Goal: A d-c voltage V_o is applied across a cylindrical capacitor of length L. The radii of the inner and outer conductors are a and b, respectively. The space between the conductors is filled with two different lossy dielectrics having, respectively, permittivity ε_1 and conductivity σ_1 in the region $a < r < c$, and permittivity ε_2 and conductivity σ_2 in the region $c < r < b$. Find the current density in each region and the surface charge density on the two conductors and the interface between the dielectrics.

Steps:

1. Find the resistance of the first layer of dielectric ($a < r < c$).
   Solution:
$$R_1 = \frac{\ln(c/a)}{2\pi\sigma_1 L}$$

2. Find the resistance of the second layer of dielectric ($c < r < b$).
   Solution:
$$R_2 = \frac{\ln(b/c)}{2\pi\sigma_2 L}$$

3. From the two resistance values to find the resistance of the bilayered dielectric present between the conductors ($a < r < b$).
   Solution:
$$R_{\text{total}} = R_1 + R_2 = \frac{1}{2\pi L}\left(\frac{1}{\sigma_1}\ln\left(\frac{c}{a}\right) + \frac{1}{\sigma_2}\ln\left(\frac{b}{c}\right)\right)$$

4. Find the current density that flows between the two conductors, and the $\mathbf{E}$ field between the conductors.
   Solution:
$$\mathbf{J}_1 = \mathbf{J}_2 = \frac{\sigma_1\sigma_2 V_o}{r(\sigma_1\ln(b/c) + \sigma_2\ln(c/a))}$$
$$\begin{aligned}\mathbf{E}_1 &= \frac{1}{\sigma_1}\mathbf{J}_1 = \frac{\sigma_2 V_o}{r(\sigma_1\ln(b/c) + \sigma_2\ln(c/a))}\\ \mathbf{E}_2 &= \frac{1}{\sigma_2}\mathbf{J}_2 = \frac{\sigma_1 V_o}{r(\sigma_1\ln(b/c) + \sigma_2\ln(c/a))}\end{aligned}$$

5. Find the surface charge density on the surface of the two conductors ($r = a$ and $r = b$).
   Solution: Using boundary condition for $\mathbf{D}$.
$$\begin{aligned}\rho_{s,a} &= \varepsilon_1 E_1\Big|_{r=a} = \frac{\varepsilon_1\sigma_2 V_o}{a(\sigma_1\ln(b/c) + \sigma_2\ln(c/a))}\\ \rho_{s,b} &= -\varepsilon_2 E_2\Big|_{r=a} = -\frac{\varepsilon_2\sigma_1 V_o}{b(\sigma_1\ln(b/c) + \sigma_2\ln(c/a))}\end{aligned}$$

6. Find the surface charge density at the interface between the two dielectrics ($r = c$).
   Solution:
$$\rho_{s,c} = -(\varepsilon_1 E_1 - \varepsilon_2 E_2)\Big|_{r=c} = \frac{(\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2)V_o}{c(\sigma_1\ln(b/c) + \sigma_2\ln(c/a))}$$

Answer:

$$\rho_{s,a} = \frac{\varepsilon_1 \sigma_2 V_o}{a(\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a))}$$

$$\rho_{s,b} = -\frac{\varepsilon_2 \sigma_1 V_o}{b(\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a))}$$

$$\rho_{s,c} = \frac{(\varepsilon_2 \sigma_1 - \varepsilon_1 \sigma_2) V_o}{c(\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a))}$$