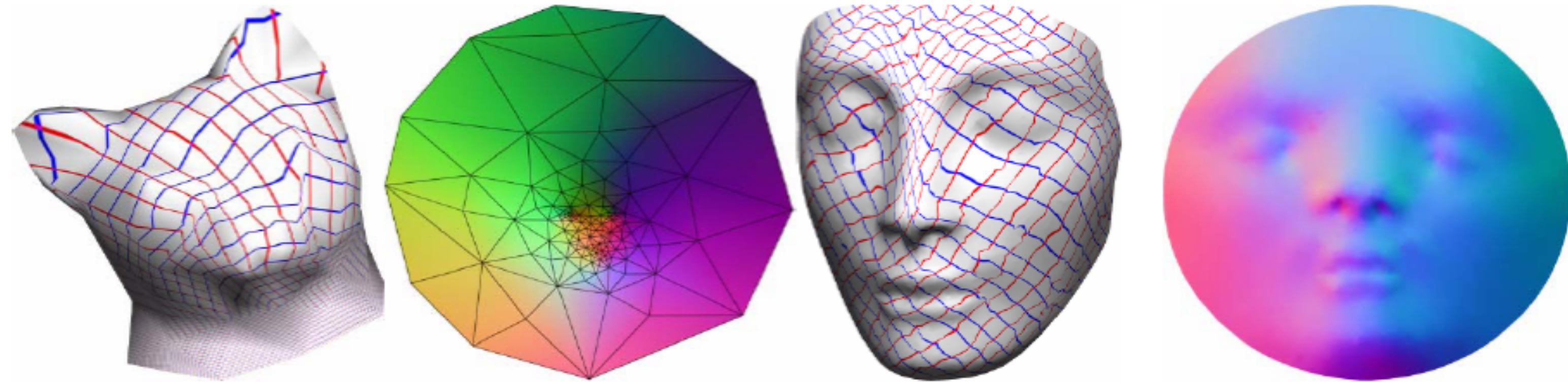


Tutte's Theorem

- If the weights are **nonnegative**, and the boundary is fixed to a **convex** polygon, the parameterization is **bijective**
- (Tutte'63 proved for uniform weights, Floater'97 extended to arbitrary nonnegative weights)
- W.T. Tutte. "How to draw a graph". Proceedings of the London Mathematical Society, 13(3):743-768, 1963.

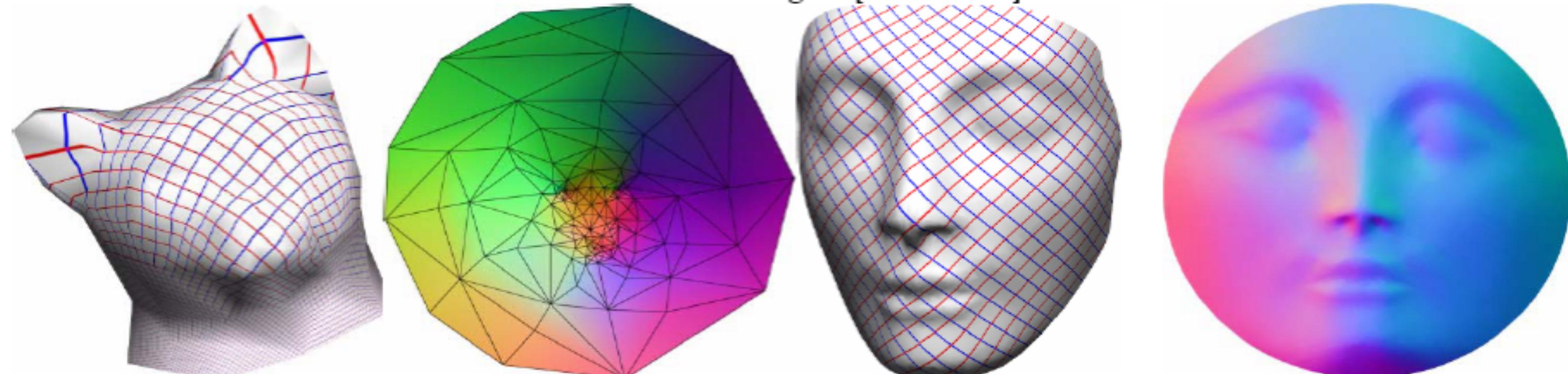
Comparison of Weights

uniform
weights



Parameterization with uniform weights [Tutte 1963] on a circular domain.

cotan
weights



Parameterization with harmonic weights [Eck et al. 1995] on a circular domain.

