1、

(1) $f = x_1^2 + 4x_2^2 + 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 12x_2x_3$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 1 & 6 & 2 \end{pmatrix} \qquad C_1 = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_1^T A C_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \qquad C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$C_2^T C_1^T A C_1 C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_3^T C_2^T C_1^T A C_1 C_2 C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} = \Lambda$$

令 $X = C_1 C_2 C_3 Y$ $\quad X^T A X = Y^T \Lambda Y$

(2) $A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}_{n+m}$ $\forall X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}$

$$X^T A X = (x_1 \cdots x_n) A_1 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + (x_{n+1} \cdots x_{n+m}) A_2 \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} > 0$$

即A正定

2、

(1) $V_1 = \{A \in P^{2\times 2} \mid A^T = A\}$

$\forall A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \in V_1 \quad A^T = A \quad \Rightarrow a_2 = a_3$

$\Rightarrow V_1 = L\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = L(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

(2) $U_2 = \{A \in P^{2\times 2} \mid trA\}$

$\forall A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \quad a_1 + a_4 = 0$

$\Rightarrow V_2 = L\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = L(\eta_1, \eta_2, \eta_3)$

$V_1 + V_2 = L(\varepsilon_1, \varepsilon_2, \eta_1, \eta_2)$

3、

(1) $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$AX = 0 \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} X = 0$

$$X = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\sigma^{-1}(\theta) = L((-\varepsilon_1+\varepsilon_2), (-\varepsilon_3+\varepsilon_4))$$

(2) $$(\lambda I - A) = \begin{pmatrix} \lambda-1 & -1 & 0 & 0 \\ -1 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda-1 & -1 \\ 0 & 0 & -1 & \lambda-1 \end{pmatrix}$$

$$D_1 = 1 \quad D_2 = 1 \quad D_3 = 1$$

$$D_4 = (\lambda-1)^4$$

$$\Rightarrow d_1 = 1 \quad d_2 = 1 \quad d_3 = 1 \quad d_4 = (\lambda-1)^4$$
$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1$$

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

4、 $\sigma\varepsilon_1 = \lambda_1\varepsilon_1 \qquad \sigma\varepsilon_2 = \lambda_2\varepsilon_2$

(1) $k_1\varepsilon_1 + k_2\varepsilon_2 = 0$

$k_1\lambda_1\varepsilon_1 + k_2\lambda_2\varepsilon_2 = 0$

$\Rightarrow k_2 \underset{\neq 0}{(\lambda_2-\lambda_1)} \varepsilon_2 = 0$

$\Rightarrow k_2 = 0 \quad \Rightarrow k_1 = 0$

(2) 若 $\underset{\parallel}{\sigma(\varepsilon_1+\varepsilon_2)} = \lambda\varepsilon_1 + \lambda\varepsilon_2$

有 $\sigma\varepsilon_1 + \sigma\varepsilon_2 = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2$

$\Rightarrow (\lambda-\lambda_1)\varepsilon_1 + (\lambda-\lambda_2)\varepsilon_2 = 0$

$\Rightarrow \begin{cases} \lambda-\lambda_1 = 0 \\ \lambda-\lambda_2 = 0 \end{cases}$ X

5、

令 $\eta_1 = \varepsilon_1$

$\eta_2 = \varepsilon_2 - \frac{(\eta_1, \varepsilon_2)}{(\eta_1, \eta_1)}\eta_1$

$= \varepsilon_2 - \frac{0}{1}\eta_1 = \varepsilon_2$

$\eta_3 = \varepsilon_3 - \frac{(\eta_1, \varepsilon_3)}{(\eta_1, \eta_1)}\eta_1 - \frac{(\eta_2, \varepsilon_3)}{(\eta_2, \eta_2)}\eta_2$

$= \varepsilon_3 - \frac{1}{1}\varepsilon_1 - \frac{1}{1}\varepsilon_2$

$= -\varepsilon_1 - \varepsilon_2 + \varepsilon_3$

$|\eta_1| = 1 \quad |\eta_2| = 1$

$|\eta_3| = (-1, -1, 1)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$= 5 - 2 - 2 = 1$

$\gamma_1 = \frac{\eta_1}{|\eta_1|} = \varepsilon_1$

$\gamma_2 = \frac{\eta_2}{|\eta_2|} = \varepsilon_2$

$\gamma_3 = \frac{\eta_3}{|\eta_3|} = -\varepsilon_1 - \varepsilon_2 + \varepsilon_3$

(2) $\sigma(\alpha) = \alpha - 2(\alpha, \varepsilon_1)\varepsilon_1$

$\sigma(\varepsilon_1) = \varepsilon_1 - 2\varepsilon_1 = -\varepsilon_1$

$\forall \alpha \in L(\varepsilon_2, \varepsilon_3) \qquad \varepsilon_2 \perp \varepsilon_1 \quad \varepsilon_3 \perp \varepsilon_1$

$\alpha = k_2\varepsilon_2 + k_3\varepsilon_3$

$\sigma(\alpha) = \alpha - 0 = \alpha$

即 $L(\varepsilon_2, \varepsilon_3) \subset V_1$

对 $\forall \alpha \in V_1$

$$\alpha = k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3$$

$$\sigma\alpha = -k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3$$

$\Rightarrow k_1 = 0$

$\Rightarrow \alpha \in L(\varepsilon_2, \varepsilon_3)$

$\Rightarrow V_1 = L(\varepsilon_2, \varepsilon_3)$

$\Rightarrow \lambda_1 = -1 \quad \lambda_{2,3} = 1$

$\Rightarrow |A| = -1$ 第二类

或 $\sigma(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon_1, \varepsilon_2, \varepsilon_3)\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$\Rightarrow |A| = -1$ 第二类

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 1 & 6 & 1 \end{pmatrix} \qquad C_1 = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_1^T A C_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 4 \\ 1 & 4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_2^T C_1^T A C_1 C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{pmatrix} \qquad C_3 = \begin{pmatrix} 1 & & \\ & \frac{1}{2\sqrt{2}} & \\ & & \frac{1}{2\sqrt{2}} \end{pmatrix}$$

$$C_3^T C_2^T C_1^T A C_1 C_2 C_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} = \Lambda$$

$$X = C_1 C_2 C_3 Y, \quad X^T A X = Y^T \Lambda Y$$