Discrete Mathematics

Adil M. Khan

Professor of Computer Science

Innopolis University

Do not worry about your difficulties
in Mathematics. I can assure you mine are still
greater!

-Albert Einstein-

Discrete Mathematics

- Study of Discrete Objects

  Consisting of Distinct Objects

- Problems Solved Using Discrete Math
  - How many ways are there to choose a valid password?
  - What's the probability of winning a lottery?
  - How to encrypt a message?
  - What is the shortest path b/w two cities?
  - How to sort a list of integers?
  - How to prove that an algorithm works correctly?
  - .
  - .
  - .

Why Study Discrete Mathematics???

- Ability to understand and create mathematical arguments

- Gateway to more advanced courses
  - Algorithms
  - Database theory
  - Automata theory
  - Compiler theory
  - Computer security
  - Operating system

Topics we'll study

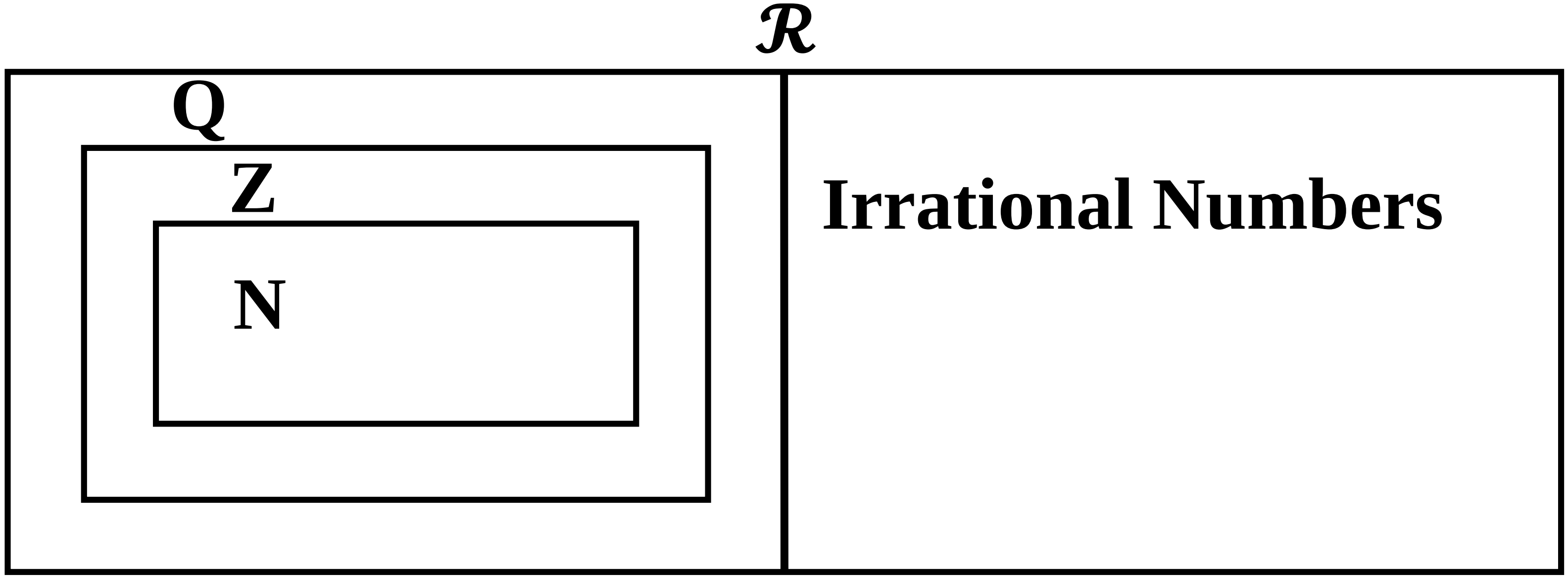
- Logic and Proofs
- Sequences and Recursion
- Mathematical Induction
- Set Theory
- Functions
- Relations
- Counting and Probability
- Graphs and Trees

Today’s Lecture

- **Integers:**
  - Arithmetic Properties
  - Powers
  - Divisibility
  - Primes of Composite Numbers
- **Rational Numbers**
  - Equivalent fractions
  - Operating with fractions
  - Decimals
- **Irrational Numbers**
- **Real Numbers**
  - Square roots
  - N-th roots
  - Logarithms
  - Inequalities
- **Oder of Operations**

