Theory of Computation

Lab Session 8

March 17, 2016

INNOPOLIS UNIVERSITY

Agenda

Turing Machine:

- formal definition;
- example;
- exercises.

Formal definition

Turing Machine

A Turing Machine (TM) with k-tapes is a tuple

$$T = \langle Q, I, \Gamma, \delta, q_0, Z_0, F \rangle$$

where

Q is a finite set of states;
I is the input language;
Γ is the memory alphabet;
$\delta :$ is the transition function;
$q_0 \in Q$ is the initial state;
$Z_0 \in \Gamma$ is the initial memory symbol;
$F \subseteq Q$ is the set of final states.

Transition function

The transition function is defined as

$$\delta : (Q - F) \times (I \cup \{\_\}) \times (\Gamma \cup \{\_\})^k \rightarrow Q \times (\Gamma \cup \{\_\})^k \times \{R, L, S\}^{k+1}$$

where elements of $\{R, L, S\}$ indicate “directions” of the head of the TM:

$$\begin{aligned} R &: \text{ move the head one position to the right;} \\ L &: \text{ move the head one position to the left;} \\ S &: \text{ stand still.} \end{aligned}$$

Remarks:

- the transition function can be partial;
- no transition outgoing from the final states;
- the symbol $\_ \notin \Gamma \cup I$ is a special blank symbol on the tapes.

Moves

Moves are based on

- one symbol read from the input tape,
- k symbols, one for each memory tape,
- state of the control device.

Actions

- Change state.
- Write a symbol replacing the one read on each memory tape.
- Move the $k+1$ heads.

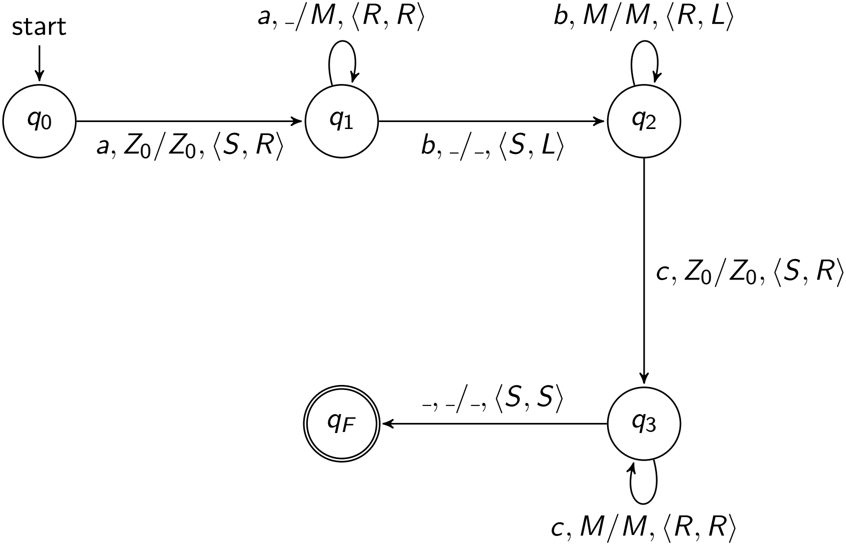
Moves: graphically

$$q \xrightarrow{i, \langle A_1, A_2, \ldots, A_k \rangle / \langle A'_1, A'_2, \ldots, A'_k \rangle, \langle M_0, M_1, \ldots, M_k \rangle} q'$$

- $q \in Q - F$ and $q' \in Q$
- i is the input symbol,
- A_j is the symbol read from the j^{th} memory tape,
- A'_j is the symbol replacing A_j,
- M_0 is the direction of the head of the input tape,
- M_j is the direction of the head of the j^{th} memory tape.

where $1 \leq j \leq k$

Example



start
q_0
q_1
q_2
q_3
q_F
$a, Z_0/Z_0, \langle S, R\rangle$
$a, \_/M, \langle R, R\rangle$
$b, \_/\_, \langle S, L\rangle$
$b, M/M, \langle R, L\rangle$
$c, Z_0/Z_0, \langle S, R\rangle$
$c, M/M, \langle R, R\rangle$
$\_, \_/\_, \langle S, S\rangle$

Configuration

A configuration (a snapshot) c of a TM with k memory tapes is the following $(k+2)$-tuple:

$$c = \langle q, x{\uparrow}y, \alpha_1{\uparrow}\beta_1, \ldots, \alpha_k{\uparrow}\beta_k \rangle$$

where

- $q \in Q$
- $x \in (I \cup \{\_\})^*$, $y = y' \cdot \_$ with $y' \in I^*$
- $\alpha_r \in (\Gamma \cup \{\_\})^*$ and $\beta'_r = \beta'_r \cdot \_$ with $\beta'_r \in \Gamma^*$ and $1 \leq r \leq k$
- $\uparrow \notin I \cup \Gamma$

