In a certain region, the electric field is given by:

$$\mathbf{E} = 4xy\mathbf{a}_x + 2x^2\mathbf{a}_y + \mathbf{a}_z$$

(a) Calculate the volume charge density.

(b) From the result of (a), find the total charge enclosed in a cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

(c) Confirm the validity of Gauss' law in integral form by evaluating the net outward electric flux through the cube, using the cube as the volume where Gauss' law is applied.

Solution:

(a) The volume charge density can be found using the differential form of Gauss' Law

$$\rho_v = \varepsilon_0 \nabla \cdot \mathbf{E} = 4\varepsilon_0 y$$

(b) The total charge can be found by integrating the charge density over the volume

$$\begin{aligned} Q &= \int_0^1 \int_0^1 \int_0^1 \rho_v dxdydz \\ &= 4\varepsilon_0 \int_0^1 ydy \\ &= 2\varepsilon_0 \end{aligned}$$

(c) The flux through the cube is

$$\begin{aligned} \oint_S D \cdot dS =& \varepsilon_0 \int_0^1 \int_0^1 (E(1,y,z) - E(0,y,z)) \cdot \mathbf{a}_x dydz \\ &+ \varepsilon_0 \int_0^1 \int_0^1 (E(x,1,z) - E(x,0,z)) \cdot \mathbf{a}_y dxdz \\ &+ \varepsilon_0 \int_0^1 \int_0^1 (E(x,y,1) - E(x,y,0)) \cdot \mathbf{a}_z dxdy \\ =& \varepsilon_0 \int_0^1 \int_0^1 4ydydz \\ =& 2\varepsilon_0 \end{aligned}$$

Answer:

(a)

$$\rho_v = 4\varepsilon_0 y$$

(b)

$$Q = 2\varepsilon_0$$

(c)

$$\oint_S D \cdot dS = 2\varepsilon_0$$