Section 1: Light as an electromagnetic wave

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1 How to describe a vector field? — Electric field as an example.

Consider two charges distributed in the space, the force between them can be calculated with the Coulomb's law.

The concept of electric field emerges.

$\vec{E}(\vec{r},t) \equiv \hat{e}|E(\vec{r},t)|$.
$\vec{E}(\vec{r},t) \equiv (E_x(\vec{r},t), E_y(\vec{r},t), E_z(\vec{r},t))$.

2 What rules does electromagnetic wave have to follow? — The Maxwell's equations.

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0, \\ \vec{\nabla} \times \vec{E} &= -\partial_t \vec{B}, \\ \vec{\nabla} \cdot \vec{E} &= 0, \\ \vec{\nabla} \times \vec{B} &= \mu_0 \varepsilon_0 \partial_t \vec{E}. \end{aligned} \tag{1}$$

The divergence of the curl of any vector field is 0, i.e., $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{w}) = 0$.

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{2}$$

$$\vec{\nabla} \times (\vec{E} + \partial_t \vec{A}) = 0. \tag{3}$$

The curl of the gradient of any scalar field is 0, i.e., $\vec{\nabla} \times (\vec{\nabla}\Lambda) = 0$.

$$\vec{E} + \partial_t \vec{A} = -\vec{\nabla}\phi, \tag{4}$$

We only require the curl of $\vec{A}$ to be $\vec{B}$, so in the condition that $\vec{E}$ and $\vec{B}$ are invariance, we can make gauge transformation[1]:

$$\begin{aligned}\vec{A} &\to \vec{A}' = \vec{A} + \vec{\nabla}\psi \\ \phi &\to \phi' = \phi - \partial_t \psi\end{aligned} \tag{5}$$

In Coulomb gauge, i.e., $\vec{\nabla} \cdot \vec{A} = 0$,

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla}^2 \phi - \partial_t (\vec{\nabla} \cdot \vec{A}) = -\vec{\nabla}^2 \phi = 0 \Rightarrow \phi = 0. \tag{6}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\vec{\nabla}^2 \vec{A} + \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) = \mu_0 \varepsilon_0 \partial_t (-\partial_t \vec{A} - \vec{\nabla}\phi), \tag{7}$$

i.e., the wave equation:

$$(c^2 \vec{\nabla}^2 - \partial_t^2) \vec{A}(\vec{r}, t) = 0. \tag{8}$$

3 Real life wave is complicated, but it can be factorized.

The plane wave $\hat{a}|A|e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ is a valid solution of the wave equation. What more, any real function can be expanded into a series of fourier series. [2] So any solution $\vec{A}(\vec{r}, t)$ of the wave equaion above can be expressed as

$$\vec{A}(\vec{r}, t) = \hat{a} \sum_k |A_k| e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}. \tag{9}$$

[1] Check 's for more details.

[2] This is the problem of the completeness of the fourier series, check Chap. 9 of Hassani's *Mathematical Physics* for more details.

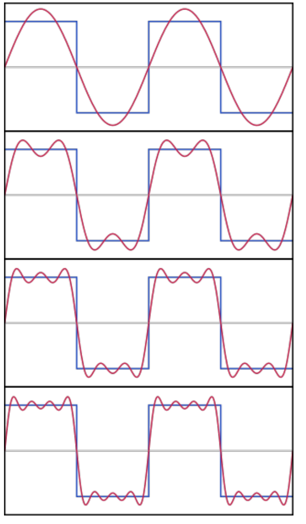

Figure 1: Use a series of trigonometric functions to simulate the square wave

Due to the properties of the wave function, it can be factorized into three wave equations belong to the x, y and z direction seperately, i.e.,

$$(c^2\partial_i^2 - \partial_t^2)\vec{A}(i,t) = 0, \quad i = x,\ y \text{ or } z. \tag{10}$$

And due to $\vec{A} = 0$, light is a transverse wave. If we choose the propagation direction to be z, then the polarization direction can only be x or y. So, in the most easy situation, we may deal with a wave which propagetes along the z direction and its polarization direction is along the x direction. Then we have

$$(c^2\partial_z^2 - \partial_t^2)\vec{A}(z,t) = 0, \tag{11}$$

$$\vec{A}(z,t) = \hat{e}_x \sum_k |A_k| e^{i(kz-\omega_k t)} = \hat{e}_x \sum_k |A_k| q_k(t) e^{ikz}. \tag{12}$$

If we consider one mode at a time, i.e., the term with same $|k|$, we have

$$\vec{A}(z,t) = \hat{e}_x (|A_k| q_k(t) e^{ikz} + |A_{-k}| q_{-k}(t) e^{-ikz}). \tag{13}$$