Goal: Show that for large z, the electric field created on the z-axis (observation point $(0, 0, z)$) by a semi-circular line charge with density ρ_l at $z = 0$, $r = a$ and $0 \leq \phi < \pi$, is equivalent to the field of a point charge with the same amount of charge, located at the origin.

Steps:

1. Choose a coordinate system
   Solution: The charge distribution can be easily described in terms of cylindrical coordinates. In particular, the semi-circle is expressed as:
   $$r = a, \quad 0 \leq \phi \leq \pi, \quad z = 0.$$
   (Compare this with the Cartesian system; how would it be expressed in Cartesian coordinate?). Hence, we choose cylindrical system.

2. Find dQ'.
   Solution: Here $dQ' = \rho_l \, dl'$, where $dl' = a \, d\phi'$, a differentially small arc-length on the semi-circle. Note the use of primed coordinate for the source points.

3. Find the vectors $\mathbf{R}$, $\mathbf{R}'$, $\mathbf{R} - \mathbf{R}'$, and $|\mathbf{R} - \mathbf{R}'|$.
   Solution:

   - $\mathbf{R}$ is the position vector of the observation point: $\mathbf{R} = z\mathbf{a}_z$.
   - $\mathbf{R}'$ is the position of dQ', $\mathbf{R}' = a\mathbf{a}_r$. Whenever this is expressed in terms of non-cartesian unit vectors (like here), express all these vectors in terms of the Cartesian unit vectors. You will see why in a moment. Here:
     $$\mathbf{R}' = a\mathbf{a}_{r'} = a\left(\cos\phi'\mathbf{a}_x + \sin\phi'\mathbf{a}_y\right)$$
     Note how this vector depends on ϕ' (i.e. it is different at different points at the semi-circle).
   - $\mathbf{R} - \mathbf{R}' = z\mathbf{a}_z - a\cos\phi'\mathbf{a}_x - a\sin\phi'\mathbf{a}_y$
   - $|\mathbf{R} - \mathbf{R}'| = \sqrt{z^2 + a^2}$ (the distance from any point on the semi-circle to $(0, 0, z)$ is the same).

4. Find the electric field due to the semi-circular line charge.
   Solution: For the semi-circular charge distribution, we apply the superposition formula:
   $$\mathbf{E} = \int_{\text{semi-circle}} \frac{dQ'}{4\pi|\mathbf{R} - \mathbf{R}'|^3}\left(\mathbf{R} - \mathbf{R}'\right)$$
   $$\begin{aligned}
   \mathbf{E} &= \int_0^\pi \frac{\rho_l a \, d\phi'}{4\pi\varepsilon_o(z^2 + a^2)^{3/2}}(z\mathbf{a}_z - a\cos\phi'\mathbf{a}_x - a\sin\phi'\mathbf{a}_y) \\
   &= \frac{\rho_l a}{4\pi\varepsilon_o(z^2 + a^2)^{3/2}}\left(z\mathbf{a_z}\int_0^\pi d\phi' - a\mathbf{a}_x\int_0^\pi \cos\phi' d\phi' - a\mathbf{a}_y\int_0^\pi \sin\phi' d\phi'\right) \\
   &= \frac{\rho_l a}{4\pi\varepsilon_o(z^2 + a^2)^{3/2}}\left(\pi z\mathbf{a}_z - 2a\mathbf{a}_y\right)
   \end{aligned}$$

   Note how the introduction of the Cartesian unit vectors instead of the r'-unit vector clarified the variables of the integration. Had we not done that, the dependence of $\mathbf{a}_{r'}$ on ϕ' would have remained implicit. Many times people tend to forget this dependence and derive totally different results.

5. Find the electric field in the limit that $z \gg a$.
Solution: For $|z| \gg a$, the first term is much larger than the second, and also, $z^2 + a^2 \approx z^2$. Hence,

$$\mathbf{E} = \frac{\rho_l\, a}{4\pi\varepsilon_o (z^2)^{3/2}} \pi z \mathbf{a}_z = \frac{\rho_l a \pi}{4\pi\varepsilon_o z^2}\mathbf{a}_z$$

6. Compare this electric field to that of a point charge.
Solution: The total charge on the line is: $Q = \int_0^\pi a\rho_l d\phi = \rho_l \pi a$. If that were considered as a point charge at the origin, it would produce at $(0, 0, z)$ an electric field intensity:

$$\mathbf{E} = \frac{\rho_l \pi a}{4\pi\varepsilon_o z^2}\mathbf{a}_z$$

which is the same equation we found for the semi-circular line charge for large z.

Answer: Proof Problem