Discrete Mathematics

Functions

Adil M. Khan

Professor of Computer Science

Innopolis University

"Go Down Deep Enough into Anything and You Will Find Mathematics!"

-Dean Schlicter-

Functions

- A way of transforming objects of one type into objects of another type.

- Imagine

{Set of Strings}

Length(w) → produces its length

Where w is a string.

e.g. length(*smile*)=*5*.

Functions

- Question that we might want to discuss when studying functions.

  - ❑ Means by which a function is computed?
  - ❑ Are there different ways?
  - ❑ Are some worse than others?
  - ❑ Are there different ways to represent a function?

Functions

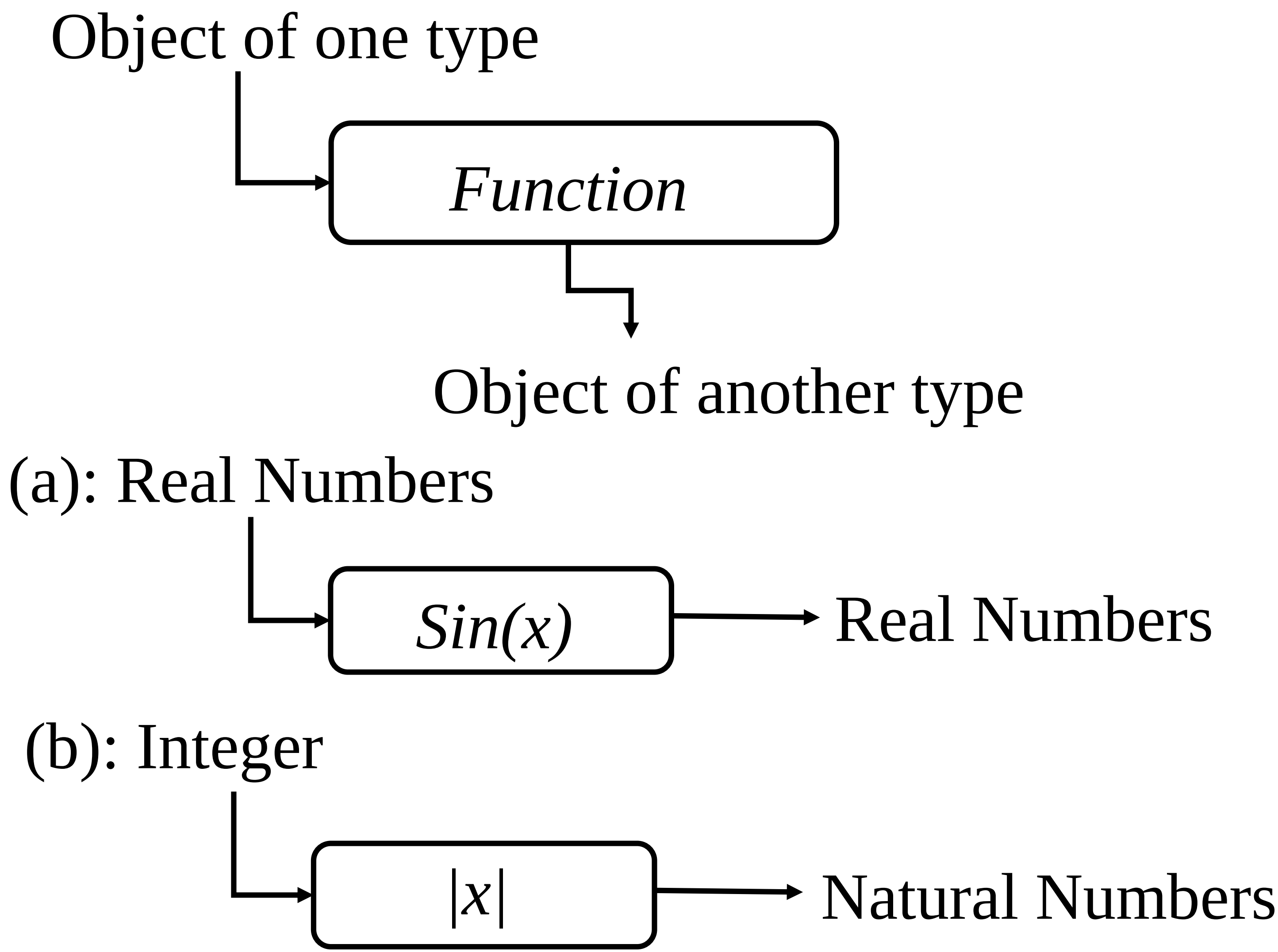
- But we are mainly interested in the following.

