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Handle Theory in the Piecewise-Linear Setting

This writeup is heavily based on *Introduction to Piecewise-Linear Topology* by Rourke and Sanderson. In particular, many proofs are reproduced almost verbatim. We will refer to this book as R&S.

Unless otherwise stated, all maps, embeddings, homeomorphisms, etc should be thought to be piecewise linear.

1 Introduction

A goal of topology is to be able to distinguish between different spaces. We do this by finding topological invariants of spaces, values that remain unchanged under homeomorphism. The homology groups of a space are an important invariant that greatly generalizes the Euler characteristic of polyhedra to arbitrary topological spaces. Homotopy type is a stronger invariant, giving rise to the homotopy groups, of which the most important is the fundamental group of a space.

The (topological) classification of surfaces shows that the homology of a surface is enough to determine the surface up to homeomorphism. A natural question to ask is whether the same holds for higher dimensional manifolds. It turns out that the answer is negative. There exist non-homeomorphic manifolds that have isomorphic homology groups. The Poincaré theorem tackles a weaker version of this question, does a homotopy sphere have to be homeomorphic (or PL-isomorphic, diffeomorphic) to the standard sphere? (A homotopy sphere must be a homology sphere.)

The topological Poincaré conjecture has been proved in all dimensions, with Grigori Perelman proving the 3-dimensional case, the last to go unsolved. The differentiable Poincaré conjecture is known to be false in general but true in dimensions 1,2,3,5, and 6 among others. The 4-dimensional case is unsolved. The piecewise linear Poincaré conjecture has been proved in all dimensions except 4. In this writeup we will prove the piecewise linear Poincaré conjecture for dimensions > 5.

2 Overview

The aim of this writeup is to prove the h-cobordism theorem, from which the high dimensional Poincaré theorem will follow. We lead with the statement of the h-cobordism theorem.

Theorem 1 (h-cobordism theorem)**.** *Let (W^w, M_0, M_1) be a cobordism and that*
(1) $\pi_1(M_0) = \pi_1(M_1) = \pi_1(W) = 0$
(2) $H_*(W, M_0) = 0$
(3) $w \geq 6$
Then $W \cong M_0 \times I$.

We will prove the h-cobordism theorem by considering handle decompositions of cobordisms.
1. Prove that every cobordism is a union of handles (has a handle decomposition).
2. Rearrange the handles to obtain a ”nice” decomposition (a decomposition where the handles are arranged in increasing order of index)
3. Prove cancellation theorems.
4. Eliminate all handles.
5. The theorem follows.

3 Definitions and Terminology

For a manifold W^w and a w-ball H such that $W \cap H \subset \partial W$, we say that (H, h) is a **handle of index p on** W, or simply a **p-handle**, if there is a homeomorphism $h : I^p \times I^q \to H$ such that $h(\dot{I}^p \times I^p) = H \cap W$.

For a p-handle (H, h), we call $h(I^p \times 0)$ the **core** of H and $h(0 \times I^q)$ the **cocore**. $h(\dot{I}^p \times 0)$ is the **attaching sphere** (a-sphere) and $h(0 \times \dot{I}^q)$ the **belt sphere** (b-sphere). We also define the **a-tube** $h(\dot{I}^p \times I^q)$ and the **b-tube** $h(I^p \times \dot{I}^q)$. We call h the **characteristic map** of H, and $f = h|\dot{I}^p \times I^q$ the **attaching map**.

Note that $W' = W \cup H$ is also a w-manifold, since a point on the intersection has a neighbourhood which is the union of two balls with a common face. We can identify W' with $W \cup_f I^w$, where $W \cup_f I^w$ is the space obtained by identifying $\dot{I}^p \times I^q$ with ∂W by the attaching map f. We say that W' is formed from W by **attaching a handle by** f. We write variously $W' = W \cup H = W \cup H^{(p)} = W \cup_f H$.

