Goal: *Infinite cylinder with circular magnetization.* An infinitely-long ferromagnetic cylinder of radius a in air has a nonuniform magnetization. In a cylindrical coordinate system whose z-axis coincides with the cylinder axis, $\mathbf{M} = M_o(r/a)\mathbf{a}_\phi$ $(0 \le r \le a)$, where M_o is a constant. Find the current densities $\mathbf{J}_m$ in the cylinder and $\mathbf{J}_{ms}$ on the cylinder, as well as the magnetic flux density $\mathbf{B}$ inside and outside the cylinder.

Steps:

1. Find the volume magnetization current density vector in the cylinder.
   Solution: Use the formula
   $$\mathbf{J}_m = \nabla \times \mathbf{M} = \frac{1}{r}\frac{\partial}{\partial r}\left(rM_\phi(z)\right)\mathbf{a}_z = \frac{2M_o}{a}\mathbf{a}_z$$

2. Find the surface magnetization current density on the cylinder surface.
   Solution: Use the formula
   $$\mathbf{J}_{ms} = M_\phi(a^-)\mathbf{a}_\phi \times \mathbf{a}_r = -M_o\mathbf{a}_z.$$

3. Find the magnetic flux density vector in the cylinder.
   Solution: Because of symmetry, the $\mathbf{B}$-field in the cylinder (due to its magnetization currents assume to reside in a vaccum) is circular (magnetic-field lines are circles centered at the cylinder axis). To find the $\mathbf{B}$-field, we apply Ampère's law as if the magnetization currents found in (a) and (b) were conduction currents in a nonmagnetic medium, to the circular contour C of radius r, to give
   $$B2\pi r = \mu_o J_m \pi r^2 \rightarrow \mathbf{B} = \frac{\mu_o J_m r}{2}\mathbf{a}\phi = \frac{\mu_o M_o r}{a}\mathbf{a}_\phi = \mu_o\mathbf{M}, \quad (0 \le r \le a)$$

4. Find the magnetic flux density vector outside the cylinder.
   Solution: For the observation point outside the cylinder, the right-hand side of Ampère's law includes the surface magnetization current density $\mathbf{J}_{ms}$ as well, and this current amounts to J_{ms} times the circumference of the cylinder. Hence, $\mathbf{B}$ outside the ferromagnetic cylinder is computed as
   $$\begin{aligned} B2\pi r &= \mu_o(J_m\pi a^2 + J_{ms}2\pi a) \\ B &= \frac{\mu_o a}{r}\left(\frac{J_m a}{2} + J_{ms}\right) \\ &= \frac{\mu_o a}{r}(M_o - M_o) = 0, \quad (a < r < \infty) \end{aligned}$$

Answer:

$$\mathbf{J}_m = \frac{2M_o}{a}\mathbf{a}_z$$
$$\mathbf{J}_{ms} = -M_o\mathbf{a}_z$$
$$\mathbf{B} = \mu_o\mathbf{M}, \quad (0 \le r \le a)$$
$$\mathbf{B} = 0, \quad (a < r < \infty)$$