Theory of Computation

Lab Session 1

January 28 , 2016

INNOPOLIS UNIVERSITY

News

Technical Report

- Students will write a technical report on a specific topic
- Groups are composed of 4 students
- Report cannot exceed 5 pages
- We will suggest a list of topics

The report might include a presentation. We will notify it by next week.

Agenda

- Introduction (rules of the game!)
- Preliminaries - Sets
- Exercises on Formal Languages

Rules of the Game

Lecture: Weekly Lecture.

Laboratory Exercises: Weekly laboratory exercises.

Assessment: Mid-term Exam (40%), Final Exam (40%), and Technical Report by Students (20%).

Rules of the Game

Policy and Procedures on Cheating and Plagiarism

Exam policy: If two or more students are caught communicating for any reason during exams (including mid-terms) they will be asked to leave the room and their exam will be failed. Same will happen for unauthorized use of devices.

Rules of the Game

Policy and Procedures on Cheating and Plagiarism

Report policy: If a submitted report contains work other than student's one it is necessary to explicitly acknowledge the source. It is encouraged to refer and quote other works, but it has to made clear which words and ideas are property and creation of the student, and which ones have come from others (which must not correspond to more than 30% of the work). If two or more reports show evidence of being produced by unauthorized cooperative work, i.e. copied from fellow students, they will be all failed without further investigation on who produced the results and who actually copied.

Preliminaries - Sets

Sets

A finite set can be described, at least in principle, by listing its elements: $A = \{1, 2, 4, 8\}$ says that A is the set whose elements are $1, 2, 4$, and 8.
For infinite (even for finite sets if they have more than just a few elements) sets ellipses $(\ldots)$ are sometimes used to describe how the elements might be listed:

$$B = \{0, 3, 6, 9, \ldots\}$$
$$C = \{13, 14, 15, \ldots, 71\}$$

Sets

We can define a set as a subset of another set characterised by some property. Sets $B = \{0, 3, 6, 9, \ldots\}$ and $C = \{13, 14, 15, \ldots, 71\}$ can be described as

$$B = \{x \in \mathbb{N} \mid \exists_{y \in \mathbb{N}} : 3y = x\}$$
$$C = \{x \in \mathbb{Z} \mid 13 \leq x \leq 71\}$$

It reads: "B is the set of all x in $\mathbb{N}$ such that x is a non-negative integer multiple of 3"

Sets (exercise)

What are the sets D and E?:

$$D = \{\{x\} \mid x \in \mathbb{N} \wedge x \leq 4\}$$
$$E = \{2i \mid i \in \mathbb{N} \wedge i \leq 30\}$$

Sets (exercise)

What are the sets D and E?:

$$D = \{\{x\} \mid x \in \mathbb{N} \wedge x \leq 4\}$$
$$E = \{2i \mid i \in \mathbb{N} \wedge i \leq 30\}$$

$$D = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}\}$$

Sets (exercise)

What are the sets D and E?:

$$D = \{\{x\} \mid x \in \mathbb{N} \wedge x \leq 4\}$$
$$E = \{2i \mid i \in \mathbb{N} \wedge i \leq 30\}$$

$$D = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}\}$$
$$E = \{0, 2, 4, 6, 8, 10, 12, \ldots, 60\}$$

Sets (operations)

- For any set A, the statement that x is an element of A is written $x \in A$.
- $A \subseteq B$ means that A is a subset of B: every element of A is an element of B.
- $\emptyset$ denotes the empty set: the set with no elements.

To show that two sets A and B are the same, we must show that A and B have exactly the same elements, i.e. $A \subseteq B$ and $B \subseteq A$.

Sets (operations)

- For any set A, the statement that x is an element of A is written $x \in A$.
- $A \subseteq B$ means that A is a subset of B: every element of A is an element of B.
- $\emptyset$ denotes the empty set: the set with no elements.

