Hamiltonian Origin of an Effective ZX Interaction and Driven Bell-State Generation

Abstract

We present a Hamiltonian-level derivation of an effective ZX interaction starting from two two-level systems (TLS) dispersively coupled to a common cavity and subject to a resonant drive. Beginning from a microscopic Jaynes–Cummings model, we eliminate the cavity via a Schrieffer–Wolff transformation, obtaining a cavity-mediated $XX + YY$ interaction between the qubits. We show that the application of a resonant drive on one qubit, followed by a transformation to the drive rotating frame and a dressed-state analysis, gives rise to an emergent conditional ZX interaction. This effective Hamiltonian is then used to generate a maximally entangled Bell state, verified by the von Neumann entropy of the reduced density matrix. The derivation demonstrates that the ZX interaction is not a primitive Hamiltonian term but an emergent, drive-induced coupling.

1 Microscopic Hamiltonian

We consider two two-level systems (TLS), labeled $i = 1, 2$, coupled to a single electromagnetic mode of a cavity. The microscopic Hamiltonian of the system is written as

$$H = H_0 + V, \tag{1}$$

where

$$H_0 = \sum_{i=1}^{2} \frac{\omega_i}{2}\sigma_{z,i} + \omega_r a^\dagger a, \tag{2}$$

describes the free Hamiltonian of the qubits and the cavity, and

$$V = \sum_{i=1}^{2} g_i \left(a\sigma_{+,i} + a^\dagger \sigma_{-,i} \right) \tag{3}$$

is the Jaynes–Cummings interaction between each qubit and the cavity mode.

Here ω_i are the qubit transition frequencies, ω_r is the cavity frequency, and g_i are the qubit–cavity coupling strengths. Throughout this work, we treat the qubits as ideal two-level systems; weak anharmonicity effects may be incorporated at a later stage if required.

At this point, no external control fields are applied. The Hamiltonian above defines the microscopic starting point from which effective qubit–qubit interactions will be derived. The inclusion of a classical drive will be introduced only after the cavity degrees of freedom have been eliminated and the relevant low-energy qubit Hamiltonian has been identified.

2 Dispersive Regime and Schrieffer–Wolff Transformation

We work in the dispersive regime, defined by

$$|\Delta_i| = |\omega_i - \omega_r| \gg g_i, \tag{4}$$

where real excitation exchange between the qubits and the cavity is strongly suppressed. In this regime, the effects of the cavity can be treated perturbatively and incorporated into an effective qubit-only Hamiltonian.

To eliminate the cavity degrees of freedom, we perform a Schrieffer–Wolff (SW) transformation of the form

$$\tilde{H} = e^{S} H e^{-S}, \tag{5}$$

where the generator S is anti-Hermitian and expanded perturbatively in the small parameters g_i/Δ_i.

The generator is chosen such that the qubit–cavity interaction is removed to first order in the dispersive expansion. This requirement fixes S through the condition

$$[S, H_0] = -V, \tag{6}$$

which ensures that all terms linear in g_i cancel in the transformed Hamiltonian.

With this choice, the leading effects of the qubit–cavity coupling appear only at second order in g_i/Δ_i, giving rise to effective qubit frequency renormalization and a cavity-mediated qubit–qubit interaction. The explicit form of these terms is obtained by expanding the transformation using the Baker–Campbell–Hausdorff formula, as shown in the following section.

2.1 BCH Expansion and Choice of Generator

To construct the Schrieffer–Wolff transformation explicitly, we choose an anti-Hermitian generator S that is linear in the qubit–cavity coupling and removes the interaction V to first order in perturbation theory. A convenient choice is

$$S = \sum_{i=1}^{2} \frac{g_i}{\Delta_i} \left(a^\dagger \sigma_{-,i} - a \sigma_{+,i} \right), \tag{7}$$

where $\Delta_i = \omega_i - \omega_r$ are the qubit–cavity detunings.

This generator is chosen such that it satisfies the condition 6 which ensures that the qubit–cavity interaction is eliminated at first order in the transformed Hamiltonian. Indeed, using

$$[H_0, a] = -\omega_r a, \qquad [H_0, \sigma_{+,i}] = \omega_i \sigma_{+,i}, \tag{8}$$

and their Hermitian conjugates, one finds

$$[S, H_0] = -\sum_{i=1}^{2} g_i \left(a\sigma_{+,i} + a^\dagger \sigma_{-,i} \right) = -V. \tag{9}$$

