Discrete Mathematics

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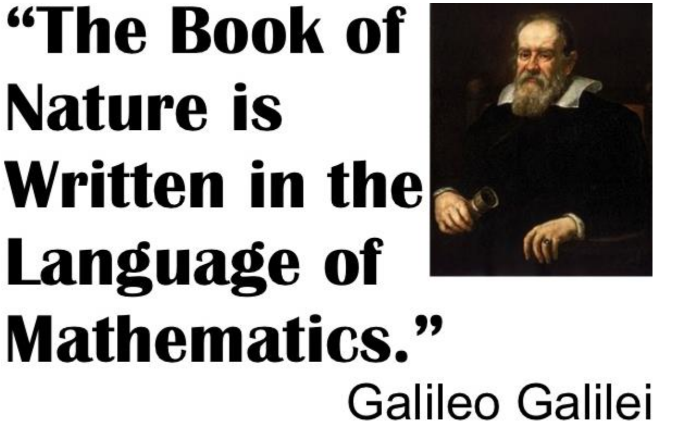

- ❑ **Logic of Compound Statements**

Logic

- Branch of mathematics used for precise reasoning
- Helpful in doing proof's
- Design of computer circuits
- Construction of computer programs
- Verification of the correctness of programs

And in many more ways….

Propositions (Statements)

- Basic building blocks of logic
- A **declarative sentence** – a sentence that declares a **fact**
- Is either <u>True</u> or <u>False</u> but <u>Not Both</u>

❑**Examples**:

- *Two plus two is equal to four.*
- *Toronto is the capital of Canada.*
- *2+2 = 3*
- *"He is a college Student"* ⟶ *???*
- *"What time is it?"* ⟶ *???*
- *"x + y = z"* ⟶ *???*

❑**Propositional Variables**

- Variables that represent propositions
- Conventional letters are : p, q, r, s, . . .
- Truth values: T(true), F(false)

Compound Statement (Propositions)

- Complicated logical statements build out of simple ones
- Three Symbols
  - ~ (not) --- ~ p (not p)
  - Λ (and) --- p Λ q (p and q)
  - V (or) --- p V q (p or q)
- ~ p (Negation), pΛq (Conjunction), pVq (Disjunctions)
- English words to logic
  - "p **but** q", "p **although** q" means "p **and** q"
  - "**neither** p **nor** q" means "~ p **and** ~ q"

Examples

1. Let **p** = "*Adil's PC runs Linux*"

   • **~p** ?

2. Let **H** = "It is hot"

   S = "It is Sunny"

   (i). "It is **not** hot **but** it is Sunny"

   "________________________"

   (ii). "It is **neither** hot **nor** Sunny"

   "________________________"

Truth Tables

Truth table for ~ p

p	~ p
T	F
F	T

Truth table for p Λ q

p	q	p Λ q
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for pVq

p	q	p V q
T	T	F
T	F	T
F	T	T
F	F	F

Logical Equivalence

(1) "*Dogs bark and Cats meow*",

(2) "*Cats meow and Dogs bark*"

- Two different ways of saying the same thing.
- Reason has nothing to do with words
- It has to do with the **logical from**

- Any two statements whose logical forms are related in the same way as (1) and (2) would be logical equivalent.
- Can be confirmed by **examining the truth tables**.

For instance, consider the 2 following statements:

If Sally wakes up late or if she misses the bus, she will be late for work. Therefore, if Sally arrives at work on time, she did not wake up late and did not miss the bus.

If x is a real number such that x < -2 or x > 2, then x2 > 4. Therefore, if x2 < 4, then x > -2 and x < 2.

While the content of the two above statements is different, their logical form is similar.

Let p stand for the statements "Sally wakes up late" and "x is a real number such that x < -2".
Let q stand for the statements "Sally misses the bus" and "x is a real number such that x > -2".
***Let r* stand for the statements "Sally is late for work" and "x2 > 4".**

Then the common form for both of the above arguments is:

If p or q, then r.
Therefore, if not r, then not p and not q.

Conditional Statements

- “if p, then q” $p \rightarrow q$
- False when p is true and q is false
- True otherwise
- Also called **implication**
- “If $\underbrace{\text{4686 is divisible by 6}}_{\substack{\text{Hypothesis} \\ p}}$, then $\underbrace{\text{4686 is divisible by 3}}_{\substack{\text{Conclusion} \\ q}}$”
- Truth of q is conditional on the truth of p.

Conditional Statements

- Truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- An easy way to understand $p \rightarrow q$ is to view it as a contract
- Imagine the following statement
- "If $\underbrace{\text{you get 100\% on the final}}_{p}$, then $\underbrace{\text{you will get an A}}_{q}$"
- Under what circumstances are you justified in saying that the professor spoke falsely?

Contrapositive of a Conditional Statement $p \to q$

- "If $\sim q$ then $\sim p$" ----- $\sim q \to \sim p$
- A conditional statement is logical equivalent to its contrapositive.
- Example:

If $\underbrace{\text{today is Easter}}_{p}$, then $\underbrace{\text{tomorrow is Monday}}_{q}$.

- Contrapositive = ???

