Goal: In certain experiments it is desirable to have a region of constant magnetic flux density. This can be created in an off-center cylindrical cavity that is cut in a very long cylindrical conductor carrying a uniform current density. Refer to the cross section in figure below.

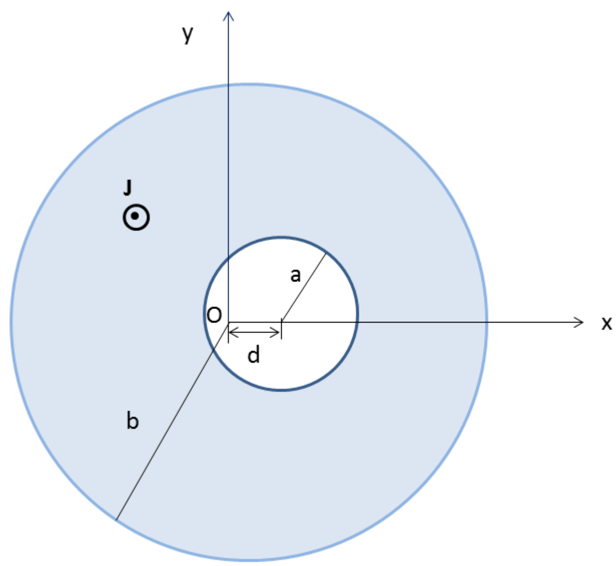

Figure 1 Region with an off-centered cylindrical cavity

The uniform axial current density is $\mathbf{J} = \mathbf{a}_z J$. Find the magnitude and direction of $\mathbf{B}$ in the cylindrical cavity whose axis is displaced from that of the conducting part by a distance d.

Steps:

1. This problem can be solved by superposition. Using Ampere's law, determine the magnetic flux density $\mathbf{B}_1$ that we would have inside the cylinder if the cavity was *not* present.
   Solution: Using a circular Amperean loop with radius $r < b$:

$$\begin{aligned}\oint \mathbf{B}_1 \cdot d\mathbf{l} &= \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S} \\ 2\pi r B_{1,\phi} &= \mu_0 (\pi r^2) J_z \\ B_{1,\phi} &= \frac{\mu_0}{2} r J_z \\ \mathbf{B}_1 &= \mathbf{a}_\phi \frac{\mu_0}{2} r J_z\end{aligned}$$

2. Write $\mathbf{B}_1$ using the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$.

Solution:

$$\begin{aligned}\mathbf{B}_1 &= B_{1,\phi}\mathbf{a}_\phi \\ &= \frac{\mu_0}{2}J_z r\mathbf{a}_\phi \\ &= \frac{\mu_0}{2}J_z\left(-y\mathbf{a}_x + x\mathbf{a}_y\right)\end{aligned}$$

3. Using Ampere's law, determine the magnetic flux density $\mathbf{B}_2$ inside a cylindrical conductor of radius a carrying a uniform current with density $-\mathbf{J}$.
Solution: Assume that this conductor is centered at the origin O' of a new coordinate system. The location of O' with respect to the origin O of the original coordinate system is $(d, 0, 0)$. Primed variables will be used to indicate coordinates with respect to O'.

$$\begin{aligned}\oint \mathbf{B}_2 \cdot d\mathbf{l}' &= -\mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}' \\ B_{2,\phi} &= -\frac{\mu_0}{2}r' J_z \\ \mathbf{B}_2 &= -\mathbf{a}'_\phi \frac{\mu_0}{2}r' J_z\end{aligned}$$

4. Write $\mathbf{B}_2$ using the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$.
Solution: We will first write $\mathbf{B}_2$ using the unit vectors $\mathbf{a}'_x$, $\mathbf{a}'_y$ and $\mathbf{a}'_z$ of the new coordinate system, O'. Then we can use the fact that the unit vectors $\mathbf{a}'_x$, $\mathbf{a}'_y$ and $\mathbf{a}'_z$ are equal, respectively, to the original unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, because O' was simply translated from O.

$$\begin{aligned}\mathbf{B}_2 &= B_{2,\phi}\mathbf{a}'_\phi \\ &= \frac{\mu_0}{2}J_z r'\mathbf{a}_\phi \\ &= \frac{\mu_0}{2}J_z\left(+y'\mathbf{a}_x - x'\mathbf{a}_y\right)\end{aligned}$$

5. Using the results of parts (2) and part (4), determine the total magnetic flux density $\mathbf{B}$ inside the cavity.
Solution: In order to use superposition, we need to use the same coordinate system. So, we must convert x' and y' to the original coordinate system. Firstly, $y' = y$ because O' is only shifted from O along the x axis. Secondly, notice that $x' = x - d$. Therefore, we can write

$$\mathbf{B}_2 = \frac{\mu_0}{2}J_z\left(+y\mathbf{a}_x - (x-d)\mathbf{a}_y\right)$$

Finally,

$$\begin{aligned}\mathbf{B} &= \mathbf{B}_1 + \mathbf{B}_2 \\ &= \frac{\mu_0}{2}J_z\left(d\mathbf{a}_y\right)\end{aligned}$$

Answer:

$$\mathbf{B} = a_y \frac{\mu_o}{2}J_z d$$