  - What exactly is a function

  - How can we classify function into different kinds

  - How can we build new functions from the existing ones?

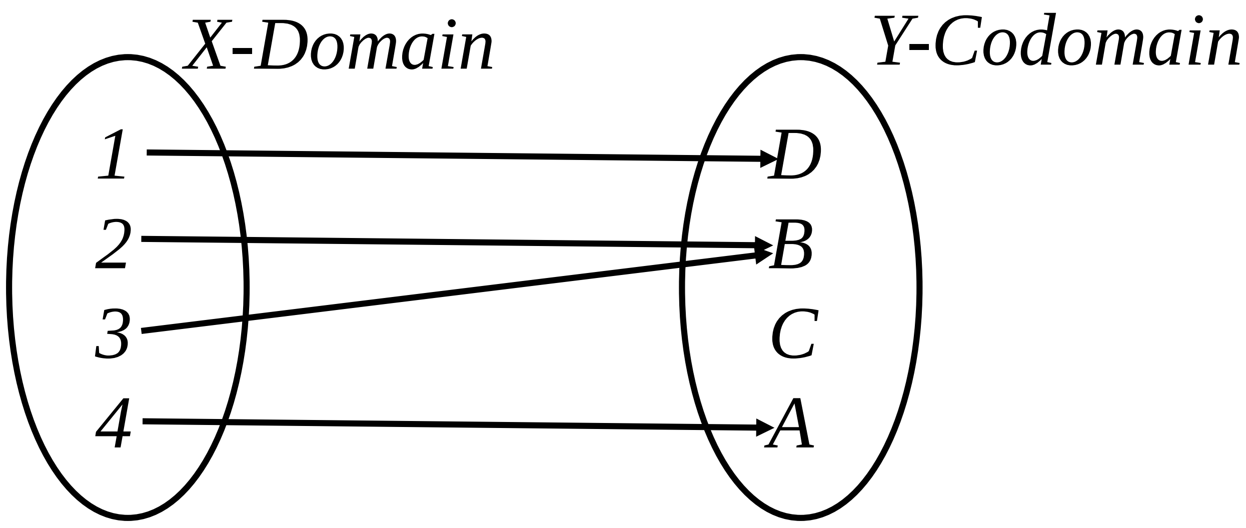
Functions – Basic Definitions



Functions – Basic Definitions—Cont.

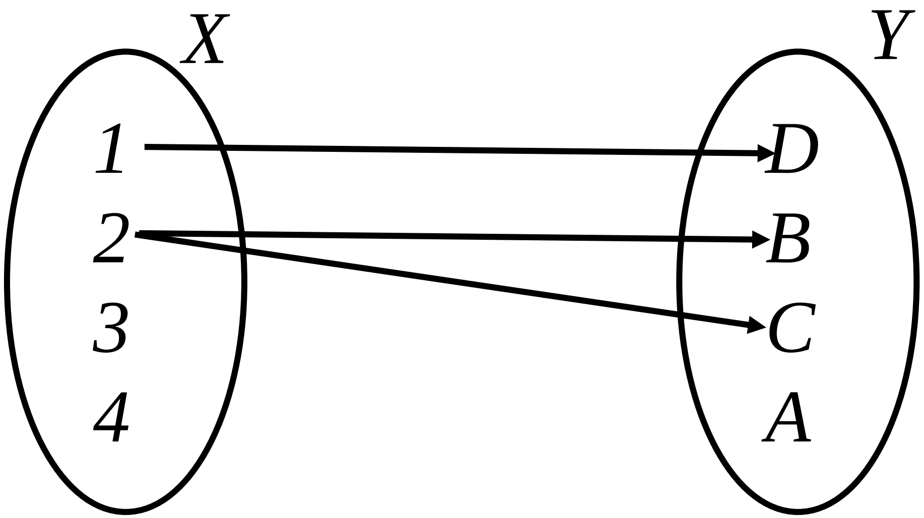
- Let ***A* and *B*** be arbitrary sets
- f: a function from ***A* to *B***.
- Associates $\boxed{\text{every}}$ element of *A* with a $\boxed{\text{single}}$ element in *B*.

