Probabilistic automata

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April 6, 2016

Abstract

This paper contains a short description of probabilistic automata theory. The main source is the article "Probabilistic automata" by M. Rabin.

1 Introduction

Probabilistic automata (p.a.) are a generalization of finite deterministic automata. We follow the formulation of finite automata in Rabin and Scott (1959) where the automata M have two-valued output and thus can be viewed as defining the set $T(M)$ of all tapes accepted by M. This involves no loss of generality. A p.a. is an automaton which, when in state s and when input is σ, has a probability $p_i(s, \sigma)$ of going into any state s_i . With any cut-point $\lambda < 1$, there is associated the set $T(M, \lambda)$ of tapes accepted by M with cup-point λ.

2 Description

2.1 The definition of the probabilistic automaton.

Probabilistic automata are like the usual automata except that now the transition table $\mathcal{M}$ assigns to each pair $(s, \sigma) \subset C \times \Sigma$ certain transition probabilities.

The probabilistic automaton over the alphabet Σ is a tuple $< S, M, s_0, F >$ where $S = \{s_0, ..., s_n\}$ is a set of states, $M : S \times \Sigma \to [0, 1]^{n+1}$ - the transition function such that $(s, \sigma) \in C \times \Sigma$

$$M(s, \sigma) = (p_0(s, \sigma), ..., p_n(s, \sigma)),$$
$$0 \leq p_i(s, \sigma), \sum_i p_i(s, \sigma) = 1,$$
$$s_0 \subset S \text{ and } F \subseteq S.$$

Probabilistic automata are models for systems (such as sequential circuits) capable of a finite number of states $s_o, ..., s_n$. The system may receive inputs $\sigma \in \Sigma$. When in state s and if the input is a then the system can go into any one of the states $s_i \in S$ and the probability of going into s_i is the (i + 1)th coordinate $p_i(s, \sigma)$ of $M(s, \sigma)$. These transition probabilities $p_i(s, \sigma)$ are assumed to remain fixed and be independent of time and previous inputs. Thus the system also has definite transition probabilities for going from state s to state s_i by a sequence $x \in \Sigma^*$ of inputs. These probabilities are calculated by means of products of certain stochastic matrices. :

For $\sigma \in \Sigma$ and $x = \sigma_1\sigma_2...\sigma_m$ define the $n+1$ by $n+1$ matrices $A(\sigma)$ and $A(x)$ by

$$A(\sigma) = [p_j(s_i, \sigma)]0 \leq i \leq n, 0 \leq j \leq n$$
$$A(x) = A(\sigma_1)A(\sigma_2)...A(\sigma_m) = [p_j(s_i, x)]0 \leq i \leq n, \leq j \leq n.$$

REMARK. An easy calculation (involving induction on m) will show the $(i+1, j+1)$ element $p_j(s_i, x)$ is the probability of the automaton for moving from state s_i to state s_j by the input sequence x.

If $M =< S, M, s_0, F >$ and $F = s_{i_0}, ..., s_{i_r}$, $I = i_0, ..., i_r$, define

$$p(x) = \sum_{i \in I} p_i(s_0, x).$$

$p(x)$ clearly is the probability for T, when started in s_0, to enter into a state which is member of F by the input sequence x.

2.2 Cut-point

A p.a. M may be used to define sets of tapes in a manner similar to that of deterministic automata except that now the set of tapes will depend not just on M but also on a parameter X.

Let M be p.a.and λ be a real number, $0 \leq X < 1$. The set of tapes $T(M, \lambda)$ is defined by

$$T(T, x) = \{x | x \in \Sigma, \lambda < p(x)\}.$$

If $x \in T(T, \lambda)$ we say that x is *accepted* by M with cut-point λ. $T(M, \lambda)$ will also be called the set defined by M with *cut-point* λ.

3 Applications

Intensive development of the theory of probabilistic automata in 60-70 years of last century and its applications to problems of synthesis and analysis of probability and stochastic processes converters generators contributed for development of computing devices which are focused on methods of statistical tests and probabilistic algorithms.

Schemes of probabilistic automata are applying:

1.To determine the possibilities of algorithmic systems

2.In the design of discrete systems displaying a statistically regular random behavior

3.In the simulation of adaptive behavior (living organisms, etc.)

3.1 Definition of algorithmic systems capabilities

The use of probabilistic automata schemes (P - schemes) is important. Probabilistic automata can be used as generators of Markov chain which are required in the construction and implementation of systems functioning processes of S or environmental influences E. Models and simulations implemented, for example, a method of statistical modeling can be used to evaluate the performance of different systems represented as P - schema, except the case of analytical models.

