① $\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!}\left[(z-a)^m f(z)\right]^{(m-1)}$

② $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$ 令 $z = e^{i\theta}$ $d\theta = \frac{dz}{iz}$

③ $\int_0^{+\infty} f(z)\, dz$ $f(z) = \frac{P(z)}{Q(z)}$

$f(z) = \frac{P_1(z)}{Q(z)} e^{imz}$

④ Roché

习题

2、

(2) $f(z) = \frac{z^{2m}}{1+z^m}$

① 极点处 $1+z^m=0$

$$z^m = e^{i\pi + 2k\pi}$$

$$z = e^{i\frac{\pi}{m} + \frac{2k}{m}\pi} \quad k=0、1、\cdots、m-1$$

$$= w_k$$

为一阶极点

$$\operatorname*{Res}_{z=w_k} f(z) = \frac{z^{2m}}{m z^{m-1}}\bigg|_{z=w_k}$$

或法：$\Rightarrow \lim\limits_{z\to w_k} \frac{z-z_0}{1+z^m} \cdot z^{2m}$

$$= \frac{1}{m} z^{m+1}\Big|_{z=w_k}$$

$$= -\frac{1}{m} w_k$$

② 无穷远点：

(1) $\operatorname*{Res}_{\infty} f(z) = \operatorname*{Res}_{0} f\left(\frac{1}{w}\right)\cdot\left(-\frac{1}{w^2}\right)$

(2) 定理：$\left(\operatorname*{Res}_{z=\infty} + \operatorname*{Res}_{z\neq\infty}\right) f(z) = 0$

有 $\sum\limits_{z_0\neq\infty} \operatorname*{Res}_{z=z_0} f(z) = -\frac{1}{m}\sum\limits_{k=0}^{m-1} w_k = -\frac{1}{m}\cdot e^{i\frac{\pi}{m}} \cdot \frac{1-\left(e^{i\frac{2\pi}{m}}\right)^m}{1-e^{i\frac{2\pi}{m}}}$

$$= 0$$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = 0$$

(4)

$$f(z)=\frac{e^z}{z^2(z-\pi i)^4}$$

$z_1=0$: 二阶

$z_2=\pi i$: 四阶

$$\operatorname*{Res}_{z=0} f(z)=\left(\frac{e^z}{(z-\pi i)^4}\right)'\Bigg|_{z=0}$$

$$=\frac{(z-\pi i)^4-4(z-\pi i)^3}{(z-\pi i)^8}e^z\Bigg|_{z=0}$$

$$=\frac{(\pi i)^4+4(\pi i)^3}{(\pi i)^8}$$

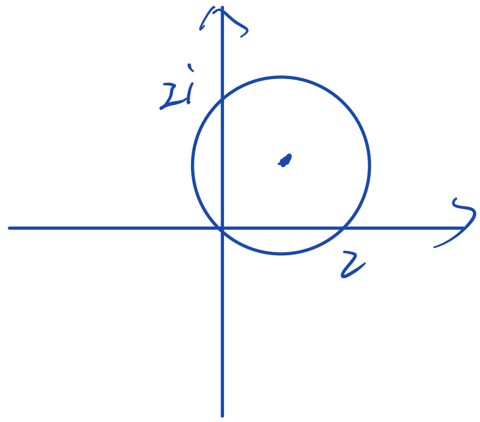
$$=\frac{\pi^4+4\pi^3 i}{\pi^8}=\frac{\pi+4i}{\pi^5}$$

$$\operatorname*{Res}_{z=\pi i} f(z)=\frac{1}{3!}\left(\frac{e^z}{z^2}\right)^{(3)}\Bigg|_{z=\pi i}=\frac{1}{6\pi^5}\left[\pi^3+6\pi^2 i-18\pi+24i\right)$$

$$\operatorname*{Res}_{z=\infty} f(z)=-\left(\operatorname*{Res}_{z=0}+\operatorname*{Res}_{z=\pi i}\right)f(z)=\cdots$$

$1 \quad \pm i$

3、



(3) $f(z) = \dfrac{dz}{(z-1)^2(z^2+1)}$

$$= \left(\operatorname*{Res}_{z=1} + \operatorname*{Res}_{z=i}\right) f(z) \cdot 2\pi i$$

$$= \left[\left(\frac{1}{z^2+1}\right)'\Bigg|_{z=1} + \frac{1}{(z-1)^2(z+i)}\Bigg|_{z=i}\right] \cdot 2\pi i$$

$$= -\frac{2z}{(z^2+1)^2}\Bigg|_{z=1} + \frac{1}{2i(i-1)^2} \cdot 2\pi i$$

$$= -\frac{1}{2} + \frac{1}{2i \times (-2i)} = \left(-\frac{1}{4} \cdot 2\pi i\right) = -\frac{\pi i}{2}$$