- **Example:**
- f: from *X to Y*



Functions – Example

- An other Example:



- What do you think?
- Can this mapping be produced by a function?

- What about the **function $\sqrt{x}$** from $R \rightarrow R$?

Functions – Basic Definitions—Cont.

- Sometimes, you will hear the term “range” of a function

  - **Range: set of all possible outputs of a function**

- We will touch this topic a little later in this lecture

Functions – Defining a Function

- The function definition should have enough details to unambiguously define

  - The domain and codomain
  - The output for every input

- **For Example**

- $f{:}N \rightarrow N$, where $f(n)=n^2$.

- On the other hand $f(x) = x^2$ is a bit less precise.

Functions – Defining a Function

- Why the function definition $f(x) = x^2$ is less precise?

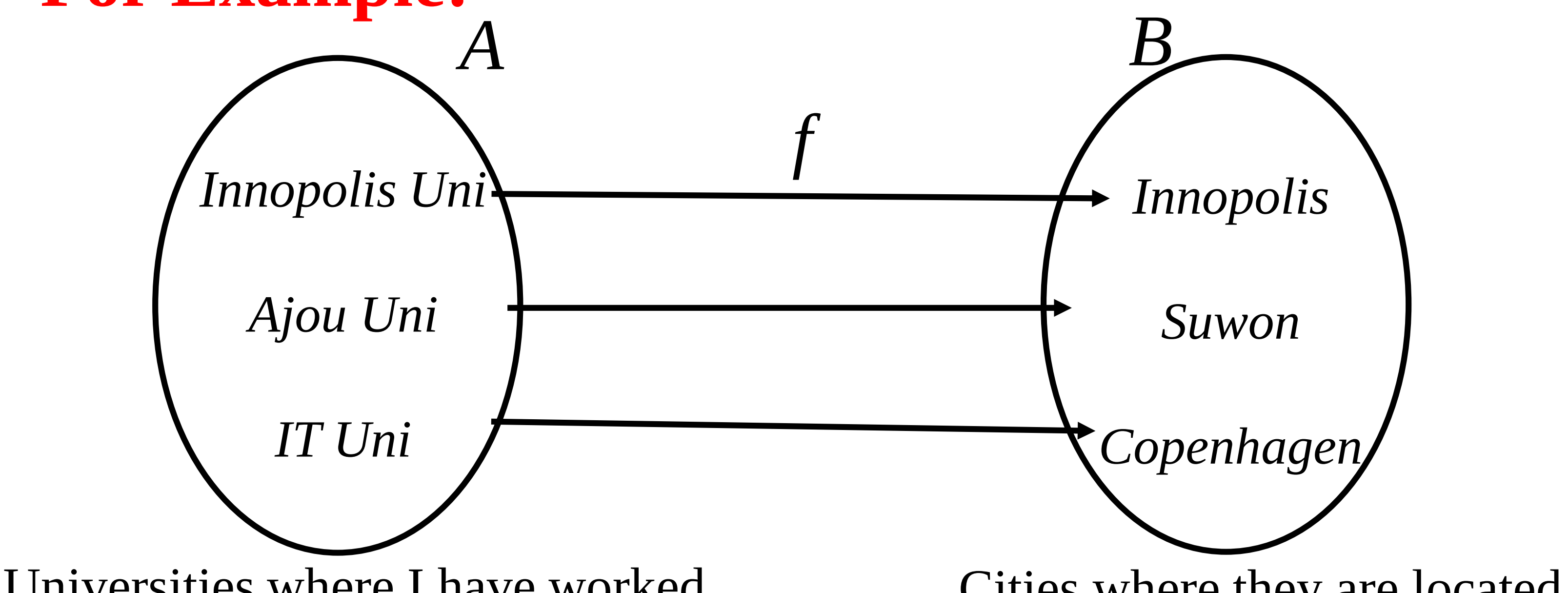
$$f{:}N \rightarrow N \qquad\qquad f{:}Z \rightarrow N$$

$$f{:}Z \rightarrow Z \qquad\qquad f{:}R \rightarrow R$$

- All four are valid, however, the properties of the function will be widely different
- For example, if $f{:}N \rightarrow N$, then f has the property that if $f(x) < f(y)$, then $x < y$
- Isn't true for $f{:}Z \rightarrow Z$

Defining Functions by a Picture

- When the domain and codomains are finite sets
- We can often define the function by drawing a picture
- **For Example:**



Defining Function – Cont.

