An introduction to λ-calculus

24th of March, 2016

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The decision problem consists in having a “procedure” that takes as an input a formula and that returns as an output

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In 1915, Leopold Löwenheim described such a procedure for the monadic predicate calculus (i.e. every predicate is of arity 0 or 1, like in the two formulas above).

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The question is: can one show that this set is RECURSIVE.

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We will write also $(t)u_1 \ldots u_{k+1}$ or even $tu_1 \ldots u_{k+1}$ instead of $(\ldots(((t)u_1)u_2)\ldots)u_{k+1}$.

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There is exactly one free occurrence of x in $(\lambda x.x)x$: it is the third one.

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Intuition: $\lambda x.u$ is like a function f that associates with x the value u (like in $f(x) = u$). Now $(\lambda x.u)v$ is the value of $f(v)$: if $f(x) = x^2 + x + 1$, then $f(3)$ is $f(x)$ in which every occurrence of x is substituted by 3, which gives $3^2 + 3 + 1$.

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One can have several occurrences of redexes.

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We have $t\beta t'$ if, and only if, t' is obtained from t by replacing some occurrence of some redex $(\lambda x.u)v$ by $v[u/x]$, which is the term v in which every free occurrence of x in v is substituted by v (we *contracted the redex* $(\lambda x.u)v$).

Intuition: $\lambda x.u$ is like a function f that associates with x the value u (like in $f(x) = u$). Now $(\lambda x.u)v$ is the value of $f(v)$: if $f(x) = x^2 + x + 1$, then $f(3)$ is $f(x)$ in which every occurrence of x is substituted by 3, which gives $3^2 + 3 + 1$.

Example: $(\lambda x.\lambda y.(x)yy)vu\beta\ (\lambda y.(v)yy)u$
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$$
\begin{tikzcd}
(I)(I)x \arrow[d, bend right, "(I)(I)x"'] \arrow[d, bend left, "(I)x"] \\
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\end{tikzcd}
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where $I = \lambda z.z$

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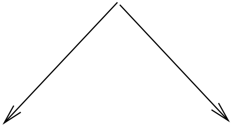
5) (More difficult) Does there exist a term whose reduction graph has the following shape?



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$$t = \lambda n.\lambda f.\lambda x.(f)(n)fx$$

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