Goal: Assume that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density of charge $\rho = A/r$ for $\alpha < r < b$, where α and b are the radius of the inner and outer conductor, respectively. The inner conductor is maintained at potential V_0 and the outer conductor is grounded. Find $V(\alpha < r < b)$ by solving Poisson's equation.

Steps:

1. Choose coordinate system.
   Solution: Cylindrical

2. Carefully state *Poisson* equation in the given space (note: there is charge density, in fact changing with r).
   Solution:

$$\nabla^2 V = -\frac{A}{\varepsilon r}$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{A}{\varepsilon r}$$

3. State the boundary conditions for the potential.
   Solution:

$$V = V_0 \quad \text{at } r = \alpha$$
$$V = 0 \quad \text{at } r = b$$

4. Solve the differential equation, subject to boundary conditions.
   Solution:

$$V = -\frac{Ar}{\varepsilon} + c_1 \ln r + c_2$$

   Applying the boundary conditions gives

$$V_0 = -\frac{A}{\varepsilon}\alpha + c_1 \ln \alpha + c_2 \quad \text{at } r = \alpha$$
$$0 = -\frac{A}{\varepsilon}b + c_1 \ln b + c_2 \quad \text{at } r = b\,.$$

$$c_1 = \frac{\frac{A}{\varepsilon}(b-\alpha) - V_0}{\ln(b/\alpha)}$$
$$c_2 = \frac{V_0 \ln b + \frac{A}{\varepsilon}(\alpha \ln b - b \ln \alpha)}{\ln(b/\alpha)}$$

5. Verification step: Determine the electric field and verify that indeed it points in the direction of decreasing potential.
Solution:

$$\begin{aligned}\mathbf{E} &= -\nabla V \\ &= -\frac{\partial V}{\partial r}\mathbf{a}_r \\ &= \left[\frac{A}{\varepsilon} - \frac{c_1}{r}\right]\mathbf{a}_r \\ &= \left[\frac{A}{\varepsilon} - \frac{\frac{A}{\varepsilon}(b-\alpha) - V_0}{\ln(b/\alpha)r}\right]\mathbf{a}_r\,.\end{aligned}$$

The electric field $\mathbf{E}$ is equal to $-\frac{\partial V}{\partial r}\mathbf{a}_r$. Since the outer conductor is grounded, the rate of change of potential with respect to the radial coordinate is negative; and the electric field points in $\mathbf{a}_r$ direction. Hence, the electric field points in the direction of decreasing potential.

Answer:

$$V = -\frac{Ar}{\varepsilon} + c_1 \ln r + c_2$$

$$c_1 = \frac{\frac{A}{\varepsilon}(b-\alpha) - V_0}{\ln(b/\alpha)}$$

$$c_2 = \frac{V_0 \ln b + \frac{A}{\varepsilon}(\alpha \ln b - b \ln \alpha)}{\ln(b/\alpha)}$$