- Another way often used to define a function is by specifying a variety of different rules to the input giving conditions under which each rule should be applied.

- These are often called piecewise functions.

- **For Example:**

$$|x| = \begin{cases} x & if\ x > 0 \\ -x & otherwise \end{cases}$$

Piecewise Functions

- When defining such functions, it is important to ensure that

- Every possible input falls into at least one of the cases

- If an input falls into multiple cases, each case produces the same output.

- **For Example:**

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if \leq 0 \end{cases} \quad \textbf{VS} \quad |x| = \begin{cases} x & if\ x > 0 \\ -x & if < 0 \end{cases}$$

Functions with multiple inputs

- When programming we often use functions like these

- ```
  int raiseToPower (int x, int y) {
          int result = 1;
          for (int i = 0; i < y, i++) {
              result *= x;
          }
          return result;
      }
  ```
- How can we define such functions mathematically? – because in our definition, a function takes only one argument, i.e., an element of the domain
  ```

Functions with multiple inputs - Cont

- Lets assume only natural numbers as input

- We can think of the above function, which appears to take in $\boxed{\text{two arguments}}$, as a functions that takes in just $\boxed{\text{one argument}}$.

  "An ordered pair of natural numbers"

- **Mathematically;**

  Raise To Power: $\mathbb{N} * \mathbb{N} \rightarrow \mathbb{N}$ where

  Raise To Power $\left((x, y)\right) = x^y$

Functions with multiple inputs-Cont

- More generally,
- We can always model an $\boxed{n-\text{argument}}$ function as a function containing a $\boxed{\text{single argument}}$.

n-way Cartesian product of an appropriate group of sets

Functions with multiple inputs-Cont

- More generally,
- We can always model an $\boxed{n - \text{argument}}$ function as a function containing a $\boxed{\text{single argument}}$.

n-way Cartesian product of an appropriate group of sets

- How will you represent a function that $\boxed{\text{adds}}$ together $\boxed{\text{three real numbers}}$ and $\boxed{\text{an integer}}$?

Functions with multiple inputs-Cont

- More generally,
- We can always model an $\boxed{\text{n} - \text{argument}}$ function as a function containing a $\boxed{\text{single argument}}$.

n-way Cartesian product of an appropriate group of sets

- How will you represent a function that $\boxed{\text{adds}}$ together $\boxed{\text{three real numbers}}$ and $\boxed{\text{an integer}}$?
- **Final Comment:**

  if f: $A_1 * \ldots * A_n \rightarrow B$, then we denote $f((x_1, \ldots, x_n))$ by $f(x_1, \ldots, x_n)$

Injection, Surjection, and Bijection

- Functions come in different shapes and size

- But these are certain types of functions that appear more frequently than others

1. Surjections (**Onto**)
2. Injections (**One to One**)
3. Bijections (**Both**)

Surjections

- Lets consider a problem

- You are in charge of distributing a bunch of fruit baskets among student groups at IU.

  Student groups {BS1, BS2, BS3, BS4, MS1, MS2}.

- You want to do it such that every group gets at least one fruit basket.

Surjections – Cont.

- Mathematically, you can think of this as a function
- $f: B \to G$, where ***B*** be the set of fruit baskets and ***G*** be the set of student groups.
- “for every $\boxed{g \in G}$, there is some fruit basket $\boxed{b \in B}$ such that $\boxed{f(b) = g}$”
- Such a function is called surjection.

Surjections – Cont.

- More generally;

- $\boldsymbol{f: A \rightarrow B}$, is a surjection if for any $b \in B$, there is some $a \in A$ such that

$$f(a) = b$$

