

18.650 – Fundamentals of Statistics

7. Generalized linear models

Linear model

A Gaussian linear model assumes

$$Y|X = x \sim \mathcal{N}(\mu(x), \sigma^2 I),$$

And¹

$$\mathbb{E}(Y|X = x) = \mu(x) = x^\top \beta,$$

\uparrow
regression function

¹Throughout we drop the boldface notation for vectors

Components of a linear model

The two model components (that we are going to relax) are

1. Random component: the response variable Y is continuous and $Y|X = x$ is Gaussian with mean $\mu(x)$.
2. Regression function: $\mu(x) = x^\top \beta$. linear

Kyphosis

The Kyphosis data consist of measurements on 81 children following corrective spinal surgery. The binary response variable, Y , indicates the presence or absence of a postoperative deformity.

The three covariates are:

- ▶ $X^{(1)}$: Age of the child in month, ↗
- ▶ $X^{(2)}$: Number of the vertebrae involved in the operation, and ↘
- ▶ $X^{(3)}$: Start of the range of the vertebrae involved. ↗

Write $X = (\mathbf{1}, X^{(1)}, X^{(2)}, X^{(3)})^\top \in \mathbb{R}^4$

Kyphosis

- The response variable is binary so there is no choice:
 $Y|X = x$ is Bernoulli with expected value
 $\mu(x) = \mathbb{E}[Y|X = x] \in (0, 1)$
- We cannot write

$$\mu(x) = x^\top \beta$$

because the right-hand side ranges through \mathbb{R}

- We need an invertible function f such that $f(x^\top \beta) \in (0, 1)$

Generalization

A generalized linear model (GLM) generalizes normal linear regression models in the following directions.

1. Random component:

$$Y|X = x \sim \text{some distribution}$$

(e.g. Bernoulli, exponential, Poisson)

2. Regression function:

$$g(\mu(x)) = x^\top \beta$$

where g called link function and $\mu(x) = \mathbb{E}(Y|X = x)$ is the regression function.

Predator/Prey

Consider the following model for the number of preys Y that a predator (Hawk) catches per day ~~a predator~~ given a number X of preys (mice) in its hunting territory.

Random component: $Y > 0$ and the variance of capture rate is known to be approximately equal to its expectation so we propose the following model:

$$Y|X = x \sim \text{Pois}(\mu(x))$$

Where $\mu(x) = \mathbb{E}[Y|X = x]$.