With this choice of generator, the transformed Hamiltonian $\tilde{H} = e^{S} H e^{-S}$ can be expanded using the Baker–Campbell–Hausdorff (BCH) formula,

$$\tilde{H} = H + [S, H] + \frac{1}{2}[S, [S, H]] + \mathcal{O}(S^3). \tag{10}$$

Substituting $H = H_0 + V$ yields

$$\tilde{H} = H_0 + V + [S, H_0] + [S, V] + \frac{1}{2}[S, [S, H_0]] + \frac{1}{2}[S, [S, V]] + \mathcal{O}(S^3). \tag{11}$$

Using the defining condition (6), the first-order interaction terms cancel exactly,

$$V + [S, H_0] = 0. \tag{12}$$

The transformed Hamiltonian therefore reduces to

$$\tilde{H} = H_0 + [S, V] + \frac{1}{2}[S, [S, H_0]] + \frac{1}{2}[S, [S, V]] + \mathcal{O}(S^3). \tag{13}$$

Next, we note that

$$[S, [S, H_0]] = [S, -V] = -[S, V], \tag{14}$$

which gives

$$\tilde{H} = H_0 + [S, V] - \frac{1}{2}[S, V] + \frac{1}{2}[S, [S, V]] + \mathcal{O}(S^3) \tag{15}$$

$$= H_0 + \frac{1}{2}[S, V] + \frac{1}{2}[S, [S, V]] + \mathcal{O}(S^3). \tag{16}$$

Since $S \sim g/\Delta$, the term $[S, [S, V]]$ scales as $\mathcal{O}(g^3/\Delta^2)$ and is neglected in the dispersive regime. Retaining terms up to second order, we obtain the effective Hamiltonian

$$\tilde{H} = H_0 + \frac{1}{2}[S, V] + \mathcal{O}\left(\frac{g^3}{\Delta^2}\right). \tag{17}$$

Equation (17) is the central result of the Schrieffer–Wolff transformation: the qubit–cavity interaction is removed at first order, and its physical consequences appear only through the second-order commutator $\frac{1}{2}[S, V]$.

2.2 Explicit Evaluation of the Commutator $[S, V]$

We now explicitly evaluate the second-order commutator $[S, V]$. Using the definitions of S and V, we write

$$[S, V] = \sum_{i,j} \frac{g_i g_j}{\Delta_i} \left[a^\dagger \sigma_{-,i} - a\sigma_{+,i},\ a\sigma_{+,j} + a^\dagger \sigma_{-,j}\right]. \tag{18}$$

Expanding the commutator gives four terms:

$$[S, V] = \sum_{i,j} \frac{g_i g_j}{\Delta_i} \Big([a^\dagger \sigma_{-,i}, a\sigma_{+,j}] + [a^\dagger \sigma_{-,i}, a^\dagger \sigma_{-,j}] - [a\sigma_{+,i}, a\sigma_{+,j}] - [a\sigma_{+,i}, a^\dagger \sigma_{-,j}] \Big). \tag{19}$$

The second and third commutators vanish identically because $[a^\dagger, a^\dagger] = 0$ and $[a, a] = 0$. We therefore retain only the first and fourth terms.

2.2.1 Nonvanishing Contributions

Using the canonical commutation relation $[a, a^\dagger] = 1$ and the fact that operators acting on different qubits commute, we evaluate the remaining terms.

First term:

$$[a^\dagger\sigma_{-,i}, a\sigma_{+,j}] = a^\dagger a\,[\sigma_{-,i},\sigma_{+,j}] + [a^\dagger, a]\sigma_{-,i}\sigma_{+,j}$$
$$= -\delta_{ij}\,\sigma_{z,i}\,a^\dagger a - \sigma_{-,i}\sigma_{+,j}. \tag{20}$$

Fourth term:

$$[a\sigma_{+,i}, a^\dagger\sigma_{-,j}] = aa^\dagger[\sigma_{+,i},\sigma_{-,j}] + [a, a^\dagger]\sigma_{+,i}\sigma_{-,j}$$
$$= \delta_{ij}\,\sigma_{z,i}\,a^\dagger a + \sigma_{+,i}\sigma_{-,j}. \tag{21}$$

2.2.2 Collecting Terms

Combining the nonvanishing contributions, we obtain

$$[S, V] = \sum_{i,j} \frac{g_i g_j}{\Delta_i}\left(-2\,\delta_{ij}\,\sigma_{z,i}\,a^\dagger a - \sigma_{-,i}\sigma_{+,j} - \sigma_{+,i}\sigma_{-,j}\right). \tag{22}$$

Separating diagonal ($i = j$) and off-diagonal ($i \neq j$) terms yields

$$[S, V] = -2\sum_i \frac{g_i^2}{\Delta_i}\sigma_{z,i}a^\dagger a$$
$$- \sum_{i\neq j}\frac{g_i g_j}{\Delta_i}\left(\sigma_{+,i}\sigma_{-,j} + \sigma_{-,i}\sigma_{+,j}\right). \tag{23}$$

2.2.3 Projection onto the Zero-Photon Subspace

In the dispersive regime, we project onto the cavity vacuum subspace $|0\rangle_r$, for which $a^\dagger a \to 0$. The photon-number–dependent terms therefore vanish.