- Also called an **Onto function**.

If we represent such a function with a picture, what will it look like?

Surjections – Cont.

- Which of these are Surjections??
- $\boldsymbol{f(x) = x}$, over real numbers
- $\boldsymbol{f(x) = x^2}$, over real numbers

Injections (One to One)

- Now suppose you are the head of a student group
- You get a fruit basket

- Now you want to distribute among students

- In other words, you want to find a function $f : F \to S$

where F and S represents the set of fruits and set of students.

Injections (One to One)

- Unfortunately, there are not enough fruits, so you want to be fair,

- Thus you define

- $\boldsymbol{f : F \rightarrow S}$
    - With the condition that every one should get at most one fruit.

- Such a function is calls injection.

Injections (One to One)

- More generally,

- $f: A \to B$ is an injection if for any $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

- Equivalently,

  for any $x_1, x_2 \in A$ if $x_1 \neq x_2$ then
  $$f(x_1) \neq f(x_2)$$

- Also called a **One to One** function.

If we represent such a function with a picture, what will it look like?

Injections – Cont.

- Which of these are Injections??
- $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$, over real numbers
- $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{2}}$, over real numbers

Some more concepts related to Surjections and Injections, before we move on to Bijections!

Functions and Sets

❖ Images:

- If $f: A \to B$ and $X \subseteq A$, the image of X under f is the set

$$f[x] = \{f(x)/x \in X\}$$

- Set of elements that we would get if we applied f to every element of X.

Functions and Sets – Cont.

❖ **Images:**

- What is the image of $X = [-1, 3]$ under

$$f : R \to R \text{ where } f(x) = x^2 \text{ ??}$$

Functions and Sets – Cont.

- **Image of the Entire Domain:**

- $f: A \to B$

$$f[A] = \{f(a) / a \in A\}$$

where $\{f(a) / a \in A\}$ consists of all the possible outputs of a function.

Functions and Sets – Cont.

- $f[A]$ is the same as codomain of f??
- Not necessarily!
- For Example: $f: R \to R$ where $f(a) = \sin(a)$

  then $f[R] = ??$

- Also referred to as the $\boxed{\text{range}}$ of the function.

<u>Range</u> $\rightarrow$ Values in the codomain that can actually be produced by the function.

Functions and Sets – Cont.

❖ **An other Important Question:**

- When are the range and codomain the same and when are they different??

  - $\text{Range} = \text{Codomain}$

- When every possible value of the codomain can be produced by the function **as its output on some input**.

Which functions have this property?

Functions and Sets – Cont.

- **Theorem:** If $f : A \to B$, then $f[A] = B$ if and only if f is surjective.

Functions and Sets – Cont.

- **Theorem:** If $f: A \to B$, then $f[A] = B$ if and only if f is surjective.

We will prove this by proving both sides of implication.

Functions and Sets – Cont.

- **Theorem:** If $f: A \to B$, then $f[A] = B$ if and only if f is surjective.

  We will prove this by proving both sides of implication.

- **Proof: (a):** If $f[A] = B$ then f is surjective.

Functions and Sets – Cont.

- **Theorem:** If $f: A \to B$, then $f[A] = B$ if and only if f is surjective.

  We will prove this by proving both sides of implication.

- **Proof: (a):** If $f[A] = B$ then f is surjective.
  consider any $b \in B$
  Since $f[A] = B$ then $b \in f[A]$

Functions and Sets – Cont.

- **Theorem:** If $f: A \to B$, then $f[A] = B$ if and only if f is surjective.

  We will prove this by proving both sides of implication.

- **Proof: (a):** If $f[A] = B$ then f is surjective. consider any $b \in B$

  Since $f[A] = B$ then $b \in f[A]$