Converse and Inverse of a Conditional Statement

- Converse is "If q then p" $q \rightarrow p$

- Inverse is "If ~ p then ~ q" $\sim p \rightarrow \sim q$

- $p \rightarrow q \not\equiv q \rightarrow p$
- $p \rightarrow q \not\equiv \sim p \rightarrow \sim q$
- $q \rightarrow p \equiv \sim p \rightarrow \sim q$

Negation of a Conditional Statement $p \rightarrow q$

- $\sim (p \rightarrow q)$ is logically equivalent to "$\sim p \wedge q$"

- **p and not q**

Bi-conditional Statement ($p \leftrightarrow q$)

- "p *if and only if* q" ---- (p ***iff*** q)
- True: when both p and q have the same truth values
- False: Otherwise

Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Bi-conditional Statement ($p \leftrightarrow q$)

- Example:

p = "You can take the flight"

q = "You buy a ticket"

$p \leftrightarrow q$ = "You can take the flight ***iff*** you buy a ticket "

Be aware!

- Bi-conditionals are not always explicit in natural language.

"if you finish your meal, **then** you can have dessert"

"John will brake the worlds record for mile run **only if** he runs the mile in under four minutes".

Bi-Conditionals are often expressed using "if, then" or "only if" construction.

Other Useful Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6.	*Double negative law:*	${\sim}({\sim}p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

Valid and Invalid Arguments

- Propositional logic can be used as a math model to investigate the validity of arguments.
- As **argument** is a **sequence of statements**.
- All but the **final statements** are called **premises**.
- Final statement is called conclusion.
- <u>Valid Argument</u>: If the premises are all true then the conclusion is also true.
  - $P_1 \land \ldots \land P_n \vdash Q$

  i.e. Premises logically implies the conclusion.

Argument Validity

- Two Ways:
  - Using truth Tables
  - Reason at a higher level using generally valid rules (inference values).

Inference Rule

- To helps showing that a conclusion follows logically from a set of premises we may apply inference rules on the form,

$$p_1 \ldots p_n \mid \therefore q$$

- The validity of the rule is ensured

  If $(p_1 \wedge \ldots \wedge p_n) \rightarrow q$ is a **Tautology**

  - A tautology is a statement which is always true. E.g. $\mathbf{p} \vee \sim \mathbf{p}$.

Inference Rule

- Modus Ponens

  - $$\begin{array}{c} p \\ \underline{p \to q} \\ \therefore q \end{array}$$ (Based on $[p \wedge (p \to q) \to q]$)

- Modus Tollens

  - $$\begin{array}{c} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$ (Based on $[(p \to q) \wedge \sim q \to \sim p]$)

- Generalization:

  - $$\begin{array}{c} \underline{p} \\ \therefore p \vee q \end{array} , \begin{array}{c} \underline{q} \\ \therefore p \vee q \end{array}$$

Inference Rule

- Specialization
$$\frac{p \wedge q}{\therefore p}\ ,\ \frac{p \wedge q}{\therefore q}$$

- Elimination
$$\frac{\begin{matrix} p \vee q \\ \sim q \end{matrix}}{\therefore p}\ ,\ \frac{\begin{matrix} p \vee q \\ \sim p \end{matrix}}{\therefore q}$$

- Conjunction
$$\frac{\begin{matrix} p \\ q \end{matrix}}{\therefore p \wedge q}$$

- Transitivity
$$\frac{\begin{matrix} p \rightarrow q \\ q \rightarrow r \end{matrix}}{\therefore p \rightarrow r}$$

Inference Rule Application

- Example: You are about to leave for University in the morning and discover that you don’t have your glasses. You know the following statements are true.

  A. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

  B. If my glasses are on the kitchen table, then I saw them at breakfast.

  C. I did not see my glasses at breakfast.

  D. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

  E. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses??

Inference Rule Application

Assume, RK= Reading the newspaper in the kitchen.

GK= Glasses are on the kitchen table.

SB= I saw my glasses at breakfast.

RL= Reading the newspaper in the living room.

GC= Glasses are on the coffee table.

So by rules of inference,

1.
$$\begin{array}{ll} RK\rightarrow GK & \text{(by A)} \\ GK\rightarrow SB & \text{(by D)} \\ \hline \therefore RK\rightarrow SB & \text{(Transitivity)} \end{array}$$

2.
$$\begin{array}{ll} RK\rightarrow SB & \text{(by 1)} \\ \sim SB & \text{(by C)} \\ \hline \therefore \sim RK & \text{(by modus tollens)} \end{array}$$

3.
$$\begin{array}{ll} RL \vee RK & \text{(by D)} \\ \sim RK & \text{(by 2)} \\ \hline \therefore RL & \text{(by elimination)} \end{array}$$

4.
$$\begin{array}{ll} RL\rightarrow GC & \text{(by C)} \\ RL & \text{(by 3)} \\ \hline \therefore GC & \text{(by modus ponens)} \end{array}$$

So the Glasses are on the Coffee table.

Contradictions and Valid Arguments

- Contradiction Rule
  - Suppose **p** is some statement whose truth you wish to deduce.
  - If you can show that the supposition that **p** is false leads logically to a contradiction, then you can conclude that **p** is true.

Contradictions and Valid Arguments

- Contradiction Rule

$$\frac{\sim p \rightarrow c}{\therefore p}$$, where c is a contradiction

p	$\sim p$	c	$\sim p \rightarrow c$	p
T	F	F	T	T
F	T	F	F	F

Logical heart of the method of proof by contradiction.

Applications of Propositional Logic

- System Specifications
  - Natural language to Logical expressions
  - Precise and unambiguous system specifications

Applications of Propositional Logic - Cont.

- For example,
  - Express the following specification using logical connectives

"The automated reply cannot be sent when the file system is full"

Applications of Propositional Logic - Cont.

Determine whether these system specifications are consistent

- "The diagnostic message is stored in the buffer or it is retransmitted"
- "The diagnostic message is not stored in the buffer"
- "If the diagnostic message is stored in the buffer, then it is transmitted"