(4) $f(z) = \dfrac{1}{(z-a)^n (z-b)^n} \quad (n \geqslant 1)$

$$\oint f(z)\,dz = \left(\operatorname*{Res}_{z=a} + \operatorname*{Res}_{z=b}\right) f(z) \cdot 2\pi i$$

$$= -\operatorname*{Res}_{z=\infty} f(z) \cdot 2\pi i$$

$$= -2\pi i \operatorname*{Res}_{w=0} f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\right)$$

$$= 2\pi i \operatorname*{Res}_{w=0} \frac{1}{(\frac{1}{w}-a)^n(\frac{1}{w}-b)^n} \cdot \frac{1}{w^2}$$

$$= 2\pi i \operatorname*{Res}_{w=0} \frac{w^{2(n-1)}}{(1-aw)^n(1-bw)^n} = 0$$

($w=0$，为可去奇点)

4、

(2)

$$I = \int_0^{2\pi} \frac{dx}{(2+\sqrt{3}\cos x)^2} \qquad z = e^{ix} \qquad dz = ie^{ix}dx$$

$$= \oint \frac{1}{(2+\sqrt{3}\,\frac{z+z^{-1}}{2})^2} \frac{dz}{iz}$$

$$= \oint \frac{4z^2}{(4z+\sqrt{3}(z^2+1))^2} \frac{dz}{iz}$$

$$= \frac{4}{i} \oint \frac{z}{(\sqrt{3}z+1)^2(z+\sqrt{3})^2} dz$$

$$= 2\pi i \cdot \frac{4}{i} \operatorname*{Res}_{z=-\frac{1}{\sqrt{3}}} \frac{z}{(z+\frac{1}{\sqrt{3}})^2(z+\sqrt{3})^2 3}$$

$$= 8\pi \left[\frac{z}{3(z+\sqrt{3})^2} \right] \Big|_{z=-\frac{1}{\sqrt{3}}}$$

$$= \frac{8}{3}\pi \frac{(z+\sqrt{3})^2 - 2z(z+\sqrt{3})}{(z+\sqrt{3})^4} \Big|_{z=-\frac{1}{\sqrt{3}}} = 4\pi$$

(3)

$$I = \int_0^{\pi} \tan(\theta + ia)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\sin(\theta+ia)}{\cos(\theta+ia)}\, d\theta$$

$$= \frac{1}{2}\oint_{|z|=1} \frac{\left(ze^{-a} - \frac{1}{z}e^{a}\right)}{i\left(ze^{-a}+\frac{1}{z}e^{a}\right)} \frac{dz}{iz}$$

$$= -\frac{1}{2}\oint_{|z|=1} \frac{z - \frac{1}{z}e^{2a}}{z^2 + e^{2a}}\, dz$$

$$= -\frac{1}{2}\oint_{|z|=1} \underbrace{\frac{z^2 - e^{2a}}{z(z+ie^{a})(z-ie^{a})}}_{f(z)}\, dz = -\pi i \operatorname{Res}$$

$$e^{2a}$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{-e^{2a}}{e^{2a}} = -1$$

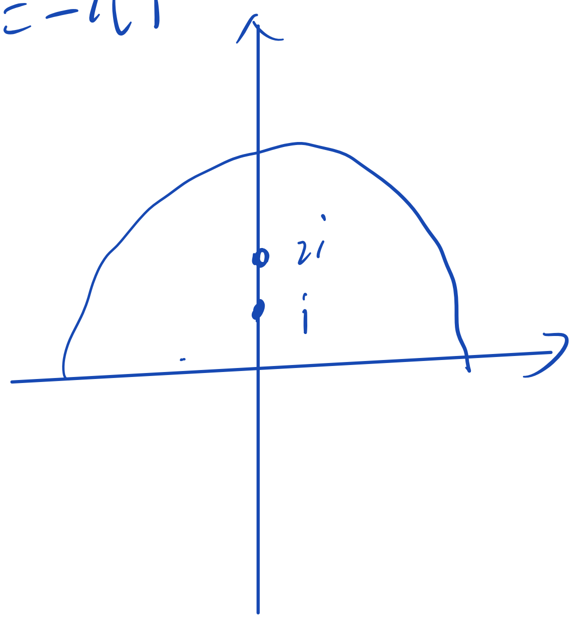
$$\operatorname*{Res}_{z=ie^a} f(z) = \frac{-2e^{2a}}{ie^a \cdot 2ie^a} = 1$$

