[Cheng P.3-15] Three charges ($+q$, $-2q$, and $+q$) are arranged along the z-axis at $z = d/2$, $z = 0$, and $z = -d/2$, respectively.

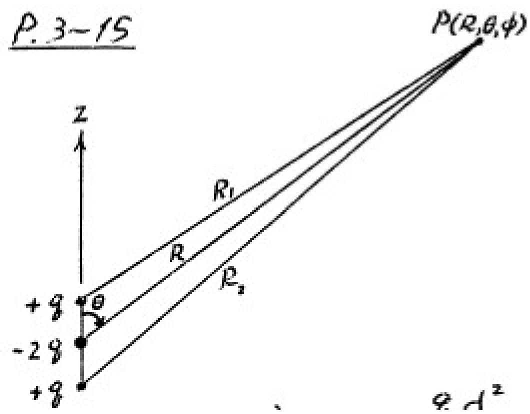
(a) Determine V and $\mathbf{E}$ at a distant point $P(R, \theta, \phi)$.

(b) Find the equations for equipotential surfaces and streamlines.

(c) Sketch a family of equipotential lines and streamlines.

(Such an arrangement of three charges is called a ***linear electrostatic quadrupole***).

Solution: We will use the diagram below to solve this problem



$$V = \frac{q}{4\pi\varepsilon_0 R}\left(\frac{R}{R_1} + \frac{R}{R_2} + 2\right)$$

$$R_1^2 = R^2 + (\frac{d}{2})^2 - Rd\cos\theta$$

$$\frac{R}{R_1} \approx \left[1 + (\frac{d}{2R})^2 - \frac{d}{R}\cos\theta\right]^{-0.5}$$

$$\approx 1 + \frac{d}{2R}\cos\theta + \frac{d^2}{4R^2}\frac{3\cos^2\theta - 1}{2}$$

$$\frac{R}{R_2} \approx 1 - \frac{d}{2R}\cos\theta + \frac{d^2}{4R^2}\frac{3\cos^2\theta - 1}{2}$$

(a)

$$V = \frac{qd^2}{16\pi\varepsilon_0 R^3}\left(3\cos^2\theta - 1\right), R^3 \gg d^3$$

$$\mathbf{E} = -\nabla V$$

$$\mathbf{E} = \frac{3qd^2}{16\pi\varepsilon_0 R^4}\left[\mathbf{a}_R\left(3\cos^2\theta - 1\right) + \mathbf{a}_\theta \sin 2\theta\right]$$

(b) Equipotential surfaces:

$$R^3 = \mathrm{c}_T(3\cos^2\theta - 1)$$

Streamlines:

$$\begin{aligned}
\frac{dR}{E_A} &= \frac{R\mathrm{d}\theta}{E_\theta} \; or \; \frac{dR}{3\cos^2\theta - 1} = \frac{R\mathrm{d}\theta}{\sin 2\theta} \\
\frac{dR}{R} &= \frac{3}{2}\frac{\mathrm{d}(\sin\theta)}{2\sin\theta} - \frac{\mathrm{d}\theta}{\sin 2\theta} \\
R^2 &= \mathrm{c}_E \sin^2\theta\cos\theta.
\end{aligned}$$

Answer:

(a)

$$\begin{aligned}
V &= \frac{qd^2}{16\pi\varepsilon_0 R^3}\left(3\cos^2\theta - 1\right),\, R^3 \gg d^3 \\
\mathbf{E} &= \frac{3qd^2}{16\pi\varepsilon_0 R^4}\left[\mathbf{a}_R\left(3\cos^2\theta - 1\right) + \mathbf{a}_\theta \sin 2\theta\right]
\end{aligned}$$

(b) Equipotential surfaces:

$$R^3 = \mathrm{c}_T(3\cos^2\theta - 1)$$

Streamlines:

$$R^2 = \mathrm{c}_E \sin^2\theta\cos\theta.$$