4、

$$\frac{\partial \varphi}{\partial z} = \frac{\varphi_1}{y} + \frac{\varphi_2}{x}$$

$$\frac{\partial \varphi}{\partial x} = \varphi_1 - \frac{z}{x^2}\varphi_2$$

$$\frac{\partial \varphi}{\partial y} = -\frac{z}{y^2}\varphi_1 + \varphi_2$$

$$\frac{\partial z}{\partial y} = \frac{-\varphi_y}{\varphi_z}$$

$$-\frac{x\varphi_x + y\varphi_y}{\varphi_z} = -\frac{x\varphi_1 + y\varphi_2 - \frac{z}{x}\varphi_2 - \frac{z}{y}\varphi_1}{\frac{x\varphi_1 + y\varphi_2}{xy}}$$

$$= z - xy$$

5、

(2) $\begin{cases} F_1 = xu + yv = 0 \\ F_2 = yu + xv - 1 = 0 \end{cases}$

$\partial(u, v)$ $\partial(F_1, F_2)$

$$\frac{\partial(F_1,F_2)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(F_1,F_2)}{\partial(x,y)}$$

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = -\begin{pmatrix} u & v \\ v & u \end{pmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{ux-vy}{y^2-x^2} \qquad \frac{\partial u}{\partial y} = \frac{vx-uy}{y^2-x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(u+x\frac{\partial u}{\partial x}-y\frac{\partial v}{\partial x}\right)(y^2-x^2)+(ux-vy)\cdot 2x}{(y^2-x^2)^2}$$

(3)

$$\begin{cases} u = f(ux, v+y) \\ v = g(u-x, v^2y) \end{cases}$$

$$\begin{cases} F_1 = f-u = 0 \\ F_2 = g-v = 0 \end{cases}$$

$$\frac{\partial(F_1,F_2)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(F_1,F_2)}{\partial(x,y)}$$

$$\begin{pmatrix} xf_1-1 & f_2 \\ \cdots & \cdots \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \cdots & \cdots \end{pmatrix} = -\begin{pmatrix} uf_1 & f_2 \\ \cdots & \cdots \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots \\ g_1 & 2vg_2-1 \end{pmatrix}\begin{pmatrix} \cdots & \cdots \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \quad \begin{pmatrix} \cdots & \cdots \\ -g_1 & vg_2 \end{pmatrix}$$

$$\frac{\partial u}{\partial x} = -\frac{uf_1(2vg_2-1)+f_2g_1}{(xf_1-1)(2vg_2-1)-f_2g_1}$$

$$\frac{\partial v}{\partial x} = -\frac{-g_1(xf_1-1)-uf_1g_1}{(xf_1-1)(2vg_2-1)-f_2g_1}$$

7.

$$\frac{\partial(F,G)}{\partial(x,z)}\cdot\frac{\partial(x,z)}{\partial y} = -\frac{\partial(F,G)}{\partial y}$$

$$\begin{pmatrix} -F_1 & -F_2 \\ yG_1 & \frac{1}{y}G_2 \end{pmatrix}\begin{pmatrix} \frac{dx}{dy} \\ \frac{dz}{dy} \end{pmatrix} = -\begin{pmatrix} F_1+F_2 \\ xG_1-\frac{z}{y^2}G_2 \end{pmatrix}$$

$$\frac{dx}{dy} = -\frac{\frac{1}{y}G_2(F_1+F_2)+F_2\left(xG_1-\frac{z}{y^2}G_2\right)}{-\frac{F_1G_2}{y}+yF_2G_1}$$

$$\frac{dz}{dy} = \frac{F_1\left(\frac{z}{y^2}G_2-xG_1\right)-(F_1+F_2)yG_1}{\cdots}$$

$$- y F_2 G_1 + \frac{1}{y} F_1 G_2$$

12、

(1) $\ln z = w + x + y$

$$z = e^{w+x+y}$$

$$\frac{\partial z}{\partial x} = e^{w+x+y}\left(1 + \frac{\partial w}{\partial x}\right)$$

$$\frac{\partial z}{\partial y} = e^{w+x+y}\left(1 + \frac{\partial w}{\partial y}\right)$$

$$e^{w+x+y}\left[y - x + y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y}\right] = (y-x)e^{w+x+y}$$

$$\Rightarrow y\frac{\partial w}{\partial x} = x\frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$= 2x\frac{\partial w}{\partial u} - \frac{1}{x^2}\frac{\partial w}{\partial v}$$

$$\frac{\partial w}{\partial y} = 2y\frac{\partial w}{\partial u} - \frac{1}{y^2}\frac{\partial w}{\partial v}$$

$$\Rightarrow 2xy\frac{\partial w}{\partial u} - \frac{y}{x^2}\frac{\partial w}{\partial v} = 2xy\frac{\partial w}{\partial u} - \frac{x}{y^2}\frac{\partial w}{\partial v}$$

$$\Rightarrow \frac{\partial w}{\partial v} = 0$$

(2)

$$\frac{\partial z}{\partial x} = \frac{\partial w}{\partial x} - 1$$

$$= \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} - 1$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial w}{\partial y} - 1 = \frac{\partial w}{\partial v} - 1$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 w}{\partial v^2}$$

$$\Rightarrow \frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2} - 2\frac{\partial^2 w}{\partial u \partial v} - 2\frac{\partial^2 w}{\partial v^2} + \frac{v}{u}\frac{\partial^2 w}{\partial v^2} = 0$$

$$\frac{\partial^2 w}{\partial u^2} + \left(\frac{v}{u} - 1\right)\frac{\partial^2 w}{\partial v^2} = 0$$

