CS771 Assignment 1

Tensor Trailblazers

Task 1

To derive the mathematical relation of a linear model to predict the time of the upper signal of an arbiter PUF :

$$t_{31}^{u}(c) = \vec{W}^{\top} \cdot \vec{X} + b$$

Proof : We know that for an i^{th} puf

$$t_i^u = (1 - c_i)\left(t_{i-1}^u + p_i\right) + c_i\left(t_{i-1}^l + s_i\right)$$

$$t_i^l = (1 - c_i)\left(t_{i-1}^l + q_i\right) + c_i\left(t_{i-1}^u + r_i\right)$$

$$t_0^u = (1 - c_0)(p_0) + c_0(s_0)$$

$$t_0^l = (1 - c_0)(q_0) + c_0(r_0)$$

$$t_1^u = p_0 + p_1 + c_0(s_0 - p_0) + c_1((q_0 - p_0) + (s_1 - p_1)) + c_0c_1((r_0 - s_0) + (p_0 - q_0))$$

$$t_1^l = q_0 + q_1 + c_0(r_0 - q_0) + c_1((p_0 - q_0) + (r_1 - q_1)) + c_0c_1((s_0 - r_0) + (q_0 - p_0))$$

$$\begin{aligned} t_2^u = {} & p_0 + p_1 + p_2 + c_0(s_0 - p_0) + c_1((q_0 - p_0) + (s_1 - p_1)) + c_2((q_0 - p_0) + (q_1 - p_1) + (s_2 - p_2)) \\ & + (c_0c_1 + c_0c_1 - 2c_0c_1c_2)((r_0 - s_0) + (p_0 - q_0)) + c_1c_2(2(p_0 - q_0) + (p_1 - q_1) + (r_1 - s_1)) \end{aligned}$$

$$\begin{aligned} t_2^l = {} & q_0 + q_1 + q_2 + c_0(r_0 - q_0) + c_1((p_0 - q_0) + (r_1 - q_1)) + c_2((p_0 - q_0) + (p_1 - q_1) + (r_2 - p_2)) \\ & + (c_0c_1 + c_0c_1 - 2c_0c_1c_2)((s_0 - r_0) + (q_0 - p_0)) + c_1c_2(2(q_0 - p_0) + (q_1 - p_1) + (s_1 - r_1)) \end{aligned}$$

$$\begin{aligned} t_3^u = {} & p_0 + p_1 + p_2 + p_3 + c_0(s_0 - p_0) + c_1((q_0 - p_0) + (s_1 - p_1)) + c_2((q_0 - p_0) + (q_1 - p_1) + (s_2 - p_2)) \\ & + c_3((q_0 - p_0) + (q_1 - p_1) + (q_2 - p_2) + (s_3 - p_3)) \\ & + (c_0c_1 + c_0c_2 + c_0c_3 - 2c_0c_1c_2 - 2c_0c_1c_3 - 2c_0c_2c_3 + 4c_0c_1c_2c_3)((r_0 - s_0) + (p_0 - q_0)) \\ & + (c_1c_2 + c_1c_3 - 2c_1c_2c_3)(2(p_0 - q_0) + (p_1 - q_1) + (r_1 - s_1)) \\ & + c_2c_3(2(p_0 - q_0) + 2(p_1 - q_1) + (p_2 - q_2) + (r_2 - s_2)) \end{aligned}$$

$$\begin{aligned} t_3^l = {} & q_0 + q_1 + q_2 + q_3 + c_0(r_0 - q_0) + c_1((p_0 - q_0) + (r_1 - q_1)) + c_2((p_0 - q_0) + (p_1 - q_1) + (r_2 - q_2)) \\ & + c_3((p_0 - q_0) + (p_1 - q_1) + (p_2 - q_2) + (r_3 - s_3)) \\ & + (c_0c_1 + c_0c_2 + c_0c_3 - 2c_0c_1c_2 - 2c_0c_1c_3 - 2c_0c_2c_3 + 4c_0c_1c_2c_3)((s_0 - r_0) + (q_0 - p_0)) \\ & + (c_1c_2 + c_1c_3 - 2c_1c_2c_3)(2(q_0 - p_0) + (q_1 - p_1) + (s_1 - r_1)) \\ & + c_2c_3(2(q_0 - p_0) + 2(q_1 - p_1) + (q_2 - p_2) + (s_2 - r_2)) \end{aligned}$$

Preprint. Under review.

