Goal: Assume the Earth to be a large conducting sphere (radius $R_0 = 6.37 \times 10^3$ km) surrounded by air (dielectric strength 3×10^6 V/m). Find the capacitance of the Earth and the maximum charge that can exist on it before the air breaks down.

Steps:

1. Using Gauss' law compute the electric field in air when charge $+Q$ is introduced on the surface of the earth. In this case $-Q$ is introduced at infinity.

   Solution:

   $$\mathbf{E} = \mathbf{a}_r \frac{Q}{4\pi\varepsilon_0 R^2}\,.$$

2. Compute the potential at the surface of the earth.

   Solution:

   $$V = \frac{Q}{4\pi\varepsilon_0 R_0}\,.$$

3. Compute the capacitance of the earth.

   Solution:

   $$\begin{aligned} C &= 4\pi\varepsilon_0 R_0 \\ &= 7.08 \times 10^{-4}\,\mathrm{F}\,. \end{aligned}$$

4. What is the maximum charge that can exist on the earth before air breaks down.

   Solution:

   $$\begin{aligned} E_b &= \frac{Q_{\max}}{4\pi\varepsilon_0 R_0^2} \\ Q_{\max} &= 4\pi\varepsilon_0 E_b R_0^2 \\ &= 1.35 \times 10^{10}\,\mathrm{C}\,. \end{aligned}$$

Answer: C = 7.08×10^{-4} F
Q_{max}=1.35×10^{10} C