Goal: A ferromagnetic sphere of radius b is magnetized uniformly with magnetization $\mathbf{M} = \mathbf{a}_z M_0$. Find the current densities $\mathbf{J}_m$, $\mathbf{J}_{ms}$, and flux density $\mathbf{B}$.

Steps:

1. Show that the equivalent magnetization volume current density $\mathbf{J}_m$ is 0.
   Solution:

$$\begin{aligned} \mathbf{J}_m &= \nabla \times \mathbf{M} \\ &= \mathbf{0}\,. \end{aligned}$$

2. Determine the equivalent surface current density $\mathbf{J}_{ms}$ over the sphere.
   Solution:

$$\begin{aligned} \mathbf{J}_{\mathbf{m,s}} &= \mathbf{M} \times \mathbf{a}_n \\ &= M_0 \mathbf{a}_z \times \mathbf{a}'_R \\ &= M_0 \left(\mathbf{a}'_R \cos\theta' - \mathbf{a}'_\theta \sin\theta'\right) \times \mathbf{a}'_R \\ &= M_0 \sin\theta' \mathbf{a}'_\phi \end{aligned}$$

3. Next, apply Biot-Savart law to determine the magnetic flux density $\mathbf{B}$ due to the $\mathbf{J}_{ms}$ at the center of the sphere.

   a) What is the observation position vector $\mathbf{R}$?
      Solution:

$$\mathbf{R} = 0\,.$$

   b) What is the source position vector $\mathbf{R}'$?
      Solution:

$$\begin{aligned} \mathbf{R}' &= b\,\mathbf{a}'_R \\ &= b\left(\sin\theta' \cos\phi' \mathbf{a}_x + \sin\theta' \sin\phi' \mathbf{a}_y\right)\,. \end{aligned}$$

   c) What is the differential magnetic flux density $\mathrm{d}\mathbf{B}$?

Solution:

$$
\begin{aligned}
d\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{\mathbf{J}dS \times (\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^3} \\
&= \frac{\mu_0}{4\pi}\frac{\mathbf{J}_{m,s} b^2 \sin\theta' d\theta' d\phi' \times (-b\mathbf{a}_R)}{b^3} \\
&= \frac{\mu_0}{4\pi}\frac{M_0 b^2 \sin^2\theta' d\theta' d\phi' \mathbf{a}'_\phi \times (-b\mathbf{a}'_R)}{b^3} \\
&= \frac{\mu_0}{4\pi} M_0 \sin^2\theta' d\theta' d\phi' \mathbf{a}'_\phi \times \left(-\mathbf{a}'_R\right) \\
&= \frac{\mu_0}{4\pi} M_0 \sin^2\theta' d\theta' d\phi' \left(\sin\theta' \mathbf{a}_z - \cos\phi' \cos\theta' \mathbf{a}_x - \sin\phi' \cos\theta' \mathbf{a}_y\right)
\end{aligned}
$$

d) Integrate.

Solution:

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} M_0 \int_0^\pi \int_0^{2\pi} \sin^2\theta' \left(\sin\theta' \mathbf{a}_z - \cos\phi' \cos\theta' \mathbf{a}_x - \sin\phi' \cos\theta' \mathbf{a}_y\right) d\phi' d\theta' \\
&= \frac{\mu_0}{2} M_0 \int_0^\pi \sin^3\theta' \mathbf{a}_z d\theta' \\
&= \frac{2\mu_0}{3} M_0 \mathbf{a}_z
\end{aligned}
$$

Answer:

$$
\begin{aligned}
J_m &= 0 \\
J_{m,s} &= a_\phi^{'} \, M_o \, \sin\theta^{'} \\
\mathbf{B} &= a_z \frac{2\mu_o}{3} M_o
\end{aligned}
$$