Trisections and Morse 2-Functions

Chan Bae

October 3, 2020

Let M^n be a smooth manifold of dimension n. All functions will be smooth unless noted otherwise. The appendix may or may not be written some day. Ideas, figures, and words have all been shamelessly plundered from a multitude of sources.

1 Morse Functions

1.1 Vocabulary

Take a real-valued function $f : M \to \mathbb{R}$. We say that a point $p \in M$ is a **critical point** of f if the induced map $f^* : TM_p \to T\mathbb{R}$ is singular. This is equivalent to the gradient $(\frac{\partial f}{\partial x^i})$ being zero.

We can learn a lot about the structure of a manifold by studying the critical points of a generic function on the manifold. The **Hessian matrix** $(\frac{\partial^2 f}{\partial x^i x^j})$ provides further information about the behaviour of a function near a critical point.

If the Hessian at a critical point p is singular, we say that p is a **degenerate critical point**. At a nondegenerate critical point, the local behaviour of f is determined by the the eigenvalues of the Hessian.

Lemma 1.1 (Morse Lemma)**.** *Let p be a nondegenerate critical point of $f : M \to \mathbb{R}$. Then there exist coordinates (x^i) such that*

$$f(x) = f(p) - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{(\lambda+1)})^2 + \cdots + (x^n)^2$$

near p.

We call λ the **index** of p, and it is equal to the number of negative eigenvalues of the Hessian at p.

Definition 1.1. We say $f : M \to \mathbb{R}$ is a **Morse Function** every critical point $p \in M$ is nondegenerate.

Theorem 1.2. *Morse functions exist on every manifold. Furthermore, they are stable and generic. That is, a small perturbation of a Morse function is Morse, and there exists a small function of any function that makes it Morse.*

Proof. See appendix. □

2 Heegaard Splittings

Definition 2.1. A **genus g Heegaard Splitting** of a closed, connected, oriented 3-manifold M is a decomposition $M = M_1 \cup M_2$ where M_1, M_2 are handlebodies[1] of genus g such that $M_1 \cap M_2 = \partial M_1 = \partial M_2 = \Sigma_g$, where Σ_g is the surface of genus g.

Definition 2.2. A **Heegaard Splitting Morse function** on a 3-manifold M is a Morse function $f : M \to \mathbb{R}$ such that:

- 0 is a regular value of f with $f^{-1}(0)$ diffeomorphic to the surface of genus g.
- There are exactly g critical values of index 2 and one critical value of index 3, all distinct, on each of the two rays $((-\infty, 0]$ and $f^{-1}([0, \infty))$.

Theorem 2.1. *Heegaard splitting Morse functions exist.*

Proof. See appendix. □

Corollary 2.1.1. *Heegaard splittings exist.*

Proof. Take a Heegaard splitting Morse function f. Then

$$M = f^{-1}((-\infty, 0]) \cup f^{-1}([0, \infty)).$$

□

Heegaard splittings can be visualised by looking at its central genus g surface and the belt/attaching spheres of the 1-/2-handles, respectively.

Definition 2.3. A **Heegaard diagram** of a Heegaard splitting $M = M_1 \cup M_2$, $\Sigma_g = M_1 \cap M_2$ is a tuple $(\Sigma_g, \alpha, \beta)$, where α and β are collections of g closed curves in Σ_g that bound disks in M_1, M_2, respectively, such that after cutting $Sigma_g$ by the curves in α or β, the result is a sphere with disks removed.

Theorem 2.2. *Heegaard diagrams "are" Heegaard splittings.*

Example 1. $S^1 \times S^2$ and S^3 can both be described by Heegaard diagrams of genus 1 as in Figure 1.

[1] In 3-manifold topology, a handlebody is the g-fold boundary connect sum $\natural^g(S^1 \times B^2)$.

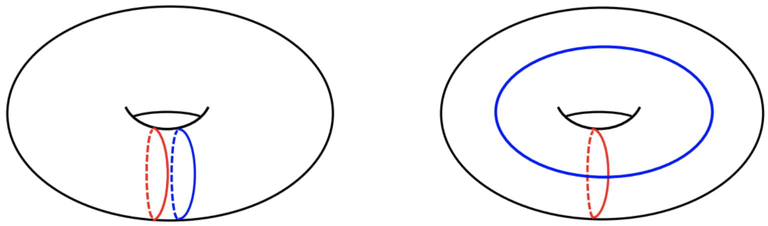

Figure 1: Heegaard diagrams for $S^1 \times S^2$(left), and S^3(right)

3 Morse 2-functions

As Heegaard Splittings can be interpreted as arising naturally from Morse functions, Trisections arise from Morse 2-functions.

