[Cheng P.4-2] Prove that the scalar potential V in Eq. (3-61)

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|} \, dv'$$

satisfies Poisson's equation, Eq. (4-6)

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

Solution: We need to prove that

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|} \, dv'$$

satisfies

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

Substituting the latter in the Poisson equation, and keeping in mind that the ∇^2 operator applies to observation coordinates only, and therefore can be taken into the integral, we get

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho(\mathbf{R}') \, \nabla^2 \left(\frac{1}{|\mathbf{R} - \mathbf{R}'|} \right) dv'.$$

Since $\nabla^2(1/R) = 0$ for $R \neq 0$ but its volume integral over all space is -4π, we can infer that

$$\nabla^2 \left(\frac{1}{|\mathbf{R} - \mathbf{R}'|} \right) = -4\pi\delta(\mathbf{R} - \mathbf{R}')$$

so that

$$\begin{aligned} \nabla^2 V &= \frac{1}{4\pi\epsilon_0} \int_{V'} \rho(\mathbf{R}') \, (-4\pi)\delta(\mathbf{R} - \mathbf{R}') \, dv' \\ &= -\frac{1}{\epsilon_0} \int_{V'} \rho(\mathbf{R}') \, \delta(\mathbf{R} - \mathbf{R}') \, dv'. \end{aligned}$$

By the sifting property of the δ function, the integrand is nonzero only when $\mathbf{R} = \mathbf{R}'$, and we can write

$$\nabla^2 V = -\frac{\rho(\mathbf{R})}{\epsilon_0},$$

which is the desired result.
Answer: Proof Problem