By observing the pattern we have :-

$$t_k^u = \sum_{i=0}^{k} p_i + c_0(s_0 - p_0) + \sum_{i=1}^{k}\left(c_i\left(\sum_{j=0}^{i-1}(q_j - p_j) + (s_i - p_i)\right)\right)$$
$$\sum_{i=0}^{k-1} c_i(\sum_{j=i+1}^{k}(-2)^{j-i-1} \sum_{(i+1)<=k_0<k_1<...<k_{(j-i-1)}<31} (c_{k_0}c_{k_1}...c_{k_{j-i-1}}))(f_i)$$
$$where f_i = 2\sum_{l=0}^{i}(p_l - q_l) - (p_i - q_i) + (r_i - s_i)$$

And by mathematical induction, we can prove that this pattern holds for t_{31}^u.

Now analyzing,

$$c_0(\sum_{j=1}^{31}(-2)^{j-1} \sum_{(1)<=k_0<k_1<...<k_{(j-1)}<31} (c_{k_0}c_{k_1}...c_{k_{j-1}}))$$

if we expand it we get,

$$c_0((c_1 + c_2 + ... + c_{31}) - 2(c_1c_2 + ...) + 4(c_1c_2c_3 + ...) - 8(c_1c_2c_3c_4 + ...) + ...)$$

from

$$(x - c_1)(x - c_2)...(x - c_{31}) = x^{31} - (c_1 + c_2 + ... + c_{31})x^{30} + (c_1c_2 + ...)x^{29} - (c_1c_2c_3 + ..)x^{28} + ... - (c_1c_2...c_{31})$$

put x = 1/2 we get

$$(1 - 2c_1)(1 - 2c_2)...(1 - 2c_{31}) = 1 - 2(c_1 + (c_2 + ... + c_{31}) + 4(c_1c_2 + ...) - 8(c_1c_2c_3 + ..) + ...$$

from above

$$(c_1 + c_2 + ... + c_{31}) - 2(c_1c_2 + ...) + 4(c_1c_2c_3 + ...) - .. = \alpha((1 - 2c_1)...(1 - 2c_{31})) + \beta$$

where α and β are some constants.

Therefore our equation can be rewritten as,

$$t_k^u = b + w_0c_0 + w_1c_1 + w_2c_2 + ... + w_{31}c_{31} + w_{32}((1 - 2c_1)...(1 - 2c_{31})) + w_{33}((1 - 2c_2)...(1 - 2c_{31})) + w_{34}((1 - 2c_3)...(1 - 2c_{31})) + ...w_{63}(1 - 2c_{30})(1 - 2c_{31}))$$

$$t_k^u(c) = \vec{W}^\top \phi(\vec{c}) + b$$

where,

$$\phi(\vec{c}) = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_{31} \\ (1 - 2c_0)(1 - 2c_1)...(1 - 2c_{31}) \\ (1 - 2c_1)(1 - 2c_2)...(1 - 2c_{31}) \\ \vdots \\ \vdots \\ (1 - 2c_{30})(1 - 2c_{31}) \end{bmatrix}$$

Task 2

Dimension of above map is 63.

Task 3

From task 1 we can say that,

$$
\begin{aligned}
t^u_{31_1} &= b_1 + w_{0_1}c_0 + w_{1_1}c_1 + w_{2_1}c_2 + ... + w_{31_1}c_{31} + w_{32_1}((1-2c_1)...(1-2c_{31})) \\
&+ w_{33_1}((1-2c_2)...(1-2c_{31})) + w_{34_1}((1-2c_3)...(1-2c_{31})) \\
&+ ...w_{63_1}(1-2c_{30})(1-2c_{31}))
\end{aligned}
$$

$$
\begin{aligned}
t^u_{31_0} &= b_0 + w_{0_0}c_0 + w_{1_0}c_1 + w_{2_0}c_2 + ... + w_{31_0}c_{31} + w_{32_0}((1-2c_1)...(1-2c_{31})) \\
&+ w_{33_0}((1-2c_2)...(1-2c_{31})) + w_{34_0}((1-2c_3)...(1-2c_{31})) \\
&+ ...w_{63_0}(1-2c_{30})(1-2c_{31}))
\end{aligned}
$$

