Goal: Assume that the outer conductor of the cylindrical capacitor in Example 3-18 (page 124 Cheng) is grounded and that the inner conductor is maintained at a potential V_o. Find the value of a that minimizes the electric field field at the surface of the inner conductor $\mathbf{E}(a)$. Then find the electric field at the surface of the inner conductor $\mathbf{E}(a)$ and the capacitance for this value of a.

Steps:

1. For the capacitor in Example 3-18, obtain an expression for the electric field between the inner and outer conductors.

   Solution:

   $$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{Q}{2\pi L r}$$

2. Determine an expression for the voltage between the inner conductor and the outer conductor.

   Solution:

   $$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = \frac{Q}{2\pi\varepsilon L} \ln\frac{b}{a}$$

3. Obtain an expression for the capacitance of this capacitor.

   Solution:

   $$C = \frac{Q}{V_{ab}} = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

4. Find the electric field at $r = a$, at the surface of the inner conductor.

   Solution:

   $$E(a) = \frac{V_o}{a\ln(b/a)}$$

5. With the inner radius, b, of the outer conductor fixed, find a so that $\mathbf{E}(a)$ is minimized.

   Solution: We maximum the expression $a\ln(b/a)$ by taking its derivative with respect to a and setting it to zero, which yields

   $$\begin{aligned} \ln\left(\frac{b}{a}\right) &= 1 \\ a &= \frac{b}{e} \end{aligned}$$

6. Determine the capacitance per unit length under the conditions of the previous question.

   Solution:

   $$C' = 2\pi\varepsilon \text{ (C/m)}$$

Answer:

$$a = \frac{b}{e}$$

$$\mathbf{E}(a) = \frac{V_o}{a\ln(\frac{b}{a})}$$

$$C^{'} = 2\pi\varepsilon \text{ C/m}$$