Goal: Find the resistance of two concentric spherical surfaces of radii R_1 and R_2 ($R_1 < R_2$). The space in between is filled with a material of conductivity σ.

Steps:

1. Choose a coordinate system.
   Solution: Spherical.

2. Assume a charge Q on the inner conductor, use Gauss law to find the field it creates. Which Gauss surface are you going to choose? Alternatively, assume potential V_0 on the one conductor and zero on the other. From Laplace equation, find V and then E.
   Solution: Using sphere of radius R as a Gaussian surface:

$$\begin{aligned} \iint \mathbf{E} \cdot d\mathbf{S} &= \frac{Q}{\varepsilon} \\ (4\pi r^2) E_r &= \frac{Q}{\varepsilon}, \\ E_r &= \frac{Q}{4\pi\varepsilon R^2}. \end{aligned} \tag{1}$$

   From Laplace equation:

$$\begin{aligned} \nabla^2 V &= 0 \\ \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) &= 0 \end{aligned}$$

   Solution to this equation is

$$V = -\frac{c_1}{R} + c_2$$

   Applying the boundary conditions ($V(R_1) = V_0$ and $V(R_2) = 0$) gives:

$$\begin{aligned} c_1 &= \frac{V_0}{\frac{1}{R_2} - \frac{1}{R_1}} \\ c_2 &= \frac{1}{R_2}\frac{V_0}{\frac{1}{R_2} - \frac{1}{R_1}} \end{aligned}$$

   From scalar potential,

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= \left(\frac{V_0}{\frac{1}{R_1} - \frac{1}{R_2}}\right)\frac{1}{R^2}\mathbf{a}_R. \end{aligned} \tag{2}$$

   Solution: Note: the electric field obtained using Gauss' law (1) and Laplace equation (2) are equivalent.

We can relate the two by performing a line integral on (1):

$$\begin{aligned} V_0 &= -\int_{R_2}^{R_1} E_r dR\,, \\ &= -\int_{R_2}^{R_1} \frac{Q}{4\pi\varepsilon R^2} dR\,, \\ &= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \end{aligned} \tag{3}$$

3. Having $\mathbf{E}$, find $\mathbf{J}$. Can you find the total current I that this $\mathbf{J}$ creates? Choose the surface that you need to use to apply the formula: $I = \iint_S \mathbf{J} \cdot \mathbf{dS}$.
Solution:

$$\begin{aligned} \mathbf{J} &= \sigma \mathbf{E} \\ \mathbf{J} &= \sigma \left(\frac{V_0}{\frac{1}{R_1} - \frac{1}{R_2}}\right) \frac{1}{R^2} \mathbf{a}_R\,. \end{aligned}$$

In order to calculate total current I integrate $\mathbf{J}$ over a sphere:

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi} \sigma \left(\frac{V_0}{\frac{1}{R_1} - \frac{1}{R_2}}\right) \frac{1}{R^2} \mathbf{a}_R \cdot \mathbf{a}_R R^2 sin\theta d\theta d\phi\,, \\ &= \frac{4\pi\sigma V_0}{\frac{1}{R_1} - \frac{1}{R_2}} \end{aligned}$$

4. Having $\mathbf{E}$, find the voltage between the conductors (if you did not assume it already). Then, $R = V/I$. Confirm that $RC = \epsilon/\sigma$ (capacitance for this geometry was found in the previous problem set).
Solution:

$$\begin{aligned} R &= \frac{V_0}{I} \\ &= \frac{1}{4\pi\sigma}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ C &= \frac{Q}{V_0} \\ &= \frac{Q}{\frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \\ &= \frac{4\pi\varepsilon}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \\ RC &= \varepsilon/\sigma \end{aligned}$$

Answer:

$$R = \frac{1}{4\pi\sigma}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$