  Therefore, there exists some $a \in A$ where $f(a) = b$

  Since our choice of b was arbitrary, thus f is surjective.

Functions and Sets – Cont.

- **Theorem:** If $f: A \to B$, then $f[A] = B$ if and only if f is surjective.

We will prove this by proving both sides of implication.

- **Proof: (a):** If $f[A] = B$ then f is surjective. consider any $b \in B$

  Since $f[A] = B$ then $b \in f[A]$

  Therefore, there exists some $a \in A$ where $f(a) = b$

  Since our choice of b was arbitrary, thus f is surjective.

Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$

Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$
  - We have $f: A \to B$,

Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$
  - We have $f: A \to B$,
  - We know that every element of $f[A]$ is an element of B. What we need to show now is that $B \subseteq f[A]$.

Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$
  - We have $f: A \to B$,
  - We know that every element of $f[A]$ is an element of B. What we need to show now is that $B \subseteq f[A]$.
  - Consider any $b \in B$

  Since f is surjective, there is some $a \in A$ such that $f(a) = b$,

Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$
  - We have $f: A \to B$,
  - We know that every element of $f[A]$ is an element of B. What we need to show now is that $B \subseteq f[A]$.
  - Consider any $b \in B$

  Since f is surjective, there is some $a \in A$ such that $f(a) = b$,
  - This means that $b \in f[A]$

Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$
  - We have $f: A \rightarrow B$,
  - We know that every element of $f[A]$ is an element of B. What we need to show now is that $B \subseteq f[A]$.
  - Consider any $b \in B$

  Since f is surjective, there is some $a \in A$ such that $f(a) = b$,
  - This means that $b \in f[A]$
  - Since our choice of b was arbitrary, so $B \subseteq f[A]$.

Functions and Sets – Cont.

- **Proof: (b):** If *f* is surjective, then $f[A] = B$
  - We have $f: A \to B$,
  - We know that every element of $f[A]$ is an element of B. What we need to show now is that $B \subseteq f[A]$.
  - Consider any $b \in B$

  Since f is surjective, there is some $a \in A$ such that $f(a) = b$,
  - This means that $b \in f[A]$
  - Since our choice of b was arbitrary, so $B \subseteq f[A]$.

From **(a)** and **(b)**: If $f: A \to B$, then $f[A] = B$ if and only if *f* is surjective.

Functions and Sets – Cont.

- **Proof: (b):** If f is surjective, then $f[A] = B$
  - We have $f: A \to B$,
  - We know that every element of $f[A]$ is an element of B. What we need to show now is that $B \subseteq f[A]$.
  - Consider any $b \in B$

  Since f is surjective, there is some $a \in A$ such that $f(a) = b$,
  - This means that $b \in f[A]$
  - Since our choice of b was arbitrary, so $B \subseteq f[A]$.

From **(a)** and **(b)**: If $f: A \to B$, then $f[A] = B$ if and only if f is surjective.

Functions and Sets – Cont.

- **Preimage:**

- If $f: A \rightarrow B$ and $Y \subseteq B$, then the preimage of Y under f is the set

  - $f^{-1}[Y] = \{x \in A / f(x) \in Y\}$

    where $\{x \in A / f(x) \in Y\}$ is a set of all the element of A (domain) that map into set Y, where $Y \subseteq B$.

Functions and Sets – Cont.

❖ Preimage – Cont.

- What is $f^{-1}[Y]$ in the following case?

  - If $f: R \to R$, where $f(x) = 2x$,

    $$Y = [1, 3]$$
  - If $f: R \to R$, where $f(x) = x^2$,

    $$Y = [4, 9]$$
  - If $f: R \to R$, where $f(x) = x^2 + 2$,

    $$Y = [0, 1], Y = [0, 2]$$

Functions and Sets – Cont.

- **Preimage and Injections**

- Just as images and surjections are related, so are preimages and injections.

Functions and Sets – Cont.