Definition 3.1. A Morse 2-function on a 4-manifold X is a smooth map $f : X \to \Sigma$ such that at every point $p \in X$ and under a suitable choice of coordinates (t, x, y, z), f has one of the following forms near p.

- $(t, x, y, z) \mapsto (t, x)$. Here p is called a **regular point**.
- $(t, x, y, z) \mapsto (t, \pm x^2 \pm y^2 \pm z^2)$. Here p is called a **fold point** and **definite** or **indefinite** depending on whether the quadratic form $\pm x^2 \pm y^2 \pm z^2$ is definite or indefinite.
- $(t, x, y, z) \mapsto (t, x^3 - tx \pm y^2 \pm z^2)$. Here p is called a **cusp point**.

Remark. Intuitively, a Morse 2-function is a function that locally looks like a 1-parameter family of functions that are Morse at all but finitely many times. These finitely many exceptions (cusp points), represent "births" and "deaths" of pairs of critical points.

3.1 Homotopies between Morse Functions

4 Trisections

Definition 4.1. A **trisection** of a closed, oriented, connected 4-manifold M is a tuple (M, X_0, X_1, X_2) with $M = X_0 \cup X_1 \cup X_2$ satisfying the following conditions.

- Each X_i is diffeomorphic to a 4-dimensional handlebody $\natural^{g_i}(S^1 \times B^3)$.
- The handlebodies X_i have pairwise disjoint interiors.
- The intersection $X_i \cap X_j$ is a 3-dimensional handlebody $H_{i,j}$ of genus g.
- The triple intersection $\Sigma_g = X_0 \cap X_1 \cap X_2$ is a closed connected surface of genus g.

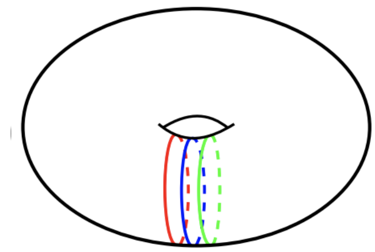

Figure 2: A trisection diagram of a trisection of $S^1 \times S^3$.

Note that $H_{i-1,i}$ and $H_{i,i+1}$ (indices taken mod 3) form a Heegaard splitting of ∂X_i with Σ. Naturally we may try to understand trisections by studying these three Heegaard splittings. They turn out to be entirely sufficient. The trisection is completely determined by the central surface and the H_{ij}s. This leads to the following definition.

Definition 4.2. A (g, k)-**trisection diagram** of a trisection $M = X_0 \cup X_1 \cup X_2$ is a tuple $(\Sigma_g, \alpha, \beta, \gamma)$ such that

- Σ is a genus g surface.
- $(\Sigma_g, \alpha, \beta)$, $(\Sigma_g, \beta, \gamma)$, $(\Sigma_g, \gamma, \alpha)$ are Heegaard diagrams for $\sharp^k(S^1 \times S^2)$.

Example 2. S^4 has an explicit trisection given by the pieces $\{(re^{i\theta}, x^2, x^3, x^4) \in \mathbb{C} \times \mathbb{R}^3 : r^2 + \Sigma(x^i)^2 = 1, \theta \in [2\pi j/3, 2\pi(j+1)/3]\}$, where $j = 0, 1, 2$. The trisection diagram for this trisection is simply S^2 with no curves.

Example 3. $S^1 \times S^3$ has a genus 1 trisection as shown in Figure 2.

Definition 4.3. A **trisecting Morse 2-function** is...

Theorem 4.1. *Trisections exist.*

Proof. □

4.1 Stabilisation

Definition 4.4. Given a (g, k)-trisection (M, X_0, X_1, X_2), we can construct a new trisection of M as follows. Recall that $H_{ij} = X_i \cap X_j$ is a three-dimensional handlebody with boundary $F = X_0 \cap X_1 \cap X_2$. Take a_{ij} to be properly embedded boundary parallel arcs in each H_{ij}, such that the endpoints of the three arcs are disjoint in F. Let N_{ij} be disjoint closed regular neighbourhoods of the a_{ij}s in M. Then

$$X_0' = (X_0 \cup N_{12}) \setminus (\mathring{N}_{20} \cup \mathring{N}_{01})$$
$$X_1' = (X_0 \cup N_{20}) \setminus (\mathring{N}_{01} \cup \mathring{N}_{12})$$

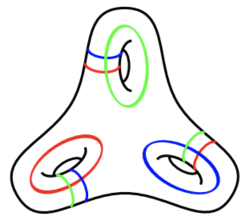

Figure 3: The simplest nontrivial trisection of S^4

$$X_2' = (X_0 \cup N_{01}) \setminus (\mathring{N}_{12} \cup \mathring{N}_{20})$$

form a new $(g{+}3, k{+}1)$-trisection of M called the **stabilisation** of (M, X_0, X_1, X_2).