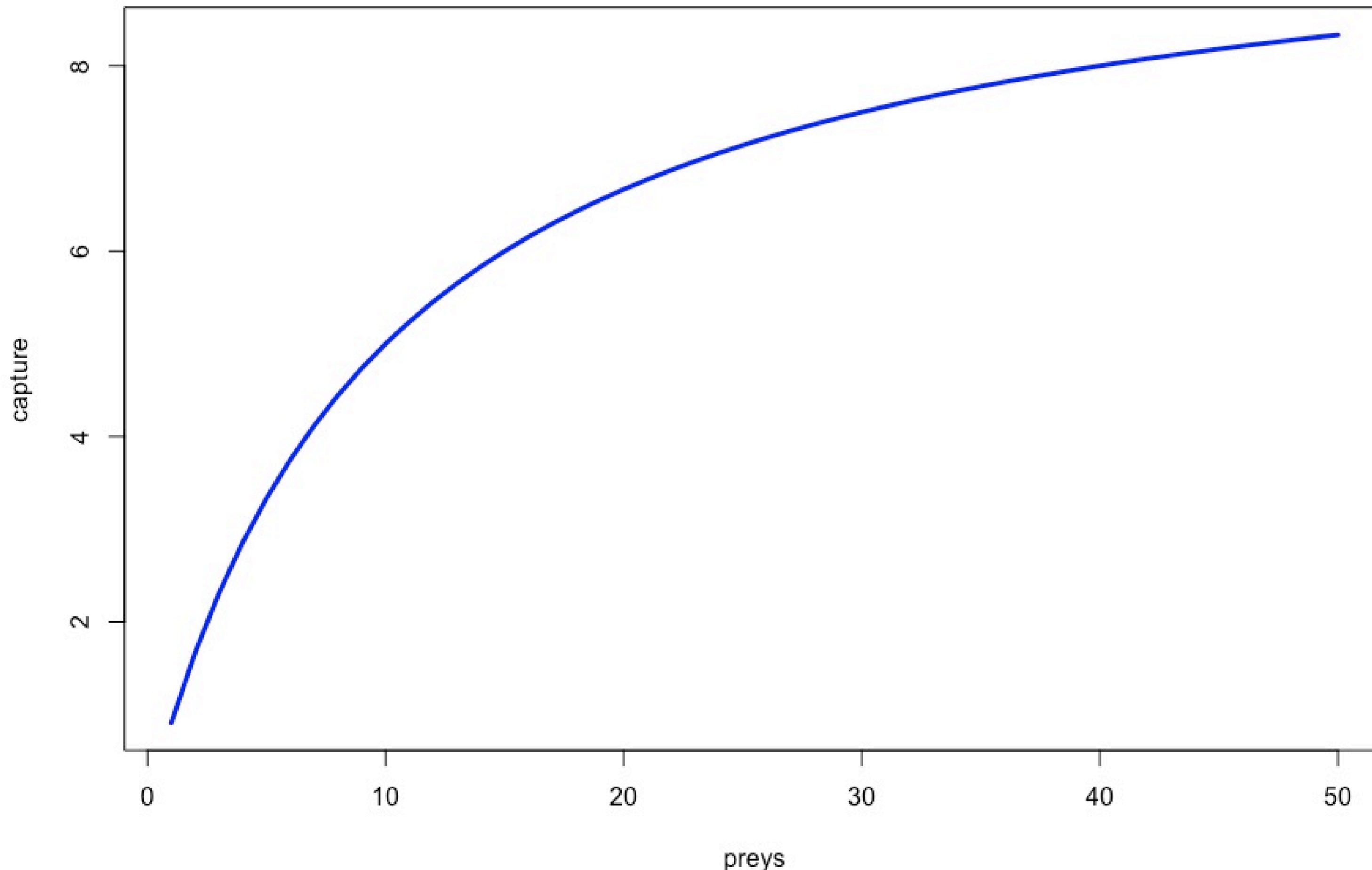
Regression function: We assume

$$\mu(x) = \frac{mx}{h+x}, \quad \text{for some unknown } m, h > 0.$$

where:

- ▶ m is the max expected daily preys the predator can cope with
- ▶ h is the number of preys such that $\mu(h) = \frac{m}{2}$

The regression function $m(x)$ for $m = h = 10$



Example 2: Prey Capture Rate

$$\mu(x) = \frac{m x}{h + x}$$

$$\frac{1}{\mu(x)} = \frac{h + x}{m x}$$

Obviously $\mu(x)$ is not linear but using **reciprocal link**: $g(x) = \frac{1}{x}$,
the right-hand side can be made linear in the parameters:

$$g(\mu(x)) = \frac{1}{\mu(x)} = \frac{1}{m} + \frac{h}{m} \cdot \frac{1}{x} = \beta_0 + \beta_1 \frac{1}{x}.$$

$$\mu(x) = \beta_0 + \beta_1 x$$

Exponential Family

A family of distribution $\{\mathbb{P}_\theta : \theta \in \Theta\}$, $\Theta \subset \mathbb{R}^k$ is said to be a **k -parameter exponential family** on \mathbb{R}^q , if there exist real valued functions:

- ▶ $\eta_1, \eta_2, \dots, \eta_k$ and B of θ ,
- ▶ T_1, T_2, \dots, T_k , and h of $y \in \mathbb{R}^q$ such that the density function (pmf or pdf) of \mathbb{P}_θ can be written as

$$f_\theta(y) = \exp \left[\sum_{i=1}^k \eta_i(\theta) T_i(y) - B(\theta) \right] h(y)$$

(Handwritten notes: $f_\theta(y) = \exp(\frac{\partial}{\partial \theta} (\eta(\theta) - B(\theta))) h(y)$)

$$f_\theta(y) = \exp \left(\eta(\theta)^T T(y) - B(\theta) + \eta(\theta)^T h(y) \right)$$
$$f_\theta(y) = \exp(\eta(\theta)^T T(y)) e^{-B(\theta)} h(y)$$