For a compact manifold W^w, we say that (W^w, M_0, M_1) is a **cobordism** if there exist embeddings $i : M_0 \to \partial W$,$j : M_1 \to \partial W$ such that ∂W is the disjoint union of $i(M_0)$ and $j(M_1)$ for manifolds M_0 and M_1. We call W a cobordism if it is clear what M_0 and M_1 are. W is an **h-cobordism** if both inclusions $M_0 \subset W$ and $M_1 \subset W$ are homotopy equivalences.

We can extend the notion of handles on a manifold to handles on a cobordism. For a cobordism (W, M_0, M_1) and a handle H on W, we say that H is **a handle on the cobordism** W if $H \cap W \subset M_1$. As in the case of a handle on a manifold, we can obtain a new cobordism $(W', M_0, \partial W - M_0)$.[1]

We will develop tools to work with handles. These will prove useful because every

[1]This concept reduces to that of a handle on a manifold if $M_0 = \emptyset$

cobordism can be regarded as a ball with a series of handles attached.

4 Handle Decompositions

Let W be a closed manifold. We define a **handle decomposition** of W to be a presentation $W = H_0 \cup H_1 \cup \cdots \cup H_t$, where H_0 is a w-ball and H_i is a handle on $W_{i-1} = \bigcup\{H_j | j \leq i-1\}$.

For a cobordism (W, M_0, M_1), we define a **handle decomposition of** W **on** M_0 to be a presentation $W = C_0 \cup H_1 \cup \cdots \cup H_t$, where C_0 is a collar on M_0 in W (which is a cobordism), and H_i is a handle on the cobordism $W_{i-1} = C_0 \cup \bigcup\{H_j | j \leq i-1\}$

We show that every cobordism has a handle decomposition. For a cobordism (W, M_0, M_1), let (K, K_0) be a triangulation of (W, M_0) and let $A_0, A_1, \ldots, A_r$ be the simplices of $K - K_0$ in order of increasing dimension. Let K'' be a second derived.

We define $C_0 = |N(K_0, K'')|$, $A_1^{**} = |st(a_i, K'')|$, where $a_i = \hat{A}_i$. Then

Theorem 2 (Existence of handle decompositions)**.** *$W = C_0 \cup A_1^{**} \cup \cdots \cup A_r^{**}$ is a handle decomposition of W on M_0 with index$(A_i^{**}) = dim(A_i)$*

Proof. C_0, being a regular neighbourhood, is a collar on M_0. We will find for each A_i^{**} a characteristic map h_i attaching A_i^{**} to W_{i-1}.

Consider the simplicial isomorphism $f_i : A_i^{**} \to st(a_i, K')'$ defined by pseudo-radial projection from a_i. f_i carries $A_i^{**} \cap W_{i-1}$ onto $N = |N(\dot{A}_i, lk(a_i, K')')|$ which is a derived neighbourhood and hence a regular neighbourhood.

Since $(A_i, st(A_i, K))$ is an unknotted ball pair (as the join of (A_i, A_i) and $(lk(A_i, K), \emptyset)$.), there exists a homeomorphism $g_i : I^p \times I^q \to st(A_i, K)$ such that $g_i(I^p \times 0) = A_i$ where dim$A_i = p$ and $p + q = w$. By the Simplicial Neighbourhood Theorem, we may assume that $g_i(\dot{I}^p \times I^q) = N$. Therefore $h_i = f_i^{-1} \circ g_i$ gives us the desired characteristic map. $\square$

Note that the collaring theorem lets us easily obtain the **symmetrical decomposition** $W = C_1 \cup H_1 \cup \cdots H_t \cup C_1$ by adding a collar C_1 to M_1 without altering M_1.