Numbers

- **N**= {0,1,2,3, . . .} The set of Natural Numbers
- **Z**= {. . . , -2,-1,0,1,2, . . .} The set of Integers
- **Q**= {p/q | p, q ϵ Z, and q ≠ 0} The set of rational numbers
- $\mathcal{R}$, the set of real number. e.g. Real Space



Integers

❑ Simple rule of Addition

- For an integer **a**,
- $0+a = a+0 = a$
- $a+(-a)=0$, and $(-a)+a=0$
- -a is the **<u>additive inverse</u>** of a.

We use "**<u>Minus a</u>**" rather than "**<u>Negative a</u>**"

Integers

- **Rules of Addition**
  - **Commutativity**
    - If **a** and **b** are integers, then
      - $a + b = b + a$
  - **Associativity**
    - If **a**, **b** and **c** are integers, then
    - $(a + b) + c = a + (b + c)$

Integers

❑ Rules of Addition

- If **$a + b = 0$,** then **$b = -a$** and **$a = -b$**

- Proof

$$a + b = 0$$

Add **$-a$** to both sides

$$a + b + (-a) = 0 + (-a) \quad \text{// commutativity, identity}$$

$$-a + a + b = -a \quad \text{//associativity, additive inverse}$$

$$0 + b = -a \quad \text{//identity}$$

$$\mathbf{b = -a}$$

As desired.

Similarly we can find: $a = -b$

Integers

- **Rules of Addition**
  - If **a**, **b** are positive integers, then **a + b** is also positive integer.
  - If **a**, **b** are negative integers, then **a** + **b** is also negative integer.

  - If we have the relationship b/w three integers.
    - $a + b = c$

Then we can drive other relationships b/w them.

$$\mathbf{a = c - b} \qquad \mathbf{b = c - a}$$

- Example: *Solve for **x***.

$$\mathbf{x + 3 = 5}$$

$$\mathbf{x = 5 - 3}$$

$$\mathbf{x = 2}$$

Integers

❑ Rules of Addition

- Cancellation rule for addition
  - If **a + b = a + c**, then **b = c**

Exercise:

Prove that if **a + b = a**, then b = 0?

Integers

- **Rules of Multiplication**

  - **Commutativity**

    - If **a** and **b** are integers, then
      - a * b = b * a

  - **Associativity**

    - If a , b and c are integers, then
      - (a * b) * c = a * (b * c)

  - For any integer a
    - 1 * a = a and 0 * a = 0

Integers

- **Rules of Multiplication**

  - **Distributivity**

    - **a * (b + c) = a * b + a * c**
    - **(b + c) * a = b * a + c * a**

    Using all these properties

    - **-1 * a = -a**
    - **-(a * b) = (-a) * (b) or -(a * b) = a * (-b)**
    - **(-a) * (-b) = a * b**

Integers

- **Powers**
  - An exponent is used to indicate repeated multiplication.
  - Tells how many times the base is used as a factor.
    - $\mathbf{a * a = a^2}$
    - $\mathbf{a * a * a = a^3}$

In general if n is a positive integer,

- $\mathbf{a^n = a * a * a \ldots a}$ (product is taken n times)

We say a^n is the *n-th power of* ***a***.

If **m, n** are positive integers, then

- $\mathbf{a^{m+n} = a^m * a^n}$

Integers

- **Powers**

  - $(a^m)^n = a^{m * n}$

  Some important formulas

  - $(a + b)^2 = a^2 + b^2 + 2ab$

  - $(a - b)^2 = a^2 + b^2 - 2ab$

  - $(a + b)(a - b) = a^2 – b^2$

Integers

- **Even and Odd integers**

  - An **even** integer is an integer which can be written in the form **2n** for some integer **n**
    - $2 = 2 * 1$
    - $4 = 2 * 2$
    - $6 = 2 * 3$
  - An odd integer is an integer that differs from an even integer by 1.
  - It can be written in the form **2m ± 1** for some integer **m.**
    - $1 = (2 * 1) - 1$
    - $3 = (2 * 2) - 1$
    - $7 = (2 * 3) + 1$

