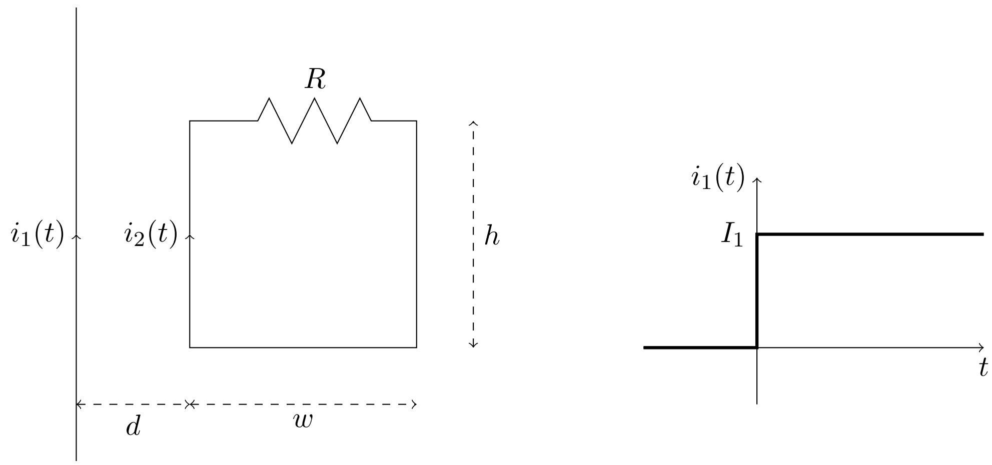
Goal: A rectangular loop of width w and height h is situated near a very long wire carrying a current $i_1(t)$, as shown in the figure below. Assume i_1 to be a step function as shown in the right panel of the figure below. Find the current $i_2(t)$ induced in the rectangular loop as a function of the mutual inductance L_{12} between the two circuits, the self-inductance L_{22} of the square loop, and the resistance R (note: you don't have to calculate L_{12} and L_22).



Steps:

1. Write the magnetic flux Φ_2 through the rectangular loop caused by currents $i_1(t)$ and $i_2(t)$. Express the flux as a function of L_{12} and L_{22}.

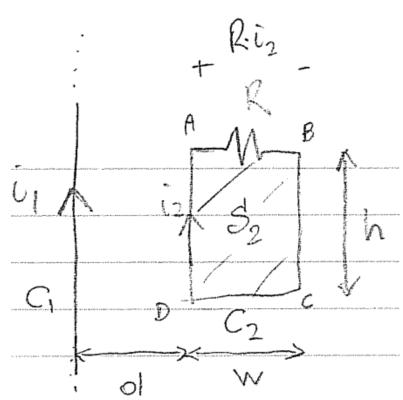
   Solution:

$$\Phi_2 = L_{12} i_1(t) + L_{22} i_2(t)$$

2. Use Faraday's law to show that $i_2(t)$ satisfies the differential equation

$$R i_2(t) + L_{22} \frac{di_2}{dt} = -L_{12} \frac{di_1}{dt}$$

   Solution: Defining the points ABCD as shown:



Faraday's law on C_2

$$\begin{aligned}\oint_{C_2} \mathbf{E}\cdot d\mathbf{l} &= -\frac{d}{dt}\Phi_2(t) = -L_{12}\frac{di_1}{dt} - L_{22}\frac{di_2}{dt} \\ &= \int_{BCDA}\mathbf{E}\cdot d\mathbf{l} + \int_{A\to B}\mathbf{E}\cdot d\mathbf{l} = 0 - \underbrace{\int_B^A \mathbf{E}\cdot d\mathbf{l}}_{\text{Voltage across R}} = Ri_2(t)\end{aligned}$$

Hence,

$$Ri_2(t) = -L_{12}\frac{di_1}{dt} - L_{22}\frac{di_2}{dt}$$

3. Solve the differential equation to find $i_2(t)$ (you can use the Laplace transform method).

Solution: The $i_1(t)$ is given by

$$i_1(t) = I_1 U(t)$$

where $U(t)$ is the unit-step function. Taking the Laplace transform of

$$Ri_2(t) + L_{22}\frac{di_2}{dt} = -L_{12}\frac{di_1}{dt}$$

to get

$$\begin{aligned} RI_2(s) &+ sL_{22}I_2(s) = -L_{12}I_1 \\ I_2(s) &= \frac{L_{12}I_1}{sL_{22}+R} = -\frac{L_{12}I_1}{L_{22}}\frac{1}{s+\frac{R}{L_{22}}}\end{aligned}$$

Taking the inverse Laplace transform of $I_2(s)$ to get

$$i_2(t) = -\frac{L_{12}I_1}{L_{22}}\,\mathrm{e}^{-\frac{R}{L_{22}}t} \quad \text{for} \quad t \geq 0$$

Answer:

$$i_2(t) = -\frac{L_{12}I_1}{L_{22}}\,\mathrm{e}^{-\frac{R}{L_{22}}t} \quad \text{for} \quad t \geq 0$$