Section 04: States Of Quantum Harmonic Oscillator

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Contents

1 Coherent State

1.1 Definition

We will start with the definition that the coherent state $|\alpha\rangle$ is the eigenstate of the annihilator operater $\hat{a}$ with eigenvalue α. Thus

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{1}$$

Then we express the coherent state as the linear combinition of the number state, i.e., $|\alpha\rangle \equiv \sum_{n=0}^{\infty} c_n\,|n\rangle$, and substituet it into the left and right side of the equaton above. What we will obtain after some calculation is the recurrence relation of c_n, which is

$$c_{n+1}\sqrt{n+1} = c_n\alpha. \tag{2}$$

Then we knew,

$$c_n = \frac{\alpha^n}{\sqrt{n!}}c_0, \tag{3}$$

and

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}c_0\,|n\rangle\,. \tag{4}$$

To determine the value of c_0, we employ the property $\langle\alpha\rangle = 1$, and get

$$|c_0| = e^{-(\alpha\alpha^*)^{1/2}}, \tag{5}$$

and

$$\begin{aligned}
|\alpha\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
&= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |n\rangle \\
&= e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} |0\rangle \\
&= e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} \cdot e^{-\alpha^{**}\hat{a}} |0\rangle \\
&= e^{\alpha\hat{a}^\dagger - \alpha^{**}\hat{a}} |0\rangle ,
\end{aligned} \tag{6}$$

in which, we use the expansion rule of the exponent of a operator $e^{\hat{A}} = \sum_{n=0}^{\infty} \hat{A}^n/n!$ and the weaker form of the Baker-Campbell-Haussdorf lemma, which says if operator $\hat{A}$ and $\hat{B}$ both commute with their commutator, then

$$e^{\hat{A}+\hat{B}} = e^{-[\hat{A},\hat{B}]/2} e^{\hat{A}} e^{\hat{B}}. \tag{7}$$

By convention, we define the displacement operator which generate a coherent state with eigenvalue α from the vacuum state as follows

$$\hat{D}(\alpha) \equiv e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}. \tag{8}$$

1.2 Expectation and variance

Define the quadratures as follows,

$$X_1 = \frac{x}{\hbar/2m\omega} \quad \text{and} \quad X_2 = \frac{p}{\sqrt{m\omega\hbar/2}}, \tag{9}$$

now we want to calculate the expectation value and the variance of the quadratures of the coherent state.

Recall that $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, the expectation value of X_1 can be written as

$$\left\langle \hat{X}_1 \right\rangle_{|\alpha\rangle} = \left\langle \hat{D}^\dagger(\alpha)\hat{X}_1\hat{D}(\alpha) \right\rangle_{|0\rangle}. \tag{10}$$

Using the Baker-Campbell-Haussdorf lemma, which says

$$e^{\hat{A}}\hat{B}e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{A}, [\hat{A}, \ldots, [\hat{A}, \hat{B}], \ldots,],], \tag{11}$$

we can prove that

$$\hat{D}^\dagger(\alpha)\hat{X}_1\hat{D}(\alpha) = \hat{X}_1 + 2\mathrm{Re}\{\alpha\}. \tag{12}$$

Thus,

$$\left\langle \hat{X}_1 \right\rangle_{|\alpha\rangle} = \left\langle \hat{X}_1 \right\rangle_{|0\rangle} + 2\mathrm{Re}\{\alpha\}, \tag{13}$$

which means the expectation value of X_1 of the coherent state is just the expectation value of X_1 of the vacuum state plus a constant relates to the real part of the eigenvalue of the coherent state.