Acceptance condition

If $T = \langle Q, I, \Gamma, \delta, q_0, Z_0, F\rangle$ is a TM and $s \in I^*$, s is accepted by T if

$$c_0 \vdash^* c_F$$

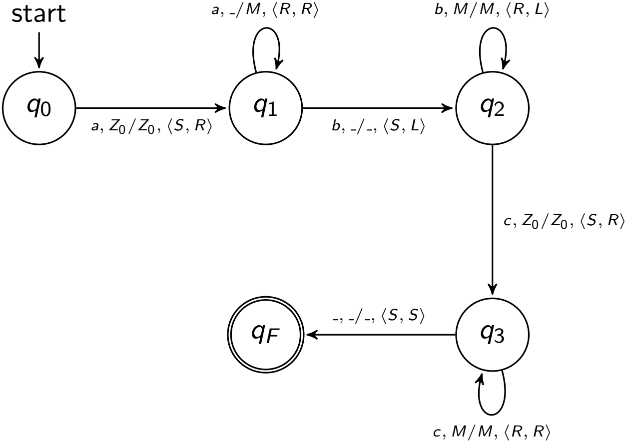
where

1. c_0 is an initial configuration defined as $c_0 = \langle q_0, \uparrow s, \uparrow Z_0, \ldots, \uparrow Z_0\rangle$ where
   - $x = \epsilon$
   - $y = s\_$
   - $\alpha_r = \epsilon$, $\beta_r = Z_0$, for any $1 \leq r \leq k$.
2. c_F is a final configuration defined as $c_F = \langle q, s'\uparrow y, \alpha_1\uparrow\beta_1, \ldots, \alpha_k\uparrow\beta_k\rangle$ where
   - $q \in F$
   - $x = s'$

$L(T) = \{s \in I^* \mid x \text{ is accepted by } T\}$

Example: language *AnBnCn*

A TM T that recognises the language $AnBnCn = \{a^n b^n c^n \mid n > 0\}$



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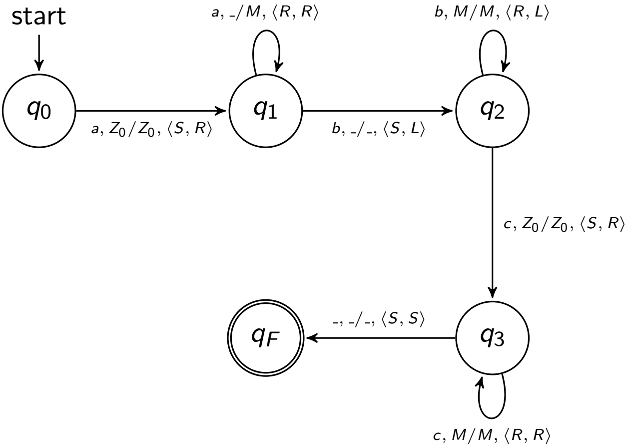
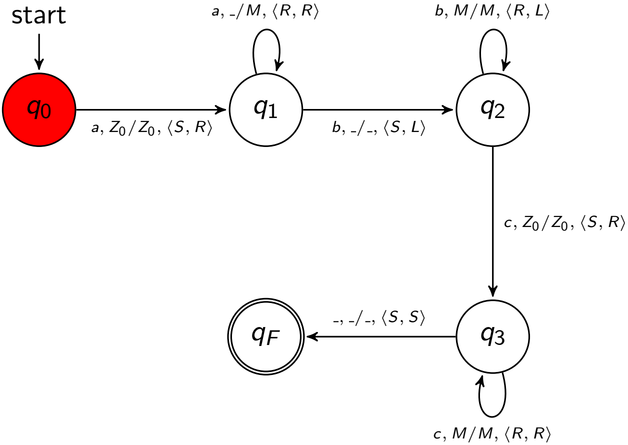


Is the string *aabbcc* recognised by T?

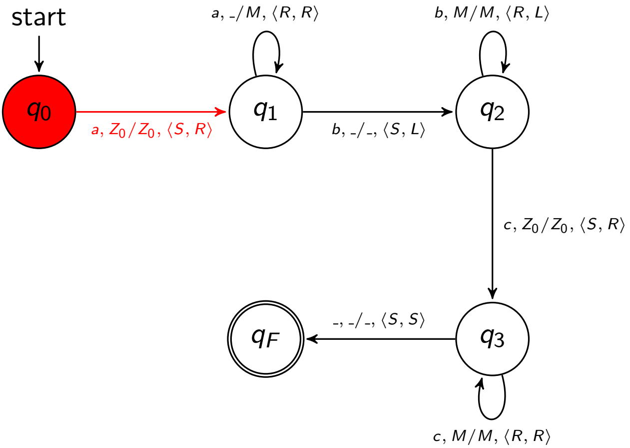
Example: language *AnBnCn*



Initial Configuration:

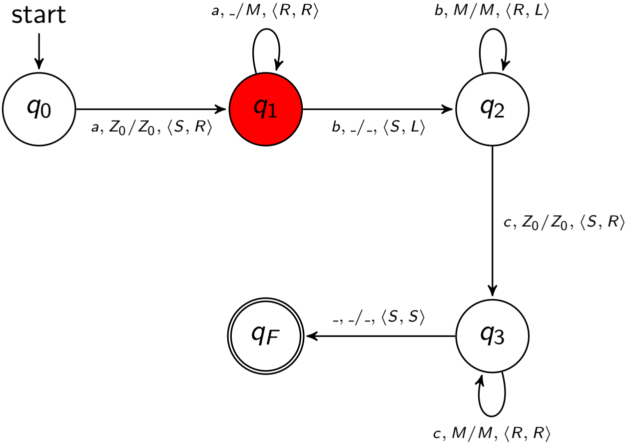
$$\langle q_0, \uparrow aabbcc, \uparrow Z_0\rangle$$

