Goal: Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\mathbf{P} = P_0 \mathbf{a}_z$ exists.

Steps:

1. Determine an equivalent surface charge density ρ_s based on the polarization vector.

   Solution: The surface charge density is

   $$\rho_s = \mathbf{a}_n \cdot \mathbf{P} = P_0 \mathbf{a}_z \cdot (-\mathbf{a}_R) = -P_0 \cos\theta \,.$$

   Note that the normal vector points to the exterior of the dielectric. Now we have a standard problem of a surface charge distribution creating an electric field, which we solve with the steps introduced previously.

2. Choose an appropriate coordinate system to compute the electric field at the center of the sphere using Coulomb's law.

   Solution: Spherical.

3. What is the distance vector $|\mathbf{R} - \mathbf{R}'|$?

   Solution:

   $$\begin{aligned}
   \mathbf{R} &= \mathbf{0} \\
   \mathbf{R}' &= R\,\mathbf{a}_{R'} \\
   &= R\left(\cos\phi' \sin\theta' \mathbf{a}_x + \sin\phi' \sin\theta' \mathbf{a}_y + \cos\theta' \mathbf{a}_z\right) \,. \\
   \mathbf{R} - \mathbf{R}' &= -R\,\mathbf{a}_{R'} \\
   &= -R\left(\cos\phi' \sin\theta' \mathbf{a}_x + \sin\phi' \sin\theta' \mathbf{a}_y + \cos\theta' \mathbf{a}_z\right) \\
   \left|\mathbf{R} - \mathbf{R}'\right| &= R\,.
   \end{aligned}$$

4. What is differential surface element dS'?

   Solution:

   $$dS' = R^2 \sin\theta' d\theta' d\phi'$$

5. Integrate to find the electric field.

   Solution:

   $$\begin{aligned}
   dQ' &= (-P_0 \cos\theta') dS' = (-P\cos\theta') R^2 \sin\theta' d\theta' d\phi' \\
   d\mathbf{E} &= \frac{-P\cos\theta'}{4\pi\varepsilon_0 R^3} R^2 \sin\theta' d\theta' d\phi' (-R)\left(\cos\phi' \sin\theta' \mathbf{a}_x + \sin\phi' \sin\theta' \mathbf{a}_y + \cos\theta' \mathbf{a}_z\right) \\
   &= \frac{P_0 \cos\theta' \sin\theta' d\theta' d\phi'}{4\pi\varepsilon_0}\left(\cos\phi' \sin\theta' \mathbf{a}_x + \sin\phi' \sin\theta' \mathbf{a}_y + \cos\theta' \mathbf{a}_z\right)
   \end{aligned}$$

Note that all terms except the z-component are integrated to zero with respect to ϕ'. The z-component is:

$$\begin{aligned}\mathbf{E} &= \mathbf{a}_z \int_0^{2\pi} \int_0^{\pi} \frac{P\,(\cos\theta')^2 \sin\theta' d\theta' d\phi'}{4\pi\varepsilon_0} \\ &= \mathbf{a}_z \frac{P}{3\varepsilon_0}\,.\end{aligned}$$

Answer:

$$\mathbf{E} = a_z\ \frac{P}{3\varepsilon_o}$$