Goal: Find the resistance between the surfaces R_1 and R_2 of a truncated conical block defined by $R_1 \leq R \leq R_2$ and $0 \leq \theta \leq \theta_0$. The two spherical surfaces ($R = R_1$ and $R = R_2$) are perfect electric conductors (PECs), while the rest of the block has conductivity σ. You can neglect edge effects.

Steps:

1. Choose a coordinate system.
   Solution: Spherical

2. Assume potential V_0 on the one conductor and zero on the other. From Laplace equation, find V and then E.

   Note: Alternatively, you can assume a charge Q on the inner conductor and use Gauss' law to find the field it creates
   Solution: Solving Laplace's equation assuming scalar potential of V_0 on inner surface and 0 V on outer surface we get

$$\nabla^2 V = 0$$
$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) = 0$$

   The solution to this equation is

$$V = -\frac{c_1}{R} + c_2$$

   Applying the boundary conditions ($V(R_1) = V_0$ and $V(R_2) = 0$) gives:

$$c_1 = \frac{V_0}{\frac{1}{R_2} - \frac{1}{R_1}}$$
$$c_2 = \frac{1}{R_2}\frac{V_0}{\frac{1}{R_2} - \frac{1}{R_1}}$$

   Solution: From scalar potential,

$$\begin{aligned}\mathbf{E} &= -\nabla V \\ &= \left(\frac{V_0}{\frac{1}{R_1} - \frac{1}{R_2}}\right)\frac{1}{R^2}\mathbf{a}_R\,. \end{aligned} \tag{1}$$

   Alternatively, on a Gaussian surface of a cone with $0 \leq \theta \leq \theta_0$ and radius R we apply Gauss' law:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\theta_0} \mathbf{E}\cdot\mathbf{a}_R R^2 \sin\theta\, d\theta\, d\phi &= \frac{Q}{\varepsilon} \\ 2\pi R^2 E_R(1-\cos\theta) &= \frac{Q}{\varepsilon} \\ E_R &= \frac{Q}{2\pi\varepsilon R^2(1-\cos\theta_0)}\,. \end{aligned} \tag{2}$$

The potential is given by:

$$V_0 = -\int_{R_2}^{R_1} \frac{Q}{2\pi\varepsilon R^2(1-\cos\theta_0)} dR$$
$$= \frac{Q}{2\pi\varepsilon(1-\cos\theta_0)}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \tag{3}$$

Note: the electric field obtained from the Laplace equation and Gauss' law are equivalent.

3. Having $\mathbf{E}$, find $\mathbf{J}$. Can you find the total current I that this $\mathbf{J}$ creates? Choose the surface that you need to use to apply the formula: $I = \iint_S \mathbf{J}\cdot\mathbf{dS}$.
   Solution:

$$\mathbf{J} = \sigma\mathbf{E}$$
$$J_R = \sigma\left(\frac{V_0}{\frac{1}{R_1}-\frac{1}{R_2}}\right)\frac{1}{R^2}$$

Using conic surface, we can compute the total current I:

$$I = \int_0^{2\pi}\int_0^{\theta_0} J_R R^2 \sin\theta \, d\theta \, d\phi$$
$$= \sigma\left(\frac{V_0}{\frac{1}{R_1}-\frac{1}{R_2}}\right) 2\pi(1-\cos\theta_0)$$

4. Having $\mathbf{E}$, find the voltage between the conductors. Then, $R = V/I$. *Solution:*

$$R = \frac{V_0}{I}$$
$$= \frac{\frac{1}{R_1}-\frac{1}{R_2}}{2\pi\sigma(1-\cos\theta_0)}$$

Answer:

$$R = \frac{\frac{1}{R_1}-\frac{1}{R_2}}{2\pi\sigma(1-\cos\theta_0)}$$