Example: language *AnBnCn*



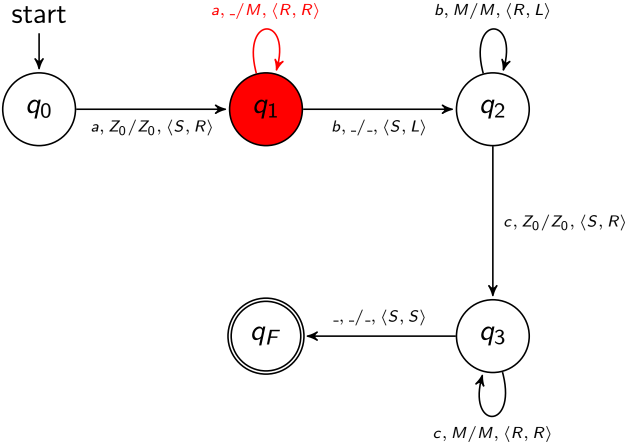
$$\langle q_0, \uparrow aabbcc, \uparrow Z_0\rangle \vdash$$

Example: language *AnBnCn*



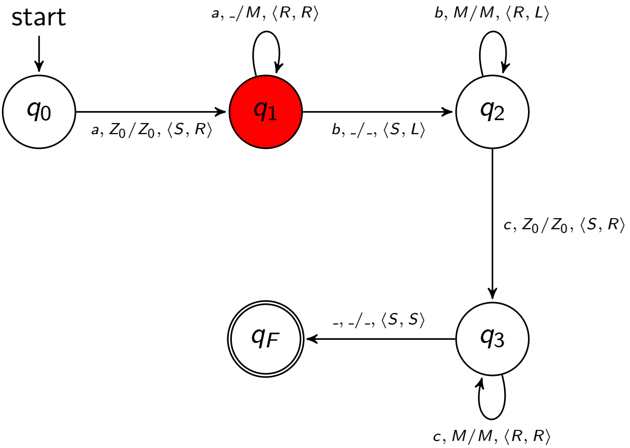
$$\langle q_0, \uparrow aabbcc, \uparrow Z_0\rangle \vdash \langle q_1, \uparrow aabbcc, Z_0\uparrow\rangle$$

Example: language *AnBnCn*



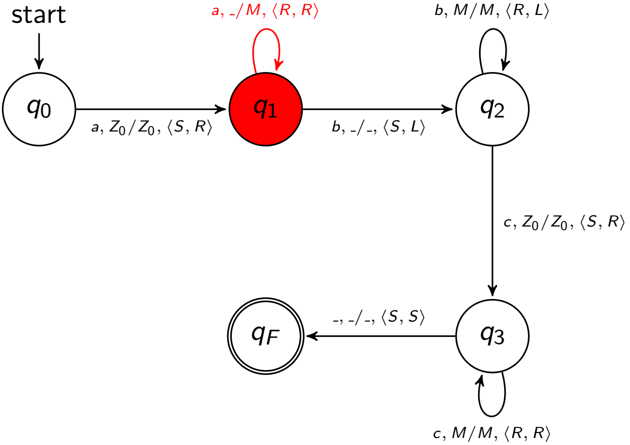
$$\ldots \vdash \langle q_1, \uparrow aabbcc, Z_0\uparrow\rangle \vdash$$

Example: language *AnBnCn*



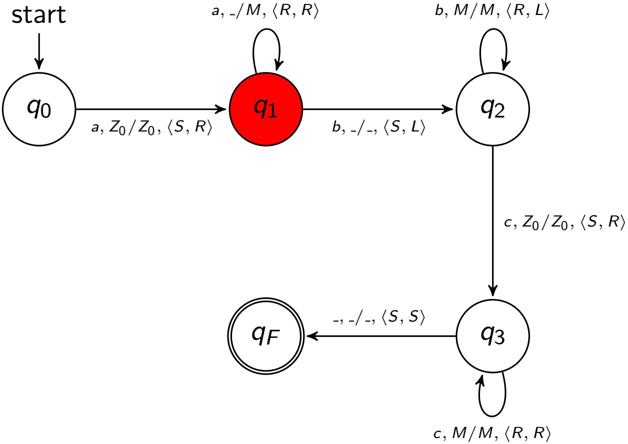
$$\ldots \vdash \langle q_1, \uparrow aabbcc, Z_0\uparrow\rangle \vdash \langle q_1, a\uparrow abbcc, Z_0M\uparrow\rangle$$

Example: language *AnBnCn*



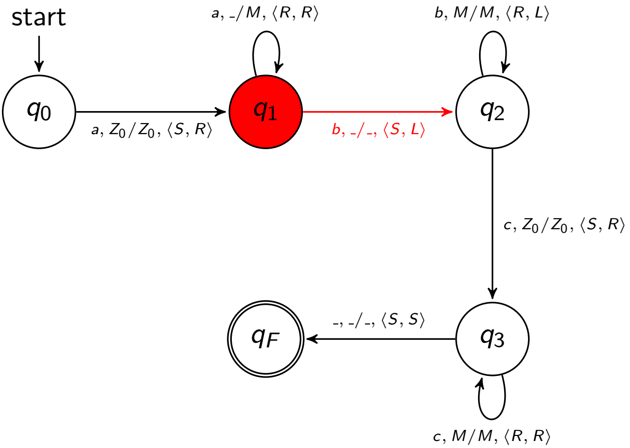
$$\ldots \vdash \langle q_1, a{\uparrow}abbcc, Z_0M{\uparrow}\rangle$$

