A positive point charge q of mass m is injected with a velocity $\mathbf{u}_0 = \mathbf{a}_y u_0$ into the $y > 0$ region where a uniform magnetic field $\mathbf{B} = \mathbf{a}_x B_0$ exists. Obtain the equation of motion of the charge, and describe the path that the charge follows.

Solution: The force on the point charge due to its velocity and the magnetic field will be $\mathbf{F} = m\mathbf{a} = q\mathbf{u} \times \mathbf{B}$. This gives two coupled differential equations

$$\begin{cases} \frac{du_y}{dt} &= \omega_o u_z \\ \frac{du_z}{dt} &= -\omega_o u_y \end{cases}$$

These equations can be combined to get

$$\frac{d^2 u_z}{dt^2} = -\omega_0^2 u_z$$
$$u_z = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

Applying the boundary condition that at $t = 0$, $u_z = 0$ gives $A = 0$ so

$$u_z = B\sin(\omega_0 t).$$

Substituting this into the second of the coupled differential equations,

$$u_y = -B\cos(\omega_0 t)$$

Applying the boundary condition that at $t = 0$, $u_y = u_0$ gives $B = -u_0$ so

$$\begin{cases} u_y &= u_0\cos(\omega_0 t) \\ u_z &= -u_0\sin(\omega_0 t) \end{cases}$$

Integrating these equations and applying the boundary condition that at $t = 0$, $y = z = 0$ gives the equations of motion for the charge

$$\begin{cases} y &= \frac{u_0}{\omega_0}\sin(\omega_0 t) \\ z &= \frac{u_0}{\omega_0}\cos(\omega_0 t) - \frac{u_0}{\omega_0} \end{cases}$$

These equations can be combined into a single equation of motion

$$y^2 + \left(z + \frac{u_0}{\omega_0}\right)^2 = \left(\frac{u_0}{\omega_0}\right)^2$$

which is the equation of a circle in the yz plane of radius u_0/ω_0 centred at $y = 0$, $z = -u_0/\omega_0$. *Answer:*

$$y^2 + \left(z + \frac{u_0}{\omega_0}\right)^2 = \left(\frac{u_0}{\omega_0}\right)^2$$