The Torsion Linking Form of a 3-manifold

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The material is mostly from *Embeddings of 3-manifolds in S^4 from the point of view of the 11-tetrahedron census* by Ryan Budney and Benjamin Burton at `https://arxiv.org/abs/0810.2346`

1 Torsion Linking Form

Let M be a closed, oriented manifold.

Definition 1.1. Let $\tau H_i(M, \mathbb{Z})$ be the torsion subgroup of $H_i(M, \mathbb{Z})$, and let $fH_i(M, \mathbb{Z}) \equiv H_i(M, \mathbb{Z})/\tau H_i(M, \mathbb{Z})$ be the free part of $H_i(M, \mathbb{Z})$.

Theorem 1.1 (Poincaré Duality)**.** *There is a canonical, natural isomorphism*

$$H_i(M, \mathbb{Z}) \cong H^{n-i}(M, \mathbb{Z}),$$

for $i \in \{0, \cdots, n\}$

Theorem 1.2 (Universal Coefficient Theorem)**.** *There is a short exact sequence*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{i-1}(M, \mathbb{Z}), \mathbb{Z}) \to H^i(M, \mathbb{Z}) \to \mathrm{Hom}(H_i(M, \mathbb{Z}), \mathbb{Z}) \to 0,$$

for $i \in \{0, \cdots, n\}$*, and a canonical isomorphism*

$$\mathrm{Ext}_{\mathbb{Z}}(H_i(M, \mathbb{Z}), \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\tau H_i(M, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}),$$

As a consequence of the above two theorems, we have

Theorem 1.3. *There are bilinear maps* $I : fH_i(M) \otimes fH_{n-i}(M) \to \mathbb{Z}$ *and* $T : \tau H_i(M) \otimes \tau H_{n-i-1}(M) \to \mathbb{Q}/\mathbb{Z}$ *such that the maps* $c \mapsto I(c, \_)$ *and* $c \mapsto T(c, \_)$ *are isomorphisms* $fH_i(M) \cong fH_{n-i}(M)$*,* $\tau H_i(M) \cong \tau H_{n-i-1}(M)$*.*

TODO. □

The pairing I is usually called the *intersection product*, and the pairing T the *torsion linking form*. From now on, our focus will be on the torsion linking form.

The torsion linking form is straightforward to compute. Take $[x] \in \tau H_i(M)$ and $[y] \in \tau H_{n-i-1}(M)$. The homology class $[y]$ is a torsion element so we have $ky = \partial Y$ for some k, Y. Then the torsion linking form $T([x], [y]) = \frac{x \pitchfork Y}{k}$ where $x \pitchfork Y$ is the (signed) (transverse) intersection number between x and Y.

Example 1. RP^3 has homology groups $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, \mathbb{Z}$. Of these, the first homology group $\mathbb{Z}/2\mathbb{Z}$ has torsion. In computing the torsion linking form $T : H_1(RP^3) \otimes H_1(RP^3) \to \mathbb{Q}/\mathbb{Z}$, the only nontrivial case is $T(1,1)$. Take RP^3 as $S^3 \subset \mathbb{R}^4$ with antipodal points identified. Consider two 1-homology classes x, y represented by transverse half-equators. Then $2y$ lifts to an equator in S^3 and bounds the intersection of S^3 with the obvious 2-plane, which projects to Y as defined above. It is clear that $x \pitchfork Y = 1$, with the single intersection ocurring at the intersection of x and y. We therefore have $T(1,1) = \frac{1}{2}$.

The same method can also compute the torsion linking groups of other manifold quotients of S^3, that is, the lens spaces excluding $S^1 \times S^2$.

2 Embedding 3-manifolds in S^4

The following theorem gives a necessary condition for a 3-manifold to embed in a homology S^4.

Theorem 2.1. *If M is a closed oriented 3-manifold which embeds in a homology S^4, there must be a splitting $H_1(M,\mathbb{Z}) = A \oplus B$ which induces a splitting*

$$\mathrm{Hom}_{\mathbb{Z}}(\tau H_1(M,\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$$

which is reversed by Poincaré duality, in the sense that the PD isomorphism

$$\tau H_1(M,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\tau H_1(M,\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

restricts to isomorphisms $A \to \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ and $B \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.

In the previous theorem, we are identifying $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ with the submodule of $\mathrm{Hom}_{\mathbb{Z}}(\tau H_1(M,\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ which is zero on B and $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ with the submodule which is zero on A.

Proof. M separates the homology S^4 into two parts. Mayer-Vietoris gives the result. [TODO: flesh out proof] □

Corollary 2.1.1. *The lens spaces excluding S^3 and $S^1 \times S^2$ cannot embed into a homology S^4.*

[TODO: Prove corollary]