Example: language *AnBnCn*



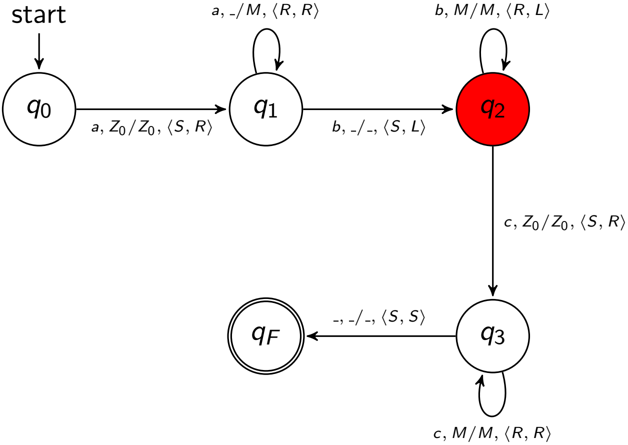
$$\ldots \vdash \langle q_1, a{\uparrow}abbcc, Z_0M{\uparrow}\rangle \vdash \langle q_1, aa{\uparrow}bbcc, Z_0MM{\uparrow}\rangle$$

Example: language *AnBnCn*



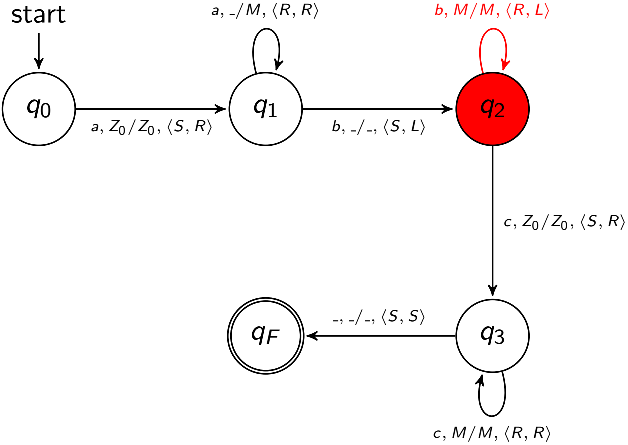
$$\ldots \vdash \langle q_1, aa{\uparrow}bbcc, Z_0MM{\uparrow}\rangle$$

Example: language *AnBnCn*



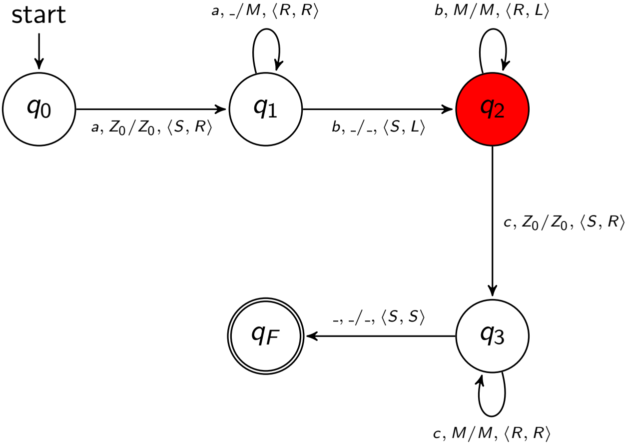
$$\ldots \vdash \langle q_1, aa{\uparrow}bbcc, Z_0MM{\uparrow}\rangle \vdash \langle q_2, aa{\uparrow}bbcc, Z_0M{\uparrow}M\rangle$$

Example: language $AnBnCn$



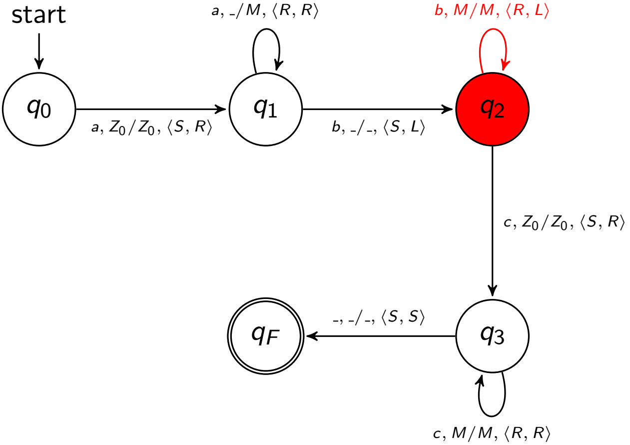
$$\ldots \vdash \langle q_2, aa{\uparrow}bbcc, Z_0M{\uparrow}M \rangle$$

Example: language *AnBnCn*



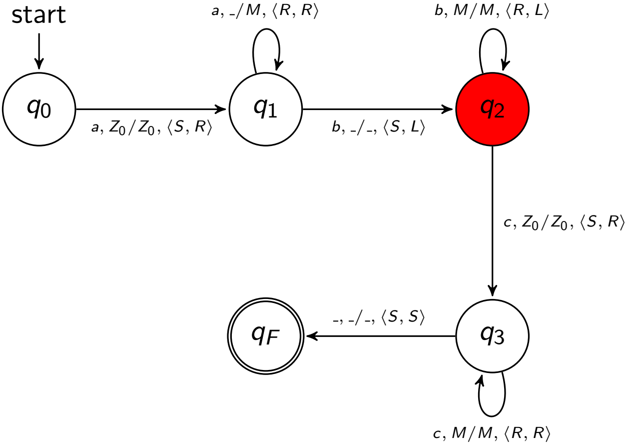
$$\ldots \vdash \langle q_2, aa{\uparrow}bbcc, Z_0M{\uparrow}M\rangle \vdash \langle q_2, aab{\uparrow}bcc, Z_0{\uparrow}MM\rangle$$