Normal distribution example

- Consider $Y \sim \mathcal{N}(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$. The density is

$$f_\theta(y) = \exp\left(\frac{\mu}{\sigma^2}y - \frac{1}{2\sigma^2}y^2 - \frac{\mu^2}{2\sigma^2}\right) \frac{1}{\sigma\sqrt{2\pi}},$$

which forms a two-parameter exponential family with

$$\eta_1 = \frac{\mu}{\sigma^2}, \quad \eta_2 = -\frac{1}{2\sigma^2}, \quad T_1(y) = y, \quad T_2(y) = y^2,$$

$$B(\theta) = \frac{\mu^2}{2\sigma^2} + \log(\sigma\sqrt{2\pi}), \quad h(y) = 1.$$

- When σ^2 is known, it becomes a one-parameter exponential family on \mathbb{R} :

$$\eta = \frac{\mu}{\sigma^2}, \quad T(y) = y, \quad B(\theta) = \frac{\mu^2}{2\sigma^2}, \quad h(y) = \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}.$$

Examples of discrete distributions

The following distributions form **discrete** exponential families of distributions with **pmf**

- ▶ Bernoulli(p): $p^y(1 - p)^{1-y}$, $y \in \{0, 1\}$
- ▶ Poisson(λ): $\frac{\lambda^y}{y!}e^{-\lambda}$, $y = 0, 1, \dots$

Examples of Continuous distributions

The following distributions form **continuous** exponential families of distributions with **pdf**:

- ▶ Gamma(a, b): $\frac{1}{\Gamma(a)b^a}y^{a-1}e^{-\frac{y}{b}}$;
 - ▶ above: a : shape parameter, b : scale parameter
 - ▶ reparametrize: $\mu = ab$: mean parameter

$$\frac{1}{\Gamma(a)} \left(\frac{a}{\mu}\right)^a y^{a-1} e^{-\frac{ay}{\mu}}.$$

- ▶ Inverse Gamma(α, β): $\frac{\beta^\alpha}{\Gamma(\alpha)}y^{-\alpha-1}e^{-\beta/y}$.
- ▶ Inverse Gaussian(μ, σ^2): $\sqrt{\frac{\sigma^2}{2\pi y^3}} e^{\frac{-\sigma^2(y-\mu)^2}{2\mu^2 y}}$.

Others: Chi-square, Beta, Binomial, Negative binomial distributions.

One-parameter canonical exponential family

- **Canonical exponential family** for $k = 1$, $y \in \mathbb{R}$

$$f_\theta(y) = \exp\left(\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right)$$

for some *known* functions $b(\cdot)$ and $c(\cdot, \cdot)$.

- If ϕ is known, this is a one-parameter exponential family with θ being the canonical parameter.
- If ϕ is unknown, this may/may not be a two-parameter exponential family.
- ϕ is called **dispersion parameter**.
- In this class, we always assume that ϕ is *known*.

$$f_\theta(y) = \exp\left(\frac{y\theta - b(\theta)}{\phi}\right) \cdot h(y) \quad \text{where } h(y) = e^{c(y, \phi)}$$

Normal distribution example

- ▶ Consider the following Normal density function with known variance σ^2 ,

$$\begin{aligned}f_{\theta}(y) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \\&= \exp\left\{\frac{y\mu - \frac{1}{2}\mu^2}{\sigma^2} - \frac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right)\right\},\end{aligned}$$

- ▶ Therefore $\theta = \underline{\mu}$, $\phi = \underline{\sigma^2}$, $b(\theta) = \underline{\frac{\theta^2}{2}}$, and

$$c(y, \phi) = -\frac{1}{2}\left(\frac{y^2}{\phi} + \log(2\pi\phi)\right).$$

Other distributions

Table 1: Exponential Family

	Normal	Poisson	Bernoulli
Notation	$\mathcal{N}(\mu, \sigma^2)$	$\mathcal{P}(\mu)$	$\mathcal{B}(p)$
Range of y	$(-\infty, \infty)$	$[0, -\infty)$	$\{0, 1\}$
ϕ	σ^2	1	1
$b(\theta)$	$\frac{\theta^2}{2}$	e^θ	$\log(1 + e^\theta)$
$c(y, \phi)$	$-\frac{1}{2} \left(\frac{y^2}{\phi} + \log(2\pi\phi) \right)$	$-\log y!$	0

Likelihood

Let $\ell(\theta) = \log f_\theta(Y)$ denote the log-likelihood function.

