Discrete Mathematics

Inductions I

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"Drama is imagination limited by logic. Mathematics is logic limited by imagination!"

\- Nathan Campbell -

Sequences and Mathematical Induction

One of the most interesting and important tasks in mathematics is to discover and characterize regular patterns

Especially, those that are associated with processes that are repeated

Main mathematical tool to study such processes is "Sequences"

Main mathematical tool used to verify conjectures about sequences is "Mathematical Induction".

Sequences and Mathematical Induction

Example: Let's consider "Selection Sort"

How does it work? 5 2 1 4 3

Start:

1	5 2 4 3
1 2	5 4 3
1 2 3	5 4
1 2 3 4	5
1 2 3 4 5	End:

Question: How efficient this sorting algorithm is?

Sequences Example – Cont.

Selection Sort:

- While there are still elements to be sorted
  - Scan across of all of them to find the smallest of what remains
  - Append that to the output

You can already tell that “finding the smallest of what remains” is more time consuming of the two.

Let’s see it for **n**

Start: scan **n** elements

scan **n - 1** elements

scan **n - 2** elements

…

scan **1** element end:

Sequences Example – Cont.

This means that the total number of elements that are scanned can be written as

$$n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$$

Question: what is this value equal to?

$n = 1$, the sum is 1

$n = 2$, the sum is $1 + 2 = 3$

$n = 3$, the sum is $1 + 2 + 3 = 6$

$n = 4$, the sum is $1 + 2 + 3 + 4 = 10$

$n = 5$, the sum is $1 + 2 + 3 + 4 + 5 = 15$

That is: 1, 3, 6, 10, 15, …

You already know that it is equal to $n(n+1)/2$ – (if not, please see me or your TAs after the class)

Explicit Formula

$$n + (n-1) + (n-2) + \ldots + 3 + 2 + 1 = n(n+1) / 2$$

Sequences Example – Cont.

So,

$$n + (n-1) + (n-2) + \ldots + 3 + 2 + 1 = n(n+1) / 2$$

or,

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = n(n+1) / 2$$

Sequences Example – Cont.

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or,

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But the question is: how would we rigorously establish this? How to prove that it is true for all **n**?

Sequences Example – Cont.

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or,

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = n(n+1) / 2$$

But the question is: how would we rigorously establish this? How to prove that it is true for all **n**?

And this is where **mathematical induction** comes into play.

Mathematical Induction

You have already seen ways to prove a statements that are true for all objects of some type: natural numbers, real numbers, etc.

- Direct proof
- Proof by contradiction
- Proof by contrapositive

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Suppose that we restrict ourselves to proving facts about natural numbers

Then we can use another power proof technique called mathematical induction.

Mathematical Induction

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- Direct proof
- Proof by contradiction
- Proof by contrapositive

Suppose that we restrict ourselves to proving facts about natural numbers

Then we can use another power proof technique called mathematical induction.

Notes: The natural numbers have many nice properties – no two adjacent natural numbers have any values between them, every natural number is even or odd, etc. – which makes it possible to prove things about the natural numbers using techniques that do not apply to other structures like the real numbers, pairs of natural numbers.

Principal of Mathematical Induction

Let $P(n)$ be a property that applies to natural numbers (here I am assuming natural numbers to be a set of non-negative integers).

If the following are true:

$$P(0) \text{ is true}$$

For any $k > 0$, $k \in N$, $P(k) \rightarrow P(k + 1)$

Then for any $n \in N$, $P(n)$ is true.

Understanding Mathematical Induction Logically

$P(0) \wedge \forall k, [P(k) \Rightarrow P(k+1)] \Rightarrow \forall n, P(n)$

In words, this means as follows:

Suppose that we have proved the following two statements:

(1) [The basis Step] P(0) is true, and

(2) [The induction step] For all k, if P(k) is true, then P(k + 1) is true

Then for all n, P(n) is true.

Understanding Mathematical Induction Logically cont.

$P(0) \land \forall n,[P(n) \Rightarrow P(n+1)] \Rightarrow \forall n, P(n)$

The induction principle makes sense because upon establishing the statements $P(0)$, $P(0) \Rightarrow P(1)$, $P(1) \Rightarrow P(2)$, etc.,

we can use a sequence of Modus Ponenses to establish $P(2)$, $P(3)$ and so on.

Understanding Mathematical Induction Philosophically

Induction is very different from the other proof techniques we have seen before.

It gives us a way to show that some property is true for all natural numbers **n** not by directly showing that it must be true, but instead by showing that we could incrementally build up the result one piece at a time.