Example: language *AnBnCn*



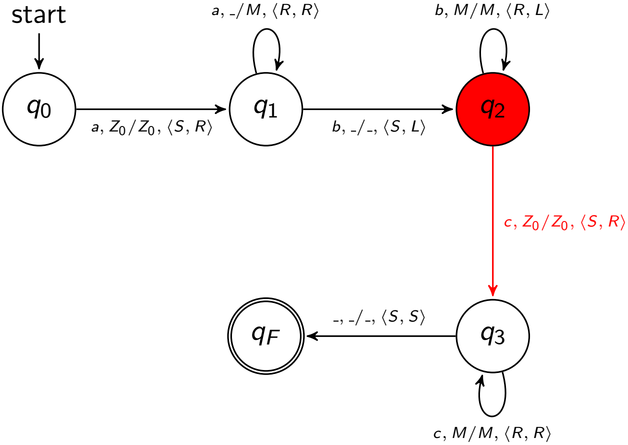
$$\ldots \vdash \langle q_2, aab{\uparrow}bcc, Z_0{\uparrow}MM\rangle$$

Example: language *AnBnCn*



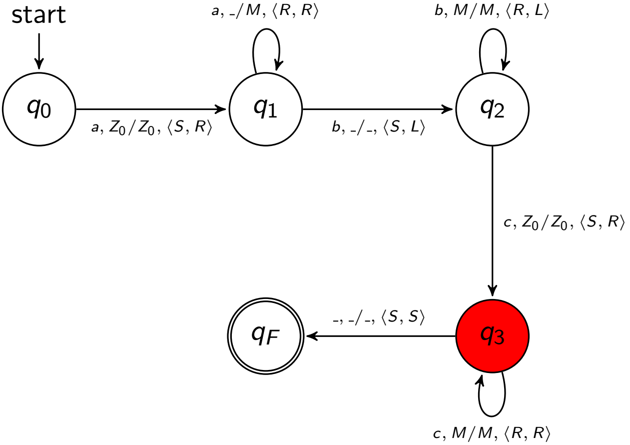
$$\ldots \vdash \langle q_2, aab{\uparrow}bcc, Z_0{\uparrow}MM\rangle \vdash \langle q_2, aabb{\uparrow}cc, {\uparrow}Z_0MM\rangle$$

Example: language *AnBnCn*



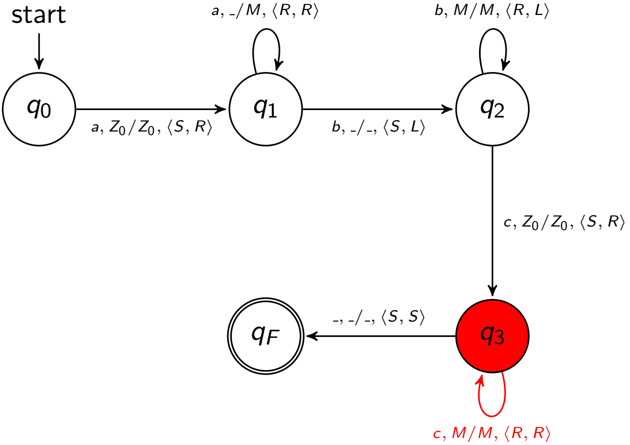
$$\ldots \vdash \langle q_2, aabb{\uparrow}cc, {\uparrow}Z_0MM\rangle$$

Example: language *AnBnCn*



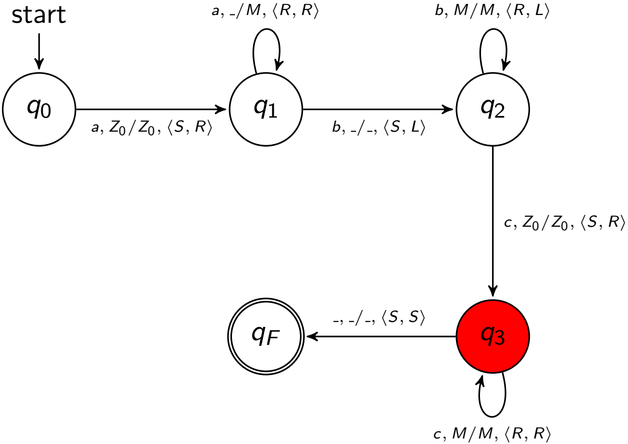
$$\ldots \vdash \langle q_2, aabb{\uparrow}cc, {\uparrow}Z_0MM \rangle \vdash \langle q_3, aabb{\uparrow}cc, Z_0{\uparrow}MM \rangle$$