We can go on to define $W_{i+1}^c = C_1 \cup \bigcup\{H_j | j \geq i+1\}$. In this case, we see that H_i can be regarded as a handle H_i^* on W_{i+1}^c with characteristic map $h_i^* = h_i \circ t$ where t is the automorphism of $I^p \times I^q$ interchanging the first p coordinates with the last q. This gives us the **dual decomposition** $W = C_1 \cup H_t^* \cup \cdots \cup H_1^* \cup C_0$ **of** W **on** M_1**.**

5 Moving Handles

We now go on to prove theorems to "control" these handles. First we prove a lemma that allows us to move handles up to ambient isotopy.

Lemma 1. *For ambient isotopic embeddings $f, g : \dot{I}^p \times I^q \to M_1$, there is a homeomorphism h between $W \cup_f H$ and $W \cup_g H$ which is the identity outside a collar on M_1 in W.*

To prove this, we will need another lemma.

Lemma 2. *Let M be a manifold with compact boundary. Then any isotopy H of ∂M extends to an isotopy of M with support in a collar of ∂M.*

Proof. Take a collar $c : \partial M \times I \to M$. We can extend H to $im(C)$ by

$$H'_t(x,s) = \begin{cases} (H_{t-s}(x), s), & \text{for } s \leq t \\ (x,s), & \text{for } s \geq t \end{cases}$$

This extends to the rest of M by the identity, giving us the desired isotopy of M. □

We now proceed to prove Lemma 1.

Proof. There is a covering isotopy $H_t : M_1 \to M_1$. Let c be a collar on M_1. Then H_t extends to ∂W by letting H_t be the identity on M_2. We can then further extend H_t to the whole of W by Lemma 2 so that it is the identity outside c. Let H_1 be the finishing homeomorphism and

$$h = \begin{cases} H_1, & \text{on } W \\ id, & \text{on } I^p \times I^q \end{cases}$$

gives us the desired homeomorphism. □

Using the previous lemma, we show that we can change the order in which certain handles are attached.

Lemma 3 (Reordering Lemma)**.** *Let $W' = W \cup H^{(r)} \cup H^{(s)}$ with $s \leq r$. Then W' is homeomorphic to $W \cup H^{(s)} \cup H^{(r)}$ with $H^{(r)}$ and $H^{(s)}$ disjoint.*

Proof. Consider the attaching map $f : \dot{I}^s \times I^{w-s} \to M_2$ of $H^{(s)}$ where $M_2 = \partial(W \cup H^{(e)}) - M_0$. We will ambient isotope f so that its image is disjoint from $H^{(r)}$. Then by the reordering lemma we can attach the handles in reverse order.

Let S^{s-1} be the a-sphere of $H^{(s)}$ and S^{w-r-1} the b-sphere of $H^{(r)}$. By general position in M_2^{w-1}, we can assume that S^{s-1} and S^{w-r-1} are disjoint. Furthermore, we can choose disjoint regular neighbourhoods N_a and N_b of S^{s-1} and S^{w-r-1}, respectively. But by the S.N.T, the a-tube N_a' and the b-tube N_b' are also regular neighbourhoods, so we may assume that $N_a = N_a'$. Furthermore, there exists an ambient isotopy of M_2 carrying N_b onto N_b'. This isotopy carries N_a off N_b', and therefore carries the image of f off $H^{(r)}$, as we intended.

$\square$

We call a handle decomposition **nice** if the handles are arranged in order of increasing index and handles of the same index are disjoint. Every handle decomposition can be turned into a nice decomposition by successively applying the reordering lemma.

6 Cancelling Handles

We will prove a theorem that asserts that two handles of index i and $i+1$ can be 'cancelled' if they intersect in a nice way. To make this statement precise, we first define the **intersection number** of two manifolds.

Let P^p, Q^q be locally flat submanifolds of M^m such that $p+q=m$, which are transverse. Let $p \in P \cap Q$. By transversality, there exists an embedding $h : I^m \to M$ such that $h(0) = p$, $h^{-1}(P) = I^p \times 0$, and $h^{-1}(Q) = 0 \times I^q$.