Integers

- **Theorem**

  - Let **a**, **b** be integers,
    - If **a** is even and **b** is also even, then **a + b** is also even
    - If **a** is even and **b** is odd, then **a + b** is odd
    - If **a** is odd and **b** is even, then **a + b** is odd
    - If a is odd and b is also odd, then a + b is also even

- **Exercise.**

  - Let’s prove the Second statement

Integers

- **Divisibility**

  - Given two integers a and b, with $a \neq 0$, we say that **a divides b**, or that **b is divisible by a** if there is an integer c, such that **b = a * c**.

  - Remember that every integer is divisible by 1 because we can always write

    - n = 1 * n

- Also, every positive integer is divisible by itself.

Rational Numbers

- By a rational numbers, we mean a fraction as $\frac{\boldsymbol{m}}{\boldsymbol{n}}$,

  where **m** and **n** are integers, n ≠ 0.

  - m is called **numerator**
  - n is called **denominator**

- Improper fraction

  - **m** larger than or equal to **n**

- Proper fraction

  - **m** smaller than **n**

Rational Numbers

❑ Equivalent Fractions

- ❑ Two fractions that represent the same value.

  - $\frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{2}}{\mathbf{4}}$

  How can we know whether two fractions are equivalent?

- ❑ Rule for cross-Multiplication

  - Let ***m, n, r, s*** be integers and assume that $n \neq 0$ and $s \neq 0$. Then

  - $\left(\frac{m}{n}\right) = \left(\frac{r}{s}\right)$, ***iff*** $m * s = r * n$

Rational Numbers

❑ Simplifying Fractions

- ❑ We can simplify four special fractions forms
  - ❑ Fractions that have the same numerator and denominator.
    - $1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \ \ldots$
  - ❑ Fractions that have a denominator of 1.
    - $\frac{5}{1} = 5,\ \frac{24}{1} = 24,\ \frac{-6}{1} = -6$
  - ❑ Fractions that have a numerator of 0.
    - $\frac{0}{8} = 0,\ \frac{0}{71} = 0,\ \frac{0}{-10} = 0$
  - ❑ Fractions that have a denominator of 0
    - $\frac{7}{0} = \infty,\ \frac{-17}{0} = \infty,$ (∞=Infinity=Undefined Value)

Rational Numbers

- **Simplifying Fractions**

  - Cancellation Rule for Fractions

    - Let a be a non-zero integer. Let **m**, **n** be integers, and n $\neq$ 0, then

      - $\frac{am}{an} = \frac{m}{n}$

    - Proof: By applying the rule for cross-multiplication and using the associativity and commutativity laws.

Rational Numbers

- **Simplifying Fractions**

  - A fraction is in <u>**simplest form**</u> when the numerator and denominator have no common factors (or divisors) other than 1.

  - Theorem:

    - "Any positive rational number has an expression as a fraction in the lowest form."

Rational Numbers

- **Operating with Fractions**
  - Addition (or Subtraction) with same denominator.
    - $\frac{\mathrm{a}}{\mathrm{d}}+\frac{\mathrm{b}}{d}=\frac{\mathrm{a+b}}{\mathrm{d}}$ **<u>or</u>** $\frac{\mathrm{a}}{\mathrm{d}}-\frac{\mathrm{b}}{d}=\frac{\mathrm{a-b}}{\mathrm{d}}$
  - With different denominator:
    - $\frac{\mathrm{m}}{\mathrm{n}}+\frac{\mathrm{r}}{s}=\frac{\mathrm{ms+rn}}{\mathrm{ns}}$ **<u>or</u>** $\frac{\mathrm{m}}{\mathrm{n}}-\frac{\mathrm{r}}{s}=\frac{\mathrm{ms-rn}}{\mathrm{ns}}$
    - Follows the same basic rules as addition of integers (commutativity and association)

Rational Numbers

- **Multiplication:**

  - Let $a = \frac{\mathrm{m}}{\mathrm{n}}$

    - Then for any positive integer **k**, such that

    - $$a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}$$

  - Follows the same basic rules as multiplication of integers.

