Домашнее задание 3
по Квантовой механике

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№1

$$S = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}$$

1. $\det(S - \lambda E) = 0$

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$-(\cos^2 - \lambda^2) - \sin^2\theta) = \lambda^2 - \cos^2\theta - \sin^2\theta = 0$$

$$\lambda = \pm 1$$

а) $\lambda_1 = 1$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} \sin\theta e^{-i\varphi} \\ 1 - \cos\theta \end{pmatrix} \qquad |\vec{x}| = \sqrt{\sin^2\theta + \cos^2\theta - 2\cos\theta + 1}$$

$$|\vec{x}| = \sqrt{2(1 - \cos\theta)}$$

$$\vec{e}_x = \frac{1}{\sqrt{2(1-\cos\theta)}} \begin{pmatrix} \sin\theta e^{-i\varphi} \\ 1 - \cos\theta \end{pmatrix}$$

б) $\lambda_{-1} = -1$

$$\begin{pmatrix} \cos\theta + 1 & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & 1 - \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} \cos\theta - 1 \\ \sin\theta e^{i\varphi} \end{pmatrix} \qquad |\vec{y}| = \sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} =$$

$$|\vec{y}| = \sqrt{4(-2\cos\theta)}$$

$$\vec{e}_y = \frac{1}{\sqrt{2(1-\cos\theta)}} \begin{pmatrix} \cos\theta - 1 \\ \sin\theta\, e^{i\varphi} \end{pmatrix}$$

Орт-сть:

$$\langle e_x | e_y \rangle = e_1^* e_2 = \frac{\sin\theta\, e^{-i\varphi}(\cos\theta - 1) + (1-\cos\theta)\sin\theta\, e^{i\varphi}}{2(1-\cos\theta)}$$

$= 0 \Rightarrow$ орт-сть выполняется!

2.

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \alpha_1 |e_x\rangle + \alpha_2 |e_y\rangle$$

$$P_1 = |\alpha_1|^2 \qquad P_2 = |\alpha_2|^2$$

$$\alpha_1 = \langle e_x | \psi \rangle = \frac{\sin\theta\, e^{i\varphi} + 1 - \cos\theta}{\sqrt{2(1-\cos\theta)} \cdot \sqrt{2}}$$

$$P_1 = |\alpha_1|^2 = \frac{\sin^2 + 1 - 2\cos\theta + \cos^2\theta}{4(1-\cos\theta)} = \frac{2(1-\cos\theta)}{4(1-\cos\theta)} = \frac{1}{2}$$

$$\cancel{P_2} \to \alpha_2 = \langle e_y | \psi \rangle = \frac{\cos\theta - 1 + \sin\theta\, e^{-i\varphi}}{2\sqrt{1-\cos\theta}}$$

$$P_2 = |\alpha_2|^2 = \frac{1}{2}$$

Ответ: $P_1 = \frac{1}{2}$; $P_2 = \frac{1}{2}$

№2

$$U(x,y) = \frac{m\omega^2}{2}(x^2+y^2)$$

1) $$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{m\omega^2}{2}(x^2+y^2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ddot{x} \qquad \frac{\partial L}{\partial q} = -m\omega x$$

Тем самым, подставив в ур-е Лагранжа:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

Получим систему:

$$\begin{cases} \ddot{x} + \omega x = 0 \\ \ddot{y} + \omega y = 0 \end{cases} \Rightarrow \begin{array}{l} x = A_x \sin(\omega t + \varphi_x) \\ y = A_y \sin(\omega t + \varphi_y) \end{array}$$

$$\dot{x} = \omega A_x \cos(\omega t + \varphi_x)$$

$$\dot{y} = \omega A_y \cos(\omega t + \varphi_y)$$

Условие Коши:

$x(0) = a \Rightarrow a = a \sin\varphi_x \Rightarrow \varphi_x = 90^\circ$

$y(0) = 0 \Rightarrow 0 = a \sin\varphi_y \Rightarrow \varphi_y = 0^\circ$

$\dot{x}(0) = 0 \Rightarrow 0 = \omega a_x \cos\varphi_x \Rightarrow a_x \in R$

$\dot{y}(0) = V_0 \Rightarrow V_0 = \omega A_y \cos\varphi_y \Rightarrow A_y = \frac{V_0}{\omega}$

Ответ:

$$\begin{cases} x = a\cos\omega t \\ y = \frac{V_0}{m}\sin\omega t \end{cases}$$

2.2.

