For caps in series, they must have the same current going through them. Assuming we started with no initial charge on the caps, this means that the caps must have the same charge since this is just the integral of the same current for each cap. Applying KVL, the total voltage across the series connected caps is:

$$V_s = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

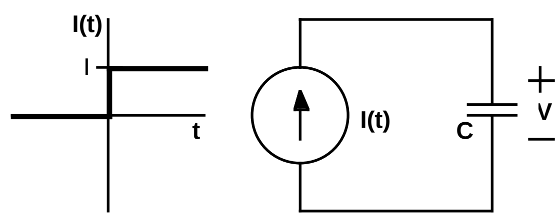
With a slight rearrangement, we get:

$$Q = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} V_s$$

So the capacitance of caps in series is the reciprocal of the sum of their reciprocals. You may have noticed that this is just the opposite of what we got for resistors. The easiest way to remember these equations is to note that capacitors in series add like resistors in parallel, and vice versa.

How Do I Use a Capacitor in a Circuit?

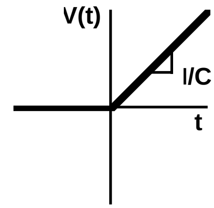
Obviously, we want to use the capacitor in a circuit. The simplest possible interesting circuit consists of a cap and a current source as shown below.



The current source puts out a current which varies over time. In particular, this current source is zero until t=0, at which point it instantaneously switches to a current I. By KCL, we know that this must be the current through the capacitor. (We'll be pretending that the current actually passes through the cap, since that's what it looks like from the outside.) Assuming that the cap has no charge across it initially, the voltage across is described by the equation:

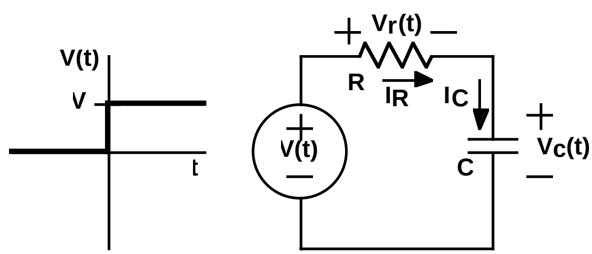
$$V = \frac{1}{C}\int_0^t I\,dt = \frac{It}{C}$$

So the voltage increases linearly with time, starting at t = 0 as shown below.



Of course, as we noted before, the Stamp outputs are well modeled as voltage sources. However, a circuit with just a cap and a voltage source quickly leads to trouble. The problem is that a step change in voltage makes dV/dt go to infinity, requiring infinite current. Caps cannot change their voltage instantaneously! Of course, if we were to hook up a cap to a Stamp output, we wouldn't get infinite current. This is where our model of the Stamp output as a simple voltage source breaks down. In fact, the current would be limited by the small output impedance of the Stamp output that we generally ignore.

To get around this problem, we next consider a circuit with a time varying voltage source, a cap and a resistor as shown below.



By KCL, we know that we have the same current flowing through all of these devices. In addition, we will need to know the initial condition of the cap. We will presume that $V_c(0) = 0$.

Writing the KCL equation at the node shared by the resistor and the capacitor yields:

$$I_R = I_C \Rightarrow \frac{V(t) - V_C(t)}{R} = C\frac{dV_C(t)}{dt}$$

For t > 0, V(t) = V, allowing us to rewrite this as:

$$V_C(t) + RC\frac{dV_C(t)}{dt} = V$$

From previous study of differential equations, you will hopefully recognize this as a separable equation. Rearranging the terms gives:

$$\frac{RC}{V - V_C(t)}dV_C(t) = dt$$

Integrating both sides and moving some terms yields:

$$\int\frac{RC}{V - V_C(t)}dV_C(t) = \int dt$$

$$-RC\log|V - V_C(t)| = t + K$$

$$\log|V - V_C(t)| = \frac{-t}{RC} + K'$$

where K and K' are constants. Next, we exponentiate both sides giving:

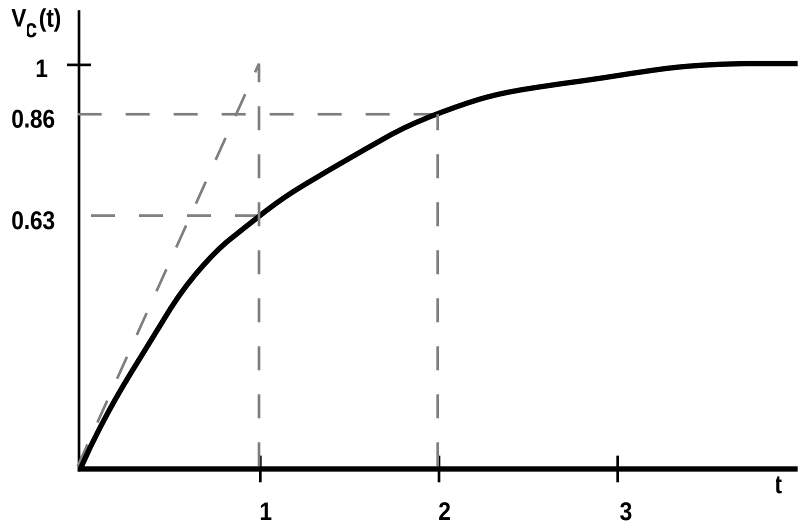
$$V - V_C(t) = K''e^{\frac{-t}{RC}}$$

$$V_C(t) = V - K''e^{\frac{-t}{RC}}$$

where K" is yet another constant. Since the initial condition tells us that $V_C(0) = 0$, then K" = V. The final result is:

$$V_C(t) = V\left(1 - e^{\frac{-t}{RC}}\right)$$

for t > 0, and $V_C(t) = 0$ for t < 0. A graph of this equation for V = 1 and RC = 1 appears below.



First of all, note the general behavior. At first, the voltage on the capacitor increases fairly rapidly. But this rate of increase slows down, and the curve approaches $V_C(t) = V$ asymptotically. If you think about this, it makes tremendous sense. At first, the resistor sees the full voltage V across it, resulting in a fairly high current to charge the cap. As the cap voltage starts to rise, this reduces the voltage across the resistor. Thus, the current decreases, and the cap voltage increases more slowly. Eventually, the cap voltage is almost identical to V, and the current flow is reduced to almost nothing. This is classic behavior for a first order system.

Second, we should note the role of the term RC. This is called the *time constant* of the system. If you plug this into the equation, you will see that after one time constant, the system reaches about 63% of its final value. After two time constants, it's up to about 86% of the final value. After ten time constants, it's up to 99.995% of its final value.

Another way of looking at the time constant is with respect to the initial slope. If you were to continue a straight line from the beginning part of the waveform, it would intersect the final value at the time constant. This is a handy way of estimating the time constant or the final value from a graph. We recommend that you take a moment to prove this to yourself - it's quite straightforward.