The mean $\mathbb{E}(Y)$ and the variance $\text{var}(Y)$ can be derived from the following identities

- ▶ First identity

$$\mathbb{E}\left(\frac{\partial \ell}{\partial \theta}\right) = \textcircled{O}$$

- ▶ Second identity

$$\mathbb{E}\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) + \mathbb{E}\left(\frac{\partial \ell}{\partial \theta}\right)^2 = 0.$$

$$\Leftrightarrow \underbrace{\mathbb{E}\left[\left(\frac{\partial \ell}{\partial \theta}\right)'\right]}_{\text{F.I.}} = - \underbrace{\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \theta^2}\right]}_{\text{T.I.}} \quad (\text{miniser})$$

Expected value

Note that

$$\ell(\theta) = \frac{Y\theta - b(\theta)}{\phi} + c(Y; \phi),$$

Therefore

$$\frac{\partial \ell}{\partial \theta} = \frac{Y - b'(\theta)}{\phi}$$

It yields

$$0 = \mathbb{E}\left(\frac{\partial \ell}{\partial \theta}\right) = \frac{\mathbb{E}(Y) - b'(\theta)}{\phi},$$

which leads to

$$\mathbb{E}(Y) = b'(\theta)$$

Variance

On the other hand we have we have

$$\frac{\partial^2 \ell}{\partial \theta^2} + \left(\frac{\partial \ell}{\partial \theta} \right)^2 = -\frac{b''(\theta)}{\phi} + \left(\frac{Y - b'(\theta)}{\phi} \right)^2$$

and from the previous result,

$$\frac{Y - b'(\theta)}{\phi} = \frac{Y - \mathbb{E}(Y)}{\phi}$$

Together, with the second identity, this yields

$$0 = -\frac{b''(\theta)}{\phi} + \frac{\text{var}(Y)}{\phi^2},$$

which leads to

$$\text{var}(Y) = b''(\theta) \cdot \phi$$

Example: Poisson distribution

$$b(\theta) = p = e^\theta$$

$$\log p = \theta \Rightarrow p = e^\theta$$

Example: Consider a Poisson likelihood,

$$f(y) = \frac{\mu^y}{y!} e^{-\mu} = \exp \left(y \underbrace{\log \mu}_{\theta} - \underbrace{\mu - \log(y!)}_{c(y, \phi)} \right)$$

$$b(\theta)$$

$$c(y, \phi), \phi = 1$$

Thus,

$$\theta = \log \mu \quad b(\theta) = e^\theta \quad \phi = 1 \quad c(y, \phi) = -\log(y!),$$

So

$$\mu = e^\theta, \quad b(\theta) = p \quad b''(\theta) = p$$

$$\mathbb{E}[X] = p, \quad \text{Var}[X] = p \cdot 1 = p$$

Link function

- ▶ β is the parameter of interest, and needs to appear somehow in the likelihood function to use maximum likelihood.
- ▶ A link function g relates the linear predictor $X^\top \beta$ to the mean parameter μ ,