For example: Climbing up a flight stairs

Climbing Up a Flight of Stairs

Let P(n) be "you can climb to top of n stairs"

We know that P(0) is true (you can climb to the top of 0 stairs by taking no steps) – Also called the "base case"

If you can climb to the top of k steps, you can climb to the top of k + 1 stairs by taking 1 more step

In other words,

For any $k \in N$, $P(k) \rightarrow P(k + 1)$ -- Also called the "inductive step"

Thus, using the principle of mathematical induction, you could conclude that you can climb a staircase of any height.

Back to Selection Sort -- Proof

Let's see how can we use mathematical induction to show that

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = n(n+1) / 2$$

Proof: By Mathematical Induction:

Let P(n) be

"the sum of ***n*** *positive natural numbers is n(n+1) / 2"*.

We must show that P(n) is true for all $n \in N$.

Back to Selection Sort – Proof – Cont.

Proof: By mathematical induction.

Let P(n) be

"the sum of ***n*** *positive natural numbers is n(n+1) / 2".*

We must show that P(n) is true for all n ∈ N.

Show that P(`0`) is true:

That is, the sum of first `0` positive natural number is

`0(0+1) / 2`

Back to Selection Sort – Proof – Cont.

Proof: By mathematical induction.

Let P(n) be

"the sum of ***n*** *positive natural numbers is n(n+1) / 2"*.

We must show that P(n) is true for all n ∈ N.

Show that P(0) is true:

That is, the sum of first 0 positive natural number is

$$0(0+1) / 2$$

The sum of zero numbers is 0. Also

$0(0+1) / 2 = 0$

Consequently P(0) is true.

Back to Selection Sort – Proof – Cont.

Proof: By mathematical induction.

Let P(n) be "the sum of n positive natural numbers is n(n+1) / 2". We must show that P(n) is true for all $n \in N$.

Show that P(0) is true:

That is, the sum of first o positive natural number is 0(0+1) / 2

The sum of zero numbers is 0. Also 0(0+1) / 2 = 0

Consequently P(0) is true.

Show that for all integers k > 0, if P(k) is true then P(k+1) is true.

Back to Selection Sort – Proof – Cont.

Proof: By mathematical induction.

Let P(n) be "the sum of n positive natural numbers is n(n+1) / 2". We must show that P(n) is true for all $n \in N$.

Show that P(0) is true:

That is, the sum of first o positive natural number is 0(0+1) / 2

The sum of zero numbers is 0. Also 0(0+1) / 2 = 0

Consequently P(0) is true.

Show that for all integers k >= 0, if P(k) is true then P(k+1) is true.

Suppose that P(k) is true. That is, the sum of first k positive natural numbers is k(k+1) / 2.

Back to Selection Sort – Proof – Cont.

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Let P(n) be "the sum of n positive natural numbers is n(n+1) / 2". We must show that P(n) is true for all $n \in N$.

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Suppose that P(k) is true. That is, the sum of first k positive natural numbers is k(k+1) / 2.

We must show that, the sum of first k+1 positive natural numbers is (k+1)(k+2) / 2.

Back to Selection Sort – Proof – Cont.

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The sum of first k+1 positive numbers = (sum of first k positive numbers) + (k+1)

Back to Selection Sort – Proof – Cont.

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The sum of first k+1 positive numbers = (sum of first k positive numbers) + (k+1)

= k(k+1) / 2 + (k+1)

Back to Selection Sort – Proof – Cont.

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$$= k(k+1) / 2 + (k+1)$$

$$= (k(k+1) + 2(k+1)) / 2$$

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The sum of first k+1 positive numbers = (sum of first k positive numbers) + (k+1)

$$= k(k+1) / 2 + (k+1)$$

$$= (k(k+1) + 2(k+1)) / 2$$

$$= (k+1)(k+2) / 2$$

Thus P(k+1) is true, completing the induction.

Back to Selection Sort – Proof – Complete

Proof: By mathematical induction.

Let P(n) be "the sum of n positive natural numbers is n(n+1) / 2". We must show that P(n) is true for all $n \in N$.

Show that P(0) is true:

That is, the sum of first o positive natural number is 0(0+1) / 2

The sum of zero numbers is 0. Also 0(0+1) / 2 = 0

Consequently P(0) is true.

Show that for all integers k >= 0, if P(k) is true then P(k+1) is true.

Suppose that P(k) is true. That is, the sum of first k positive natural numbers is k(k+1) / 2.

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The sum of first k+1 positive numbers = (sum of first k positive numbers) + (k+1)

= k(k+1) / 2 + (k+1)

= (k(k+1) + 2(k+1)) / 2

= (k+1)(k+2) / 2

Thus P(k+1) is true, completing the induction.