Example 4. Stabilising the trivial genus 0 trisection of S^4 given in Example 2 gives the genus 3 trisection shown in Figure 3.

Trisections admit a natural **connected sum** operation. This is done by removing a standardly trisected 4-ball from the interiors of each manifold and then identifying the resulting boundary spheres respecting the trisection.

Theorem 4.2. *Stabilising a trisection is equivalent to connect summing with the trisection of S^4 shown in Figure 3.*

5 Group Trisections

Given a (g, k)-trisection (M, X_0, X_1, X_2) with trisection surface $\Sigma_g = X_0 \cap X_1 \cap X_2$, the inclusions $\Sigma \hookrightarrow X_i \cap X_j \hookrightarrow X_i \hookrightarrow M$ induce a commutative cube of epimorphisms of fundamental groups such that every face is a pushout.

$$\begin{tikzcd}
 & H_g \arrow[r] \arrow[dr] & Z_k \arrow[dr] & \\
S_g \arrow[ur] \arrow[r] \arrow[dr] & H_g \arrow[ur] \arrow[dr] & Z_k \arrow[r] & \pi(M) \\
 & H_g \arrow[ur] \arrow[r] & Z_k \arrow[ur] &
\end{tikzcd}$$

Inspired by this, we define the (purely group-theoretic) notion of a group trisection.

Definition 5.1. A (g, k)-group trisection of a group G is a commutative cube of groups as shown below, such that each homomorphism is surjective and every face is a pushout. Here, we require

$$S_g = \langle a_1, b_1, \cdots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle$$

$$H_g = \langle x_1, \cdots, x_g \rangle$$

$$Z_k = \langle z_1, \cdots, z_k \rangle$$

That is, S_g is the standard genus g surface group, and H_g, Z_k are free groups.

$$\begin{tikzcd}
 & H_g \arrow[r] \arrow[dr] & Z_k \arrow[dr] & \\
S_g \arrow[ur] \arrow[r] \arrow[dr] & H_g \arrow[ur] \arrow[dr] & Z_k \arrow[r] & G \\
 & H_g \arrow[r] \arrow[ur] & Z_k \arrow[ur] &
\end{tikzcd}$$

There is a functorial one-to-one correspondence between trisections of groups and trisections of smooth, closed, orientable 4-manifolds. Furthermore, since every finitely presented group is the fundamental group of some 4-manifold, every finitely presented group admits a trisection.

Now let V be the set of vertices of the cube, and E the set of edges. We label the groups $G_v | v \in V$, and the maps $f_e | e \in E$ so we may refer to a trisection as a pair $(\{G_v\}, \{f_e\})$.

Definition 5.2. An **isomorphism of trisections** between trisections $(\{G_v\}, \{f_e\})$ and $(\{G'_v\}, \{f'_e\})$ is a collection $\{h_v : G_v \to G'_v | v \in V\}$ of isomorphisms that commute with each f_e, and f'_e.

The connected sum operation on trisected 4-manifolds has a group trisection analogue.

Definition 5.3. Given a $(g, k)-$trisection $(\{G_v\}, \{f_e\})$ of G and a $(g', k')-$trisection $(\{G'_v\}, \{f'_e\})$ of G', we define the **connected sum** $(\{G''_v\}, \{f''_e\})$ of the two trisections as follows:

- G''_v is the free product $G_v * G'_v$ generated by the disjoint union of generators of G_v, G'_v.
- For a generator g, $f''_e(g)$ is defined to be either $f_e(g)$ or $f'_e(g)$ depending on whether g is from G_v, or G'_v.

Remark. $(\{G''_v\}, \{f''_e\})$ is a group trisection of $G'' = G * G'$ of genus $(g+g', k+k')$.

We go on to define the stabilisation of a group trisection.

Definition 5.4. The **stabilisation** of a group trisection $(\{G_v\}, \{f_e\})$ is defined to be the connected sum of $(\{G_v\}, \{f_e\})$ and the (3,1)-trisection of the trivial group arising from the (3,1)-trisection of S^4 shown in Figure 3.

We are now able to state:

Conjecture 5.1. *Every* $(3k, k)$*-trisection of the trivial group is stably equivalent to the trivial trisection of the trivial group.*

This purely group-theoretic conjecture is in fact equivalent to the smooth 4-dimensional Poincaré conjecture.

Proof. A $(3k, k)-$trisection of the trivial group corresponds to a $(3k, k)$-trisection of a simply connected 4-manifold. Since the Euler characteristic of a (g, k)-trisected manifold is $3g - k + 2$, we have a simply connected 4-manifold with Euler characteristic 2. That is, a homotopy S^4. □

6 Appendix

6.1 Morse functions