1、设 $a>0$　$f(x)\in C[a,+\infty)$　$\lim\limits_{x\to+\infty} f(x)=b$

证 $y'+ay=f(x)$ 的任何解 $\varphi(x)$ 满足

$\lim\limits_{x\to+\infty}\varphi(x)=\frac{b}{a}$

Pr:

$$\varphi(x)=e^{-a\int_{x_0}^{x}dt}\int_{x_0}^{x}e^{a\int_{x_0}^{s}d\tau}f(s)\,ds$$

$$=\int_{x_0}^{x}e^{a\int_{x}^{s}d\tau}f(s)\,ds$$

$$=\frac{\int_{x_0}^{x}e^{as}f(s)\,ds}{e^{ax}}$$

$$\xRightarrow{\text{令 } x\to+\infty}\frac{e^{ax}f(x)}{a\,e^{ax}}$$

$$=\frac{f(x)}{a}\quad(x\to+\infty)$$

$$=\frac{b}{a}$$

2、 证：$\begin{cases} y' = f(x,y)(y^2-1) \\ y(\tau) = \xi \quad |\xi| < 1 \end{cases}$ 的解存在区间为$(-\infty, +\infty)$

Pr：设$\varphi(x)$为解，存在区间为(α, β)

若 $\beta < +\infty$，由 $D = \mathbb{R}^2$，$\partial D = \varnothing$

由延拓定理，$x \to \beta^-$ 时 $\varphi(x)$ 无界

$\Rightarrow \exists x_1 > \tau$ s.t. $|\varphi(x_1)| > 1$

不妨设 $\varphi(x_1) > 1$ $\Rightarrow \exists x_0 \in (\tau, x_1)$ 使 $\varphi(x_0) = 1$

与 $\begin{cases} y' = f(x,y)(y^2-1) \\ y(x_0) = 1 \end{cases}$ 只有 $y = 1$ 解矛盾

3、证：若 $|f(x,y)| \le P(x)|y| + Q(x)$、$D(f(x,y)) = \mathbb{R}^2$、$x \in \mathbb{R}$

则 $y' = f(x,y)$ 的$\forall$解在$(-\infty, +\infty)$上存在

Pr：若$\exists$ $\varphi(x)$ 只在 $x \in (\alpha, \beta)$ 上存在，且 $\beta \neq +\infty$

$D(f(x,y))=\mathbb{R}^2$，由延拓定理，$\lim\limits_{x\to\beta}|\varphi(x)|=+\infty$

$$|\varphi(x)| = \left|y_0+\int_{x_0}^{x} f(t,\varphi(t))\,dt\right|$$

$$\le |y_0| + \left|\int_{x_0}^{x}\left(P(t)|\varphi(t)|+Q(t)\right)dt\right|$$

$$\le |y_0| + \int_{x_0}^{x}|Q(t)|\,dt + \int_{x_0}^{x}|P(t)|\,|\varphi(t)|\,dt$$

由 Gronwall 不等式：

$$|\varphi(x)| \le \left(|y_0|+\int_{x_0}^{x}|Q(t)|\,dt\right)\exp\left(\int_{x_0}^{x}|P(t)|\,dt\right)$$

$$\lim_{x\to\beta}|\varphi(x)| \le \left(|y_0|+\int_{x_0}^{\beta}Q(t)\,dt\right)\exp\left(\int_{x_0}^{\beta}|P(t)|\,dt\right)$$

矛盾

4、设 $y''+a_1y'+a_2y=0$ 解为 u, v，且 $\begin{pmatrix} u(x_0) & v(x_0) \\ u'(x_0) & v'(x_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

求 $y''+a_1y'+a_2y=f(x)$ 常数变易公式

解：

设 $W(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}$ 有 $W'+a_1W=0$

$$\Rightarrow W = W(x_0)e^{a_1x} = e^{a_1x}$$

设特解 $\tilde{x} = c_1u+c_2v$，有

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

$$\Rightarrow c_1' = \frac{-f(x)v(x)}{W(x)} \qquad c_2' = \frac{f(x)u(x)}{W(x)}$$

$$\tilde{x} = c_1u+c_2v$$

$$= \int_{x_0}^{x} \frac{-v(t)u(x)+u(t)v(x)}{W(t)} f(t)\,dt$$

$$= \int_{x_0}^{x} \left(-v(t)u(x)+u(t)v(x)\right)e^{a_1t}\,dt$$

5、 $x''' + 5x'' + 6x' = f(t)$，$f(t)$ 在 $-\infty < t < +\infty$ 上连续，$t \to +\infty$ 时特解是否有极限，若存在，求其极限，若不存在，试对 $f(t)$ 添加条件使其存在极限

解：设 $y = x'$　$y'' + 5y' + 6y = f(t)$

$\Rightarrow$ 齐次基本解　$y_1 = e^{-2t}$　　$y_2 = e^{-3t}$

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

特解：

$$y^* = \int_{t_0}^{t} \frac{-e^{-3s-2t} + e^{-2s-3t}}{W(s)} f(s)\,ds$$

$$= \frac{\int_{t_0}^{t} e^{2s} f(s)\,ds}{e^{2t}} + \frac{\int_{t_0}^{t} e^{3s} f(s)\,ds}{e^{3t}} = x'$$

$$\xrightarrow[\text{洛}]{t \to +\infty} \frac{e^{2t} f(t)}{2e^{2t}} + \frac{e^{3t} f(t)}{3e^{3t}} = \frac{5}{6} f(t) = x'$$

$t \to +\infty$，$x''' + 5x' + 6x = f$ 特解有极限 $\Leftrightarrow$ $y'' + 5y' + 6y = f$ 特解 $\to 0$

$\Rightarrow \lim_{t\to+\infty} f(t)=0$ 时 原方程有极限

6、设 $\varphi(x)$ $x\in(-h,h)$ $h>0$ 是初值

问题 $y'=x^2+y^2$ $y(0)=0$ 的解，证 $\varphi(x)$ 是奇函数

Pr: $\varphi(x)$ s.t. $\varphi'(x)=x^2+\varphi^2(x)$ $\varphi(0)=0$

对 $-\varphi(-x)$ 有

$$\frac{d(-\varphi(-x))}{dx}=\varphi'(-x)=x^2+(-\varphi(-x))^2$$

$$-\varphi(-0)=0$$

即 $-\varphi(-x)$ 也是 $\begin{cases} y'=x^2+y^2 \\ y(0)=0 \end{cases}$ 的解

由存在唯一性 $\varphi(x)=-\varphi(-x)$

即 $\varphi(x)$ 是奇函数

$\varphi_2 = \underline{C(x)}\,\varphi_1$

若 $\varphi\varphi^{-1} = C$

$(C\varphi)' = AC\varphi$

$C'\varphi + C\varphi' = AC\varphi$???

$(\varphi C)' = A\varphi C$

$\varphi C' + \varphi' C = A\varphi C = \varphi' C$

$\Rightarrow \varphi C' = 0 \quad \Rightarrow C' = 0$

设 $\psi' = A\psi$

下证 $c\psi$ 是解

$$(c\psi)' = c\psi' = cA\psi = A(c\psi)$$