As same as above, we can prove

$$\hat{D}^\dagger(\alpha)\hat{X}_2\hat{D}(\alpha) = \hat{X}_2 + 2\mathrm{Im}\{\alpha\}. \tag{14}$$

Now we have to calculate the expectation value of the quadratures of the vacuum state. Here, we use the wigner function to calculate it. Recall that the wigner function of the vacuum state is

$$W_0(x,p) = \frac{1}{\pi\hbar} e^{-\frac{x^2}{2l^2} - \frac{l^2 p^2}{\hbar^2}}, \tag{15}$$

and one property of the wigner function is

$$\left\langle \hat{x}^{(m)} \hat{p}^{(n)} \right\rangle = \iint \mathrm{d}x \mathrm{d}p W(x,p) x^{(m)} p^{(n)}, \tag{16}$$

we have the following results

$$\langle \hat{x} \rangle_{|0\rangle} = \langle \hat{p} \rangle_{|0\rangle} = 0, \quad \left\langle \hat{x}^2 \right\rangle_{|0\rangle} = \frac{\hbar}{2\omega}, \quad \left\langle \hat{p}^2 \right\rangle_{|0\rangle} = \frac{\hbar\omega}{2}. \tag{17}$$

Thus,

$$\left\langle \hat{X}_1 \right\rangle_{|\alpha\rangle} = 2\mathrm{Re}\{\alpha\} \quad \text{and} \quad \left\langle \hat{X}_2 \right\rangle_{|\alpha\rangle} = 2\mathrm{Im}\{\alpha\}, \tag{18}$$

which means for coherent state the central position of the wigner function is $(2\mathrm{Re}\{\alpha\}, 2\mathrm{Im}\{\alpha\})$.

If we calculate the variance of the quadratures of the coherent state as above, we will find that

$$\left\langle \Delta \hat{X}_1 \right\rangle_{|\alpha\rangle} = \left\langle \Delta \hat{X}_1 \right\rangle_{|0\rangle} \quad \text{and} \quad \left\langle \Delta \hat{X}_2 \right\rangle_{|\alpha\rangle} = \left\langle \Delta \hat{X}_2 \right\rangle_{|0\rangle}, \tag{19}$$

which means the uncertainty of the quadratures of the coherent state is the the same as the vacuum state's, and because the vacuum state is a minimal uncertainty state (which means $\left\langle \Delta \hat{X}_1 \right\rangle \left\langle \Delta \hat{X}_2 \right\rangle = \frac{1}{2}$), the coherent state is also a minimal uncertainty state.

1.3 Time evolution

We discuss the time evolution of the coherent state by introducing the time evolution operator

$$\hat{U} = e^{i\hat{H}t/\hbar}, \tag{20}$$

which gives out the relation between the states at different time, i.e.,

$$|\alpha(t)\rangle = \hat{U}\,|\alpha(0)\rangle\,. \tag{21}$$

It can be proved that,

$$\begin{pmatrix} \hat{U}^\dagger \hat{X}_1 \hat{U} \\ \hat{U}^\dagger \hat{X}_2 \hat{U} \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix}, \tag{22}$$

which gives out the following

$$\begin{pmatrix} \left\langle \hat{X}_1(t) \right\rangle \\ \left\langle \hat{X}_2(t) \right\rangle \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} \left\langle \hat{X}_1(0) \right\rangle \\ \left\langle \hat{X}_2(0) \right\rangle \end{pmatrix}, \tag{23}$$

$$\left\langle \Delta\hat{X}_1(t) \right\rangle = \left\langle \Delta\hat{X}_1(0) \right\rangle \quad \text{and} \quad \left\langle \Delta\hat{X}_2(t) \right\rangle = \left\langle \Delta\hat{X}_2(0) \right\rangle, \tag{24}$$