$$X^\top \beta = g(\mu) \cdot = g(\mu(X))$$

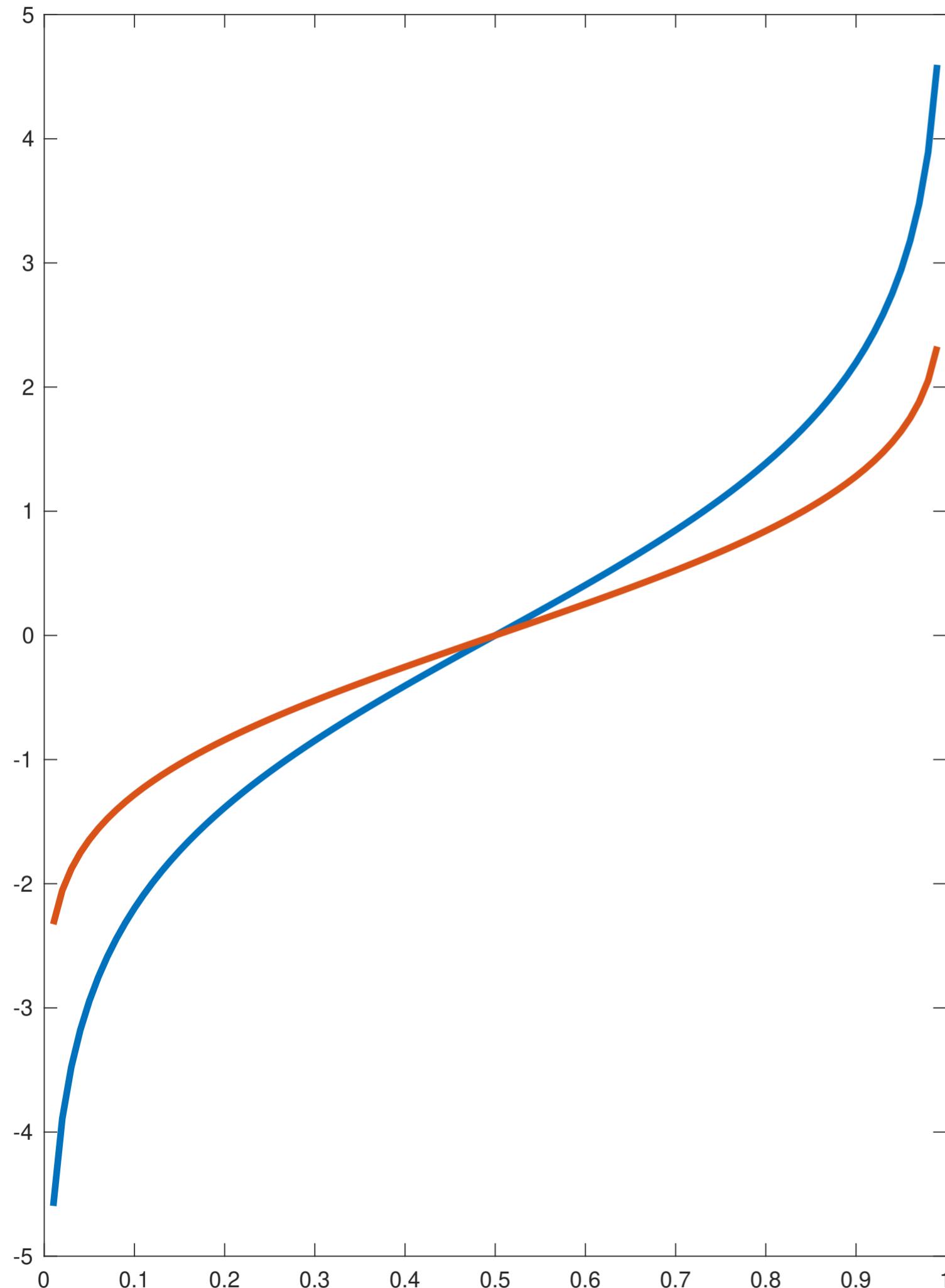
- ▶ g is required to be monotone increasing and differentiable

