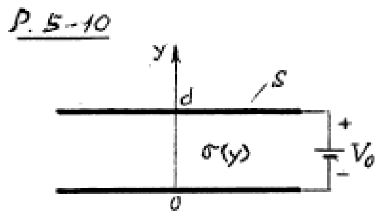
**Goal:** The space between two parallel conducting plates each having an area $S$ is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from $\sigma_1$ at one plate ($y = 0$) to $\sigma_2$ at the other plate ($y = d$). A d-c voltage $V_o$ is applied across the plates as shown below. Find the resistance between the plates, the surface charge density on the plates, and the amount of charge between the plates.




**Steps:**

1. Write an expression for $\sigma(y)$.
   *Solution:*
$$\sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{y}{d}$$

2. Write an expression relating $\mathbf{E}$ and $\sigma(y)$.
   *Solution:*
$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = -\mathbf{a}_y \frac{J_o}{\sigma(y)}$$

3. Find the total resistance between the two plates.
   *Solution:*
$$V_o = -\int_0^d \mathbf{E} \cdot \mathbf{a}_y dy = \frac{J_o d}{\sigma_2 - \sigma_1} \ln \frac{\sigma_2}{\sigma_1}$$
   Then
$$R = \frac{V_o}{I} = \frac{V_o}{J_o S} = \frac{d}{(\sigma_2 - \sigma_1)S} \ln \frac{\sigma_2}{\sigma_1}$$

4. Find the surface charge densities on the two plates.
   *Solution:* Upper plate:
$$\rho_s = \varepsilon_o E_y(d) = \frac{\varepsilon_o J_o}{\sigma_2} = \frac{\varepsilon_o(\sigma_2 - \sigma_1)V_o}{\sigma_2 d \ln(\sigma_2/\sigma_1)}$$
   Lower plate:
$$\rho_s = \varepsilon_o E_y(0) = \frac{\varepsilon_o J_o}{\sigma_1} = \frac{\varepsilon_o(\sigma_2 - \sigma_1)V_o}{\sigma_1 d \ln(\sigma_2/\sigma_1)}$$

5. Find the volume charge density and the total amount of charge between the plates.
   *Solution:*
$$\rho = \nabla \cdot \mathbf{D} = \frac{d}{dy}(\varepsilon_o \mathbf{E}) = \varepsilon_o J_o \frac{(\sigma_2 - \sigma_1)}{d\left(\sigma_1 + (\sigma_2 - \sigma_1)\frac{y}{d}\right)^2}$$

Now, integrate the above expression from $y = 0$ to $y = d$ yields

$$Q_{\text{total}} = \varepsilon_o J_o \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)$$

*Answer:*

(a)

$$R = \frac{d}{(\sigma_2 - \sigma_1) S} \ln \frac{\sigma_2}{\sigma_1}$$

(b) Upper plate:

$$\rho_s = \frac{\varepsilon_o J_o}{\sigma_2} = \frac{\varepsilon_o (\sigma_2 - \sigma_1) V_o}{\sigma_2 d \ln(\sigma_2 / \sigma_1)}$$

Lower plate:

$$\rho_s = \frac{\varepsilon_o J_o}{\sigma_1} = \frac{\varepsilon_o (\sigma_2 - \sigma_1) V_o}{\sigma_1 d \ln(\sigma_2 / \sigma_1)}$$

(c)

$$Q_{\text{total}} = \varepsilon_o J_o \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)$$