Домашняя работа №4

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№1

$$\hat{x}\,\psi(x) = x\,\psi(x)$$

$$\{\hat{x}\}_{xx'} = x\,\delta(x-x')$$

1) $\{\hat{x}\}_{pp'}$ – ?

$$\{\hat{x}\}_{pp'} = \sum_{xx'} S^{-1}_{px} A_{xx'} S_{x'p'} = \sum_{x} S^{-1}_{px}\, x\, S_{xp'}$$, где

$$S_{xp'} = \langle x|p'\rangle = \frac{1}{\sqrt{L}} e^{\frac{ixp'}{\hbar}}$$ – матрица перехода

$$\{S^{-1}\}_{px} = S^{*}_{xp} = S_{px} = \langle p|x\rangle = \langle x|p\rangle^{*} = \frac{1}{\sqrt{L}} e^{-\frac{ixp}{\hbar}}$$

тогда

$$\{\hat{x}\}_{pp'} = \frac{1}{L}\sum_{x} e^{-\frac{ixp}{\hbar}} \cdot e^{\frac{ixp'}{\hbar}} \cdot x = \frac{1}{\cancel{L}} \int e^{+\frac{ix}{\hbar}(p'-p)}\, x \cdot \frac{dx\,\cancel{L}}{2\pi\hbar} =$$

$$= \left| \frac{x}{\hbar} = k \quad \frac{dx}{\hbar} = dk \right| = \frac{\cancel{L}}{\cancel{L}} \int e^{ik(p'-p)} \frac{dk}{2\pi}\, k\hbar$$

$$\Delta\ \delta'(x) = \int \frac{dk}{2\pi} e^{ikx}\, ik$$

Тогда, зная $\delta'(x)$ и преобразуя (поделив его на i), получим

$$-i\delta'(x) = \int \frac{dk}{2\pi} e^{ikx}\, k$$

Подставим δ-функцию с аргументом $(p'-p)$ в $\{\hat{x}\}_{pp'}$

$$\{\hat{x}\}_{pp'} = -i\hbar\,\delta'(p'-p) = -i\hbar \frac{d}{dp}\delta(p'-p) = \boxed{i\hbar \frac{d}{dp}\delta(p-p')}$$

~~N1~~

2) $\hat{x}\, a_p - ?$

$$\int \{\hat{x}\}_{pp'}\, a(p')\, dp' = \int i\hbar \frac{d}{dp} \delta(p-p')\, a(p')\, dp' =$$

$$= i\hbar \frac{d}{dp} \int \delta(p-p')\, a(p')\, dp' = \boxed{i\hbar \frac{da(p)}{dp}}$$

N2

1) $f(x) = x^2 \qquad [-1;1]$

$f(1) = f(-1) \quad ; \quad e^{ik_n} = e^{-ik_n} \Rightarrow e^{2ik_n} = 1$

$2k_n = 2\pi n$

$k_n = \pi n$

$$f_n = \frac{1}{2}\int_{-1}^{1} x^2 e^{-i\pi n x} = \left| \begin{matrix} u = x^2 & dV = e^{-i\pi n x}dx \\ du = 2x\,dx & V = \frac{i}{\pi n} e^{-i\pi n x} \end{matrix} \right| =$$

$$= \frac{i}{2\pi n}\left(x^2 e^{-i\pi n x}\Big|_{-1}^{1} - \frac{i}{\pi n}\int_{-1}^{1} x e^{-i\pi n x}\, dx = \left| \begin{matrix} u = x & dV = e^{-i\pi n x}dx \\ du = dx & V = \frac{i}{\pi n} e^{-i\pi n x} \end{matrix} \right| \right.$$

$$= \frac{i}{2\pi n}\left(e^{-i\pi n} - e^{i\pi n}\right) + \frac{1}{\pi^2 n^2}\left(x e^{-i\pi n x}\Big|_{-1}^{1} + \frac{1}{\pi^2 n^2}\int e^{-i\pi n x}\, dx\right.$$

$$= \frac{i}{2\pi n}\underbrace{\cancel{\left(e^{-i\pi n} - e^{i\pi n}\right)}}_{0} + \frac{1}{\pi^2 n^2}\left(e^{-i\pi n} + e^{i\pi n}\right) - \frac{i}{\pi^3 n^3}\underbrace{\cancel{\left(e^{-i\pi n x} - e^{i\pi n x}\right)}}_{0}$$

Воспользуемся формулой:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\left.\begin{matrix} f_n = \frac{2}{\pi^2 n^2}\cos(\pi n) = \frac{2(-1)^n}{\pi^2 n^2} \\ f_0 = \int_{-1}^{1} x^2 dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{matrix}\right\} \Rightarrow$$

тогда

$$\boxed{f(x) = \frac{2}{3} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{i\pi n x}}$$

2) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ — ?