3.2 Method of statistical modeling

The probabilistic automata can be used for tasks such as the regulation of traffic through road intersection. By the way, often this complicated task is solved through the traffic piecewise stationary. According to this assumption, calculated expectations of cars assumptions suggest that the traffic streams are described by the Poisson distribution. This automaton mainly transmits traffic on the street with heavy traffic (line) and does not cover it when you see every single car moving by the crossing street. Probabilistic automata may be implemented as a system consisting of an automaton and a deterministic random-number generator, which generates signals with a predetermined probability distribution. Naturally, the numerical values of the traffic light switch probability and duration of its signaling are selected based on the actual conditions. Simple example is given. We assume that the automaton has only two states: driveway highway opened (Q_0) or closed (Q_1). Obviously, while driving on the open highway traffic the other street is prohibited to be used and vice versa. The input signal is a signal from the sensor of availability of transport to cross the street. Graph machine in this case can be represented as in Fig.1. Where, x - the input signal (x = 1, if there is traffic on the cross street);

P_{ij} - probability (transition) of the transition from state i to state j;

Q_0 - the initial state (driveway highway opened).

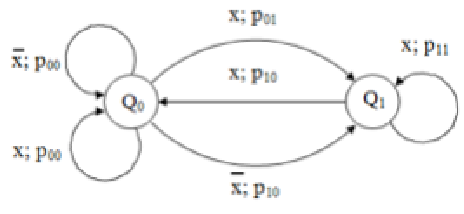

Figure 1: Graph of probabilistic automata for of the traffic light

3.3 Probabilistic language

Let A - AB, and $a \in [0, 1)$. Probabilistic language (PL) - BA A with section point a named the set of string $\subseteq X^*$ (Where X - the set of input signals A) defined as follows:

$$A_a = \{u \in X^* |\ f_A(u) > a\}.$$

The concept PL can be used for a variety of problems, in particular for the simulation of the learning process. One of the training models has the following form. A finite set S, whose elements are called reactions to some stimulus and every reaction s $\in$ S, is considered either a right or wrong. Let us consider example about simulation of adaptive behavior.

3.4 Simulation of Adaptive Behavior

Need of using the adaptive method arises in the solution of facility management tasks with incomplete initial information about the parameters and laws of their functioning. For the theory of adaptive systems it is very important to study its various methods in relation to specific classes of management objects.

Probabilistic machines are widely used in modeling behavior of robots. It is believed that a probabilistic automaton is one of the most effective and simple models of conditioned reflex behavior.

There is a device with N sensors and M effectors (actuators). Thus, the input alphabet of X = signals, and output - Y = (assuming an independent mining unit of each control action). The machine performs actions in accordance with a stochastic matrix P with size $Q \times X \times Y$, where Q − number of states. That is, in a state of being q (t) and accepting input for signal $x(t)$, the state machine transits to $q(t + 1)$. However, it commits an act y, chosen from the corresponding probability vector - rows of the matrix P (Fig. 2.):

$$y(t+1) = F(x(t), q(t), P(t)),$$
$$q(t+1) = Q(x(t), q(t)).$$

The automatic reaction to the input is evaluated - the machine is punished or encouraged. The meaning of the response to signal of penalties / promotion is to change the values of the probabilities of actions performed. Theoretically, the change of probability with the encouragement ($s = 0$) and punishment ($s = 1$) is as follows:

$$p_{ij}(t+1, s(t)) = p_{ij}(t, s(t)) + (-1)^{s(t+1)} \times g \times p_{ij}(t, s(t)) \times [1 - p_{ij}(t, s(t))]$$

$$p_{ik}(t+1, s(t)) = p_{ik}(t, s(t)) - (-1)^{s(t+1)} \times g \times p_{ik}(t, s(t)) \times p_{ij}(t, s(t))$$

Where g is a parameter determines the speed of learning. Thus, with the passage of time in the "training" Machine must create the necessary values of probabilities action.

In fact, learning machine is necessary for a long time (in addition we must not forget that this must be done in "real time"). Therefore, increasing the machine's ability to assess the situation at the expense of memory expansion - add new states - is extremely disadvantageous from the standpoint of study time. In theory, good results demonstrate a fully automatic of three states (Fig. 3):

p_{11}	p_{12}	p_{13}	...	p_{1n}
p_{21}	p_{22}	p_{23}	...	p_{2n}
p_{31}	p_{32}	p_{33}	...	p_{3n}
...				
p_{n1}	p_{n2}	p_{n3}	...	p_{nn}

Figure 2: Stochastic Action Matrix

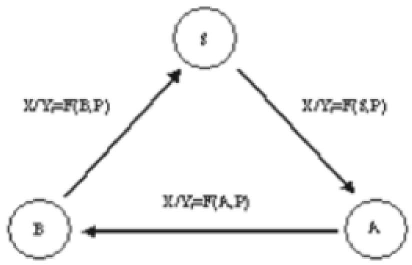

Figure 3: Machine control structure

4 Conclusion

In this paper we describe main points of *probabilistic automata*, basic definitions and applications. At first, we get a definition of probabilistic automaton. Secondly we describe properties of *transition function* and *cut-point* of transition function. Thirdly, we give some examples of applications.

5 Reference

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