ДЗ по квантовой механике

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$$I(k)=\int_{-\infty}^{+\infty}\frac{e^{-ikx}}{(x+i)^3}\,dx=\left|\begin{matrix}kx=t\\ dx=\frac{dt}{k}\end{matrix}\right|=\int_{-\infty}^{+\infty}\frac{e^{-it}}{\left(\frac{t}{k}+i\right)^3}\,\frac{dt}{k}=k^2\int_{-\infty}^{+\infty}\frac{e^{-it}}{(t+ik)^3}\,dt$$

$$=k^2\int_{-\infty}^{+\infty}\frac{e^{-iz}}{(z+ik)^3}\,dz=k^2\,2\pi i\sum_{z_i}\operatorname{res} f(z)$$

по теореме Коши о вычетах

Особые точки

1) $z_1=\infty$ — простой полюс. Не подходит к теореме, т.к. лежит на оси $\operatorname{Re} z$, т.е. не в верхней пол-сти

2) $z_2=-ik$ — полюс 3 порядка. Подходит к теореме, если $k=-|k|$, т.е. при отрицательных k

$\Downarrow$

при $k>0$ $I(k)$ ~~не~~ $=0$

при $k<0$

$$k^2\int_{-\infty}^{+\infty}\frac{e^{iz}}{(z-ik)^3}\,dz=2\pi i\sum_{z}\operatorname{res} f(z)$$

$$\operatorname{res}_{z=z_0} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{d^{m-1}z}(z-z_0)^m f(z)$$

$$\operatorname{res}_{z=ik} f(z)=\frac{1}{2}\lim_{z\to ik}\frac{d^2}{d^2z}(z-ik)^2\frac{e^{iz}}{(z-ik)^3}=$$

$$=\frac{1}{2}\lim_{z\to ik}\left(\frac{e^{iz}(z-ik)-e^{iz}}{(z-ik)^2}\right)'=\frac{1}{2}\lim_{z\to ik}\left(\frac{e^{iz}(iz+k-1)}{(z-ik)^2}\right)'$$

$$= \frac{1}{2} \lim_{z \to ik} \frac{\cancel{(ie^{iz}(iz+k-1) + ie^{iz})(z-ik)^2 - 2(z-ik)e^{iz}(iz+k-1)}}{\cancel{(z-ik)^{4}3}}$$

$$= \frac{1}{2} \lim_{z \to ik} \frac{\cancel{ie^{iz}(iz+k)(z-ik) - 2e^{iz}(iz+k-1)}}{\cancel{(z-ik)^3}} =$$

$$= \cancel{\frac{1}{2} \cdot 2 e^{-k} \cdot \infty = \infty}$$

$$\operatorname{res}_{z=ik} f(z) = \frac{1}{2} \lim_{z \to ik} \frac{d^2}{d^2 z} \frac{\cancel{(z-ik)^3} e^{iz}}{\cancel{(z-ik)^3}} = \frac{1}{2} \lim_{z \to ik} (ie^{iz})' =$$

$$= \frac{1}{2} \lim_{z \to ik} (-e^{iz}) = \left| -\frac{1}{2} e^{-k} \right| = \frac{1}{2} e^{-k}$$

тогда $k^2 \int_{-\infty}^{+\infty} f(x) = k^2 \cdot 2\pi i \frac{1}{2} e^{-k} = i\pi k^2 e^{-k}$

$\boxed{\text{ответ}}$ при $k < 0$

$$I(x) = i\pi k^2 e^{-k}$$

$$U(x) = \frac{2}{x^4+1} \qquad \text{ф-я } U(x) \text{ — четн}$$

$$U(-x) = U(x)$$

$$U(x) = \sum_n f_n e^{ik_n x}$$

$$f_n = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{e^{ik_n x}}{x^4+1} dx = \frac{2}{2} \int_{-\infty}^{+\infty} \frac{e^{ik_n x}}{x^4+1} dx = \frac{k_n^3}{2} \int_{-\infty}^{+\infty} \frac{e^{iz}}{z^4+k_n^4} dz$$

$z_1 = k\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$, $z_2 = k\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$ — подходят (полюс 1 порядка)