The effective second-order interaction is thus

$$\frac{1}{2}[S, V] = -\frac{1}{2}\sum_{i\neq j}\frac{g_i g_j}{\Delta_i}\left(\sigma_{+,i}\sigma_{-,j} + \sigma_{-,i}\sigma_{+,j}\right). \tag{24}$$

For two qubits, this reduces to

$$\frac{1}{2}[S, V] = J\left(\sigma_{+,1}\sigma_{-,2} + \sigma_{-,1}\sigma_{+,2}\right), \tag{25}$$

with

$$J = -\frac{g_1 g_2}{2}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}\right). \tag{26}$$

3 Cavity-Mediated Qubit–Qubit Interaction

Evaluating the commutator $[S, V]$ and projecting onto the zero-photon subspace yields the effective qubit Hamiltonian

$$H_{\text{eff}}^{(0)} = \sum_{i=1}^{2}\frac{\omega_i}{2}\sigma_{z,i} + J\left(\sigma_{+,1}\sigma_{-,2} + \sigma_{-,1}\sigma_{+,2}\right), \tag{27}$$

where

$$J = -\frac{g_1 g_2}{2}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}\right). \tag{28}$$

The photon-number–dependent dispersive terms proportional to $a^\dagger a \sigma_{z,i}$ vanish upon projection onto the cavity vacuum subspace. As a result, the bare qubit frequencies ω_i remain unchanged at this order.

Using Pauli operators,

$$\sigma_{+,1}\sigma_{-,2} + \sigma_{-,1}\sigma_{+,2} = \frac{1}{2}\left(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}\right). \tag{29}$$

Thus, after eliminating the cavity,

$$H_{\text{eff}} = \sum_{i=1}^{2} \frac{\omega_i}{2}\sigma_{z,i} + J\left(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}\right). \tag{30}$$

For notational simplicity, we absorbed the factor of 1/2 into the definition of J. At this stage, no ZX interaction is present.

4 Drive Rotating Frame and Rotating-Wave Approximation

Having eliminated the cavity degrees of freedom, we now introduce a classical microwave drive applied to qubit 1. The effective qubit Hamiltonian in the laboratory frame is

$$H_{\text{eff}}(t) = \sum_{i=1}^{2} \frac{\omega_i}{2}\sigma_{z,i} + J\left(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}\right) + \Omega\cos(\omega_d t)\,\sigma_{x1}, \tag{31}$$

where Ω and ω_d denote the amplitude and frequency of the drive, respectively.

To make the effect of the drive explicit and remove its fast time dependence, we transform to a rotating frame defined by the unitary operator

$$U(t) = \exp\left[-i\frac{\omega_d t}{2}(\sigma_{z1} + \sigma_{z2})\right]. \tag{32}$$

Although the drive acts only on qubit 1, we employ a global rotating frame for both qubits so that the exchange interaction retains a simple, time-independent form.

The Hamiltonian in the rotating frame is given by

$$H_{\text{rot}} = U^\dagger H_{\text{eff}}(t) U - iU^\dagger \dot{U}. \tag{33}$$

We now evaluate each contribution explicitly.

Free qubit terms. Since $[\sigma_{z,i}, U(t)] = 0$, we have

$$U^\dagger\left(\sum_{i=1}^{2} \frac{\omega_i}{2}\sigma_{z,i}\right)U = \sum_{i=1}^{2} \frac{\omega_i}{2}\sigma_{z,i}. \tag{34}$$

The second term contributes

$$-iU^\dagger \dot{U} = \frac{\omega_d}{2}(\sigma_{z1} + \sigma_{z2}), \tag{35}$$

so that the free Hamiltonian becomes

$$\sum_{i=1}^{2} \frac{\delta_i}{2}\sigma_{z,i}, \qquad \delta_i = \omega_i - \omega_d. \tag{36}$$

