[Cheng P.3-37] A capacitor consists of two concentric spherical shells of radii R_i and R_o. The space between them is filled with a dielectric of relative permittivity ε_r from R_i to b $(R_i < b < R_o)$ and another dielectric or relative permittivity $2\varepsilon_r$ from b to R_o.

(a) Determine $\mathbf{E}$ and $\mathbf{D}$ everywhere in terms of an applied voltage V.

(b) Determine the capacitance.

Solution: Assume charge Q on inner shell and $-Q$ on outer shell. For $R_i < R < R_o$:

$$\mathbf{D} = \mathbf{u}_r \frac{Q}{4\pi R^2},$$

for $R_i < R < b$:

$$\mathbf{E}_1 = \frac{\mathbf{D}}{\varepsilon_0 \varepsilon_r},$$

for $b < R < R_0$:

$$\mathbf{E}_2 = \frac{\mathbf{D}}{2\varepsilon_0 \varepsilon_r}.$$

Potential:

$$V = -\int_{R_0}^{R_1} \mathbf{E} \cdot d\mathbf{R} = -\int_{b}^{R_i} E_1 \, dR - \int_{R_0}^{b} E_2 \, dR = \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}\right).$$

a)

$$\mathbf{D} = \mathbf{a}_R \frac{\varepsilon_0 \varepsilon_r V}{R^2\left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}\right)} \quad , R_i < R_0.$$

$$\mathbf{D} = 0 \quad \mathbf{E} = 0 \quad \text{for } R < R_i \text{ and } R > R_0$$

$$\mathbf{E}_1 = \mathbf{u}_R \frac{V}{R^2\left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}\right)}$$

$$\mathbf{E}_2 = \mathbf{u}_R \frac{V}{R^2\left(\frac{2}{R_i} - \frac{1}{b} - \frac{1}{R_0}\right)}$$

b)

$$C = \frac{4\pi\varepsilon_0\varepsilon_r}{\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}}$$

Answer:

(a)

$$\mathbf{D} = \mathbf{a}_R \frac{\varepsilon_0 \varepsilon_r V}{R^2\left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}\right)} \quad , R_i < R_0.$$

$$\mathbf{D} = 0 \quad \mathbf{E} = 0 \quad \text{for } R < R_i \text{ and } R > R_0$$

$$\mathbf{E}_1 = \mathbf{u}_R \frac{V}{R^2\left(\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}\right)}$$

$$\mathbf{E}_2 = \mathbf{u}_R \frac{V}{R^2\left(\frac{2}{R_i} - \frac{1}{b} - \frac{1}{R_0}\right)}$$

b)

$$C = \frac{4\pi\varepsilon_0\varepsilon_r}{\frac{1}{R_i} - \frac{1}{2b} - \frac{1}{2R_0}}$$