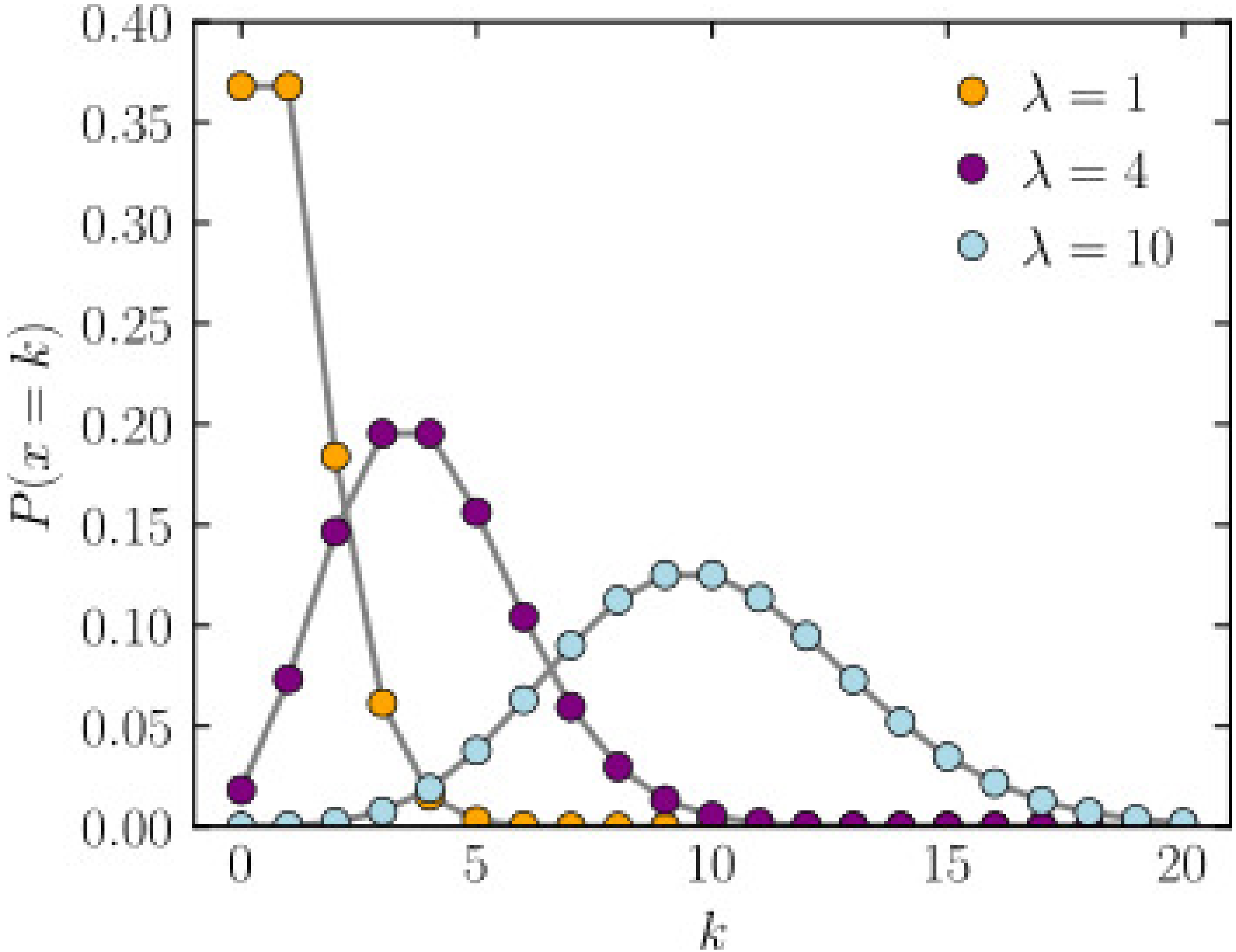
which means that the central position of the wigner function of the coherent state will make an uniform circular motion and variance of the coherent state keeps as a constant. This can be related with the image of the time evolution of the wigner function as follows,

`../figs/wigner_func-coherent_state.gif`

1.4 Properties

1. The average number is given by
$$\langle \hat{n} \rangle_{|\alpha\rangle} = \langle \alpha | \, \hat{a}^\dagger \hat{a} \, | \alpha \rangle = |\alpha|^2. \tag{25}$$

2. The state distribution in the number state presentation is
$$P_n = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}, \tag{26}$$
which is exactly the Poisson distribution $f(n, |\alpha|^2)$.


