Section 2: Light Quantization

Sollovin

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1 Light in a 1D cavity

Because the wave equation is a linear equation, we are able to employ the quasi-1D approximation and only consider one mode of the light propagate along one direction with polarization along another direction. If the system we are facing requires more complicate wave, we can build it from the simple condition we considered here, which is guaranteed by the orthogonality between the modes and between the different directions.

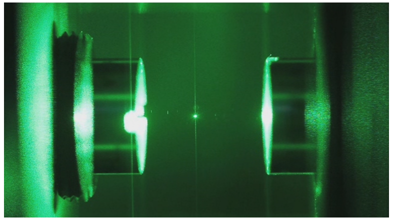

Figure 1: Real cavity

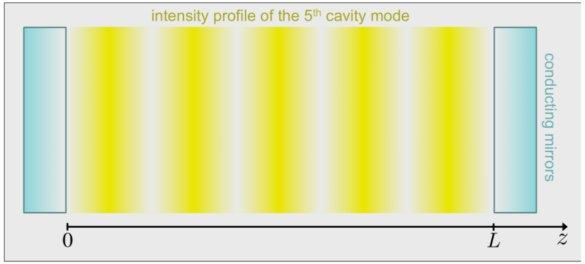

Figure 2: 1D cavity

The formal solution of mode k of the light which propagates along the z direction with polarization along x direction is

$$\vec{A} = \hat{e}_x\big(A_k q_k(t)e^{ikz} + A_{-k}q_{-k}(t)e^{-ikz}\big), \tag{1}$$

So,

$$\vec{E} = -\partial_t \vec{A} = -\hat{e}_x\big(A_k \dot{q}_k(t)e^{ikz} + A_{-k}\dot{q}_{-k}(t)e^{-ikz}\big), \tag{2}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\hat{e}_y\big(A_k k q_k(t)e^{ikz} - A_{-k}k q_{-k}(t)e^{-ikz}\big). \tag{3}$$

A simple 1D-cavity consisting on two perfectly conducting plane mirrors facing each other. Light propagate between the mirrors back and forth which implies

$$A_k = A_{-k}. \tag{4}$$

The components of the electric field parallel to the conducting mirrors must vanish. Taking $z = 0$ at the mirror on the left and $z = L$ at the mirror on the right.

The $z = 0$ boundary condidtions imply

$$\vec{E}(0, t) = 0 \Rightarrow A_k \dot{q}_k(t) = -A_{-k} \dot{q}_{-k}(t) \Rightarrow q_k(t) = -q_{-k}(t). \tag{5}$$

The $z = L$ boundary condidtions imply

$$\begin{aligned} \vec{E}(L, t) = 0 &\Rightarrow A_k \dot{q}_k(t) e^{ikL} = -A_{-k} \dot{q}_{-k}(t) e^{-ikL} \\ &\Rightarrow e^{ikL} + e^{-ikL} = i \sin(kL) = 0 \\ &\Rightarrow k = \frac{n\pi}{L}, \quad \text{with} \quad n \in N. \end{aligned} \tag{6}$$

Now we have formal solution

$$\vec{A} = \vec{e}_x A_n q_n(t) \sin(k_n z), \tag{7}$$

$$\vec{E} = -\vec{e}_x A_n \dot{q}_n(t) \sin(k_n z), \tag{8}$$

$$\vec{B} = \vec{e}_y k_n A_n q_n(t) \cos(k_n z), \tag{9}$$

with $k_n = n\pi/L$, $n = 1, 2, 3, \ldots$.

Using the expression above, we calculate the electromagnetc energy as follows

$$
\begin{aligned}
&E_{\mathrm{em}}(t)\\
&=\frac{1}{2}\int_{\mathrm{cavity}}\mathrm{d}^3\vec{r}\bigg(\varepsilon_0\vec{E}^2(z,t)+\frac{1}{\mu_0}\vec{B}^2(z,t)\bigg)\\
&=\frac{1}{2}\int_{\mathrm{cavity}}\mathrm{d}x\mathrm{d}y\cdot A_n^2\bigg(\varepsilon_0\dot{q}_n^2(t)\int_0^L\mathrm{d}z\sin^2(k_nz)+\frac{k_n^2}{\mu_0}q_n^2(t)\int_0^L\mathrm{d}z\cos^2(k_nz)\bigg)\\
&=\frac{\varepsilon_0LS}{4}A_n^2\big(\dot{q}_n^2(t)+c^2k_n^2q_n^2(t)\big),
\end{aligned}
\tag{10}
$$

in which, we use the mathematical relation

$$
\int_0^L\mathrm{d}z\sin(k_nz)\sin(k_mz)=\int_0^L\mathrm{d}z\cos(k_nz)\cos(k_mz)=\frac{L}{2}\delta_{nm}. \tag{11}
$$

In fact, if we choose more general formal solution $\vec{A}=\hat{e}_x\sum_{n=1}^{\infty}A_nq_n(t)\sin(k_nz)$, we will arrive the same place,

$$
\begin{aligned}
&E_{\mathrm{em}}(t) \\
&= \frac{1}{2}\int_{\mathrm{cavity}} \mathrm{d}^3\vec{r}\bigg(\varepsilon_0\vec{E}^2(z,t) + \frac{1}{\mu_0}\vec{B}^2(z,t)\bigg) \\
&= \frac{1}{2}\int_{\mathrm{cavity}} \mathrm{d}x\mathrm{d}y \cdot \sum_{n,m=1}^{\infty} A_n A_m\bigg(\varepsilon_0\dot{q}_n(t)\dot{q}_m(t)\int_0^L \mathrm{d}z\sin(k_n z)\sin(k_m z) + \frac{k_n k_m}{\mu_0}q_n(t)q_m(t)\int_0^L \mathrm{d}z\cos(k_n z)\cos(k_m z)\bigg) \\
&= \frac{\varepsilon_0 LS}{4}\sum_{n=1}^{\infty} A_n^2\big(\dot{q}_n^2(t) + c^2k_n^2q_n^2(t)\big),
\end{aligned}
\tag{12}
$$

in which, the $\sin(k_n z)\sin(k_m z)$ term equals to zero due to the orthogonal relation.

If we choose $A_n^2 = 2m_n/\varepsilon_0 LS$, we have

$$
E_{\mathrm{em}}(t) = \sum_{n=1}^{\infty}\left[\frac{p_n^2(t)}{2m_n} + \frac{m_n\omega_n^2}{2}q_n^2(t)\right], \tag{13}
$$

in which $m_n = \sqrt{A_n^2\varepsilon_0 LS/2}$ and $\omega_n = ck_n$.

2 The quantization of the harmonic oscillator