[Cheng P.6-2] An electron is injected with a velocity $\mathbf{u}_0 = \mathbf{a}_y u_0$ into a region where both an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ exist. Describe the motion of the electron if

(a) $\mathbf{E} = \mathbf{a}_z E_0$ and $\mathbf{B} = \mathbf{a}_x B_0$,

(b) $\mathbf{E} = -\mathbf{a}_z E_0$ and $\mathbf{B} = -\mathbf{a}_z B_0$.

Discuss the effect of the relative magnitudes of E_0 and B_0 on the electron paths in parts (a) and (b).

Solution: The electron will experience a force upon it from the electric and magnetic fields in the region, which will determine its motion.

$$\begin{aligned}
\mathbf{F} &= ma \\
q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) &= m\frac{du}{dt} \\
\frac{du}{dt} &= -\frac{e}{m}(\mathbf{E} + \mathbf{u} \times \mathbf{B})
\end{aligned}$$

(a) Here $\mathbf{E} = \mathbf{a}_z E_0$ and $\mathbf{B} = \mathbf{a}_x B_0$.

$$\begin{aligned}
\frac{\partial u_x}{\partial t} &= 0 & & & u_x &= 0 \\
\frac{\partial u_y}{\partial t} &= -\frac{e}{m} B_0 u_x & &\longrightarrow & u_y &= \left(u_0 - \frac{E_0}{B_0}\right)\cos\omega_0 t + \frac{E_0}{B_0} \\
\frac{\partial u_z}{\partial t} &= -\frac{e}{m}(E_0 - B_0 u_x) & & & u_z &= \left(\frac{E_0}{B_0} - u_0\right)\sin\omega_0 t; \quad \omega_0 = \frac{e}{m} B_0
\end{aligned}$$

If the electron is injected at the origin at $t = 0$ then $x = 0$, $y = \frac{c_2}{\omega_0}\sin\omega_0 t + \frac{E_0}{B_0}t$, $z = -\frac{c_2}{\omega_0}(1 - \cos\omega_0 t)$, and $c_2 = u_0 - \frac{E_0}{B_0}$. This yields the equation of motion

$$\left(y - \frac{E_0}{B_0}t\right)^2 + \left(z + \frac{c_2}{\omega_0}\right)^2 = \left(\frac{c_2}{\omega_0}\right)^2.$$

If $\frac{E_0}{B_0} = u_0$, $u_x = u_z = 0$, $u_y = 0$, then $x = z = 0$, and $y = u_0 t$.

(b) Here $\mathbf{E} = -\mathbf{a}_z E_0$ and $\mathbf{B} = -\mathbf{a}_z B_0$.

$$\begin{aligned}
\frac{\partial u_x}{\partial t} &= \frac{e}{m} B_0 u_y = \omega_0 u_y \\
\frac{\partial u_y}{\partial t} &= -\omega_0 u_x \\
\frac{\partial u_z}{\partial t} &= -\frac{e}{m} E_0.
\end{aligned}$$

The first two equations are for circular motion, while the last one is acceleration in the z direction. This creates a helical motion. See P6-1 for more details.

Answer:

(a)

$$\left(y - \frac{E_o}{B_o}t\right)^2 + \left(z + \frac{c_2}{\omega_0}\right)^2 = \left(\frac{c_2}{\omega_0}\right)^2 \quad where \;\; c_2 = u_o - \frac{E_o}{B_o}$$

(b)

$$\frac{\partial u_x}{\partial t} = \frac{e}{m}B_0 u_y = \omega_0 u_y$$
$$\frac{\partial u_y}{\partial t} = -\omega_0 u_x$$
$$\frac{\partial u_z}{\partial t} = -\frac{e}{m}E_0.$$