Worksheet 10
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Problems come from a variety of sources including Axler and random problems floating online. Only a few are written by me.

linear stuff

1. Suppose V is a real inner product space.

a) Show that $\langle v + w, v - w \rangle = ||v||^2 - ||w||^2$ for all $v, w \in V$.

b) Show that if v, w have the same norm, then $v + w$ and $v - w$ are orthogonal to each other.

c) Show that the diagonals of a rhombus are perpendicular to each other.

2. Suppose V is a nonzero complex vector space, and let $H \subset \mathcal{L}(V)$ be such that $ST = TS$ for all $S, T \in H$. Show that there exists a vector in V that is an eigenvector of every $S \in H$.

3. Suppose $\langle -, - \rangle_1$ and $\langle -, - \rangle_2$ are inner products on a vector space V such that $\langle v, w \rangle_1 = 0$ if and only if $\langle v, w \rangle_2 = 0$. Show that there is a positive number c such that $\langle v, w \rangle_2 = c\langle v, w \rangle_1$ for all $v, w \in V$.

4. Suppose $T \in \mathcal{L}(V)$ is such that $||Tv|| \leq ||v||$ for all $v \in V$. Show that $T - \sqrt{2}I$ is injective.