[Cheng P.4-4] Verify that

$$V_1 = C_1/R \quad \text{and} \quad V_2 = C_2 z/(x^2+y^2+z^2)^{3/2},$$

where C_1 and C_2 are arbitrary constants, are solutions of Laplace's equation.

Solution: First, we need to verify that $V_1 = C_1/R$ satisfies the Laplace equation. In spherical coordinates, the Laplacian of V_1 is

$$\begin{aligned}
\nabla^2 V_1 &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\left(\frac{C_1}{R}\right)\right) \\
&= \frac{-C_1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{1}{R^2}\right) \\
&= \frac{-C_1}{R^2}\frac{\partial}{\partial R}(1) = 0 \\
\implies \nabla^2 V_1 &= 0
\end{aligned}$$

which is the Laplace equation, as desired.

Next, we need to verify that $V_2 = C_2 z/(x^2+y^2+z^2)^{3/2}$ satisfies the Laplace equation. Converting to spherical coordinates and taking the Laplacian of V_2,

$$\begin{aligned}
\nabla^2 V_2 &= C_2\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\left(\frac{R\cos\theta}{R^3}\right)\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{R\cos\theta}{R^3}\right)\right)\right] \\
&= C_2\left[\frac{\cos\theta}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right)\right) - \frac{1}{R^4\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)\right] \\
&= C_2\left[\frac{-2\cos\theta}{R^2}\frac{\partial}{\partial R}\left(\frac{1}{R}\right) - \frac{2\sin\theta\cos\theta}{R^4\sin\theta}\right] \\
&= C_2\left[\frac{2\cos\theta}{R^4} - \frac{2\cos\theta}{R^4}\right] = 0 \\
\implies \nabla^2 V_2 &= 0
\end{aligned}$$

which is the Laplace equation, as desired.

Answer: Proof problem