Rational Numbers

- **Division:**

  - If **a** is a rational number and **a** $\neq 0$, then there exists ($\exists$) a rational number, denoted by

    - a^{-1} such that

    - $a^{-1} * a = a * a^{-1} = 1$

  - Note that if $a = \frac{m}{n}$ then $a^{-1} = \frac{n}{m}$

  - a^{-1} is called the **multiplicative inverse** of a.

Rational Numbers

- **Decimals:**

  - Finite decimals give us examples of rational numbers.

    - $1.4 \ = \frac{14}{10}$
    - $1.41 = \frac{141}{100}$
    - $0.2 \ = \frac{1}{5}$
    - $0.75 = \frac{3}{4}$
    - $0.3333\ldots = 0.\overline{3} = \frac{1}{3}$
    - $\ldots$

Irrational Numbers

- A number that cannot be expressed as fraction of $\frac{p}{q}$ for any integers p and q.

- Have decimal expressions that neither terminate nor become periodic

  - $\sqrt[2]{2} = 1.41421356237 \ldots$
  - $\sqrt[2]{3} = 1.73205080757\ldots$
  - $\pi = 3.14159265359 \ldots$
  - $. . .$

Irrational Numbers

- ❑ Is $\sqrt[2]{25}$ an irrational number?
  - No!
  - Because $\sqrt[2]{25} = \pm 5$
- ❑ Is $\sqrt[2]{-1}$ an irrational number?
  - No or Yes??? In both cases HOW?

Real Numbers

- Integers, Rational and Irrational Numbers are part of a larger system.

- Real Numbers can be described as all the numbers that consist of a decimal expansion, possibly infinite.

Real Numbers

- **Properties of Real Numbers:**
  - Addition:
    - $a + b = b + a$
    - $a + (b + c) = (a + b) + c$
    - For all ($\forall$) real numbers a, b, and c.
  - Multiplication
    - $a * b = b * a$
    - $a * (b * c) = (a * b) * c$
    - $\forall$ real numbers a, b , c.
  - Also
    - $a * (b + c) = a * b + a * c$
    - $(b + c) * a = b * a + c * a$

Real Numbers

- **Absolute Value**

  - The non-negative values of a real number without regard to it sign.

    - $|a| = a$ for a positive a.
    - $|a| = -a$ for a negative a

      (in which case –a is positive).
    - $|0| = 0$

Real Numbers

- **Square Roots**
  - If $a > 0$, then there exists ($\exists$) a number b such that (s.t).
    - $b^2 = a$
- **N-th Roots**
  - There exists a unique real number r such that
    - $r^n = a$

      It is called the n-th root of a, and is denoted by
    - $a^{1/n}$ or $\sqrt[n]{a}$

Logarithms

- Can be seen as the reverse operation of the exponentiation.
- The logarithm of a number is the exponent to which another fixed value, the **Base** must be raised to produce that number.
  - $\log_{10}(10000) = 4$, because $10^4=10000$
  - $\log_2(16) = 4$, because $2^4=16$
  - $\log_3\left(\frac{1}{3}\right) = -1$, because $3^{-1}=\frac{1}{3}$

Logarithms

- **Properties of Logarithms:**
  - Product:
    - $\log_b(x * y) = \log_b(x) + \log_b(y)$
  - Quotient:
    - $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
  - Power:
    - $\log_b(x^p) = p * \log_b(x)$
  - Change of Base:
    - $\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$

Inequalities

Symbol	Meaning	Example
$>$	Greater Than	$(X + 3) > 2$, for any X
$<$	Less Than	$(7X) < 28$, X= {…, -2, -1, 0 , 1 , 2 , 3}
$\geq$	Greater Than or Equal	$5 \geq (X - 1)$, X= {…, -2, -1, 0, 1, …, 5 ,6}
$\leq$	Less Than or Equal	$(2Y + 1) \leq 7$, Y= {…, -2, -1, 0, 1, 2, 3}

- Let **a, b, c** be real numbers,

  - If $a > b$ and $b > c$ then $a > c$. (Transitivity)
  - If $a > b$ and $c > 0$ then $a*c > b*c$.
  - If $a > b$ and $c < 0$ then $a*c < b*c$.