Nonuniformly magnetized ferrormagnetic disk. A thin ferrormagnetic disk of radius a and thickness d ($d \ll a$) in air has a nonuniform magnetization, given by $\mathbf{M} = M_o(r/a)^2\mathbf{a}_z$ (Fig. Q5.37), where M_o is a constant.

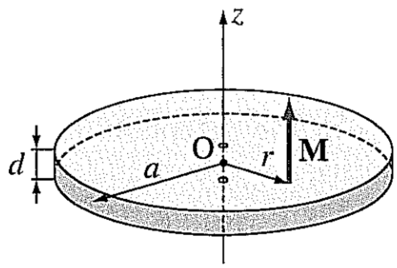


Figure 5.37 Nonuniformly magnetized thin ferromagnetic disk; for Problem 5.5.

Calculate

(a) the distribution of magnetization currents of the disk, and

(b) the magnetic flux density vector along the z-axis.

Solution:

(a) We know the volume magnetization current density is given by

$$\mathbf{J}_m = \nabla \times \mathbf{M}.$$

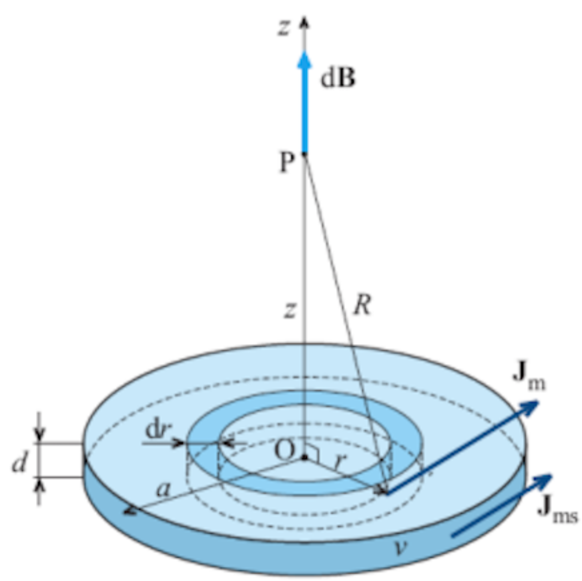
Our problem has cylindrical symmetry so using that result in cylindrical coordinates we arrive at

$$\mathbf{J}_m = -\frac{\partial M_z(r)}{\partial r}\mathbf{a}_\phi = -\frac{2M_0 r}{a^2}\mathbf{a}_\phi.$$

There is also a magnetization current density flowing circumferentially along the disk outside surface given by

$$\mathbf{J}_{ms} = M(a^-)\mathbf{a}_z \times \mathbf{a}_r = M_0\mathbf{a}_r.$$

This configuration is shown in the diagram below.



(b) To evaluate the magnetic field we subdivide the disk in Fig 5.37 into elementary hollow disks of radii r ($0 \le r \le a$), width dr, and thickness (height) d, as shown in the diagram above. As $d << a$, each such disk can be replaced by an equivalent current loop (wire) of radius r and current intensity

$$\mathrm{d}I_m(r) = J_m(r)d\mathrm{d}r$$

(cross section of the hollow disk through which the current of density $\mathbf{J}_m$ flows is a rectangle of side lengths d and $\mathrm{d}r$, and surface area $d\ \mathrm{d}r$). The magnetic flux density vector for this loop at an arbitrary point on the z-axis is given by

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 \mathrm{d}I_m(r) r^2}{2R^3}\mathbf{a}_z = -\frac{\mu_0 M_0 d r^3 \mathrm{d}r}{a^2 R^3}\mathbf{a}_z; \quad R = \sqrt{r^2 + z^2},$$

and the resultant field $\mathbf{B}$ is found by the superposition of $\mathrm{d}\mathbf{B}$ to sum the contributions of all equivalent loops over the volume of the thin disk in the diagram above. This is done by integrating the expression above.

$$\begin{aligned}
\mathrm{d}\mathbf{B} &= -\frac{\mu_0 M_0 d r^3 \mathrm{d}r}{a^2 R^3}\mathbf{a}_z; \quad R\mathrm{d}R = r\mathrm{d}r \\
\mathbf{B} &= -\frac{\mu_0 M_0 d}{a^2}\int_{r=0}^{a} r^2 \frac{dR}{R^2}\mathbf{a}_z \\
&= -\frac{\mu_0 M_0 d}{a^2}\int_{r=0}^{a} (R^2 - z^2)\frac{dR}{R^2}\mathbf{a}_z \\
&= -\frac{\mu_0 M_0 d}{a^2}\left(\int_{r=0}^{a} dR - z^2\int_{r=0}^{a}\frac{dR}{R^2}\right)\mathbf{a}_z \\
&= -\frac{\mu_0 M_0 d}{a^2}\left(\sqrt{a^2+z^2} - |z| + \frac{z^2}{\sqrt{a^2+z^2}} - \frac{z^2}{|z|}\right)\mathbf{a}_z
\end{aligned}$$

Finally, we add the $\mathbf{B}$ field due to the equivalent circular loop with radius a and current $I_m(a) = \mathbf{J}_{ms}d = M_0 d$, representing the surface current in part (a) flowing circumferentially. This field is given by

$$\mathbf{B} = \frac{\mu_0 M_0 d a^2}{2(z^2+a^2)^{3/2}}\mathbf{a}_z.$$

Adding them together we get,

$$\mathbf{B}_{\text{tot}} = -\frac{\mu_0 M_0 d}{a^2}\left[\sqrt{a^2+z^2} - |z| + \frac{z^2}{\sqrt{a^2+z^2}} - \frac{z^2}{|z|} - \frac{a^4}{2(z^2+a^2)^{3/2}}\right]\mathbf{a}_z.$$

Answer:

(a) Volume magnetization current density $\mathbf{J}_m = -\frac{2M_0 r}{a^2}\mathbf{a}_\phi$ and surface magnetization current density flowing circumferentially $\mathbf{J}_{ms} = M_0\mathbf{a}_r$.

(b) $\mathbf{B}_{\text{tot}} = -\frac{\mu_0 M_0 d}{a^2}\left[\sqrt{a^2+z^2} - |z| + \frac{z^2}{\sqrt{a^2+z^2}} - \frac{z^2}{|z|} - \frac{a^4}{2(z^2+a^2)^{3/2}}\right]\mathbf{a}_z$