❖ Preimage and Injections

- Just as images and surjections are related, so are preimages and injections.
- Let $f: A \to B$ be an injection
  - This means that every $b \in B$ has either $\boxed{0}$ or $\boxed{1}$ elements mapping to it.
  - Therefore $f^{-1}[\{b\}]$ should either contain $\boxed{0}$ or $\boxed{1}$ elements.

Functions and Sets – Cont.

❖ Preimage and Injections

- Just as images and surjections are related, so are preimages and injections.
- Let $f: A \rightarrow B$ be an injection
  - This means that every $b \in B$ has either $\boxed{0}$ or $\boxed{1}$ elements mapping to it.
  - Therefore $f^{-1}[\{b\}]$ should either contain $\boxed{0}$ or $\boxed{1}$ elements.
  - **In other words if f is injective then,**

$$|f^{-1}[\{b\}]| \leq 1$$

Functions and Sets – Cont.

❖ **Bijections**

- A function is called a bijection if it is **injection** and **surjection**.

- For every element of the codomain, there is a unique element of the domain mapping to it.

Functions and Sets – Cont.

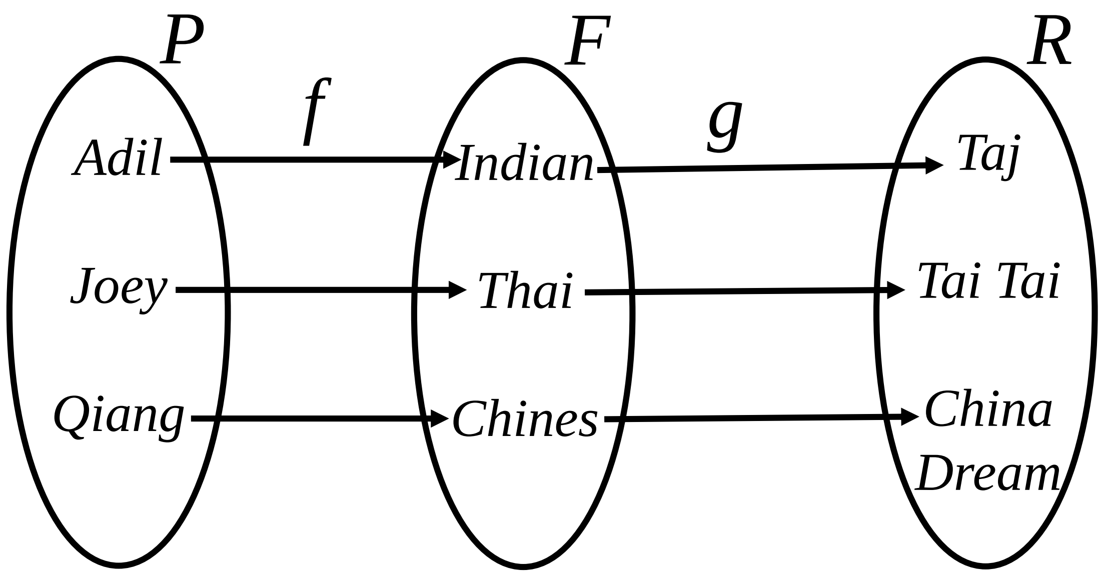
❖ Bijections

- For Example:
  - $f: R \rightarrow R$, where $f(x) = x^3$
  - $f: S \rightarrow S$, where $f(x) = x$
- What about
  - $f: R \rightarrow R$, where $f(x) = x^2$

Functions and Sets – Cont.

❖ Transformations on Functions

- P: set of people
- F: Set of different types of food
- R: Set of restaurants



- $f: P \rightarrow F$
- $g: F \rightarrow R$

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- We want to tell people in which restaurant they can find their favorite food.

- This is, we want to find a new function

  - $\boxed{m: P \rightarrow R}$

- How to define this function??

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- $\boxed{m}$ must glue together $\boxed{f}$ and $\boxed{g}$.
- That is
  - $M(p) = b\big(f(p)\big), p \in P$
- It is very common to join function like this. It is called $\boxed{\text{composition}}$ of the functions.

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- More formally,
  - Let $f: A \to B$ and $g: B \to C$.
  - Define a new function $g \circ f: A \to C$ as follows
  - $(g \circ f)(a) = g(f(a))$ for all $a \in A$

Functions and Sets – Cont.

❖ Transformations on Functions – Cont.

- Given two functions f and g, is $\boxed{g \circ f}$ or $\boxed{f \circ g}$ always guaranteed??
- NO!

- For Example:
  - Let $g: F \rightarrow R$ (form previous example)
  - Let $h: R \rightarrow R$ where $h(x) = x^3$
  - Can we do $\boxed{g \circ f}$ or $\boxed{f \circ g}$??
  - Their domains and codomains are incomparable!

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ be injections. Then $g \circ f: A \to C$ is an injection.

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be injections. Then $g \circ f: A \rightarrow C$ is an injection.

Proof: Consider any $x, y \in A$ where $x \neq y$

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ be injections. Then $g \circ f: A \to C$ is an injection.

Proof: Consider any $x, y \in A$ where $x \neq y$

- Since $x \neq y$, f$(x) \neq f(y)$ as *f* is an injection.

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ be injections. Then $g \circ f: A \to C$ is an injection.

Proof: Consider any $x, y \in A$ where $x \neq y$

- Since $x \neq y$, f$(x) \neq f(y)$ as *f* is an injection.
- Since f$(x) \neq f(y)$ and g is an injection, therefore,

$$(g \circ f)(x) \neq (g \circ f)(y)$$

Functions and Sets – Cont.

❖ **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ be injections. Then $g \circ f: A \to C$ is an injection.

Proof: Consider any $x, y \in A$ where $x \neq y$

- Since $x \neq y$, f(x) $\neq f(y)$ as *f* is an injection.
- Since f(x) $\neq f(y)$ and g is an injection, therefore,