We will need the following lemma, which we state but do not prove. A proof can be found in R&S (Lemma 5.11).

Lemma 4. *The orientation class of h is determined by the orientation classes of $h|I^p \times 0$ and $h|0 \times I^q$.*

This allows us to define the **sign** $\epsilon(p)$ of p. Take an orientation class for each of P, Q, and M and choose h so that $h|I^p \times 0$ and $h|0 \times I^q$ are in the orientation classes for P and Q. Define $\epsilon(P)$ to be 1 if h is in the orientation class for M and -1 if not. We define the **intersection number** of P and Q, $\epsilon(P,Q)$ to be $\sum\{\epsilon(p)|p \in P \cap Q\}$.

We move on to study intersections of handles. Suppose $W' = W \cup H^{(r)} \cup H^{(r+1)}$ and let $M_2 = \partial(W \cup H^{(r)}) - M_0$. Then the b-sphere S_1 of $H^{(r)}$ and the a-sphere S_2 of $H^{(r+1)}$ are in complementary dimension im M_2. [2] By general position, we can shift the attaching map of $H^{(r+1)}$ so that S_2 meets S_1 transversally in a finite number of points. The characteristic maps provide standard orientations to S_1, S_2, and the b-tube of $H^{(r)}$. We can now define the **incidence number** $\epsilon(H^{(r+1)}, H^{(r)})$ to be the intersection number $\epsilon(S_1, S_2)$.

We now see a homology interpretation of the incidence number.[3]

[2] $dim(S_1) + dim(S_2) = dim(M_2)$

[3] The definition of the homological degree of g can be found in Appendix A.7 of R&S

Lemma 5. *Let $q : W \cup H^{(r)} \to S^r$ be the topological map sending W to a basepoint $* \in S^r$, collapses $H^{(r)}$ onto its core D^r, and identifies $D^r/\partial D^r$ with $S^r/*$. Then the restriction of q, $g : S_2 \to S^r$ has homological degree $\epsilon(H^{(r+1)}, H^{(r)})$.*

Proof. Consider the attaching map f of $H^{(r+1)}$. The degree of g is unaffected by an isotopy of f. Take a point $p \in S_1 \cap S_2$. Then the characteristic map h for $H^{(r)}$ defines a standard transverse disc $D_p = h(I^r \times p)$ to S_1 at p. By the disc theorem for pairs, we can isotope S_2 rel S_1 to make it agree with D_p near p. We can di this for each intersection. After a further isotopy carrying a standard neighbourhood of S_1 onto the b-tube, we have $S_2 \cap H^{(r)} = \bigcup\{D_p | p \in S_1 \cap S_2\}$. $q|D_p$ is the standard identification of $D_p/\partial D_p$ with $S^r/*$. The result follows from the definition of degree. □

We state a new definition here. If S_1 and S_2 intersect transversally in just one point p, we say that $H^{(r)}$ and $H^{(r+1)}$ are **complementary handles**.

We can now show how to cancel handles of adjacent index.

Lemma 6 (Cancellation lemma: Mk I)**.** *For $W' = W \cup H^{(r)} \cup H^{(r+1)}$ with $H^{(r)}$ and $H^{(r+1)}$ complementary, there exists a homeomorphism $h : W' \to W$ which is the identity outside a neighbourhood of $H^{(r)} \cup H^{(r+1)}$.*

Proof. We may again assume that $S_2 \cap (b\text{-tube of } H^{(r)}) = D_p$, where $S_1 \cap S_2 = p$. By the disc theorem for pairs, we can assume that $h_1(I^r \times B) = h_1(D \times I^{w-r-1})$ where h_1 and h_2 are the characteristic maps of $H^{(r)}$ and $H^{(r+1)}$, respectively, and B, D, are neighbourhoods of p in S_1, S_2, respectively.