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{m}{2}\omega^2(x^2+y^2)$$

$$\frac{\partial L}{\partial \dot{q}} = p$$

$$\left.\begin{aligned} p_x &= m\dot{x} \\ p_y &= m\dot{y} \end{aligned}\right\} \Rightarrow \dot{x} = \frac{p_x}{m}$$

$$H = \sum p\dot{q} - L = p_x\dot{x} + p_y\dot{y} - L =$$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} - \left(\frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \frac{m}{2}\omega^2(x^2+y^2)\right) =$$

$$= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m}{2}\omega^2(x^2+y^2)$$

$$H = E = \frac{m}{2}\frac{dx^2+dy^2}{dt^2} + \frac{m}{2}\omega^2(x^2+y^2)$$

$$dt = \sqrt{\frac{m}{2}}\frac{\sqrt{dx^2+dy^2}}{\sqrt{E-\frac{m}{2}\omega^2(x^2+y^2)}}$$

Тогда импульс

$$p_x = \frac{dx}{\sqrt{dx^2+dy^2}} \cdot \sqrt{2m\left(E-\frac{m}{2}\omega^2(x^2+y^2)\right)}$$

$$p_y = \frac{dy}{\sqrt{dx^2+dy^2}} \sqrt{2m\left(E-\frac{m}{2}\omega^2(x^2+y^2)\right)}$$

$$S_0 = \sqrt{2m}\int\sqrt{E-\frac{m}{2}\omega^2(x^2+y^2)}\sqrt{dx^2+dy^2}$$

$$\underbrace{\sqrt{2m}\int\sqrt{E-\frac{m}{2}\omega^2(x^2+y^2)}\sqrt{1+\dot{y}^2}\,dx}_{L(x,y)\,dx}$$

$$\frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{\sqrt{2m}\sqrt{E-\frac{m}{2}\omega^2(x^2+y^2)}\,\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$\frac{\partial L}{\partial y} = -\frac{\sqrt{2m}\sqrt{1+\dot{y}^2}\,m\omega^2 y}{2\sqrt{E-\frac{m}{2}\omega^2(x^2+y^2)}}$$

$$\frac{d}{dx}\underbrace{\frac{\sqrt{E-\frac{m}{2}\omega^2(x^2+y^2)}\,y'}{\sqrt{1+y'^2}}}_{\varphi(y)} = -\frac{\sqrt{1+y'^2}\,m\omega^2 y}{2\sqrt{E-\frac{m}{2}\omega^2(x^2+y^2)}}$$

$$\int 2\varphi'\varphi = -\int m\omega^2 y y'$$

$$C - m\omega^2\frac{y^2}{2} = \varphi^2$$

$$\frac{y'^2\left(E-\frac{m}{2}\omega^2(x^2+y^2)\right)}{1+y'^2} = -\frac{m\omega^2 y^2}{2} + C$$

$$y'^2E - \frac{m}{2}\omega^2 x^2 y'^2 - \cancel{\frac{m}{2}\omega^2 y^2 y'^2} = -\frac{m}{2}\omega^2 y^2 \cancel{+\frac{m}{2}\omega^2 y^2} y'^2 + C$$

$$y'^2\left(E-\frac{m}{2}\omega^2x^2\right) = \frac{m}{2}\omega^2y^2 + C$$

$$\frac{dy}{dx} = 0 \quad \begin{matrix} x=a \\ y=0 \end{matrix} \Rightarrow C = 0$$

$$\sqrt{\frac{2}{m\omega^2}}\frac{dy}{y} = \frac{dx}{\sqrt{E-\frac{m}{2}\omega^2x^2}} \qquad E = \frac{m}{2}v_0^2 + \cancel{\frac{m\omega^2 a^2}{2}}$$

$$\sqrt{\frac{2}{m\omega}}\frac{dy}{y} = \frac{dx}{\sqrt{\frac{m}{2}v_0^2 + \frac{m}{2}\omega^2(a^2-x^2)}} = \frac{dx}{\sqrt{\frac{m}{2}}\sqrt{v_0^2+\omega^2(a^2-x^2)}}$$

$$\sqrt{\frac{2}{m\omega}}\,dy =$$

№3

$\hat{H} = -\mu B \sigma$

$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$ ур-е Шредингера

$\psi(t) = \psi_0 e^{\frac{\hat{H}}{i\hbar}t}$

$|\psi\rangle(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$e^{-\frac{\hat{H}t}{i\hbar}} = \sum_{n=0} -\frac{\hat{H}t}{i\hbar} = 1 - \frac{\hat{H}t}{i\hbar \cdot 1!} - \frac{\hat{H}^2 t^2}{\hbar^2 \cdot 2} + \frac{\hat{H}^3 t^3}{i\hbar^3 \cdot 3!} + \frac{\hat{H}^4 t^4}{\hbar^4 \cdot 4!} \cdot$

$= \underbrace{\sum_{n=0} (-1)\left(\frac{\hat{H}t}{\hbar}\right)^{2n} \frac{1}{(2n)!}}_{\cos\left(\hat{H}\frac{t}{\hbar}\right)} + i \underbrace{\left(\sum_{n=0} \left(\frac{\hat{H}t}{\hbar}\right)^{2n-1}\right)}_{\sin\left(\hat{H}\frac{t}{\hbar}\right)}$

