Goal: Lightning strikes a lossy dielectric sphere $\epsilon = 1.2\epsilon_0$, $\sigma = 10$ S/m, of radius 0.1 m, at time $t = 0$, uniformly depositing charge 1 mC. For all t, determine the electric field intensity and current density inside and outside the sphere, and the time it takes for the charge density to diminish to 1% of its initial value. Also find the electrostatic energy stored in the space outside the sphere. Does this energy change with time?

Steps:

1. Apply Gauss law to find the electric field. What surface are you going to choose ? What symmetries exist in this problem ?
   Solution: Volume charge density is

$$\begin{aligned}\rho_0 &= Q/(4/3\pi R^3)\\ &= 0.239\ (\text{C/m}^3)\end{aligned}$$

   Use spherical Gaussian surface to find electric field. Electric field inside the sphere ($R < 0.1$ m)

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho(t)/\varepsilon\\ \int\!\!\int \mathbf{E} \cdot d\mathbf{S} &= (4/3)\pi R^3 \rho(t)/\varepsilon\\ E_r &= \frac{R}{3\varepsilon}\rho(t)\,.\end{aligned}$$

   But, $\rho(t) = \rho_0\, \mathrm{e}^{-\sigma/\varepsilon}$. Hence,

$$\begin{aligned}E_r &= \frac{R}{3\varepsilon}\rho_0\, \mathrm{e}^{-\sigma/\varepsilon t}\,,\\ &= 7.498 \cdot 10^9 R\, \mathrm{e}^{-9.41\cdot 10^{11} t}\end{aligned}$$

   Electric field outside the sphere ($R > 0.1$ m):

$$\begin{aligned}\int\!\!\int \mathbf{E} \cdot d\mathbf{S} &= Q/\varepsilon\\ E_r &= \frac{1}{4\pi\varepsilon R^2}Q\,,\\ &= \frac{8.99 \cdot 10^6}{R^2}\,.\end{aligned}$$

2. Having $\mathbf{E}$, find $\mathbf{J}$ inside the sphere ($R < 0.1$ m)
   Solution:

$$\begin{aligned}\mathbf{J} &= \sigma\mathbf{E}\\ &= \sigma\frac{R}{3\varepsilon}\rho_0\, \mathrm{e}^{-\sigma/\varepsilon t}\mathbf{a}_R\,,\\ &= 7.498 \cdot 10^{10} R\, \mathrm{e}^{-9.41\cdot 10^{11} t}\mathbf{a}_R\,.\end{aligned}$$

   Outside the sphere ($R > 0.1$ m)

$$\mathbf{J} = 0\,.$$

3. Apply continuity equation to find ρ from $\mathbf{J}$. How is this decay of the charge density compatible with charge conservation ?
Solution:

$$\begin{aligned}\nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \\ \frac{1}{R^2}\left(\frac{\partial}{\partial R}\left(7.498 \cdot 10^{10} R^3 \mathrm{e}^{-9.41 \cdot 10^{11} t}\right)\right) &= -\frac{\partial \rho}{\partial t} \\ 0.239 \mathrm{e}^{-9.41 \cdot 10^{11} t} \text{ (C/m)} &= \rho(t)\,.\end{aligned}$$

4. Calculate the electrostatic energy inside and outside the sphere as a function of time. First, find $w_e = \frac{1}{2}\epsilon|\mathbf{E}|^2$ inside and outside the sphere.
Solution: Inside the sphere $R < 0.1\,\mathrm{m}$

$$w_e = \frac{1}{2}\varepsilon(7.498 \cdot 10^9)^2 \mathrm{e}^{(-2)(9.41 \cdot 10^{11} t)} R^2\,.$$

Outside the sphere $R > 0.1\,\mathrm{m}$

$$w_e = \frac{1}{2}\varepsilon_0\left(\frac{8.99 \cdot 10^6}{R^2}\right)^2$$

5. Then, integrate the two w_e's in their respective volumes. Does each energy change with time ? Does (or can) the total energy change with time?
Solution: Energy inside the sphere:

$$\begin{aligned}W_e &= \int_0^{2\pi}\int_0^{\pi}\int_0^{0.1} \frac{1}{2}\varepsilon(7.498 \cdot 10^9)^2 \mathrm{e}^{(-2)(9.41 \cdot 10^{11} t)} R^4 \sin\theta dR d\theta d\phi \\ &= 7.506 \mathrm{e}^{(-2)(9.41 \cdot 10^{11} t)} \text{ (kJ)}\,.\end{aligned}$$

Energy outside the sphere:

$$\begin{aligned}W_e &= \int_0^{2\pi}\int_0^{\pi}\int_{0.1}^{\infty} \frac{1}{2}\varepsilon_0\left(\frac{8.99 \cdot 10^6}{R^2}\right)^2 R^2 \sin\theta dR d\theta d\phi \\ &= 45.06 \text{ (kJ)}\end{aligned}$$

Energy outside the sphere is constant. Energy inside the sphere changes with time (this energy is lost as heat).

Answer:

(a)

$$\begin{aligned}E_{\text{inside}} &= 7.498 \times 10^9 R\, e^{-9.41 \times 10^{11} t} \\ E_{\text{outside}} &= \frac{8.99 \times 10^6}{R^2}\end{aligned}$$

(b)

$$J_{\text{inside}} = a_R\ 7.498\ \times\ 10^{10} R\, e^{-9.41\,\times\,10^{11} t}$$
$$J_{\text{outside}} = 0$$

(c)

$$W_e = 45.06\, kJ$$