To show that two sets A and B are the same, we must show that A and B have exactly the same elements, i.e. $A \subseteq B$ and $B \subseteq A$. Are the following statements true?

$$\{0,1\} = \{1,0\}$$
$$\{0,1,2,1,0\} = \{1,1,1,1,0,2,2\}$$

Sets (operations)

For two sets A and B, we can define their union $A \cup B$, their intersection $A \cap B$, and their difference $A \backslash B$ (sometimes denoted as $A - B$), as follows[1]:

$$
\begin{aligned}
A \cup B &= \{x \mid x \in A \vee x \in B\} \\
A \cap B &= \{x \mid x \in A \wedge x \in B\} \\
A \backslash B &= \{x \mid x \in A \wedge x \notin B\}
\end{aligned}
$$

If we assume that a set A is a subset of some "universal" set U, then we say that the complement of A is $U \backslash A$. It is often denoted as $\overline{A}$.

$$\overline{A} = U \backslash A = \{x \in U \mid x \notin A\}$$

[1] $\vee$ and $\wedge$ denote the logical 'or' and logical 'and' respectively.

Sets (Union of any number of sets) - Notation

If $A_0, A_1, A_2, \ldots$ are sets, the union of these sets can be denoted as

$$\bigcup\{A_i \mid i \geq 0\} = \{x \mid x \in A_i \text{for at least one } i \text{ with } i \geq 0\}$$

or

$$\bigcup_{i=0}^{\infty} A_i$$

Sets (Power Sets)

For a set A, the set of all subsets of A is called the power set. sometimes denoted as 2^A.

Sets (Power Sets)

For a set A, the set of all subsets of A is called the power set.
sometimes denoted as 2^A.
What is the power set of set $\{a, b, c\}$?

Sets (Power Sets)

For a set A, the set of all subsets of A is called the power set. sometimes denoted as 2^A.
What is the power set of set $\{a, b, c\}$?

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Sets (Power Sets)

For a set A, the set of all subsets of A is called the power set. sometimes denoted as 2^A.
What is the power set of set $\{a, b, c\}$?

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

For a set A, how many elements does the set 2^A have exactly?

Sets (Power Sets)

For a set A, the set of all subsets of A is called the power set.
sometimes denoted as 2^A.
What is the power set of set $\{a, b, c\}$?

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

For a set A, how many elements does the set 2^A have exactly?
2^n elements, where n is the cardinality of A.

Exercise Session - Languages

Notation and Terminology

Alphabet: a finite set of symbols, e.g. $\{a, b\}$, or $\{0, 1\}$. Normally denoted by Σ

String: a string over an alphabet (Σ) is a finite sequence of symbols in Σ.

length: the length of a string x, denoted as $|x|$, is the number of occurrences of symbols in x.

empty string: is the null string over Σ. It is denoted as ϵ. By definition, $|\epsilon| = 0$

Concatenation of strings

If x and y are two string over an alphabet, the concatenation xy (sometimes denoted as $x \cdot y$) consists of the symbols of x followed by those of y:

$$\begin{aligned} x &= ab \\ y &= bab \\ xy &= abbab \end{aligned}$$

Concatenation is an associative operation: $(xy)z = x(yz)$ for all possible strings x, y, and z.

Kleene Star

- It is a unary operator that applies to a set of symbols or a set of strings. It is denoted as $*$.
- If Σ is an alphabet then Σ^* is the set of all strings over symbols in Σ, including the empty string.

If A and B are the sets $\{a, b, c\}$ and $\{0, 1\}$, respectively, then

$$A^* = \{\epsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, \ldots\}$$
$$B^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, \ldots\}$$

Languages

- A language is a set of strings over an alphabet.
- Formally, a language L over an alphabet Σ is a subset of Σ^*.

If A is the alphabet $\{a, b, c\}$. Then
$A^* = \{\epsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, \ldots\}$
Examples of languages over A are

$$\begin{aligned}
L_1 &= \{c, bb, ca\} \\
L_2 &= \{a\} \\
L_3 &= \{\epsilon, bb, aaaaa\} \\
L_4 &= \{aac, bbbb, ccaab\}
\end{aligned}$$

Constructing new Languages

Languages are sets.