(3) $\begin{cases} u = x+y \\ v = \frac{y}{x} \end{cases}$ $z = xw$ $x = \frac{u}{v+1}$

$y = \frac{uv}{u+1}$

$$\frac{\partial z}{\partial x} = w + x\frac{\partial w}{\partial x}$$

$$= w + x\left(\frac{\partial w}{\partial u} - \frac{y}{x^2}\frac{\partial w}{\partial v}\right)$$

$$= w + x\frac{\partial w}{\partial u} - \frac{y}{x}\frac{\partial w}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} + x\left(\frac{\partial^2 w}{\partial u^2} - \frac{y}{x^2}\frac{\partial^2 w}{\partial u \partial v}\right) + \frac{y}{x^2}\frac{\partial w}{\partial v}$$

$$- \frac{y}{x}\left(\frac{\partial^2 w}{\partial u \partial v} - \frac{y}{x^2}\frac{\partial^2 w}{\partial v^2}\right)$$

$$= 2\frac{\partial w}{\partial u} - \frac{y}{x^2}\frac{\partial w}{\partial v} + x\frac{\partial^2 w}{\partial u^2} - 2\frac{y}{x}\frac{\partial^2 w}{\partial u \partial v} + \frac{y^2}{x^3}\frac{\partial^2 w}{\partial v^2}$$

$$+ \frac{y}{x^2}\frac{\partial w}{\partial v}$$

$$= 2\frac{\partial w}{\partial u} + x\frac{\partial^2 w}{\partial u^2} - \frac{2y}{x}\frac{\partial^2 w}{\partial u \partial v} + \frac{y^2}{x^3}\frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial w}{\partial y} + x\left(\frac{\partial^2 w}{\partial u^2} + \frac{1}{x}\frac{\partial^2 w}{\partial u \partial v}\right) - \frac{1}{x}\frac{\partial w}{\partial v} - \frac{y}{x}\left(\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{x}\frac{\partial^2 w}{\partial v^2}\right)$$

$$= \frac{\partial w}{\partial u} + \frac{1}{x}\frac{\partial w}{\partial v} + x\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial u \partial v} - \frac{y}{x}\frac{\partial^2 w}{\partial u \partial v} + \frac{y}{x^2}\frac{\partial^2 w}{\partial v^2}$$

$$-\frac{1}{x}\frac{\partial w}{\partial v}$$

$$= \frac{\partial w}{\partial u} + x\frac{\partial^2 w}{\partial u^2} + \frac{x-y}{x}\frac{\partial^2 w}{\partial u \partial v} + \frac{y}{x^2}\frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial z}{\partial y} = x\frac{\partial w}{\partial y}$$

$$= x\left(\frac{\partial w}{\partial u} + \frac{1}{x}\frac{\partial w}{\partial v}\right) = x\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

$$\frac{\partial^2 z}{\partial y^2} = x\left(\frac{\partial^2 w}{\partial u^2} + \frac{1}{x}\frac{\partial^2 w}{\partial u \partial v}\right) + \frac{\partial^2 w}{\partial u \partial v} + \frac{1}{x}\frac{\partial^2 w}{\partial v^2}$$

$$= x\frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{x}\frac{\partial^2 w}{\partial v^2}$$

2	x	$-\frac{2y}{x}$	$\frac{y^2}{x^3}$
-2	$-2x$	$\frac{2y-2x}{x}$	$-\frac{2y}{x^2}$
0	x	2	$\frac{1}{x}$
0	0	0	$\frac{y^2-2xy+x^2}{x^3}$

$$\Rightarrow \frac{\partial^2 w}{\partial v^2} = 0$$

13、 $y = f(x, t)$

$$\frac{dy}{dx} = f_1 + f_2\left(\frac{\partial t}{\partial x} + \frac{\partial t}{\partial y}\frac{dy}{dx}\right)$$

$$\frac{\partial t}{\partial x} = -\frac{F_x}{F_t}$$

$$\left(1 + \frac{f_2 F_y}{F_t}\right)\frac{dy}{dx} = f_1 - f_2\frac{F_x}{F_t}$$

$$\frac{dy}{dx} = \frac{f_1 F_t - f_2 F_x}{F_t + f_2 y}$$