$$\mu = g^{-1}(X^\top \beta).$$

Examples of link functions

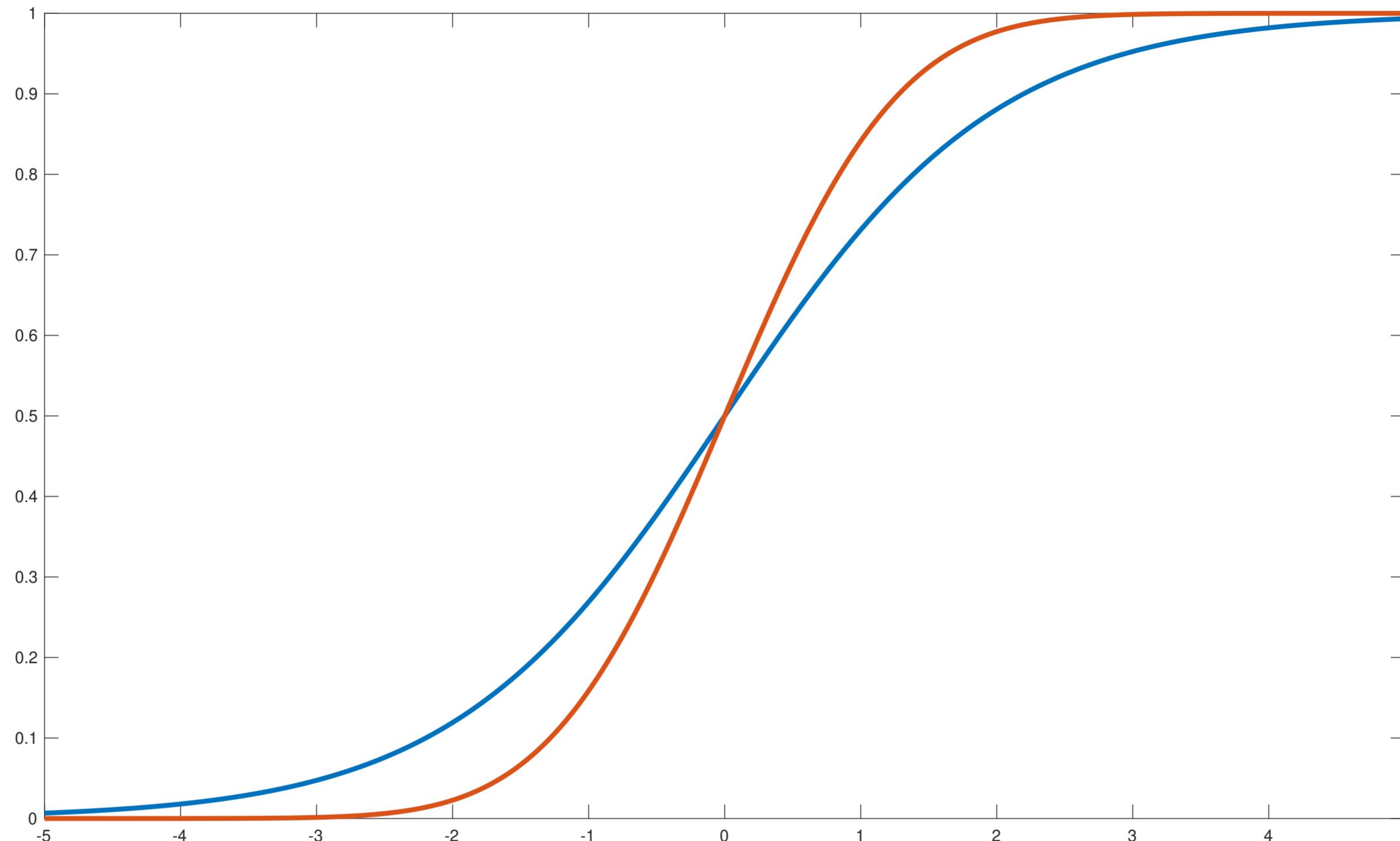
- ▶ For LM, $g(\cdot) = \text{identity}$. ✓
- ▶ Poisson data. Suppose $Y|X \sim \text{Poisson}(\mu(X))$.
 - ▶ $\mu(X) > 0$;
 - ▶ $\log(\mu(X)) = X^\top \beta$; *log-link*
 - ▶ In general, a link function for the count data should map $(0, +\infty)$ to \mathbb{R} .
 - ▶ The log link is a natural one.
- ▶ Bernoulli/Binomial data.
 - ▶ $0 < \mu < 1$;
 - ▶ g should map $(0, 1)$ to \mathbb{R} :
 - ▶ 3 choices:
 1. logit: $\log\left(\frac{\mu(X)}{1-\mu(X)}\right) = X^\top \beta$;
 2. probit: $\Phi^{-1}(\mu(X)) = X^\top \beta$ where $\Phi(\cdot)$ is the normal cdf;
 - ▶ The logit link is the natural choice.

Examples of link functions for Bernoulli response



- in blue:
$$g_1(x) = f_1^{-1}(x) = \log\left(\frac{x}{1-x}\right)$$
 (logit link)
- in red:
$$g_2(x) = f_2^{-1}(x) = \Phi^{-1}(x)$$
 (probit link)

Examples of link functions for Bernoulli response



- in blue: $f_1(x) = \frac{e^x}{1 + e^x}$
- in red: $f_2(x) = \Phi(x)$ (Gaussian CDF)

Canonical Link

- The function g that links the mean μ to the canonical parameter θ is called **Canonical Link**:

$$g(\mu) = \theta$$

- Since $\mu = b'(\theta)$, the canonical link is given by

$$g(\mu) = (b')^{-1}(\mu).$$

- If $\phi > 0$, the canonical link function is **strictly increasing**.
Why?