- Operations on languages are ways of constructing new languages: for two languages L_1 and L_2 over the alphabet Σ, $L_1 \cup L_2$, $L_1 \cap L_2$, and $L_1 \setminus L_2$ are also languages over Σ.
- String operation of concatenation is also used to construct new languages: if L_1 and L_2 are both languages over Σ, the concatenation of L_1 and L_2 is the language

$$L_1 L_2 = \{xy \mid x \in L_1 \wedge y \in L_2\}$$

Constructing new Languages

Languages are sets.

- Operations on languages are ways of constructing new languages: for two languages L_1 and L_2 over the alphabet Σ, $L_1 \cup L_2$, $L_1 \cap L_2$, and $L_1 \backslash L_2$ are also languages over Σ.
- String operation of concatenation is also used to construct new languages: if L_1 and L_2 are both languages over Σ, the concatenation of L_1 and L_2 is the language

$$L_1 L_2 = \{xy \mid x \in L_1 \wedge y \in L_2\}$$

Example:

$$\{a, aa\}\{\emptyset, b, ab\} = \{a, ab, aab, aa, aaab\}$$

Constructing new Languages

Languages are sets.

- Operations on languages are ways of constructing new languages: for two languages L_1 and L_2 over the alphabet Σ, $L_1 \cup L_2$, $L_1 \cap L_2$, and $L_1 \backslash L_2$ are also languages over Σ.
- String operation of concatenation is also used to construct new languages: if L_1 and L_2 are both languages over Σ, the concatenation of L_1 and L_2 is the language

$$L_1 L_2 = \{xy \mid x \in L_1 \wedge y \in L_2\}$$

Example:

$$\{a, aa\}\{\emptyset, b, ab\} = \{a, ab, aab, aa, aaab\}$$

Is this statement true?

$$L_1 L_2 = L_2 L_1$$

Exponential notation

The concatenation of k copies of a single symbol a, a single string s, or a single language L is defined as:
If $k > 0$, then

$$a^k = aa \ldots a$$

where there are k occurrences of a, similarly for s^k and L^k. In the case where L is simply the alphabet Σ,

$$\Sigma^k = \{x \in \Sigma^* \mid |x| = k\}$$

Exponential notation

The concatenation of k copies of a single symbol a, a single string s, or a single language L is defined as:
If $k > 0$, then

$$a^k = aa \ldots a$$

where there are k occurrences of a, similarly for s^k and L^k. In the case where L is simply the alphabet Σ,

$$\Sigma^k = \{x \in \Sigma^* \mid |x| = k\}$$

Example:

$$\Sigma = \{0, 1\}$$
$$\Sigma^2 = \{00, 01, 10, 11\}$$

Exercises (1)

Construct the power set for the following sets:

i $\{a, b\}$

ii $\{0, 1\} \cup \{1, 2\}$

iii $\{z\}$

iv $\{0, 1, 2, 3, 4\} \cap \{1, 3, 5, a\}$

v $\{0, 1, 2, 3\} \setminus \{1, 3, 5, a\}$

vi $\emptyset$

Determine the following languages over the alphabet $\Sigma = \{0, 1\}$

vii Σ^0

viii Σ^4

ix 2^Σ

Solution (1)

Construct the power set for the following sets:

Solution (1)

Construct the power set for the following sets:

i $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution (1)

Construct the power set for the following sets:

i $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

ii $2^{\{0,1\} \cup \{1,2\}} = 2^{\{0,1,2\}} =$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Solution (1)

Construct the power set for the following sets:

i $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

ii $2^{\{0,1\}\cup\{1,2\}} = 2^{\{0,1,2\}} =$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

iii $2^{\{z\}} = \{\emptyset, \{z\}\}$

Solution (1)