subtracting above equation we get :-

$$
\begin{aligned}
(t^u_{31_1} - t^u_{31_0}) &= (b_1 - b_0) + (w_{0_1} - w_{0_0})c_0 + (w_{1_1} - w_{1_0})c_1 + (w_{2_1} - w_{2_0}c_2 + ... + (w_{31_1} - w_{31_0})c_{31} \\
&+ (w_{32_1} - w_{32_0})((1-2c_1)...(1-2c_{31})) + (w_{33_1} - w_{33_0})((1-2c_2)...(1-2c_{31})) \\
&+ (w_{34_1} - w_{34_0})((1-2c_3)...(1-2c_{31})) + ...(w_{63_1} - w_{63_0})(1-2c_{30})(1-2c_{31}))
\end{aligned}
$$

above equation can be converted to another linear equation -

$$
\begin{aligned}
(\triangle t^u_{31}) &= \triangle b + \triangle w_0 c_0 + \triangle w_1 c_1 + \triangle w_2 c_2 + ... + \triangle w_{31} c_{31} \\
&+ \triangle w_{32}((1-2c_1)...(1-2c_{31})) + \triangle w_{33}((1-2c_2)...(1-2c_{31})) \\
&+ \triangle w_{34}((1-2c_3)...(1-2c_{31})) + ...\triangle w_{63}(1-2c_{30})(1-2c_{31}))
\end{aligned}
$$

now we can clearly see that if

$$\triangle t^u_{31} > 0 \quad \text{then we have} \quad r_1 = 1 \tag{1}$$

$$\triangle t^u_{31} < 0 \quad \text{then we have} \quad r_1 = 0 \tag{2}$$

therefore we can say that,

$$\frac{1 + \text{sign}\left(\vec{W}^\top \cdot \phi(\vec{c}) + b\right)}{2} = r_1$$

where,

$$
\phi(\vec{c}) = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_{31} \\ (1-2c_0)(1-2c_1)...(1-2c_{31}) \\ (1-2c_1)(1-2c_2)...(1-2c_{31}) \\ \vdots \\ \vdots \\ (1-2c_{30})(1-2c_{31}) \end{bmatrix}
$$

Task 4

Dimension that we need for the linear model to predict the response remains same as required in task 2 i.e. 63.

Task 6

Report outcomes of experiments with both the `sklearn.svm.LinearSVC` and `sklearn.linear_model.LogisticRegression` methods when used to learn the linear model. In particular, report how various hyperparameters affected training time and test accuracy using tables and/or charts.

1. Changing the loss hyperparameter in `LinearSVC` (hinge vs squared hinge)
2. Setting C in `LinearSVC` and `LogisticRegression` to high/low/medium values
3. Changing `tol` in `LinearSVC` and `LogisticRegression` to high/low/medium values
4. Changing the penalty (regularization) hyperparameter in `LinearSVC` and `LogisticRegression` (`l2` vs `l1`)

Experimental Results

LinearSVC

We have reported the training time obtained after running the Google Collab Script while keeping certain settings for Response0 and Response1 individually, so the training time for any hyperparameter includes the time taken to calculate both Response0 and Response1 as the script calculates them simultaneously.

Table 1: Effect of setting C for Response0

C Value	Training Time (s)	Test Accuracy (%)
High	60.6s	98.414
Medium	11s	99.2625
Low	4s	98.225

Table 2: Effect of setting C for Response1

C Value	Training Time (s)	Test Accuracy (%)
High	60.6s	99.886
Medium	11s	99.2625
Low	4s	98.225

Table 3: Effect of setting tol for Response0

tol Value	Training Time (s)	Test Accuracy (%)
High	0.8s	93.0375
Medium	12s	98.00
Low	60.6s	98.414

Table 4: Effect of setting tol for Response1

tol Value	Training Time (s)	Test Accuracy (%)
High	60.6s	99.886
Medium	12s	98.00
Low	12s	99.075

LogisticRegression

Table 5: Effect of setting C for Response0

C Value	Training Time (s)	Test Accuracy (%)
High	58.7s	98.412
Medium	56.53s	98.412
Low	56.3s	98.414

Table 6: Effect of setting C for Response1

C Value	Training Time (s)	Test Accuracy (%)
High	58.7s	99.882
Medium	56.53s	99.884
Low	56.3s	99.88

Table 7: Effect of setting tol for Response0

tol Value	Training Time (s)	Test Accuracy (%)
High	0.2s	87.3875
Medium	0.3s	98.9125
Low	58.7s	98.412

Table 8: Effect of setting tol for Response1

tol Value	Training Time (s)	Test Accuracy (%)
High	0.2s	87.3875
Medium	0.3s	98.9125
Low	58.7s	99.882