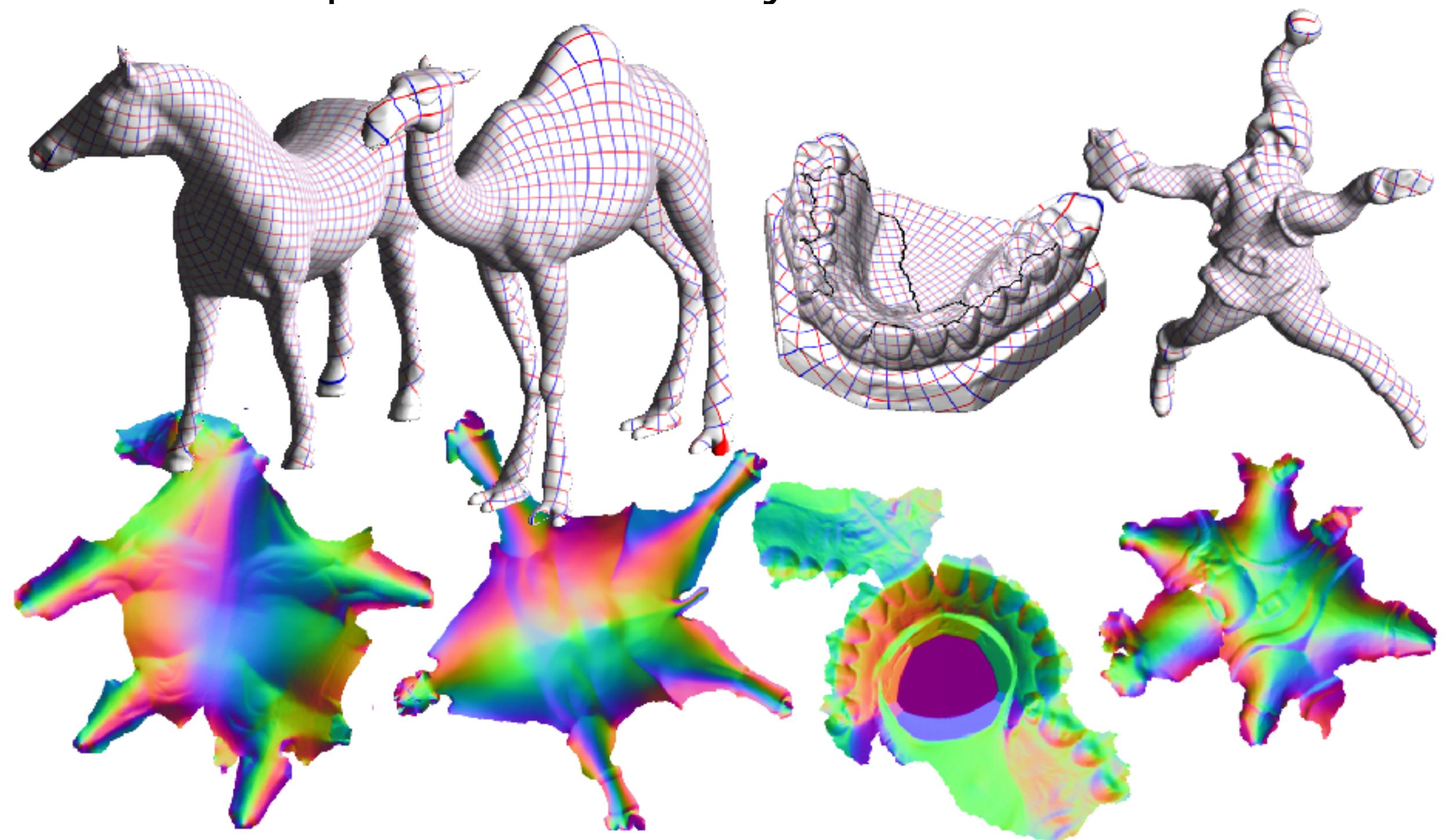
Eck et al. 1995, "Multiresolution analysis of arbitrary meshes", SIGGRAPH 1995

Discussion

- The results of cotan-weights mapping are better than those of uniform convex mapping (local area and angles preservation).
- But: the mapping *is not always legal* (the cotan weights can be negative for badly-shaped triangles...)
- In any case: sparse system to solve. Robust and efficient numerical solvers exist (Eigen Sparse LDLT)

Discussion

- Both mappings have the problem of **fixed boundary** – it constrains the minimization and causes **distortion**.
- More advanced methods do not require boundary conditions.



ABF++ method,
Sheffer et al. 2005

<http://www.cs.ubc.ca/~sheffa/ABF++/abf.htm>

References

Fundamentals of Computer Graphics, Fourth Edition

4th Edition by **Steve Marschner, Peter Shirley**

Chapter 11

<https://open.gl>

Polygon Mesh Processing

Mario Botsch, Leif Kobbelt, Mark Pauly, Pierre Alliez, Bruno Levy