$$\operatorname*{Res}_{z=-ie^a} f(z) = \frac{-2e^{2a}}{-ie^a(-2ie^a)} = 1$$

① $a>0$

$$I = -\pi i(-1+1) = \pi i$$

② $a<0$ $\quad I = -\pi i(-1+1+1) = -\pi i$



5、

(1) $$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

$$= \frac{1}{2}\oint \frac{z^2}{(z^2+1)(z^2+4)}\,dz$$

极点：$z = \pm i$、$\pm 2i$

$$= \pi i\left(\operatorname*{Res}_{z=i} + \operatorname*{Res}_{z=2i}\right) f(z)$$

z^4+5z^2+4

$$= \pi i\left(\frac{1}{6} + \frac{1}{-6}\right)$$

$4z^3+10z$

$$= 0$$

$$\frac{1}{4z^2+10}$$

(4)

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x\sin mx}{x^4+a^4}\,dx$$

$$\tilde{I} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{z e^{imz}}{z^4+a^4}\,dz \quad (\text{收敛})$$

极点

$$z_k = a\cdot e^{\frac{\pi}{4}+\frac{2k}{4}\pi}$$
$$= a\cdot \varepsilon_k$$

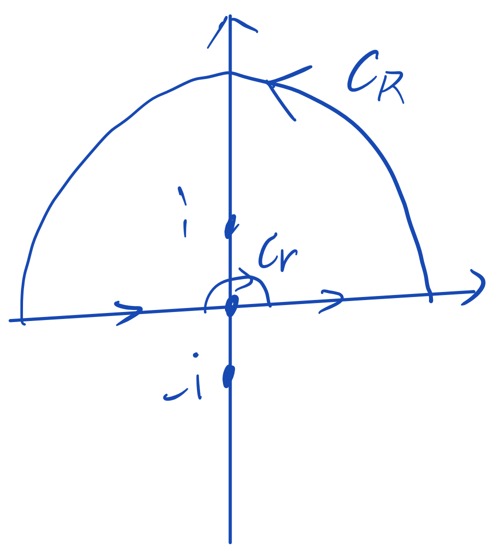
$$= \pi i\left(\operatorname*{Res}_{z=z_0} + \operatorname*{Res}_{z=z_1}\right) f(z)$$

$$= \pi i \sum \left.\frac{e^{imz}}{4z^2}\right|_{z=z_0,\, z_1}$$

6、

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin z}{z(z^2+1)^2}\,dz$$

$$\tilde{I} = \frac{1}{2}\int_{-\infty}^{+\infty} \underbrace{\frac{e^{iz}}{z(z^2+1)^2}}_{=f(z)}\,dz \quad (\text{收敛})$$



$$= \frac{1}{2}\left(\oint - \int_{C_R} + \int_{C_r}\right) f(z)\,dz$$

$$= \pi i \operatorname*{Res}_{z=i} f(z) - \frac{1}{2}\int_{C_R} f(z)\,dz + \frac{1}{2}\int_{C_r} f(z)\,dz$$

$$= \pi i \left(\frac{e^{iz}}{z(z+i)^2}\right)'\Bigg|_{z=i} + \left(\cdots\cdots \right)$$

$$\lim_{z\to\infty} z\,\frac{1}{z(z^2+1)^2} = 0 \Rightarrow \int_{C_R} f(z)\,dz = 0$$

(Jordan)

$$\lim_{z\to 0} z f(z) = 1 \Rightarrow \int_{C_r} f(z)\,dz = \pi i$$

$$= \left(-\frac{3}{4e} + \frac{1}{2}\right)\pi i$$

$$I = \operatorname{Im} \tilde{I} = \frac{2e-3}{4e}\pi$$

15、

(3)

$$I = \oint 2\,\frac{f'(z)}{f(z)}\,dz \qquad f(z) = z^3 - 7z + 6$$

$|z|=4$ $\qquad |z^3| = 64 > 15 \geq |\ldots|$

$$= 4\pi i\,(N_{f,C} - 0) = 12\pi i$$

$N_{f,C} = 3$