Example: language $AnBnCn$



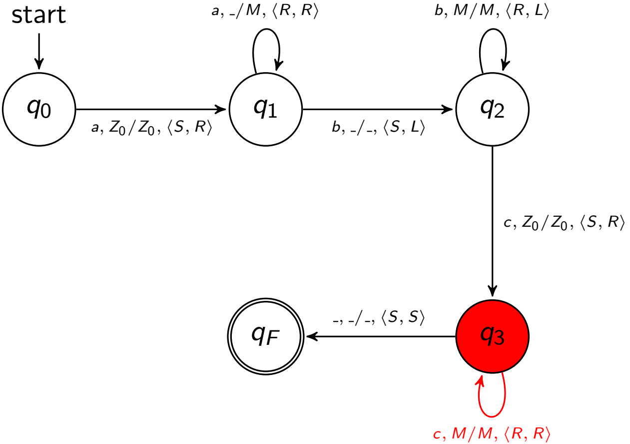
$$\ldots \vdash \langle q_3, aabb{\uparrow}cc, Z_0{\uparrow}MM\rangle$$

Example: language *AnBnCn*



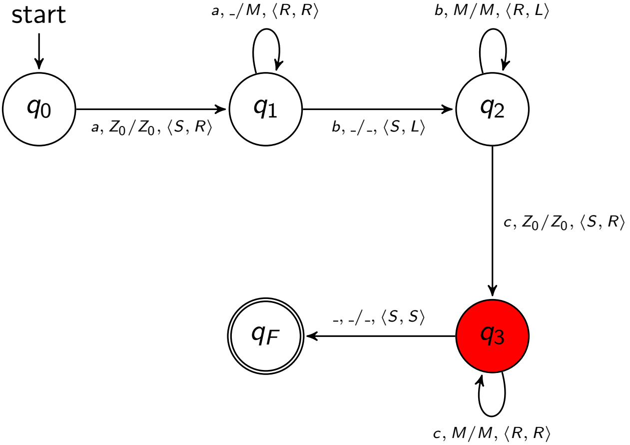
$$\ldots \vdash \langle q_3, aabb{\uparrow}cc, Z_0{\uparrow}MM\rangle \vdash \langle q_3, aabbc{\uparrow}c, Z_0M{\uparrow}M\rangle$$

Example: language *AnBnCn*



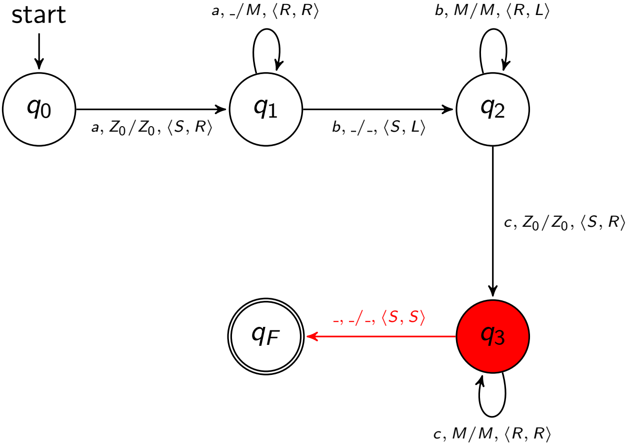
$$\ldots \vdash \langle q_3, aabbc{\uparrow}c, Z_0M{\uparrow}M\rangle$$

Example: language *AnBnCn*



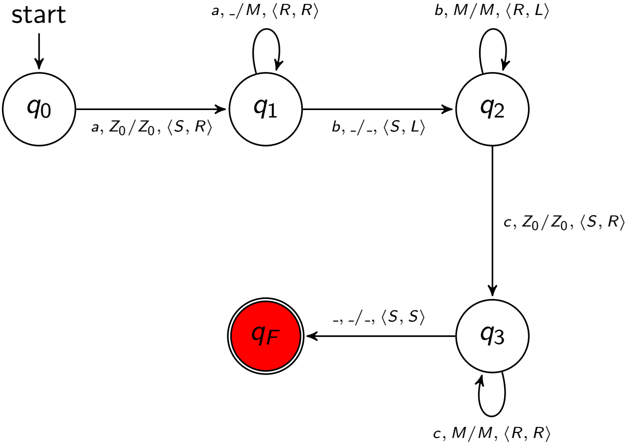
$$\ldots \vdash \langle q_3, aabbc{\uparrow}c, Z_0M{\uparrow}M\rangle \vdash \langle q_3, aabbcc{\uparrow}, Z_0MM{\uparrow}\rangle$$

Example: language *AnBnCn*



$$\ldots \vdash \langle q_3, aabbcc\uparrow, Z_0MM\uparrow\rangle$$

Example: language *AnBnCn*



$$\ldots \vdash \langle q_3, aabbcc\uparrow, Z_0MM\uparrow\rangle \vdash \langle q_F, aabbcc\uparrow, Z_0MM\uparrow\rangle$$

Example: language $AnBnCn$

Is the string $aabbcc$ recognised by T?
Yes, we found:

$$\langle q_0, \uparrow aabbcc, \uparrow Z_0 \rangle \vdash^* \langle q_F, aabbcc\uparrow, Z_0 MM\uparrow \rangle$$

