Obtain 4th order method:

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + h\left[b_0 y'_{n+1} + b_1 y'_n + b_2 y'_{n-1}\right]$$

$$E_{n+1} = y(x_{n+1}) - y_{n+1}$$

$$= y(x_{n-1}+2h) - \Big[a_1 y(x_{n-1}+h) + a_2 y(x_{n-1}) + h b_0 y'(x_{n-1}+2h) + h b_1 y'(x_{n-1}+h) + h b_2 y'(x_{n-1})\Big]$$

$$= y(x_{n-1}) + 2h y'(x_{n-1}) + \frac{(2h)^2}{2} y''(x_{n-1}) + \frac{(2h)^3}{6} y'''(x_{n-1}) + \frac{(2h)^4}{24} y^{iv}(x_{n-1}) + \cdots$$

$$- a_1 \left[y(x_{n-1}) + h y'(x_{n-1}) + \frac{h^2}{2} y''(x_{n-1}) + \frac{h^3}{6} y'''(x_{n-1}) + \frac{h^4}{24} y^{iv}(x_{n-1}) + \cdots\right]$$

$$- a_2\, y(x_{n-1})$$

$$- h b_0 \left[y'(x_{n-1}) + 2h\cdot y''(x_{n-1}) + \frac{(2h)^2}{2} y'''(x_{n-1}) + \frac{(2h)^3}{6} y^{iv}(x_{n-1}) + \cdots\right]$$

$$- h b_1 \left[y'(x_{n-1}) + h y''(x_{n-1}) + \frac{h^2}{2} y'''(x_{n-1}) + \frac{h^3}{6} y^{iv}(x_{n-1}) + \cdots\right]$$

$$- h b_2\, y'(x_{n-1})$$

$$= y(x_{n-1})\cdot\left[1 - a_1 - a_2\right] + h\, y'(x_{n-1})\left[2 - a_1 - b_0 - b_1 - b_2\right]$$

$$+ h^2\, y''(x_{n-1})\left[\frac{4}{2} - \frac{a_1}{2} - 2b_0 - b_1\right]$$

$$+ h^3\, y'''(x_{n-1})\left[\frac{8}{6} - \frac{a_1}{6} - \frac{b_0 \times 4}{2} - \frac{b_1}{2}\right]$$

$$+ h^4\, y^{iv}(x_{n-1})\cdot\left[\frac{16}{24} - \frac{a_1}{24} - \frac{8}{6} b_0 - \frac{b_1}{6}\right] + O(h^5)$$

$\because$ LTE $\simeq O(h^5)$

$\therefore$

$1 - a_1 - a_2 = 0 \rightarrow (1)$

$2 - a_1 - b_0 - b_1 - b_2 = 0 \rightarrow (2)$

$2 - \frac{a_1}{2} - 2b_0 - b_1 = 0 \rightarrow (3)$

$\frac{8}{6} - \frac{a_1}{6} - \frac{4b_0}{2} - \frac{b_1}{2} = 0 \rightarrow (4)$

$\frac{16}{24} - \frac{a_1}{24} - \frac{8}{6}b_0 - \frac{b_1}{6} = 0 \rightarrow (5)$

$$a_1 + a_2 = 1 \rightarrow (1)$$

$$a_1 + b_0 + b_1 + b_2 = 2 \rightarrow (2)$$

$$a_1 + 4b_0 + 2b_1 = 4 \rightarrow (3)$$

$$a_1 + 12b_0 + 3b_1 = 8 \rightarrow (4)$$

$$a_1 + 32b_0 + 4b_1 = 16 \rightarrow (5)$$

$$a_1 = 4 - 4b_0 - 2b_1 \rightarrow (6)$$

$$4 - 4b_0 - 2b_1 + 12b_0 + 3b_1 = 8$$

$$\Rightarrow 8b_0 + b_1 = 4 \rightarrow (7)$$

$$4 - 4b_0 - 2b_1 + 32b_0 + 4b_1 = 16$$

$$\Rightarrow 28b_0 + 2b_1 = 12$$

$$\Rightarrow 14b_0 + \cancel{b_1} = 6 \rightarrow (8)$$

$$8b_0 \cancel{+b_1} = 4 \rightarrow (7)$$

$$(-) \quad (-) \quad (-)$$

$$6b_0 = 2$$

$$\Rightarrow b_0 = 1/3$$

$$b_1 = 4 - 8b_0 = 4 - \frac{8}{3} = \frac{4}{3}$$

$$a_1 = 4 - 4 \times \frac{1}{3} - 2 \times \frac{4}{3} = 4 - \frac{4}{3} - \frac{8}{3}$$

$$= \frac{12-4-8}{3} = 0$$

$$\therefore b_2 = 2 - a_1 - b_0 - b_1$$

$$= 2 - 0 - \frac{1}{3} - \frac{4}{3}$$

$$= \frac{6-1-4}{3} = 1/3$$

$$a_2 = 1 - a_1 = 1 - 0 = 1$$

$$\therefore y_{n+1} = y_{n-1} + h\left[\frac{1}{3}y'_{n+1} + \frac{4}{3}y'_n + \frac{1}{3}y'_{n-1}\right]$$

$$= y_{n-1} + \frac{h}{3}\left[y'_{n+1} + 4y'_n + y'_{n-1}\right]$$

Milne - Simpson - method of order 4.