Summation Notation

In the previous example, we kept referring to the sum as "the sum of n positive natural numbers" -- a long way of explaining the sum

When working with summations, mathematicians typically use Σ notation

In general, we can describe the sum of $a_1 + a_2 + \ldots + a_n$ as follows:

$$\sum_{i=1}^{n} a_i$$

Summation Notation – Cont.

More formally,

$$\sum_{i=1}^{n} a_i \text{ is the sum of all } a_i \text{ where } i \in \mathbb{N} \text{ and } m \leq i \leq n$$

Summation Notation – Cont.

For Example: The sum of 1+2+3+…+n can be written as,

$$\sum_{i=1}^{n} i$$

For n=5,

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$$

Summation Notation – Cont.

The sum of no numbers is call the empty sum and has a value 0,

$$\sum_{i=1}^{0} 2^i = 0, \sum_{i=137}^{42} (i+1) = 0, \sum_{i=-1}^{-2} i = 0, \sum_{i=5}^{0} i^i = 0$$

What about this one??

$$\sum_{i=0}^{0} 2^i$$

Previous Proof with Summation Notation

Theorem: For any $n \in N$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof: By induction. Let P(n) be the statement that,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We prove that P(n) is true for all $n \in N$ by induction on n.

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We prove that P(n) is true for all $n \in N$ by induction on n.

As our base case, we prove P(0), that is

$$\sum_{i=1}^{0} i = \frac{0(0+1)}{2}$$

The left hand side of this equality is empty sum, which is 0. The right hand side of the equality is also 0, so P(0) holds.

Previous Proof with Summation Notation –Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Previous Proof with Summation Notation –Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

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Now show that P(k+1) holds

Previous Proof with Summation Notation –Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Now show that P(k+1) holds

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + k + 1$$

Peeling off the last term!

Previous Proof with Summation Notation –Cont.

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$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + k + 1$$
$$= \frac{k(k+1)}{2} + k + 1$$

Previous Proof with Summation Notation –Cont.

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Now show that P(k+1) holds

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + k + 1$$
$$= \frac{k(k+1)}{2} + k + 1$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+2)(k+1)}{2}$$

Previous Proof with Summation Notation –Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Now show that P(k+1) holds

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

Thus P(k+1) holds, completing the induction.

Shortcoming of Mathematical Induction

- Induction can tell you that a certain result is correct
- But it does not tell you why that result is correct.
- Think about the proof that you just did
- To elaborate further, let’s look at another example

Summing Odd Numbers

Theorem: For any natural number n,

$$\sum_{i=0}^{n-1} (2i + 1) = n^2$$

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We prove that P(n) is true for all $n \in N$ by induction on n.

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As our base case, we prove P(0) , that,

$$\sum_{i=0}^{-1}(2i+1)=0^2$$

The left hand side of this equality is empty sum, which is 0. The right hand side of the equality is also 0. So P(0) holds.

Summing Odd Numbers – Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

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Summing Odd Numbers – Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

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We will prove P(k+1), meaning that

$$\sum_{i=0}^{k}(2i+1) = (k+1)^2$$

Summing Odd Numbers – Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

$$\sum_{i=0}^{k-1}(2i+1)=k^2$$

We will prove P(k+1), meaning that

$$\sum_{i=0}^{k}(2i+1)=(k+1)^2$$

To see this, note that,

$$\sum_{i=0}^{k}(2i+1)=\sum_{i=0}^{k-1}(2i+1)+2k+1$$

Summing Odd Numbers – Cont.

For the inductive step, assume that for some natural number k, P(k) holds, meaning that

$$\sum_{i=0}^{k-1}(2i+1) = k^2$$

We will prove P(k+1), meaning that

$$\sum_{i=0}^{k}(2i+1) = (k+1)^2$$

To see this, note that,

$$\sum_{i=0}^{k}(2i+1) = \sum_{i=0}^{k-1}(2i+1) + 2k + 1$$

$$= k^2 + 2k + 1 = (k+1)^2$$

Summing Odd Numbers – Cont.

Once again, using induction we showed that the result is true

But once again, we do not know why this is true

For this, you need to learn how to manipulate summations

Manipulating Summations

Separating off a final term and adding on a final term

a) Rewrite

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \text{ by separating off the final term}$$

b) Rewrite

$$\sum_{k=0}^{n} 2^k + 2^{n+1} \text{ as a single summation}$$

Solution:

Manipulating Summations

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a) Rewrite

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \text{ by separating off the final term}$$

b) Rewrite

$$\sum_{k=0}^{n} 2^k + 2^{n+1} \text{ as a single summation}$$

Solution:

a): $$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

b): $$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

Manipulating Summations – Cont.

