Embedding graphs in $\mathbb{R}^3$ without self-intersections

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February 9, 2016

Introduction

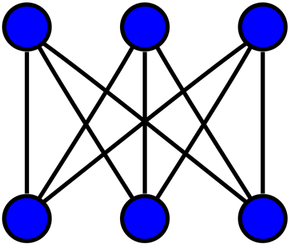


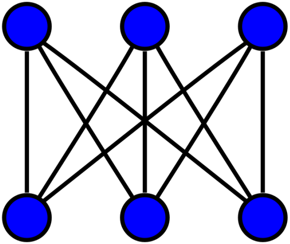
Figure: The Utility graph

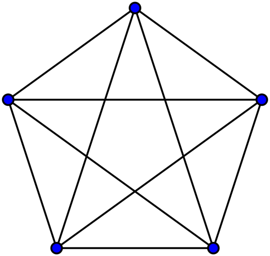
- Can we move the vertices so that edges don't intersect?

Kuratowski's theorem

Theorem (Kuratowski)

A finite graph is planar if and only if it does not contain a Kuratowski subgraph.





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- **Proof.**
  Suppose for a contradiction that four distinct points of C lay on the plane $ax + by + cz = d$. Then the cubic equation $at + at^2 + at^3 = d$ would have four roots. $\square$

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- TOO EASY! Not very interesting.

Definitions

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- More restrictive than usual definition.

A constraint

- Let's consider graphs that are only slight distortions of the original graph.
- We'll move each vertex by distance at most ϵ
- Can we eliminate all self-intersections?

Baire spaces

- A **Baire space** is a topological space in which any countable union of closed sets, each with empty interior, also has empty interior.

Theorem (Baire Category Theorem)

Every complete metric space is a Baire space. In particular, $\mathbb{R}^3$ *is a Baire space.*

Corollary

Any countable union of planes in $\mathbb{R}^3$ *has empty interior.*

The Main Theorem

Theorem

For any subset V *of* $\mathbb{R}^3$, *and for any* $\epsilon > 0$, *there exists a bijective map* $f : V \to X \subset \mathbb{R}^3$ *such that* $|p - f(p)| < \epsilon$ *for all* $p \in V$ *and no four points of* X *lie on the same plane. In particular, no distinct lines between points intersect.*

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Proof.

We will inductively build up X. Let $V = \{v_1, v_2, \cdots, v_n\}$. First, we place v_1, v_2, v_3 so that they are not collinear. For each $m \in \{1, \cdots, n\}$, suppose we have found for each $k < m$ a set X_k such that:

- Every plane passing through three points of X_k intersects X_k at only those points.
- If $l < k$, then $X_l \subset X_k$

The Main Theorem

Proof (continued).

Let P_k be the set of planes passing through three points of X_k. Since each plane is closed with empty interior, $\bigcup_{k<m} P_k$ has empty interior for all m. We can therefore find a point p in a ϵ-neighbourhood of v_m that isn't on $\bigcup_{k<m} P_k$.

Define $f(v_m) = p$ and $X_m = \bigcup_{k<m} X_k \cup \{p\}$. By construction, X_m satisfies the inductive hypotheses. Letting $X = X_n$ completes the proof.

$\square$

A detour: Transfinite induction

Transfinite induction is induction generalized from $\mathbb{N}$ to **Ordinal numbers**.
Ordinal numbers have the following properties.

- 0 is an ordinal and there is no smaller ordinal.
- Given any ordinal α, the set $\{\beta : \beta < \alpha\}$ is a well-ordered set.
- For every set there exists a well-ordering of the elements of X
- Given any well-ordered set X, there is exactly one ordinal such that X is order-isomorphic to $\{\beta : \beta < \alpha\}$

A detour: Transfinite induction

Given any set Y, there exists an ordinal α with the same cardinality as Y. And since $\{\beta : \beta < \alpha\}$ is well-ordered, there is a least such ordinal. Therefore, every set Y can be well-ordered in such a way that the set of predecessors of any element of Y has strictly smaller cardinality than the cardinality of Y itself.

Theorem (Transfinite Induction)

Suppose we have a set X in one-to-one correspondence with an ordinal θ. For each $\alpha < \theta$, write x_α for the element of X corresponding to α. Let P be a statement about elements of X. If $P(x_0)$ is true, and $P(x_\alpha)$ is true whenever $P(x_\beta)$ is true for every $\beta < \alpha$, then $P(x)$ is true for every $x \in X$

The Continuum Hypothesis

Continuum Hypothesis.

There is no set with cardinality strictly between that of the natural numbers and the real numbers

□

Independent of ZFC!

The Main Theorem

Theorem

For any subset V of $\mathbb{R}^3$, and for any $\epsilon > 0$, there exists a bijective map $f : V \to X \subset \mathbb{R}^3$ such that $|p - f(p)| < \epsilon$ for all $p \in V$ and no four points of X lie on the same plane. In particular, no distinct lines between points intersect.

Proof.

We start by well-ordering V. Let v_α be the point associated with the ordinal α. We may assume that each element has strictly less than c predecessors in the well ordering.

As last time, we place v_1, v_2, v_3 so that they are not collinear.

$\square$

The Main Theorem

Proof (continued).

Suppose for each $\beta < \alpha$, we have found a set X_β such that:

- Every plane passing through three points of X_β intersects X_β at only those points.
- If $\gamma < \beta$, then $X_\gamma \subset X_\beta$

Let P_β be the set of planes passing through three points of X_β. By the continuum hypothesis, $\bigcup_{\beta<\alpha} P_\beta$ is a countable union of planes! Since each plane is closed with empty interior, $\bigcup_{\beta<\alpha} P_\beta$ has empty interior for all α. We can therefore find a point p in a ϵ-neighbourhood of v_α that isn't on $\bigcup_{\beta<\alpha} P_\beta$.

□

Proof (continued).

Define $f(v_\alpha) = p$ and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta \cup \{p\}$. By construction, X_α satisfies the inductive hypotheses. Letting $X = X_c$ completes the proof. □

We have proved:

Theorem

For any subset V of $\mathbb{R}^3$, and for any $\epsilon > 0$, there exists a bijective map $f : V \to X \subset \mathbb{R}^3$ such that $|p - f(p)| < \epsilon$ for all $p \in V$ and no four points of X lie on the same plane. In particular, no distinct lines between points intersect.

References

- The idea of using the curve (t, t^2, t^3), and the proof of the finite case of the main theorem is from **Topology** by James Munkres.
- The treatment of transfinite induction follows the one at the Tricki (http://www.tricki.org/article/Transfinite_induction)