0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
$P(x = k)$
0
5
10
15
20
k
$\lambda = 1$
$\lambda = 4$
$\lambda = 10$

3. The coherent state is a minimal uncertainty state, which is impied by the equal

$$\left\langle \Delta\hat{X}_1 \right\rangle \left\langle \Delta\hat{X}_2 \right\rangle = \frac{1}{2}. \tag{27}$$

4. The coherent state is complete, i.e.,

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| = 1. \tag{28}$$

5. The overlaps of two coherent state is given by

$$\begin{aligned}\langle\alpha|\alpha'\rangle &= \langle 0|\hat{D}^\dagger(\alpha)\hat{D}(\alpha')|0\rangle = \langle 0|\hat{D}(\alpha'-\alpha)|0\rangle = \langle 0|\alpha'-\alpha\rangle \\ &= \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha'\alpha^* - \frac{1}{2}|\alpha'|^2\right) = \exp\left(-|\alpha-\alpha'|^2\right).\end{aligned} \tag{29}$$

6.

$$|\alpha\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha'\, |\alpha'\rangle\,\langle\alpha'|\alpha\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha'\, |\alpha'\rangle \exp\left(-|\alpha'-\alpha|^2\right), \tag{30}$$

which means that a coherent state $|\alpha\rangle$ can be represented by other coherent state $|\alpha'\rangle$, thus the coherent states is overcomplete.

2 Squeezing state

2.1 Definition

As the displacement operator generate a coherent state from the vacuum state, the squeezing state is generated by a squeezing operator which has the following form

$$\hat{S}(z) = e^{(z^*\hat{a}^2 - z(\hat{a}^\dagger)^2)/2}, \quad z \equiv re^{i\theta} \in C. \tag{31}$$

It's easy to see that

$$\hat{S}^\dagger(z)\hat{a}\hat{S}(z) = \hat{a}\cosh r - e^{i\theta}\hat{a}^\dagger \sinh r. \tag{32}$$

Now we define the squeezed coherent state as

$$|\alpha, z\rangle = \hat{S}(z)\hat{D}(\alpha)\,|0\rangle\,, \tag{33}$$

and we can check that

$$\begin{aligned}\langle\hat{a}\rangle &= \langle\alpha, z|a|\alpha, z\rangle \\ &= \langle 0|\hat{D}^\dagger(\alpha)\hat{S}^\dagger(z)\hat{a}\hat{S}(z)\hat{D}(\alpha)|0\rangle \\ &= \langle\alpha|\hat{a}\cosh r - e^{i\theta}\hat{a}^\dagger \sinh r|\alpha\rangle \\ &= \alpha\cosh r - \alpha^* r^{i\theta}\sinh r,\end{aligned} \tag{34}$$

$$\begin{aligned}\left\langle\hat{a}^2\right\rangle &= \langle\alpha|\hat{S}^\dagger(z)\hat{a}\hat{S}(z)\hat{S}^\dagger(z)\hat{a}\hat{S}(z)|\alpha\rangle \\ &= \alpha^2\cosh^2 r + (\alpha^*)^2 e^{i\theta}\sinh^2 r - 2|\alpha|^2 e^{i\theta}\sinh r\cosh r - e^{i\theta}\cosh r\sinh r,\end{aligned} \tag{35}$$

and

$$\left\langle\hat{a}^\dagger\hat{a}\right\rangle = |\alpha|^2\left(\cosh^2 r + \sinh^2 r\right) - (\alpha^*)^2 e^{i\theta}\cosh r\sinh r - \alpha^2 e^{-i\theta}\sinh r\cosh r + \sinh^2 r. \tag{36}$$

Recall that $\hat{X}_1 = \hat{a}^\dagger + \hat{a}$ and $\hat{X}_2 = i(\hat{a}^\dagger - \hat{a})$, consider a $\frac{\theta}{2}$ rotation of $\hat{X}_1$ and $\hat{X}_2$, we have

$$\begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix}, \tag{37}$$

i.e.,

$$\begin{aligned} &\hat{Y}_1 + i\hat{Y}_2 = (\hat{X}_1 + i\hat{X}_2)e^{-i\theta/2} \\ \Rightarrow &\hat{Y}_1 + i\hat{Y}_2 = \hat{a}e^{-i\theta/2} \\ \Rightarrow &\hat{Y}_1 = \hat{a}e^{-i\theta/2} + \hat{a}^\dagger e^{i\theta/2} \quad \text{and} \quad \hat{Y}_2 = \cdots . \end{aligned} \tag{38}$$