$$g \xrightarrow{\text{str.}} \tilde{g}^{-1} \xrightarrow{\text{str.}} (\tilde{g}^{-1})' > 0 \Leftrightarrow b'' > 0 \Leftrightarrow \text{Var} > 0$$

$\phi > 0$

Example: the Bernoulli distribution

- We can check that

$$b(\theta) = \log(1 + e^\theta)$$

- Hence we solve

$$b'(\theta) = \frac{\exp(\theta)}{1 + \exp(\theta)} = \mu \quad \Leftrightarrow \quad \theta = \log\left(\frac{\mu}{1-\mu}\right)$$

- The canonical link for the Bernoulli distribution is the *logit link*

Other examples

	$b(\theta)$	$g(\mu)$
Normal	$\theta^2/2$	μ
Poisson	$\exp(\theta)$	$\log \mu$
Bernoulli	$\log(1 + e^\theta)$	$\log \frac{\mu}{1-\mu}$
Gamma	$-\log(-\theta)$	$-\frac{1}{\mu}$

exercise

Model and notation

- Let $(X_i, Y_i) \in \mathbb{R}^p \times \mathbb{R}$, $i = 1, \dots, n$ be independent random pairs such that the conditional distribution of Y_i given $X_i = x_i$ has density in the **canonical exponential family**:

$$f_{\theta_i}(y_i) = \exp \left\{ \frac{y_i \theta_i - b(\theta_i)}{\phi} + c(y_i, \phi) \right\}.$$

- $\mathbf{Y} = (Y_1, \dots, Y_n)^\top$, $\mathbf{X} = (X_1, \dots, X_n)^\top = \begin{pmatrix} x_1 & \dots \\ x_2 & \dots \\ \vdots & \ddots \end{pmatrix} \in \mathbb{R}^{n \times p}$
- Here the mean $\mu_i = \mathbb{E}[Y_i | X_i]$ is related to the canonical parameter θ_i via

$$\mu_i = b'(\theta_i)$$

- and μ_i depends linearly on the covariates through a link function g :

$$g(\mu_i) = \mathbf{X}_i^\top \boldsymbol{\beta}.$$

Back to β

- Given a link function g , note the following relationship between β and θ :

$$\begin{aligned}\theta_i &= (b')^{-1}(\mu_i) \\ &= (b')^{-1}(g^{-1}(X_i^\top \beta)) \equiv \underline{h(X_i^\top \beta)},\end{aligned}$$

where h is defined as

$$h = (b')^{-1} \circ g^{-1} = (g \circ b')^{-1}.$$

- Remark: if g is the **canonical** link function, h is *the identity*

$$g = (b')^{-1}$$

Log-likelihood

- The log-likelihood is given by

$$\begin{aligned}\ell_n(\mathbf{Y}, \mathbb{X}, \beta) &= \sum_i \frac{Y_i \theta_i - b(\theta_i)}{\phi} + cst \\ &= \sum_i \frac{Y_i h(X_i^\top \beta) - b(h(X_i^\top \beta))}{\phi} + cst\end{aligned}$$

up to a constant term.

- Note that when we use the canonical link function, we obtain the simpler expression

$$\begin{aligned}\ell_n(\mathbf{Y}, \mathbb{X}, \beta) &= \sum_i \frac{Y_i X_i^\top \beta - b(X_i^\top \beta)}{\phi} + cst \\ -H_\beta \ell_n &\stackrel{\text{def}}{=} \sum_i H_\beta b'(X_i^\top \beta) X_i X_i^\top = \frac{1}{\phi} \sum_{i=1}^n b''(X_i^\top \beta) (X_i^\top X_i)\end{aligned}$$

Strict concavity

- ▶ The log-likelihood $\ell(\theta)$ is **strictly concave** using the canonical function when $\phi > 0$. Why? if Rank(\mathbb{X}) = p
- ▶ As a consequence the maximum likelihood estimator is unique
- ▶ On the other hand, if another parameterization is used, the likelihood function may not be strictly concave leading to **several local maxima**.

Concluding remarks

- ▶ Maximum likelihood for Bernoulli Y and the logit link is called *logistic regression*.
- ▶ In general, there is no closed form for the MLE and we have to use *optimization algorithms (gradient ascent)*
- ▶ The asymptotic normality of the MLE also applies to GLMs.