By expanding a standard neighbourhood of S_1 onto the b-tube of $H^{(r)}$, we can assume that these are the only intersections of $H^{(r)}$ and $H^{(r+1)}$. W' now shells to W in two steps:
(1) shell $H^{(r)}$ from $h_1(I^r \times (S_1 - B))$,
(2) shell $H^{(r+1)}$ onto $h_2((S_2 - D^0) \times I^{w-r-1})$.
The lemma follows.

□

We can prove a stronger theorem in high dimensions if M_1 is simply connected.

Lemma 7 (Cancellation lemma: Mk II)**.** *Let $W' = W \cup H^{(r)} \cup H^{(r+1)}$, M_1 be simply connected, $w - r \geq 4$, $r \geq 2$, and $w \geq 6$. If $\epsilon(H^{(r+1)}, H^{(r)}) = \pm 1$, then $W' \cong W$.*

Using the same notation as in the last proof, we prove this by finding an ambient isotopy of S_2 that carries S_2 onto S_2' such that $S_2' \cap S_1$ is a point. To do this, we will need the following lemma.

Lemma 8 (Whitney lemma). *Suppose P^p, Q^q are connected locally flat submanifolds of M^m and transverse to each other. Further suppose that either*
(1) $p \geq 3$, $q \geq 3$ and $\pi_1(M) = 0$, or
(2) $p = 2$, $q \geq 3$, and $\pi_1(M - Q) = 0$.
If we have $\epsilon(p) = -\epsilon(q)$ for $p, q \in P \cap Q$, there is an isotopy of M carrying P to P' with P' transverse to Q in M and with $P' \cap Q = P \cap Q - p - q$. The isotopy has support in a compact set which does not meet any other intersection points.

A proof of the Whitney lemma can be found in R&S.

We now prove Lemma 7.

Proof. S_1 is in codimension ≥ 2, and S_2 is in codimension ≥ 3. $M_2 - S_1$ and $M_1 -$ (a-sphere of $H^{(r)}$) deformation retracts onto $M_1 -$ (a-tube of $H^{(r)}$) by the deformation retractions given by using the product structure of $I^r \times I^{w-s}$.

By general position, $M_1 -$ (a-sphere of $H^{(r)}$) is simply connected. We can now use the Whitney lemma to eliminate pairs of oppositely oriented intersection points until S_1 and S_2 intersect transversally in just one point.

The result follows from Lemma 6. □

We can also go the other way, and turn a ball attached to W into a pair of complementary handles of any index.

Lemma 9 (Introduction Lemma). *Suppose $W' = W \cup B^w$ where $B^w \cap W = B \cap M_1 =$ face B_1 of B. Then we can write $W ;= W \cup H^{(r)} \cup H^{(r+1)}$ with $H^{(r)}$ and $H^{(r+1)}$complementary.*
Moreover if $B^r \subset B_1$ is a locally flat disc then we can assume that the a-sphere of $H^{(r)}$ is ∂B^r and that (a-sphere of $H^{(r+1)}$)$\cap W \subset B^r$.

A proof can be found in R&S (6.6 Introduction Lemma).

7 Homology and Handle Decompositions

Given a cobordism (W, M_0, M_1) and a handle decomposition of W on M_0, we can construct a CW complex K attached to M_0 that has the same homotopy type as W, and has one p-cell for each p-handle by the following construction.

We may assume that the decomposition is nice. Suppose inductively that we have defined K_{i-1} and a homotopy equivalence $l_{i-1} : W_{i-1} \to K_{i-1}$, rel M_0.
Let $r_i : H_i \to \mathrm{core}(H_i) \cup a\text{-tube}(H_i)$ be the obvious deformation retraction. Then we can construct $l_i : W_i \to K_i$ by noting that $W_{i-1} \cup_{f_i} H_i$ is homotopy equivalent to $K_{i-1} \cup_{g_i} H_i$, where $g_i = l_{i-1} \circ f_i$, which deformation retracts on $K_{i-1} \cup_{g_i|} I^p$ (where

p = index H_i). We can define $K_i = K_{i-1} \cup_{g_i|} I^p$, which is a CW complex, and we are done.