$f(x) = x^2$

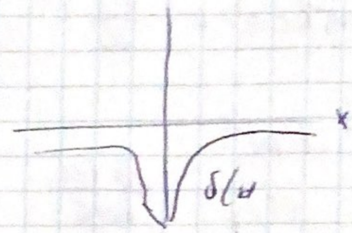
$f(x) = \frac{2}{3} + \frac{2}{\pi^2} \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{i\pi n x}$, чтобы получить ряд $\sum \frac{1}{n^2}$ нам

надо под суммой $+(-1)^n$, а получить могу только при $f(1)$:

$$f(1) = 1 = \frac{2}{3} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1 \cdot \pi^2}{3 \cdot 2} = \boxed{\frac{\pi^2}{6}}$$

№ 5

$$U(x) = -\alpha\,\delta(x)$$



$\{U(x)\}_{pp'}$ — ?

$$\{U(x)\}_{pp'} = \sum_{xx'} \{S^{-1}\}_{px} U(x)_{xx'} S_{x'p'} = \frac{1}{L} \sum_x U(x)\, e^{\frac{ix}{\hbar}(p'-p)} =$$

$$= -\frac{\alpha}{L} \sum_x \delta(x)\, e^{\frac{ix}{\hbar}(p'-p)} = -\frac{\alpha}{\cancel{L}} \underbrace{\int \delta(x)\, e^{\frac{ix}{\hbar}(p'-p)} \frac{dx \cancel{L}}{2\pi\hbar}}_{1}$$

$$= \boxed{-\frac{\alpha}{2\pi\hbar}}$$

№ 4

1) $\psi(x) = \frac{A}{x + ia}$, $a > 0$

$$\int_{-\infty}^{+\infty} \langle \psi | \psi \rangle dx = A^2 \int_{-\infty}^{+\infty} \frac{dx}{x^2 + a^2} = A^2 \frac{1}{a} \operatorname{arctg} \frac{x}{a} \Big|_{-\infty}^{+\infty} = \frac{A^2}{a} \pi = 1$$

$$\Rightarrow A = \sqrt{\frac{a}{\pi}}$$

4) $\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle = \int \psi^*(x) \hat{p}\, \psi(x)\, dx =$

$= \int \psi^*(x) \left(-i\hbar \frac{d}{dx}\psi(x)\right) dx = i\hbar \frac{a}{\pi} \int \frac{1}{x - ia} \frac{1}{(x+ia)^2} dx =$

$= i\hbar \frac{a}{\pi} \int_{-\infty}^{+\infty} \left(\frac{1}{4a^2(x+ia)} + \frac{i}{2a(x+ia)^2} - \frac{1}{4a^2(x-ia)} \right) dx$

$= \frac{i\hbar}{\pi} \left(\frac{\ln|x+ia|}{4a^2} - \frac{\ln(|x-ia|)}{4a} - \frac{i}{2(x+ia)} \right) \Big|_{-\infty}^{+\infty} = \frac{i\hbar}{\pi} \frac{i\pi}{2a} = -\frac{\hbar}{2a}$

ответ: $-\frac{\hbar}{2a}$

2) $W_p = |a_p|^2$

$dW_p = |a_p|^2 \frac{dp L}{2\pi\hbar}$

$\psi(x) = \langle x | \psi \rangle = \sum a_p \langle x | p \rangle$

$a_p = \int \psi(x) S_{xp}\, dx = \sqrt{\frac{a}{\pi L}} \int_{-\infty}^{+\infty} \frac{e^{\frac{ipx}{\hbar}}}{x + ia} dx = \ldots$

3) ...

№ 3

$U(t')\psi(t) = U(t)\psi(t)$

$\{U\}_{pp'} = (S^{-1})_{px} U(t)_{xx'} S_{x'p'} = \frac{1}{L} \sum_x U(x) e^{\frac{ix}{\hbar}(p'-p)} =$

$= \frac{1}{L} \int U(x) e^{\frac{ix}{\hbar}(p'-p)} \frac{dx L}{2\pi\hbar}$

$\Delta \quad U(\hat{x}) = \hat{H} - \frac{\hat{p}^2}{2m} = -i\hbar \frac{\partial}{\partial t} - \frac{\left(-i\hbar \frac{d}{dx}\right)^2}{2m} = U(x)$

$-\frac{\hbar^2}{2m} \int \frac{i(p'-p)}{\hbar} e^{\frac{ix}{\hbar}(p'-p)} \frac{dx}{2\pi\hbar} = -\frac{\hbar i (p'-p)}{2m} \underbrace{\int e^{ik(p'-p)} \frac{dk}{2\pi}}_{\delta(p'-p)} =$

$= -\frac{i\hbar}{2m}(p'-p)\,\delta(p'-p) = 0\ ?$

№6

$$\hat{H} = i\partial_x + x^2$$

$$\hat{H}\psi(x) = \int H(x,x')\,\psi(x')\,dx' = \left(i\frac{\partial\psi(x)}{\partial x} + x^2\psi(x)\right) =$$

$$= \int H_1(x,x')\,\psi(x') + \int H_2(x,x')\,\psi(x')\,dx'$$

$$\int H_1(x,x')\,\psi(x')\,dx' = \lambda i\frac{\partial\psi(x)}{\partial x}$$

по св-ву $\int f(x)\,\delta^{(n)}(x-a)\,dx = (-1)^n f^{(n)}(a)$, тогда

$$\int \lambda i\frac{\partial}{\partial x'}\delta(x'-x)\,\psi(x')\,dx' = \frac{\partial\psi(x)}{\partial x}$$

$$\int H_2(x,x')\,\psi(x')\,dx' = x^2\psi(x) = \int x^2\delta(x'-x)\,\psi(x')\,dx' = x^2\psi(x)$$

$$\hat{H}\psi = \frac{H}{\cancel{L}}\,\psi(x) = i\frac{\partial}{\partial x}\psi + x^2\psi \quad ?$$