[Cheng P.4-6] Assume that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density of charge $\rho = A/r$ for $a < r < b$, where a and b are, the radii of the inner and outer conductors, respectively. The inner conductor is maintained at a potential V_0, and the outer conductor is grounded. Determine the potential distribution in the region $a < r < b$ by solving Poisson's equation.

Solution: Poisson's equation

$$\nabla^2 V = \frac{A}{\varepsilon r} \to \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \frac{A}{\varepsilon r}.$$

Integrating yields the solution

$$V = -\frac{A}{\varepsilon} r + c_1 \ln r + c_2.$$

Using the boundary conditions we can solve for the constants.

$$\begin{aligned} r = a: &\qquad V_0 = -\frac{A}{\varepsilon} a + c_1 \ln a + c_2 \\ r = b: &\qquad 0 = -\frac{A}{\varepsilon} b + c_1 \ln b + c_2. \end{aligned}$$

Solving these yields

$$\begin{aligned} c_1 &= \frac{\frac{A}{\varepsilon}(b-a) - V_0}{\ln(\frac{b}{a})}, \\ c_2 &= \frac{\frac{A}{\varepsilon}(a \ln b - b \ln a) + V_0 \ln b}{\ln(\frac{b}{a})}. \end{aligned}$$

Answer:

$$\begin{aligned} V_0 &= -\frac{A}{\varepsilon} r + c_1 \ln r + c_2, \\ c_1 &= \frac{\frac{A}{\varepsilon}(b-a) - V_0}{\ln(\frac{b}{a})}, \\ c_2 &= \frac{\frac{A}{\varepsilon}(a \ln b - b \ln a) + V_0 \ln b}{\ln(\frac{b}{a})}. \end{aligned}$$