1. $(x-z)\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$

转化为 $x=x(y,z)$ 表示

$F=z-z(x(y,z),y)=0$

$$\begin{cases} x \\ y \\ z=z(x(y,z),y) \end{cases}$$

$$\frac{\partial z}{\partial(x(y,z),y)}\cdot\frac{\partial(x(y,z),y)}{\partial(y,z)}=\frac{\partial z}{\partial(y,z)}$$

$$\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)\begin{pmatrix}\frac{\partial x}{\partial y} & \frac{\partial x}{\partial z}\\ 1 & 0\end{pmatrix}=(0,\ 1)$$

$$\begin{cases} \frac{\partial z}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial z}{\partial y}=0 \\ \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial z}=1 \end{cases}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}=\frac{1}{\partial x/\partial z}$$

$$\frac{\partial z}{\partial y}=-\frac{\partial x/\partial y}{\partial x/\partial z}$$

$$\Rightarrow (x-z)-y\frac{\partial x}{\partial y}=0$$

2、证：

由 $z=y+x\varphi(z)$ 确定隐函数 $z=z(x,y)$

s.t. $\dfrac{\partial z}{\partial x^2}=\dfrac{\partial}{\partial y}\left[\varphi^2(z)\dfrac{\partial z}{\partial y}\right]$

Pf

$$\frac{\partial z}{\partial x}=x\varphi'(z)\frac{\partial z}{\partial x}+\varphi(z) \qquad \frac{\partial z}{\partial y}=1+x\varphi'\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\varphi}{1-x\varphi'} \qquad \frac{\partial z}{\partial y}=\frac{1}{1-x\varphi'}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\varphi'(1-x\varphi')\frac{\partial z}{\partial x}-\varphi\left(-\varphi'-x\varphi''\frac{\partial z}{\partial x}\right)}{(1-x\varphi')^2}$$

$$=\frac{\varphi'\varphi+\varphi\left(\varphi'+x\varphi''\frac{\varphi}{1-x\varphi'}\right)}{(1-x\varphi')^2}$$

$\Vert$

$$\frac{\partial}{\partial y}\left(\frac{\varphi^2}{1-x\varphi'}\right)=\frac{2\varphi\varphi'\frac{\partial z}{\partial y}(1-x\varphi')-\varphi^2\left(-x\varphi''\frac{\partial z}{\partial y}\right)}{(1-x\varphi')^2}$$

3、将 $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2$ 变为 $w=w(u,v)$ 方程

$$\begin{cases} x=u \\ y=\frac{u}{1+uv} \\ z=\frac{u}{1+uw} \end{cases}$$

$$\frac{1}{z}=w+\frac{1}{u}$$

$$-\frac{1}{z^2}\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial u}=\frac{\partial w}{\partial u}-\frac{1}{u^2}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}=\frac{1+uw-u\left(w+u\frac{\partial w}{\partial u}\right)}{(1+uw)^2}$$

$$\frac{\partial z}{\partial u}=\frac{z^2}{u^2}-z^2\frac{\partial w}{\partial u}$$

$$=\frac{1-u\frac{\partial w}{\partial u}}{(1+uw)^2}$$

$$u=x$$

$$v=\frac{1}{y}-\frac{1}{x}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\cdot\frac{1}{u^2}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}\left(-\frac{1}{y^2}\right)$$

$$u^2\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}-\frac{\partial z}{\partial v}=z^2$$

$$z^2-z^2u^2\frac{\partial w}{\partial u}=z^2$$

$$\Rightarrow z^2u^2\frac{\partial w}{\partial u}=0 \qquad \frac{\partial w}{\partial u}=0$$