Lab-session about λ-calculus

31st of March, 2016

Syntax

The grammar of λ-terms is defined as follows. The set of λ-terms is

$$\Lambda := \mathcal{V} \mid (\Lambda)\Lambda \mid \lambda\mathcal{V}.\Lambda$$

where $\mathcal{V}$ is a denumerable set of *variables*.

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The β-reduction is the contextual closure of: $(\lambda x.v)u\beta v[u/x]$

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Drawing reduction graphs

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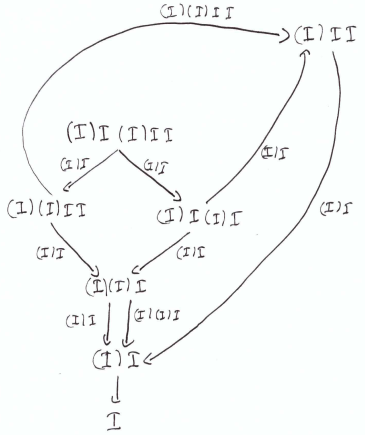
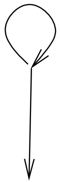


1) Find a λ-term with the following reduction graph: 

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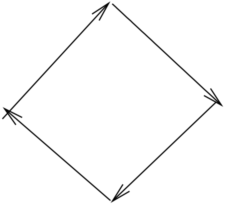

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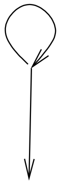

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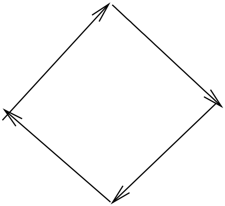

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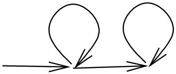

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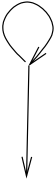

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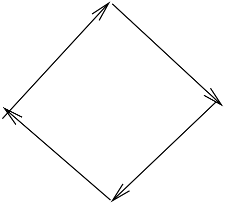

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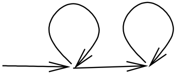

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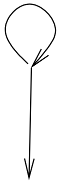
1) Find a λ-term with the following reduction graph: 

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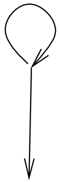

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4) Is a λ-term with a finite reduction graph necessarily normalizable?

Operations on Church numerals

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3) Find some normal λ-term that computes the square, i.e. find some normal λ-term t such that, for any $n \in \mathbb{N}$, we have $(t)\ulcorner n \urcorner \beta^* \ulcorner n^2 \urcorner$.