场论

定义：

物理量在空间的分布、变化规律 $\Longrightarrow$ 场

$\begin{cases}\text{数量场（}\Omega\text{中每点都有值 } f(x_1,x_2\cdots)\text{）} \\ \text{向量场（}\Omega\text{中每点都有向量值 } \vec{f}(x_1,x_2\cdots)\text{）}\end{cases}$

$\begin{cases}\text{稳定场} \\ \text{不稳定场}\end{cases}$

梯度：

对数量场 $f(x,y,z)$ $(x,y,z)\in\Omega$ 有连续偏导

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

方向导数：$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\langle l,\hat{x}\rangle + \frac{\partial f}{\partial y}\cos\langle l,\hat{y}\rangle + \frac{\partial f}{\partial z}\cos\langle l,\hat{z}\rangle$

$$= \nabla f \cdot \vec{l}$$

对 Σ: $f(x,y,z) = C$

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

$\hat{n} = \frac{\nabla H}{|\nabla H|}$

$$\frac{\partial f}{\partial \hat{n}} = \frac{\nabla f}{|\nabla H|} \cdot \nabla f = |\nabla f|$$

通量、散度

设流体速度场 $\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$

Σ上通量：$\phi = \iint\limits_{\Sigma} \vec{v} \cdot ds$

$$\frac{\phi}{\Delta V} = \frac{\iint \vec{v}\, d\vec{s}}{\Delta V}$$

$$\Rightarrow \phi = \iiint (V_x + V_y + V_z)\, d\Delta V$$

$$= (V_x + V_y + V_z)\big|_{P_0} \Delta V$$

当 $V \to M$ $\quad \mathrm{div} f(M) = \nabla \cdot f = V_x + V_y + V_z \big|_M$

(单位体积通量)

向量线

场：$\vec{a} = P\hat{x} + Q\hat{y} + R\hat{z}$

场：$\vec{a}=P\hat{x}+Q\hat{y}+R\hat{z}$

向量线：$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\Leftrightarrow \vec{a} \parallel d\vec{l}$

环量、旋度

速度场：$\vec{v}=v_x\hat{x}+v_y\hat{y}+v_z\hat{z}$

运动：$v=v_0+w\times r$

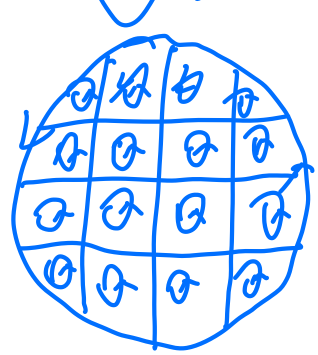
$$\Rightarrow v=\begin{pmatrix} v_{0x} \\ v_{0y} \\ v_{0z} \end{pmatrix}+\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ w_x & w_y & w_z \\ x-x_0 & y-y_0 & z-z_0 \end{vmatrix}$$

$$\Rightarrow 2\begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}=\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ v_x & v_y & v_z \end{vmatrix}=\nabla\times V=\vec{B}$$ （旋度场）

定义：给定向量场 $f=f_x\hat{x}+f_y\hat{y}+f_z\hat{z}$

$rot(\vec{f})=\nabla\times f$　$m\Sigma$为Σ面积　$\otimes\vec{B}$

环量： $\dfrac{\oint_{\partial\Sigma} \vec{f}\cdot d\vec{l}}{m\Sigma}$ $(\Sigma \to M)$



$$= \frac{\iint_{\Sigma} (\nabla\times f)\cdot d\vec{S}}{m\Sigma} \quad \Sigma \to M$$

$$= \frac{[\nabla\times f]_{\tilde{M}}\cdot m\Sigma}{(m\Sigma)} \quad (\tilde{M} \to M)$$

$$= [\nabla\times f]_M \cdot \widehat{(m\Sigma)} \le [\nabla\times f]_M$$

定义： $\nabla = \dfrac{\partial}{\partial x}\hat{x} + \dfrac{\partial}{\partial y}\hat{y} + \dfrac{\partial}{\partial z}\hat{z}$

$\nabla\cdot\nabla f = \Delta f$ $\quad\quad$ $\nabla = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

若 $\Delta f = 0$ 则 f 为调和函数

$$\nabla\cdot(g\nabla f) = \nabla\cdot\left(g\frac{\partial f}{\partial x},\ g\frac{\partial f}{\partial y},\ g\frac{\partial f}{\partial z}\right)$$

$$= \sum\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + g\frac{\partial^2 f}{\partial x^2}\right)$$

$= \nabla g \cdot \nabla f + g \Delta f$

Green 第一公式：

$I = \oiint a \cdot ds = \iiint \nabla \cdot a \, dv$ 令 $a = g \nabla f$

$\Rightarrow I = \iiint (\nabla g \cdot \nabla f + g \Delta f) \, dV$

且 $I = \iint_{\partial\Omega} g \nabla f \cdot \vec{n} \, ds = \iint_{\partial\Omega} g \frac{\partial f}{\partial n} ds$

Green 第二公式：

$\Rightarrow \iiint_{\Omega} (f \Delta g - g \Delta f) \, dV = \iint_{\partial\Omega} \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) ds$

保守场与势函数

向量场 $\vec{F} = P\hat{x} + Q\hat{y} + R\hat{z}$

若 $\exists V$ s.t. $\nabla V = \vec{F}$

则 $\vec{F}$ 为有势场 $V = -v$ 为势函数

$$E \cdot 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0} \cdot \frac{1}{r}\hat{r}$$