Goal: An electron is injected with a velocity $\mathbf{u}_0 = \mathbf{a}_y u_0$ into a region where both an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ exist. Find the velocity of the electron for all time $\mathbf{u}(t)$.

Steps:

1. Calculate total force exerted by the vector field on the electron as a function of time if $\mathbf{E} = \mathbf{a}_z E_0$ and $\mathbf{B} = \mathbf{a}_x B_0$. Note that the contribution of $\mathbf{B}$ to the force changes with time.

   Solution:

$$\begin{aligned} \mathbf{F}(t) &= q\left(\mathbf{E} + \mathbf{u}(t) \times \mathbf{B}\right) \\ &= q\left(\mathbf{a}_z E_0 - \mathbf{a}_z u_y(t) B_0 + \mathbf{a}_y u_z(t) B_0\right) \end{aligned}$$

2. Use $\mathbf{F} = m\mathbf{a} = m\dfrac{d\mathbf{u}}{dt}$ to compute the velocity of the electron.

   Solution:

$$\mathbf{F} = m\mathbf{a}$$

$$\frac{q}{m}\left(\mathbf{a}_z E_0 - \mathbf{a}_z u_y(t) B_0 + \mathbf{a}_y u_z(t) B_0\right) = \frac{d\mathbf{u}}{dt} \Rightarrow \begin{cases} \frac{du_x}{dt} = 0 \\ \frac{du_y}{dt} = \frac{q}{m} u_z(t) B_0 \\ \frac{du_z}{dt} = \frac{q}{m}(E_0 - u_y(t) B_0) \end{cases}$$

   Solving the coupled differential equation for three components gives:

$$\begin{aligned} u_x(t) &= 0 \\ u_y(t) &= \left(u_0 - \frac{E_0}{B_0}\right)\cos\left(\frac{q}{m}B_0 t\right) + \frac{E_0}{B_0} \\ u_z(t) &= \left(\frac{E_0}{B_0} - u_0\right)\sin\left(\frac{q}{m}B_0 t\right) \end{aligned}$$

3. Discuss the effect of relative magnitudes of E_0 and B_0 on the electron path.

   Solution: If, $\dfrac{E_0}{B_0}$ equals u_0 there electron is stationary. If $\dfrac{E_0}{B_0} << u_0$, then the motion is almost circular. If $\dfrac{E_0}{B_0} >> u_0$, then the electron moves back-and-forth in x-direction while moving along y-direction.

4. Now, calculate total force exerted by the vector field on the electron as a function of time if $\mathbf{E} = -\mathbf{a}_z E_0$ and $\mathbf{B} = -\mathbf{a}_z B_0$.

   Solution:

$$\begin{aligned} \mathbf{F}(t) &= q\left(\mathbf{E} + \mathbf{u}(t) \times \mathbf{B}\right) \\ &= q\left(-\mathbf{a}_z E_0 + \mathbf{a}_y u_x(t) B_0 - \mathbf{a}_x u_y(t) B_0\right) \end{aligned}$$

5. Use $\mathbf{F} = m\mathbf{a} = m\dfrac{d\mathbf{u}}{dt}$ to compute the velocity of the electron.

   Solution:

$$\mathbf{F} = m\mathbf{a}$$

$$\frac{q}{m}\left(-\mathbf{a}_z E_0 + \mathbf{a}_y u_x(t) B_0 - \mathbf{a}_x u_y(t) B_0\right) = \frac{d\mathbf{u}}{dt} \Rightarrow \begin{cases} \frac{du_x}{dt} = -\frac{q}{m} u_y B_0 \\ \frac{du_y}{dt} = \frac{q}{m} u_x(t) B_0 \\ \frac{du_z}{dt} = -\frac{q}{m} E_0 \end{cases}$$

   Solving the coupled differential gives

$$u_x(t) = -u_0 \sin\left(\frac{q}{m} B_0 t\right)$$
$$u_y(t) = u_0 \cos\left(\frac{q}{m} B_0 t\right)$$
$$u_z(t) = -\frac{q}{m} E_0 t\,.$$

6. Describe the motion of the electron based on your answer in part (5). What effect does the relative magnitudes of E_0 and B_0 have on the electron path?

   Solution: Helical motion.

Answer:

$$u_x(t) = -u_0 \sin\left(\frac{q}{m} B_0 t\right)$$
$$u_y(t) = u_0 \cos\left(\frac{q}{m} B_0 t\right)$$
$$u_z(t) = -\frac{q}{m} E_0 t\,.$$