$z_{3,4} = k\left(\pm \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$ — не лежат ~~ниже~~ в верхней полуплоскости

$$\operatorname{res}_{z_1} f(z) = \lim_{z \to k\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)} \frac{e^{iz}}{(z-z_2)(z-z_3)(z-z_4)} = \frac{e^{ik\frac{\sqrt{2}}{2} - k\frac{\sqrt{2}}{2}}}{k^3\sqrt{2} \cdot \sqrt{2} i (\sqrt{2} + \sqrt{2}i)}$$

$$= \frac{e^{ik\frac{\sqrt{2}}{2} - k\frac{\sqrt{2}}{2}}}{2k^3\, 2\sqrt{2} k^3 (i-1)}$$

$$\operatorname{res}_{z_2} f(z) = \lim_{z \to k\left(-\frac{\sqrt2}{2}, \frac{\sqrt2}{2} i\right)} \frac{e^{iz}}{(z-z_1)(z-z_3)(z-z_4)} =$$

$$= \frac{e^{ik\frac{\sqrt2}{2} - k\frac{\sqrt2}{2}}}{2\sqrt2\,(i+1)}$$

$$V_n = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{ik_n x}}{x^4+1} = 2\pi i \left(\cancel{e^{ik\left(\frac{\sqrt2}{2} i - \frac{\sqrt2}{2}\right)\frac{\sqrt2}{2}}} \right.$$

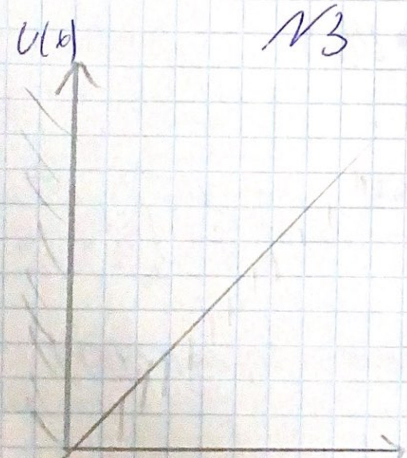
$$= \frac{2\pi i\, e^{\frac{\sqrt2}{2}k(i-1)}}{2\cdot 2\sqrt2}\left(-\frac{1}{i+1} + \frac{1}{i-1} \right) =$$

$$= \frac{2\pi\, i\cdot i\, e^{\frac{\sqrt2}{2}k(i-1)}}{2\sqrt2\,(-2)} = \frac{\pi}{2\sqrt2} e^{\frac{\sqrt2}{2}k_n(i-1)}$$

тогда $U(x)$ в Фурье

$$U(x) \sim \sum_{n=-\infty}^{\infty} \frac{\pi}{2\sqrt2} e^{\frac{\sqrt2}{2}k_n(i-1)} \cdot e^{ik_n x}$$

№3



$U(x) = \lambda x$

Ур-е Шредингера

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi(x) = E\psi(x)$$

$\psi(0) = 0$

$\hat{p} \to p$

$\hat{x} \to i\hbar \frac{\partial}{\partial p}$

$$\frac{p^2}{2m}a(p) + \alpha i\hbar\, a'(p) = E\, a(p)$$

$$\psi(x) = \frac{1}{\sqrt{L}}\int e^{\frac{ipx}{\hbar}} a(p)\frac{dp\,L}{2\pi\hbar} \to \psi(0) = \frac{1}{\sqrt{L}}\int_{-\infty}^{+\infty} a(p)\,dp\frac{L}{2\pi\hbar} = 0$$

$$\int_{-\infty}^{+\infty} a(p)\,dp = 0$$

$$a'(p) = \left(E - \frac{p^2}{2m}\right)\left(-\frac{i}{\alpha\hbar}\right) a(p)$$

$$E = \frac{k^2}{2m}$$

$$a(p) = C e^{-\frac{i}{2\alpha\hbar m}\left(pk^2 - \frac{p^3}{3}\right)}$$

$$\int_{-\infty}^{+\infty} a(p)\,dp = C\int_{-\infty}^{+\infty} e^{-\frac{i}{2\alpha\hbar m}\left(pk^2 - \frac{p^3}{3}\right)}dp = 0$$

