[Cheng P.3-44] A parallel-plate capacitor of width w, length L, and separation d is partially filled with a dielectric medium of dielectric constant ε_r as shown in Fig. 3-43. A battery of V_0 volts is connected between the plates.

(a) Find $\mathbf{D}$, $\mathbf{E}$, and ρ_s in each region.

(b) Find the distance x such that the electrostatic energy stored in each region is the same.

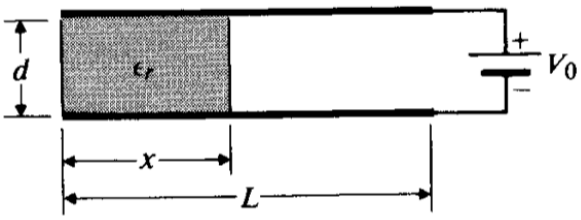

FIGURE 3–43
A parallel-plate capacitor (Problem P.3–44).

Solution: Region 1 - dielectric; region 2 – air

(a)

$$\mathbf{E}_1 = -\mathbf{a}_y \frac{V_0}{d},$$
$$\mathbf{D}_1 = -\mathbf{a}_y \varepsilon_0 \varepsilon_r \frac{V_0}{d}$$
$$\rho_{s1} = \varepsilon_0 \varepsilon_r \frac{V_0}{d}, \text{ (top plate).}$$
$$\mathbf{E}_2 = -\mathbf{a}_y \frac{V_0}{d},$$
$$\mathbf{D}_2 = -\mathbf{a}_y \varepsilon_0 \frac{V_0}{d}$$
$$\rho_{s2} = \varepsilon_0 \frac{V_0}{d}, \text{ (top plate).}$$

(b) We must now try to equate the stored energies in each region to solve for x.

$$W_{e1} = W_{e2}$$
$$\frac{1}{2} C_1 V_0^2 = \frac{1}{2} C_2 V_0^2$$
$$\frac{1}{2} \frac{Q_1}{V_0} V_0^2 = \frac{1}{2} \frac{Q_2}{V_0} V_0^2$$
$$\frac{1}{2} \frac{\rho_{s,1} x}{V_0} V_0^2 = \frac{1}{2} \frac{\rho_{s,2}(L-x)}{V_0} V_0^2$$
$$\frac{1}{2} \frac{\varepsilon_0 \varepsilon_1 V_0}{d} x V_0 = \frac{1}{2} \frac{\varepsilon_0 V_0}{d} (L-x) V_0$$
$$\varepsilon_1 x = (L-x)$$
$$x = \frac{L}{\varepsilon_r + 1}.$$

Answer:

(a)

$$\rho_{s1} = \varepsilon_0 \varepsilon_r \frac{V_0}{d} \text{ , (top plate).}$$

(b)

$$x = \frac{L}{\varepsilon_r + 1} .$$