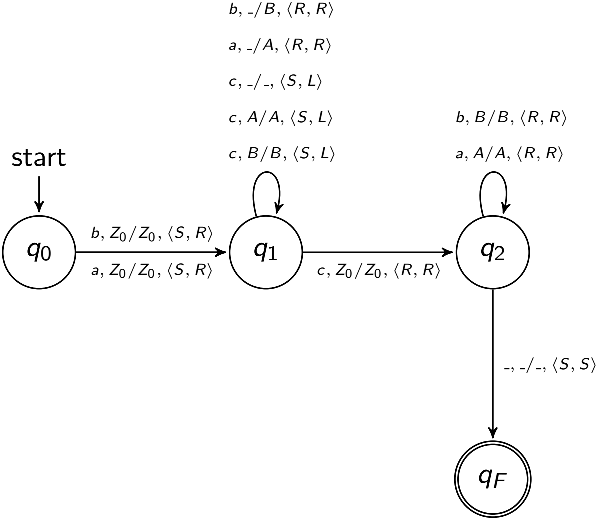
Exercises

Build TMs that recognise the following languages:

- $L_1 = \{wcw \mid w \in \{a, b\}^*\}$
- $L_2 = \{wcw^R \mid w \in \{a, b\}^*\}$, where w^R is the reversed string w.
- $L_3 = \{a^n b^n | n \geq 0\} \cup \{a^n b^{2n} | n \geq 0\}$ (the homework)

Solution (1)

TM that recognises the language $L_1 = \{wcw \mid w \in \{a, b\}^*\}$



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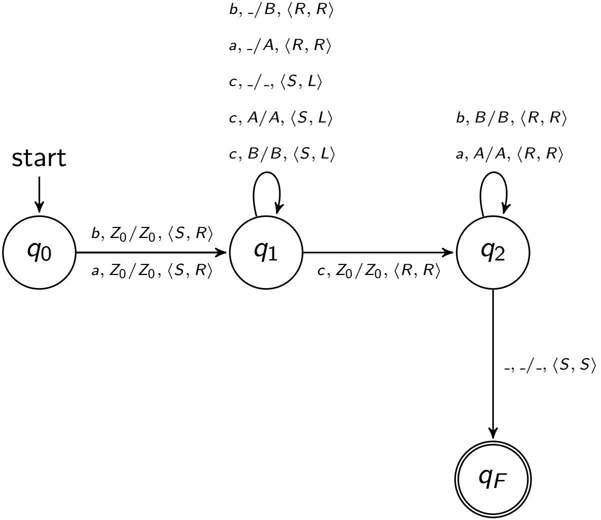


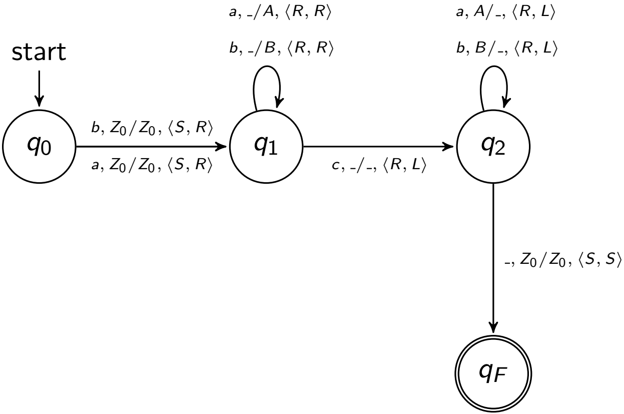
Is the string *abbcabb* recognised by the TM above?

Is the string *abbcabb* recognised by the TM?

$$
\begin{array}{l}
\langle q_0, \uparrow abbcabb, \uparrow Z_0 \rangle \vdash \\
\langle q_1, \uparrow abbcabb, Z_0 \uparrow \rangle \vdash \\
\langle q_1, a \uparrow bbcabb, Z_0 A \uparrow \rangle \vdash \\
\langle q_1, ab \uparrow bcabb, Z_0 AB \uparrow \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 ABB \uparrow \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 AB \uparrow B \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 A \uparrow BB \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 \uparrow ABB \rangle \vdash \\
\langle q_1, abb \uparrow cabb, \uparrow Z_0 ABB \rangle \vdash \\
\langle q_2, abbc \uparrow abb, Z_0 \uparrow ABB \rangle \vdash \\
\langle q_2, abbca \uparrow bb, Z_0 A \uparrow BB \rangle \vdash \\
\langle q_2, abbcab \uparrow b, Z_0 AB \uparrow B \rangle \vdash \\
\langle q_2, abbcabb \uparrow, Z_0 ABB \uparrow \rangle \vdash \\
\langle q_F, abbcabb \uparrow, Z_0 ABB \uparrow \rangle
\end{array}
$$

Solution (2)

TM that recognises the language $L_2 = \{wcw^R \mid w \in \{a, b\}^*\}$, where w^R is the reversed string w.



Solution (2)

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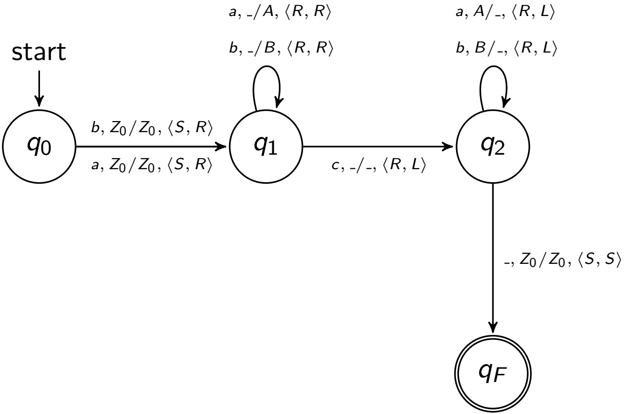