$$(g \circ f)(x) \neq (g \circ f)(y)$$

- Since our choice of x and y was arbitrary, This means that for any $x, y \in A$ where, $x \neq y, (g \circ f)(x) \neq (g \circ f)(y)$, so $g \circ f$ is injective.

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Proof: Consider any $c \in C$

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Proof: Consider any $c \in C$

- Since g is surjective, there exists, some $b \in B$ such that $g(b) = c$.

Functions and Sets – Cont.

- **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Proof: Consider any $c \in C$

- Since g is surjective, there exists, some $b \in B$ such that $g(b) = c$.
- Similarly, since f is surjective, there exists, some $a \in A$ such that $f(a) = b$.

Functions and Sets – Cont.

❖ **Composition of Injections, Surjections, and Bijections.**

Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Proof: Consider any $c \in C$

- Since g is surjective, there exists, some $b \in B$ such that $g(b) = c$.
- Similarly, since f is surjective, there exists, some $a \in A$ such that $f(a) = b$.
- Then $(g \circ f)(a) = g\big(f(a)\big) = g(b) = c$

Functions and Sets – Cont.

❖ Composition of Injections, Surjections, and Bijections.

Theorem: Let $f: A \to B$ and $g: B \to C$ are surjections. Then $g \circ f: A \to C$ is an surjection.

Proof: Consider any $c \in C$

- Since g is surjective, there exists, some $b \in B$ such that $g(b) = c$.
- Similarly, since f is surjective, there exists, some $a \in A$ such that $f(a) = b$.
- Then $(g \circ f)(a) = g\big(f(a)\big) = g(b) = c$
- Thus, for any $c \in C$, there is an $a \in A$. Therefore $\boxed{g \circ f}$ is surjective.

Functions and Sets – Cont.

❖ **Composition of Injections, Surjections, and Bijections.**

- **Theorem:** Let $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection.

- **Proof:**

"What do you guys think??"

"What is the proof?"

Relation Between Exponential And Logarithmic Functions

- For positive numbers b $\neq$ 1, the exponential function with base b, denoted exp_b, is the function from R to R+ defined as follows:

- For all real numbers x,

$$\exp_b(x) = b^x$$

- where $b^0 = 1$ and $b^{-x} = \frac{1}{b^x}$.

Relation Between Exponential And Logarithmic Functions

- When working with the exponential function, it is useful to recall the laws of exponents from elementary algebra.

Laws of Exponents

If b and c are any positive real numbers and u and v are any real numbers, the following laws of exponents hold true:

$$b^u b^v = b^{u+v} \qquad 7.2.1$$

$$(b^u)^v = b^{uv} \qquad 7.2.2$$

$$\frac{b^u}{b^v} = b^{u-v} \qquad 7.2.3$$

$$(bc)^u = b^u c^u \qquad 7.2.4$$

Relation Between Exponential And Logarithmic Functions

- Equivalently, for each positive real number x and real number y,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

- It can be shown using calculus that both the exponential and logarithmic functions are one-to-one and onto.
- Therefore, by definition of one-to-one, the following properties hold true:

For any positive real number b with $b \neq 1$,

$$\text{if } b^u = b^v \text{ then } u = v \quad \text{for all real numbers } u \text{ and } v, \qquad 7.2.5$$

and

$$\text{if } \log_b u = \log_b v \text{ then } u = v \quad \text{for all positive real numbers } u \text{ and } v. \qquad 7.2.6$$

Relation Between Exponential And Logarithmic Functions

- These properties are used to derive many additional facts about exponents and logarithms. In particular we have the following properties of logarithms.

Theorem 7.2.1 Properties of Logarithms

For any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a \log_b x$

d. $\log_c x = \dfrac{\log_b x}{\log_b c}$