$$p = kt$$

$$k\int e^{-\frac{k^3}{2m\alpha\hbar}i\left(t - \frac{t^3}{3}\right)}dt = 0$$

$$\int e^{iS(t)}dt$$

$$S(t) = \frac{t^3}{3} - t$$

$$S'(t) = t^2 - 1 = 0 \quad t = \pm 1$$

$$S''(t) = 2t \quad \begin{cases} 2 > 0, & t = 1 \\ -2, & t = -1 \end{cases}$$

$$S(1) = -\frac{2}{3}$$

$$S(-1) = \frac{2}{3}$$

$$\lambda = \boxed{\frac{k^3}{2m\alpha\hbar}} \gg 1$$ — безразмерна

$$= \sqrt{\frac{2\pi}{2\lambda}}\; e^{-\frac{2}{3}i\lambda + i\frac{\pi}{4}}\Big|_{t=1} + e^{\frac{2}{3}i\lambda + i\frac{\pi}{4}}$$

$$\sqrt{\frac{2\pi}{2\lambda}}\; 2\cos\left(\frac{2}{3}\lambda - \frac{\pi}{4}\right)$$

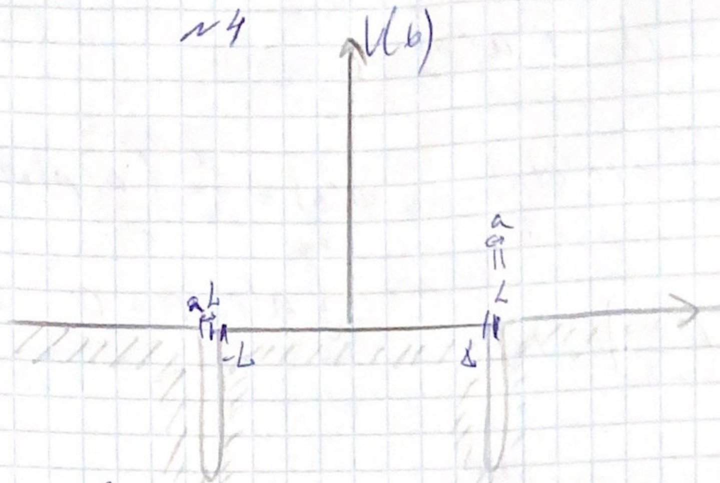
$$\frac{2}{3}\lambda - \frac{\pi}{4} = -\frac{\pi}{2} + \pi n$$

$$\lambda = \frac{3}{2}\left(\pi n - \frac{\pi}{4}\right) = \frac{3}{2}\pi\left(n - \frac{1}{4}\right)$$

$$k = \left\{3m\alpha\hbar\,\pi\left(n - \frac{1}{4}\right)\right\}^{\frac{1}{3}}$$

$$E = \frac{k^2}{2m} = \frac{\left\{3m\alpha\hbar\pi\left(n - \frac{1}{4}\right)\right\}^{\frac{2}{3}}}{2m}$$

№4



$$U(x) = -\frac{\hbar^2 k}{m}\delta(x-L) - \frac{\hbar^2}{m}\delta(x+L)$$

ур-е Шредингера:

$$-\frac{\hbar^2}{2m}\psi''(x) + U(x)\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{\hbar^2}{m}k\left(\delta(x+L)+\delta(x-L)\right)\psi(x) = E\psi(x)$$

$$\psi'' = -\underbrace{\left(\frac{2mE}{\hbar^2} + 2k\left(\delta(x+L)+\delta(x-L)\right)\right)}_{\varkappa^2}\psi$$

$$\psi = A\cos\varkappa x + B\sin\varkappa x$$

для четн $\cos\varkappa x$ $\psi_s = A\cos\varkappa x$

$kL = 1$

$$\psi(L+a) = \psi(L-a) = \psi(-L-a) = \psi(-L+a) = 0$$

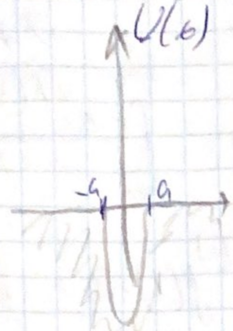
$$\psi(L+a) = A\cos\left(\sqrt{\frac{2Em}{\hbar^2} + 2k\cdot 0}\,(L+a)\right) = 0$$

$$\sqrt{\frac{2Em}{\hbar^2}}\,(L+a) = \frac{\pi}{2} + \pi n$$

$$\frac{2Em}{\hbar^2}L^2 = \frac{\pi^2}{4}(1+2n)^2$$

$$E = \frac{\pi^2\hbar^2}{8mL^2}(1+2n)^2$$

№6



$$U(x) = -\frac{\hbar^2}{m}\kappa\,\delta(x)$$

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E(x)\psi$$

$$\psi'' = -\underbrace{\left(\frac{2mE}{\hbar^2} + 2\kappa\delta(x)\right)}_{\gamma^2}\psi$$