It turns out that we can define the incidence number $\epsilon(e^{r+1}, e^r)$ for cells e^{r+1}, e^r of a CW complex, and that if e^{r+1}, e^r are cells corresponding to the handles H^{r+1}, H^r in the above construction, $\epsilon(e^{r+1}, e^r) = \epsilon(H^{r+1}, H^r)$. Furthermore, the homology groups of the CW complex can be computed from the incidence numbers. Which means that we can compute the homology groups $H_*(W, M_0)$ from the handle decomposition, since homology groups of homotopy equivalent spaces are isomorphic.[4]

We also state but do not prove a lemma concerning some relative homotopy groups of W.

Lemma 10. *Let* $W = C_0 \cup H_1 \cup \cdots \cup H_t$ *be a nice decomposition and* $W^{(s)} = C_0 \cup \bigcup\{H_i^{(p)} | p \leq s\}$. *Then we have*

$$\pi_1(W, W^{(s)}) = 0 \textit{ for } i \leq s$$

$$\pi_1(W, M^{(s)}) = 0 \textit{ for } i \leq s, n-s-1$$

where $M^{(s)} = \partial W^{(s)} - M_0$

We now proceed to look at the relationship between the homology and cohomology groups of decompositions. Recall that every symmetrical decomposition $W = C_0 \cup H_1 \cup \cdots \cup H_t \cup C_1$ has an associated dual decomposition. We may assume that the decomposition is nice, and therefore that the dual decomposition is also nice. Then we have K, the CW complex associated with the decomposition, and $K*$ the complex associated with the dual decomposition, or just the **dual complex**, (attached to M_1).

We are now ready to prove

Theorem 3. $H_*(W, M_0; \mathbb{Z}_2) \cong H^{w-*}(W, M_1; \mathbb{Z}_2)$.
If W *is orientable,* $H_*(W, M_0; \mathbb{Z}) \cong H^{w-*}(W, M_1; \mathbb{Z})$.

Proof. Using the same notation as above, let $H^{(r)}$, $H^{(r+1)}$, be successive handles, and $H^{(w-r)}$, $H^{(w-r-1)}$ their duals, and e^r, e^r+1, e^{w-r}, e^{w-r-1}, be the corresponding cells of K and $K*$. Then since the a-sphere of $H^{(w-r)}$ = b-sphere of $H^{(r)}$, and similarly for $H^{(r+1)}$, we can see that $\epsilon(H^{(r+1)}, H^{(r)}) = \epsilon(H^{(w-r)}, H^{(w-r-1)})$ mod 2.

Therefore $\epsilon(e^{r+1}, e^r) = \epsilon(e^{w-r}, e^{w-r-1})$ mod 2, and there is an isomorphism between the chain complex of K and the cochain complex of K^* with $\mathbb{Z}_2$ coefficients. This proves the first part of the theorem.

Moving on to the second part, we will use the notation of Lemma5. Suppose that W is orientable. Then each "level" manifold $M_i = \partial W_i - M_0$ is orientable and we have $\epsilon(S_1, S_2) = \pm\epsilon(S_2, S_1)$, implying $\epsilon(e^{r+1}, e^r) = \pm\epsilon(e^{w-r}, e^{w-r-1})$. But

[4] Details on CW complexes and their homology groups can be found in A.7 of R&S

since $\epsilon(S_1, S_2) = (-1)^{w(w-r-1)}\epsilon(S_2, S_1)$ in M_i, and H has the same orientation (=$(-1)^{r(w-r)}$) as $H*$, the signs are all positive and we have the second part of the theorem.

□

The case $M_0 = M_1 = \emptyset$ is called Poincaré duality, and the case $M_0 = \emptyset$ is called Lefschetz duality.

8 Eliminating Handles

We now develop the main weapons we will use to prove the h-cobordism theorem. Handles of low index can be eliminated given certain conditions.