Thus we have

$$\left\langle \Delta\hat{Y}_1^2 \right\rangle = \left\langle \hat{Y}_1^2 \right\rangle - \left\langle \hat{Y}_1 \right\rangle^2 = \frac{1}{4}e^{-2r}, \tag{39}$$

$$\left\langle \Delta\hat{Y}_2^2 \right\rangle = \frac{1}{4}e^{2r}, \tag{40}$$

$$\left\langle \Delta\hat{Y}_1 \right\rangle \left\langle \Delta\hat{Y}_2 \right\rangle = \frac{1}{4}, \tag{41}$$

which means that the a squeezing operator $\hat{S}(z)$ with $z = re^{i\theta}$ will squezze the variance along Y_1 direction and stretch the variance along Y_2, the squeezing intensity is determined by r and the direction is determined by θ.

2.2 Properties

1.

$$\hat{S}(z)\left|0\right\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \sqrt{\frac{(2n)!}{\cosh r}} e^{in\theta} \tanh^n r \left|2n\right\rangle . \tag{42}$$

2.
$$\frac{\langle\hat{n}\rangle}{\langle\Delta\hat{n}\rangle} = \frac{\sinh^2 r}{\sqrt{\sinh^4 r + \sinh^2 r + \frac{1}{4}\sinh^2 2r}} \xrightarrow{r\to\infty} 1/\sqrt{2}, \tag{43}$$

which means that the variance of the number is always the same order of their mean value.

3 Thermal state

3.1 Definition

We start with the definition that thermal state is the state with maximum entropy and find the representation of thermal state with the Lagrange multiplier method.

The Lagrange function of the density operator is defined as follows

$$L[\hat{\rho}] = -\operatorname{tr}\{\hat{\rho}\ln\hat{\rho}\} + \lambda(1 - \operatorname{tr}\{\hat{\rho}\}) + \beta(\bar{E} - \operatorname{tr}\left\{\hat{H}\hat{\rho}\right\}). \tag{44}$$

The thermal state density operator satisfies

$$\delta L = L[\hat{\rho} + \delta\hat{\rho}] - L[\hat{\rho}] = -\operatorname{tr}\left\{\delta\hat{\rho}(\ln\hat{\rho} + 1 + \lambda + \beta\hat{H})\right\} = 0, \tag{45}$$

which gives out

$$\hat{\rho}_{\text{th}} = e^{-(\lambda+1)}e^{-\beta\hat{H}}. \tag{46}$$

Notice that $\operatorname{tr}\{\hat{\rho}\} = 1$, we have

$$\hat{\rho}_{\text{th}} = e^{-\beta\hat{H}}/\operatorname{tr}\left\{e^{-\beta\hat{H}}\right\}. \tag{47}$$

3.2 Properties

1. $$\bar{E} = \frac{\mathrm{tr}\left\{\hat{H}\exp\left(-\beta\hat{H}\right)\right\}}{\mathrm{tr}\left\{\exp\left(-\beta\hat{H}\right)\right\}} = -\partial_\beta \ln\left[\mathrm{tr}\left\{e^{-\beta\hat{H}}\right\}\right] = -\partial_\beta \ln\left[\sum_n e^{-\beta E_n}\right]. \tag{48}$$

2. $$P(E_n) = \langle E_n|\hat{\rho}_{\mathrm{th}}|E_n\rangle = e^{-\beta E_n}/Z, \quad Z = \sum_n e^{-\beta E_n}, \tag{49}$$

   which is the Boltzmann distribution.

3. Thermal state has $\sqrt{2\bar{n}+1}$ times more uncertainty than vacuum state.

4. For harmonic oscillator,

   - $$E_n = \hbar\omega(n+\frac{1}{2}), \quad Z = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}. \tag{50}$$

   - $$\bar{n} = \frac{1}{e^{\beta\hbar\omega}-1}, \tag{51}$$

     which is the Bose-Einstein distribution for an equillibrium temperature $T = 1/k_{\mathrm{B}}\beta$.

   - $$\hat{\rho}_{\mathrm{th}}(\bar{n}) = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{1+n}} |n\rangle\langle n|\,. \tag{52}$$