$$\psi = A\cos\gamma x + B\sin\gamma x$$

для чет. $\psi = A\cos\gamma x$

$$\psi(-a) = \psi(a) = 0$$

$$\sqrt{\frac{2mE}{\hbar^2} + 2\kappa\underset{0}{\underset{\parallel}{\delta(a)}}}\;a = \frac{\pi}{2} + \pi n$$

$$\frac{2mE}{\hbar^2}a^2 = \frac{\pi^2}{4}(1+2n)^2$$

$$E_{чет} = \frac{\pi^2\hbar^2}{8ma^2}(1+2n)^2$$

для нечет.

$$\psi = A\sin\gamma x$$

$$\sqrt{\frac{2mE}{\hbar^2} + 2\kappa\delta(a)}\;a = \pi n$$

$$\frac{2mE}{\hbar^2}a^2 = \pi^2 n^2$$

$$E_{нечет} = \frac{\pi^2\hbar^2}{2ma^2}n^2$$

№7

$\lambda \to \infty$

$$I(\lambda) = \int_{-\infty}^{+\infty} \frac{e^{i\lambda\left(x+\frac{x^4}{4}\right)}}{x^2+1}\,dx$$

$$\int e^{i\lambda S(x)} f(x)\,dx = \begin{cases} ① \sqrt{\frac{2\pi}{\lambda |S''(x_0)|}}\, f(x_0)\, e^{i\lambda S(x_0) + i\frac{\pi}{4}}, & S''(x_0) > 0 \\ ② \sqrt{\frac{2\pi}{\lambda |S''(x_0)|}}\, f(x_0)\, e^{i\lambda S(x_0) - i\frac{\pi}{4}}, & S''(x_0) < 0 \end{cases}$$

$S(x) = x + \frac{x^4}{4} \to S'(x) = 1 + x^3$

$1 + x^3 = 0 \Rightarrow x_0 = -1$

$S''(x) = 3x^2$; $S''(-1) = 3 \Rightarrow$ решение ①

$f(x) = \frac{1}{x^2+1}$; $f(-1) = \frac{1}{2}$

$\Downarrow$

$$I(\lambda) = \frac{1}{2}\sqrt{\frac{2\pi}{\lambda \cdot 3}}\, e^{i\left(\lambda \cdot 3 + \frac{\pi}{4}\right)}$$

Ответ: $I(\lambda) = \sqrt{\frac{\pi}{6\lambda}}\, e^{i\left(3\lambda + \frac{\pi}{4}\right)}$

№8

$$I(p) = \int_0^{+\infty} e^{-pt} t^{z-1}\,dt = \begin{vmatrix} pt = r \\ dt = \frac{dr}{p} \end{vmatrix} = \int_0^{+\infty} e^{-r}\left(\frac{r}{p}\right)^{z-1}\frac{dr}{p}$$

$$= p^{1-z}\cdot p \int_0^{+\infty} e^{-r} r^{z-1}\,dr = p^{-z}\int_0^{+\infty} e^{-r} r^{z-1}\,dr = p^{-z}\Gamma(z)$$

где $\Gamma(z) = \int_0^{+\infty} e^{-r} r^{z-1}\,dr$

Рассмотрим при $z = 3$

$$\int_0^{+\infty} e^{-r} r^2\,dr = \begin{vmatrix} u = r^2 & dv = e^{-r}dr \\ du = 2r\,dr & v = -e^{-r} \end{vmatrix} =$$

$$= \cancel{-e^{-r} r^2 \Big|_0^{+\infty}} + 2\int_0^{+\infty} r e^{-r}\,dr = \cancel{-2 e^{-r} r \Big|_0^{+\infty}} + 2\int_0^{+\infty} e^{-r}\,dr$$

$$= -2\, e^{-t}\Big|_0^{+\infty} = -2\left(\lim_{b\to+\infty}\frac{1}{e^b} - \lim_{a\to 0}\frac{1}{e^a}\right) = 2$$

Из этого следует, что $\Gamma(z) = (z-1)!$

и $I(p) = p^{-z}\,\Gamma(z) = p^{-z}(z-1)!$