Drive term. Writing the drive Hamiltonian as

$$\Omega\cos(\omega_d t)\,\sigma_{x1} = \frac{\Omega}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)\sigma_{x1}, \tag{37}$$

and using $U^\dagger \sigma_{\pm,1} U = \sigma_{\pm,1} e^{\pm i\omega_d t}$, we obtain

$$U^\dagger H_d(t) U = \frac{\Omega}{2}\left(\sigma_{x1} + \sigma_{+1}e^{2i\omega_d t} + \sigma_{-1}e^{-2i\omega_d t}\right). \tag{38}$$

Within the rotating-wave approximation, the rapidly oscillating terms at $\pm 2\omega_d$ are neglected, leaving

$$U^\dagger H_d(t) U \approx \frac{\Omega}{2}\sigma_{x1}. \tag{39}$$

Exchange interaction. The exchange term may be written as

$$\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2} = \sigma_{+1}\sigma_{-2} + \sigma_{-1}\sigma_{+2}. \tag{40}$$

Under the rotating-frame transformation, the phase factors associated with the two qubits cancel, yielding

$$U^\dagger\left(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}\right)U = \sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}. \tag{41}$$

Collecting all contributions, we obtain the time-independent rotating-frame Hamiltonian

$$H_{\text{rot}} = \frac{\delta_1}{2}\sigma_{z1} + \frac{\delta_2}{2}\sigma_{z2} + \frac{\Omega}{2}\sigma_{x1} + J\left(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}\right). \tag{42}$$

5 Dressed-State Transformation of the Driven Qubit

We now diagonalize the driven control-qubit Hamiltonian appearing in the rotating-frame Hamiltonian

$$H_{\text{rot}} = \frac{\delta_1}{2}\sigma_{z1} + \frac{\delta_2}{2}\sigma_{z2} + \frac{\Omega}{2}\sigma_{x1} + J(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}). \tag{43}$$

The Hamiltonian of the driven control qubit alone is

$$H_1 = \frac{\delta_1}{2}\sigma_{z1} + \frac{\Omega}{2}\sigma_{x1}. \tag{44}$$

This Hamiltonian describes a two-level system subject to an effective static field in the xz plane.

5.1 Diagonalization of the Control-Qubit Hamiltonian

We now diagonalize the driven control-qubit Hamiltonian

$$H_1 = \frac{\delta_1}{2}\sigma_{z1} + \frac{\Omega}{2}\sigma_{x1}. \tag{45}$$

Our goal is to identify a basis in which H_1 is proportional to σ_{z1}, thereby defining the dressed eigenstates of the driven qubit.

Since H_1 contains only σ_{x1} and σ_{z1} components, it can be diagonalized by a rotation about the y axis. We therefore consider a unitary transformation of the form

$$R = \exp\left(-i\frac{\theta}{2}\sigma_{y1}\right), \tag{46}$$

where the angle θ is to be determined.

Applying this transformation yields

$$R^\dagger H_1 R = \frac{\delta_1}{2}\left(\cos\theta\,\sigma_{z1} - \sin\theta\,\sigma_{x1}\right) + \frac{\Omega}{2}\left(\cos\theta\,\sigma_{x1} + \sin\theta\,\sigma_{z1}\right). \tag{47}$$

Collecting terms proportional to σ_{z1} and σ_{x1}, we obtain

$$R^\dagger H_1 R = \frac{1}{2}\Big[(\delta_1\cos\theta + \Omega\sin\theta)\sigma_{z1} + (-\delta_1\sin\theta + \Omega\cos\theta)\sigma_{x1}\Big]. \tag{48}$$

Diagonalization is achieved by requiring that the coefficient of σ_{x1} vanish, which fixes the rotation angle through

$$-\delta_1\sin\theta + \Omega\cos\theta = 0, \qquad \tan\theta = \frac{\Omega}{\delta_1}. \tag{49}$$

With this choice of θ, the transformed Hamiltonian becomes

$$R^\dagger H_1 R = \frac{\Delta}{2}\sigma_{z1}, \qquad \Delta = \sqrt{\delta_1^2 + \Omega^2}. \tag{50}$$

Thus, in the dressed basis, the driven control qubit behaves as an effective two-level system with energy splitting Δ.

5.2 Transformation of Pauli Operators

Having diagonalized the driven control-qubit Hamiltonian by the rotation $R = \exp(-i\theta\sigma_{y1}/2)$, we now determine how the qubit–qubit interaction transforms in the dressed basis. Since the rotation acts only on qubit 1, the Pauli operators of qubit 2 remain unchanged.