Lemma 11 (Eliminating 0-handles). *Suppose we have a handle decomposition of W on M_0, and there are i_t t-handles for each t. Suppose further that each component of W meets M_0. Then there is another decomposition of W with no 0-handles $(i_1 - i_0)$ 1-handles, and i_t t-handles for $t > 1$.*

Proof. We may assume that the decomposition is nice. Attaching handles of index 2 or higher does not affect connectivity. Therefore every 0-handle is connected to another 0-handle or to C_0 by a 1-handle. But a 0-handle attached to one end of a 1-handle forms a complementary pair that can be cancelled. Therefore every 0-handle can be cancelled. □

Lemma 12 (Eliminating 1-handles). *Suppose W is connected and we are given a handle decomposition of W on M_0 with no 0-handles and i_t t-handles for $t > 0$. Suppose that $\pi_1(W, M_0) = 0$, and $w \geq 6$. Then there is another decomposition with i_t t-handles for $t \neq 1, 3$, no 1-handles and $(i_1 + i_3)$ 3-handles.*

Proof. As usual, we assume that the decomposition is nice. Let (H_1, h_1) be a 1-handle. We will replace H_1 by a 3-handle.

Let $\alpha = h_1(I^1 \times x)$ be an arc in the b-tube of H_1 "parallel" to the core. As in Lemma 3 we can assume by general position that α misses the 2-handles and so lies in $M^{(2)} = \partial W^{(2)} - M_0$. $\pi_1(W^{(2)}, C_0) = 0$, and we can find a map $f : D^2 \to W^{(2)}$ with $f(\partial D^2) = \alpha \cup \beta$ where β lies in C_0.

We can again assume as in Lemma 3, that β is embedded in $M^{(2)}$ disjoint from all 1- and 2-handles. We homotope f rel ∂D^2 into $M^{(2)}$ (We can do this by Lemma 10), and replace f by a locally flat embedded disc D^2 (via general position).

Now the introduction lemma allows us to introduce a complementary 2- and 3-handle pair (H_2, H_3) along a neighbourhood of D^2 so that the a-sphere of $H)2$

is ∂D^2. Then (H_1, H_2) are complementary and can be cancelled. We have now eliminated the 1-handle.

$\square$

Lemma 13 (Elimination of s-handles, $2 \leq s \leq w-4$)**.** *Suppose given a handle decomposition of W on M_0 with no handles of index $< s$ and i_t handles of index t for $t \geq s$. Then, if M_0 is simply connected, $2 \leq s \leq w-4$, $w \geq 6$ and $H_s(W, M_2) = 0$, we can find a new decomposition with the same number of t-handles for $t \neq s$, $s+1$, with no s-handles and with $(i_{s+1} - i_s)$ $(s+1)$-handles.*

To prove this lemma, we will need another lemma. The adding lemma asserts that we can add or subtract the incidence numbers of r-handles with $(r-1)$-handles. We will not prove this lemma, but a proof can be found in R&S (Lemma 6.7).

Lemma 14 (Adding lemma)**.** *Suppose $W' = W \cup_{f_1} H_1 \cup_{f_2} H)2$ with $im(f_1)$ and $im(f_2)$ disjoint and index $H_1 =$ index $H_2 = r$. If $w - r \geq 2$, $r \geq 2$, and M_1 is simply connected, there is an f_3 isotopic to f_2 such that $[f_3] = [f_1] + [f_2]$ and $im(f_1)$ and $im(f_3)$ disjoint. We can also find f_3 such that $[f_3] = [f_2] - [f_1]$*

We can now prove Lemma 13.