Lets look at a bit more complicated case.

We already know that,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

But what about this sum

$$\sum_{i=0}^{n-1} i$$

Manipulating Summations – Cont.

One thing that we can note is the following,

$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} i + n - n$$

Manipulating Summations – Cont.

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$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} i + n - n = \boxed{\sum_{i=0}^{n} i} - n$$

Manipulating Summations – Cont.

One thing that we can note is the following,

$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} i + n - n = \boxed{\sum_{i=0}^{n} i} - n$$

Thus gives us,

$$\frac{n(n+1)}{2} - n = \frac{n(n+1) - 2n}{2} = \frac{n(n+1-2)}{2} = \frac{n(n-1)}{2}$$

Thus

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

We will use this result, soon!

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1} (2i + 1) = n^2$$

Let's prove this by manipulating summations, so that we also know why this is true!

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1} (2i + 1) = n^2$$

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1}(2i+1) = n^2$$

$$\sum_{i=0}^{n-1}(2i+1) = \sum_{i=0}^{n-1} 2i + \sum_{i=0}^{n-1} 1$$

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1}(2i+1) = n^2$$

$$\sum_{i=0}^{n-1}(2i+1) = \sum_{i=0}^{n-1} 2i + \boxed{\sum_{i=0}^{n-1} 1}$$

$$= \sum_{i=0}^{n-1} 2i + \boxed{n}$$

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1}(2i+1) = n^2$$

$$\sum_{i=0}^{n-1}(2i+1) = \sum_{i=0}^{n-1} 2i + \sum_{i=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} 2i + n$$

$$= 2\sum_{i=0}^{n-1} i + n$$

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1}(2i+1) = n^2$$

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$$= 2\,\frac{n(n-1)}{2} + n = n(n-1) + n$$

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1}(2i+1) = n^2$$

$$\sum_{i=0}^{n-1}(2i+1) = \sum_{i=0}^{n-1} 2i + \sum_{i=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} 2i + n$$

$$= 2\sum_{i=0}^{n-1} i + n$$

$$= 2\,\frac{n(n-1)}{2} + n = n(n-1) + n$$

$$= n^2 - n + n = n^2$$

Back to Summing Odd Numbers

$$\sum_{i=0}^{n-1}(2i+1) = n^2$$

$$\sum_{i=0}^{n-1}(2i+1) = \sum_{i=0}^{n-1} 2i + \sum_{i=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} 2i + n$$

$$= 2\sum_{i=0}^{n-1} i + n$$

$$= 2\,\frac{n(n-1)}{2} + n = n(n-1) + n$$

$$= n^2 - n + n = n^2$$

An entirely new proof of the fact that the sum of the first n odd numbers is equal to $\mathbf{n^2}$.

Practice

Proof by mathematical induction:

Theorem: For any natural number **n**,

$$\sum_{i=0}^{n-1} (a_i + b_i) = \sum_{i=0}^{n-1} a_i + \sum_{i=0}^{n-1} b_i$$

Theorem: For any natural number **n** and any $r \in \mathcal{R}$

$$\sum_{i=0}^{n-1} r\ a_i = r \sum_{i=0}^{n-1} a_i$$

Product

$$\prod_{i=m}^{n} a_i$$

Is the product of all a_i where $i \in \mathbb{N}$ and m $\leq i \leq n$.

Product

For Example:

$$\prod_{i=1}^{5} i = 1 * 2 * 3 * 4 * 5 = 120$$

Just as the empty sum is defined to be 0, the empty product to be one.

A product of no numbers is called the **empty product** and is equal to 1.

There are many functions that are defined in terms of products. One special function worth noting is the factorial function, which is defined as follows:

For any number $n \in \mathbb{N}$, **n factorial**, denoted by **n!**, is defined as $\mathbf{n! = \prod_{i=1}^{n} i}$

Validity of Mathematical Induction Revisited!

Why is mathematical induction a valid proof technique?

The reason comes from the well-ordering property --

states that every nonempty subset of the set of positive integers has a least element.

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We will do this by "Proof by Contradiction"

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Assume that there is at

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Thus, by the well-ordering property, S has a least element, which will be denoted by **m**.

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We know that **m** cannot be 1, because P(1) is true.

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Because the conditional statement $P(m - 1) \rightarrow P(m)$ is

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also true, it must be the case that P(m) is true.

This contradicts the choice of m. Hence, P(n)

must be true for every positive integer n.