Construct the power set for the following sets:

i $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

ii $2^{\{0,1\} \cup \{1,2\}} = 2^{\{0,1,2\}} = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

iii $2^{\{z\}} = \{\emptyset, \{z\}\}$

iv $2^{\{0,1,2,3,4\} \cap \{1,3,5,a\}} = 2^{\{1,3\}} = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$

Solution (1)

Construct the power set for the following sets:

i $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

ii $2^{\{0,1\}\cup\{1,2\}} = 2^{\{0,1,2\}} =$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

iii $2^{\{z\}} = \{\emptyset, \{z\}\}$

iv $2^{\{0,1,2,3,4\}\cap\{1,3,5,a\}} = 2^{\{1,3\}} = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$

v $2^{\{0,1,2,3\}\setminus\{1,3,5,a\}} = 2^{\{0,2\}} = \{\emptyset, \{0\}, \{2\}, \{0, 2\}\}$

Solution (1)

Construct the power set for the following sets:

i $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

ii $2^{\{0,1\}\cup\{1,2\}} = 2^{\{0,1,2\}} =$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

iii $2^{\{z\}} = \{\emptyset, \{z\}\}$

iv $2^{\{0,1,2,3,4\}\cap\{1,3,5,a\}} = 2^{\{1,3\}} = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$

v $2^{\{0,1,2,3\}\setminus\{1,3,5,a\}} = 2^{\{0,2\}} = \{\emptyset, \{0\}, \{2\}, \{0, 2\}\}$

vi $2^{\emptyset} = \{\emptyset\}$

Solution (1)

Determine the following languages over the alphabet $\Sigma = \{0, 1\}$

Solution (1)

Determine the following languages over the alphabet $\Sigma = \{0, 1\}$

vii $\Sigma^0 = \{\epsilon\}$

Solution (1)

Determine the following languages over the alphabet $\Sigma = \{0, 1\}$

vii $\Sigma^0 = \{\epsilon\}$

viii

$$\Sigma^4 = \{0000, 0001, 0010, 0011, \\ 0100, 0101, 0110, 0111, \\ 1000, 1001, 1010, 1011, \\ 1100, 1101, 1110, 1111\}$$

Solution (1)

Determine the following languages over the alphabet $\Sigma = \{0, 1\}$

vii $\Sigma^0 = \{\epsilon\}$

viii

$$\Sigma^4 = \{0000, 0001, 0010, 0011, \\ 0100, 0101, 0110, 0111, \\ 1000, 1001, 1010, 1011, \\ 1100, 1101, 1110, 1111\}$$

ix $2^\Sigma = \{\{\epsilon\}, \{0\}, \{1\}, \{0, 1\}\}$

Exercises (2)

Find a possible alphabet for the following languages[2]

i The language $L = \{oh, ouch, ugh\}$

ii The language $L = \{apple, pear, 4711\}$

iii The language of all binary strings

Determine what the Kleene star operation over the following alphabets produces:

iv $\Sigma = \{0, 1\}$

v $\Sigma = \{a\}$

vi $\Sigma = \emptyset$ (the empty alphabet)

[2] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

ii The language $L = \{apple, pear, 4711\}$:
$\Sigma = \{a, p, l, e, a, r, 4, 7, 1\}$

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

ii The language $L = \{apple, pear, 4711\}$:
$\Sigma = \{a, p, l, e, a, r, 4, 7, 1\}$

iii The language of all binary strings: $\Sigma = \{0, 1\}$

[3]A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

ii The language $L = \{apple, pear, 4711\}$:
$\Sigma = \{a, p, l, e, a, r, 4, 7, 1\}$

iii The language of all binary strings: $\Sigma = \{0, 1\}$

Determine what the Kleene star operation over the following alphabets produces:

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

ii The language $L = \{apple, pear, 4711\}$:
$\Sigma = \{a, p, l, e, a, r, 4, 7, 1\}$

iii The language of all binary strings: $\Sigma = \{0, 1\}$

Determine what the Kleene star operation over the following alphabets produces:

iv $\Sigma = \{0, 1\}$: All binary strings

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

ii The language $L = \{apple, pear, 4711\}$:
$\Sigma = \{a, p, l, e, a, r, 4, 7, 1\}$

iii The language of all binary strings: $\Sigma = \{0, 1\}$

Determine what the Kleene star operation over the following alphabets produces:

iv $\Sigma = \{0, 1\}$: All binary strings

v $\Sigma = \{a\}$: All strings which contains nothing but a's

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

ii The language $L = \{apple, pear, 4711\}$:
$\Sigma = \{a, p, l, e, a, r, 4, 7, 1\}$

iii The language of all binary strings: $\Sigma = \{0, 1\}$

Determine what the Kleene star operation over the following alphabets produces:

iv $\Sigma = \{0, 1\}$: All binary strings

v $\Sigma = \{a\}$: All strings which contains nothing but a's

vi $\Sigma = \emptyset$ (the empty alphabet): the language that contains only the empty string

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Exercises (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$

ii $L = \Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \dots\}$

Assuming that $\Sigma = \{0, 1\}$, construct complement languages for the following:

iii $\overline{\{010, 101, 11\}}$

iv $\overline{\Sigma^* \backslash \{110\}}$

State the following languages explicitly

v $2^{\{a,b\}} \backslash 2^{\{a,c\}}$

vi $\{x \in \mathbb{N} \mid \exists_{y \in \mathbb{N}} : y < 10 \wedge (y + 2 = x)\}$ ($\mathbb{N}$ is the set of all non-negative integers)

Solution (3)

State the alphabet Σ for the following languages:

Solution (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$
  $\Sigma = \{0, 1\}$

Solution (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$
  $\Sigma = \{0, 1\}$

ii $L = \Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \dots\}$
  $\Sigma = \{a\}$

Solution (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$
$\Sigma = \{0, 1\}$

ii $L = \Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \ldots\}$
$\Sigma = \{a\}$

Assuming that $\Sigma = \{0, 1\}$, construct complement languages for the following:

Solution (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$
$\Sigma = \{0, 1\}$

ii $L = \Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \ldots\}$
$\Sigma = \{a\}$

Assuming that $\Sigma = \{0, 1\}$, construct complement languages for the following:

iii $\overline{\{010, 101, 11\}}$
$L =$
$\{\epsilon, 0, 1, 00, 01, 10, 000, 001, 011, 100, 110, 111, 0000, 0001, \ldots\}$

Solution (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$
  $\Sigma = \{0, 1\}$

ii $L = \Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \ldots\}$
  $\Sigma = \{a\}$

Assuming that $\Sigma = \{0, 1\}$, construct complement languages for the following:

iii $\overline{\{010, 101, 11\}}$
  $L =$
  $\{\epsilon, 0, 1, 00, 01, 10, 000, 001, 011, 100, 110, 111, 0000, 0001, \ldots\}$

iv $\overline{\Sigma^* \backslash \{110\}}$
  $L = \{110\}$

Solution (3)

State the following languages explicitly

Solution (3)

State the following languages explicitly

- v $2^{\{a,b\}} \backslash 2^{\{a,c\}}$

  $L = \{\{b\}, \{a, b\}\}$

Solution (3)

State the following languages explicitly

v $2^{\{a,b\}} \backslash 2^{\{a,c\}}$
$L = \{\{b\}, \{a, b\}\}$

vi $\{x \in \mathbb{N} \mid \exists_{y \in \mathbb{N}} : y < 10 \wedge (y + 2 = x)\}$ ($\mathbb{N}$ is the set of all non-negative integers)

Solution (3)

State the following languages explicitly

v $2^{\{a,b\}} \setminus 2^{\{a,c\}}$
$L = \{\{b\}, \{a, b\}\}$

vi $\{x \in \mathbb{N} \mid \exists_{y \in \mathbb{N}} : y < 10 \wedge (y + 2 = x)\}$ ($\mathbb{N}$ is the set of all non-negative integers)
$L = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$