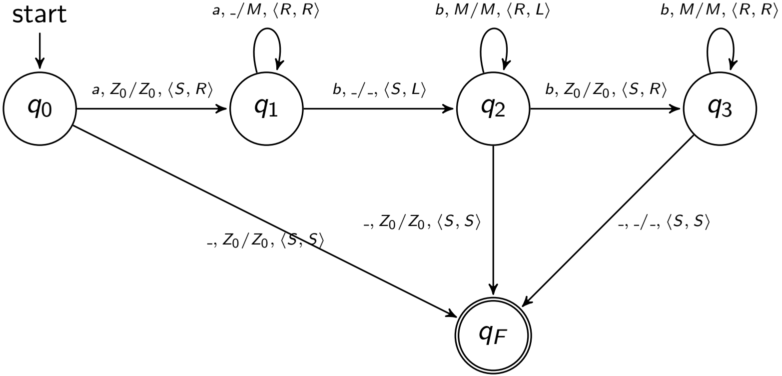
Is the string *abbcbba* recognised by the TM above?

Is the string *abbcbba* recognised by the TM?

$$
\begin{array}{l}
\langle q_0, \uparrow abbcbba, \uparrow Z_0 \rangle \vdash \\
\langle q_1, \uparrow abbcbba, Z_0 \uparrow \rangle \vdash \\
\langle q_1, a \uparrow bbcbba, Z_0 A \uparrow \rangle \vdash \\
\langle q_1, ab \uparrow bcbba, Z_0 AB \uparrow \rangle \vdash \\
\langle q_1, abb \uparrow cbba, Z_0 ABB \uparrow \rangle \vdash \\
\langle q_2, abbc \uparrow bba, Z_0 AB \uparrow B \rangle \vdash \\
\langle q_2, abbcb \uparrow ba, Z_0 A \uparrow BB \rangle \vdash \\
\langle q_2, abbcbb \uparrow a, Z_0 \uparrow ABB \rangle \vdash \\
\langle q_2, abbcbba \uparrow, \uparrow Z_0 ABB \rangle \vdash \\
\langle q_F, abbcbba \uparrow, \uparrow Z_0 ABB \rangle
\end{array}
$$

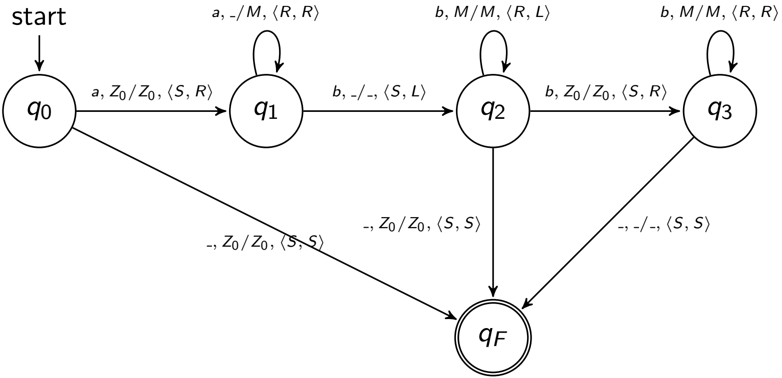
Solution (3)

TM TT that recognises the language
$L_3 = \{a^n b^n | n \geq 0\} \cup \{a^n b^{2n} | n \geq 0\}$



Solution (3)

TM TT that recognises the language
$L_3 = \{a^n b^n | n \geq 0\} \cup \{a^n b^{2n} | n \geq 0\}$



Is the string *aabb* recognised by TT? Is the string *abb* recognised by TT

Is the string *aabb* recognised by TT?

$$
\begin{array}{l}
\langle q_0, \uparrow aabb, \uparrow Z_0 \rangle \vdash \\
\langle q_1, \uparrow aabb, Z_0 \uparrow \rangle \vdash \\
\langle q_1, a \uparrow abb, Z_0 M \uparrow \rangle \vdash \\
\langle q_1, aa \uparrow bb, Z_0 MM \uparrow \rangle \vdash \\
\langle q_2, aa \uparrow bb, Z_0 M \uparrow M \rangle \vdash \\
\langle q_2, aab \uparrow b, Z_0 \uparrow MM \rangle \vdash \\
\langle q_2, aabb \uparrow, \uparrow Z_0 MM \rangle \vdash \\
\langle q_F, aabb \uparrow, \uparrow Z_0 MM \rangle
\end{array}
$$

Is the string *abb* recognised by TT?

$$\begin{aligned}
&\langle q_0, \uparrow abb, \uparrow Z_0 \rangle \vdash \\
&\langle q_1, \uparrow abb, Z_0 \uparrow \rangle \vdash \\
&\langle q_1, a \uparrow bb, Z_0 M \uparrow \rangle \vdash \\
&\langle q_2, a \uparrow bb, Z_0 \uparrow M \rangle \vdash \\
&\langle q_2, ab \uparrow b, \uparrow Z_0 M \rangle \vdash \\
&\langle q_3, ab \uparrow b, Z_0 \uparrow M \rangle \vdash \\
&\langle q_3, abb \uparrow, Z_0 M \uparrow \rangle \vdash \\
&\langle q_F, abb \uparrow, Z_0 M \uparrow \rangle
\end{aligned}$$