$\psi(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mu B \begin{pmatrix} \cos\theta \\ \sin\theta e^{i\varphi} \end{pmatrix} \frac{ti}{\hbar} - \mu^2 B^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{t^2}{\hbar^2 2!} +$

$+ \mu^3 B^3 \begin{pmatrix} \cos\theta \\ \sin\theta e^{i\varphi} \end{pmatrix} \frac{t^3 i}{\hbar^3 \cdot 3!} + \ldots =$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cancel{\sum \frac{\left(\frac{\mu B t}{\hbar}\right)^{2n}}{(2n)!}} \sum_{n=0} \left(\frac{\mu B t}{\hbar}\right)^{2n} \frac{1}{(2n)!} - \begin{pmatrix} \cos\theta \\ \sin\theta e^{i\varphi} \end{pmatrix} i \sum_{n=0} \left(\frac{t \mu B}{\hbar}\right)^{2n-1} \frac{1}{(2n-1)!}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos\frac{\mu B t}{\hbar} - i \sin\frac{\mu B t}{\hbar} \begin{pmatrix} \cos\theta \\ \sin\theta e^{i\varphi} \end{pmatrix}$

№4

$\psi(x) = const\, x^2 e^{-\frac{x}{\lambda}}$, $x \geq 0$

1. $\langle \psi | \psi \rangle = 1$

$$\int \psi^* \psi \, dx = C^2 \int x^4 e^{-\frac{2x}{\lambda}} = \left| \begin{matrix} u = x^4 & dv = e^{-\frac{2x}{\lambda}} dx \\ du = 4x^3 dx & v = -\frac{\lambda}{2} e^{-\frac{2x}{\lambda}} \end{matrix} \right|$$

$$= C^2 \left(\cancel{-\frac{\lambda}{2} x^4 e^{-\frac{2x}{\lambda}} \Big|_0^{+\infty}} \right) + \cancel{2} C^2 \frac{\lambda}{\cancel{2}} \cdot \cancel{4} \int_0^{+\infty} x^3 e^{-\frac{2x}{\lambda}} dx =$$

$$= \left| \begin{matrix} u = x^3 & dv = e^{-\frac{2x}{\lambda}} dx \\ du = 3x^2 dx & v = -\frac{\lambda}{2} e^{-\frac{2x}{\lambda}} \end{matrix} \right| = C^2 \left(-\lambda^2 x^3 e^{-\frac{2x}{\lambda}} \Big|_0^{+\infty} \right) +$$

$$+ 3 C^2 \lambda^2 \int_0^{+\infty} x^2 e^{-\frac{2x}{\lambda}} dx = \left(\quad \right)_0 + 3 C^2 \lambda^3 \int_0^{+\infty} x e^{-\frac{2x}{\lambda}} dx =$$

$$= \left(\quad \right)_0 + \frac{3}{2} C^2 \lambda^4 \int_0^{+\infty} e^{-\frac{2x}{\lambda}} dx = -\frac{3}{2} C^2 \frac{\lambda^5}{2} e^{-\frac{2x}{\lambda}} \Big|_0^{+\infty} =$$

$$= \frac{3}{4} C^2 \lambda^5 = 1 \Rightarrow C = \sqrt{\frac{4}{3\lambda^5}}$$

2. $dP(x, x+dx) = |\langle x | \psi \rangle|^2 dx$

$dP = |\psi(x)|^2 dx$, $|\psi(x)|^2 = C^2 x^4 e^{-\frac{2x}{\lambda}}$

$P = C^2 \int_0^{+\infty} x^4 e^{-\frac{2x}{\lambda}} dx = C^2 = \frac{4}{3\lambda^5} \frac{3}{4} \lambda$

3. $\langle x \rangle = C^2 \int x \cdot |\psi(x)|^2 dx = C^2 \int x^5 e^{-\frac{2x}{\lambda}} dx =$

$$= \left| \begin{matrix} u = x^5 & dv = e^{-\frac{2x}{\lambda}} dx \\ du = 5x^4 dx & v = -\frac{\lambda}{2} e^{-\frac{2x}{\lambda}} \end{matrix} \right| = C^2 \left(\cancel{-\frac{\lambda}{2} x^5 e^{-\frac{2x}{\lambda}} \Big|_0^{+\infty}} \right) +$$

$$+ \frac{5}{2} C^2 \lambda \int_0^{+\infty} x^4 e^{-\frac{2x}{\lambda}} dx = \frac{5}{2} \cancel{C^2} \lambda \cdot \cancel{\frac{3}{4} \lambda^5} = 2{,}5 \lambda$$