Using the identities derived from the y-axis rotation, we find

$$R^\dagger\sigma_{z1}R = \cos\theta\,\sigma_{z1} - \sin\theta\,\sigma_{x1}, \tag{51}$$
$$R^\dagger\sigma_{x1}R = \cos\theta\,\sigma_{x1} + \sin\theta\,\sigma_{z1}, \tag{52}$$
$$R^\dagger\sigma_{y1}R = \sigma_{y1}. \tag{53}$$

These relations show that operators originally transverse to the bare quantization axis acquire a longitudinal component in the dressed basis. In particular, σ_{x1} develops a component along σ_{z1}, a key ingredient for the emergence of a conditional interaction.

5.3 Dressed-Basis Form of the Interaction

We now apply the dressed-state transformation to the exchange interaction

$$H_{\text{int}} = J(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}). \tag{54}$$

Using the operator identities above, we obtain

$$R^\dagger H_{\text{int}} R = J\Big[(\cos\theta\,\sigma_{x1} + \sin\theta\,\sigma_{z1})\sigma_{x2} + \sigma_{y1}\sigma_{y2}\Big]. \tag{55}$$

The interaction thus contains a longitudinal component proportional to $\sigma_{z1}\sigma_{x2}$, which was absent in the original Hamiltonian. This term originates from the mixing of σ_{x1} and σ_{z1} induced by the drive and forms the basis of the cross-resonance interaction.

5.4 Secular Approximation and Emergent ZX Interaction

In the dressed basis, the control qubit has an energy splitting $\Delta = \sqrt{\delta_1^2 + \Omega^2}$, which is assumed to be large compared to the interaction strength J. We therefore move to the interaction picture with respect to the dominant term $(\Delta/2)\sigma_{z1}$.

In this interaction picture, terms that do not commute with σ_{z1}, namely those proportional to σ_{x1} and σ_{y1}, acquire rapid time dependence oscillating at frequencies of order Δ. Over gate timescales long compared to $1/\Delta$, these contributions average to zero.

By contrast, the longitudinal term $\sigma_{z1}\sigma_{x2}$ commutes with the dressed control-qubit Hamiltonian and remains time independent in the interaction picture. Retaining only this slowly varying contribution yields the effective interaction

$$H_{\mathrm{ZX}} = J_{ZX}\,\sigma_{z1}\sigma_{x2}, \tag{56}$$

with

$$J_{ZX} = J\sin\theta = J\frac{\Omega}{\sqrt{\delta_1^2+\Omega^2}} \approx J\frac{\Omega}{\delta_1}, \qquad (\Omega \ll \delta_1). \tag{57}$$

This effective ZX Hamiltonian generates conditional rotations of the target qubit, with the control qubit acting as a classical parameter through its dressed-state projection. Equation 56 is an effective Hamiltonian in the interaction picture with respect to the $(\Delta/2)\sigma_{z1}$.

Fate of the Residual XX and YY Terms. We emphasize that the original exchange interaction $J(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2})$ does not vanish identically under the dressed-state transformation. Instead, it is decomposed into components that are longitudinal and transverse with respect to the dressed quantization axis of the control qubit.

The transverse components proportional to $\sigma_{x1}\sigma_{x2}$ and $\sigma_{y1}\sigma_{y2}$ do not commute with the dressed control-qubit Hamiltonian $(\Delta/2)\sigma_{z1}$. Consequently, they acquire rapid oscillatory phases in the interaction picture and are suppressed by secular averaging when $\Delta \gg J$.

In contrast, the longitudinal term $\sigma_{z1}\sigma_{x2}$ commutes with the dominant dressed Hamiltonian and produces slow, coherent conditional dynamics. As a result, the long-time evolution is governed by the effective ZX interaction, while residual XX and YY contributions appear only as higher-order corrections.

6 Bell-State Generation

We consider the effective Hamiltonian H_{ZX} and an initial product state

$$|\psi(0)\rangle = |+\rangle_1 |0\rangle_2 . \tag{58}$$

Time evolution under H_{ZX} for

$$t = \frac{\pi}{4J_{ZX}} \tag{59}$$

produces a maximally entangled Bell state.

To quantify the entanglement, we compute the reduced density matrix of qubit 2, $\rho_2 = \mathrm{Tr}_1\,\rho$, and evaluate the von Neumann entropy

$$S = -\,\mathrm{Tr}(\rho_2 \ln \rho_2). \tag{60}$$

We find

$$S = \ln 2, \tag{61}$$

confirming maximal entanglement.

7 Conclusion

Starting from a microscopic cavity-QED Hamiltonian, we have derived an effective ZX interaction induced by a resonant drive and shown how it enables Bell-state generation. The ZX term is not a primitive Hamiltonian component but emerges from cavity-mediated exchange, control-qubit dressing, and time-scale separation. This analysis provides a Hamiltonian-level foundation for driven entangling gates.