Proof. We again assume that the decomposition is nice. As noted in section 6, we can then compute $H_*(W, M_0)$ from the incidence numbers. Take a typical s-handle $H^{(s)}$. Let $H_i^{(s+1)}$ be the $(s+1)$-handles and $n_i = \epsilon(H_i^{(s+1)}, H^{(s)})$. The adding lemma lets us add the $(s+1)$-handles to reduce $\sum |n_i|$ as far as possible.WE do this until only n_1 is nonzero and since $H_s(W, M_0) = 0$ we must have $n_1 = \pm 1$. $H^{(s)}$ and $H_1^{(s+1)}$ are therefore algebraically complementary, and we can cancel them by the cancellation lemma Mk.II.

The result follows by successively eliminating all s-handles.

$\square$

9 The proof of the h-cobordism theorem

We now have the tools to prove the h-cobordism theorem.

Theorem 4 (h-cobordism theorem)**.** *Let (W^w, M_0, M_1) be a cobordism and that*
(1) $\pi_1(M_0) = \pi_1(M_1) = \pi_1(W) = 0$
(2) $H_*(W, M_0) = 0$
(3) $w \geq 6$
Then $W \cong M_0 \times I$.

Proof. Choose a symmetric handle decomposition $W = C_0 \cup H_1 \cup \cdots \cup H_t \cup C_1$. By elimination of 0-handles, we may assume that there are no 0-handles.
By elimination of 1-handles, we may assume that there are no 1-handles (The conditions are easily checked to be met. There are no 0-handles, and by assumptions (1) and (3).
By repeating the above with the dual decomposition of W, we may assume no w- or (w-1)- handles exist.
By elimination of s-handles, ($2 \leq s \leq w-4$), we can eliminate all s-handles for $2 \leq s \leq w-4$. (The conditions for are met by assumption (2) and induction on s)
Applying 6.16 to the dual decomposition lets us eliminate the (w-2) handles. Now we only have (w-3) handles. (Here we need $H_*(W, M_1) = 0$, which follows from assumption (3) and duality.) But $H_{w-3}(W, M_0) = 0$ implies that no (w-3)-handles exist. ($Ker(\partial_{w-3}) = C_{w-3}$ since no (w-4)-handles exist, and $Im(\partial_{w-2}) = 0$ since no (w-2)-handles exist. Therefore $C_{w-3} = 0$ implying no (w-3)-handles exist)[5]
Therefore W has no handles and is homeomorphic to $M_0 \times I$.

□

The high-dimensional Poincaré theorem follows from the h-cobordism theorem as a corollary.

Theorem 5 (High-dimensional Poincaré theorem.). *Let M^n be a closed manifold of the homotopy type of S^n, for $n \geq 5$. Then M^n is (topologically) homeomorphic with S^n.*

Proof. Choose two disjoint n-cubes D_1 and D_2 in M. Define $W_1 = cl(M - D_1)$ and $W = cl(W_1 - D_2)$. Then both W_1 and W are manifolds. (We see this by taking neighbourhoods of D_1, D_2, and noting that $cl(\mathbb{R}^n - I^n)$ is a manifold.)

We will show that W is an h-cobordism between ∂D_1 and ∂D_2.

Observe that ∂D_1, ∂D_2, and M are all simply connected. And since W is homotopy equivalent to M minus two points, $\pi_1(W) = 0$.

We can see that $H_*(W, \partial D_2) \cong H_*(W_1, D_2) \cong \tilde{H}_*(W_1)$ by excision and since D_2 is contractible. But $H_*(W_1) \cong H^{n-*}(W_1, \partial D_1) \cong H^{n-*}(M, D_1)$ by Lefshetz duality and excision. And $H^{n-*}(M, D_1) \cong H^{n-*}(M) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$ since M is a homotopy sphere.

Therefore $\tilde{H}_*(W_1) = 0$, and W is an h-cobordism. We can now apply the h-cobordism theorem to find a homeomorphism $h : W :\to \dot{I}^n \times I^1$, which extends into a homeomorphism between M and $\dot{I}^{n+1}$. This proves the theorem.

□

[5]Using the formulation of H_n, C_n given in A.7 of R&S