Artificial Intelligence - Introduction to Reinforcement Learning

ENSISA 2A

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ENSISA, Université Haute-Alsace

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Une école d'ingénieurs de l'Université de Haute-Alsace

Overview

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- What is a Reinforcement Learning model, i.e. a Sequential Decision Model and its different components

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  - Slides of Dr. Mireille Sarkiss, Telecom SudParis, Institut Polytechnique de Paris, Lectures 1-2

Sequential Decision Making

Sequential Decision Making

Suppose the following scenario

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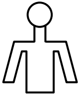
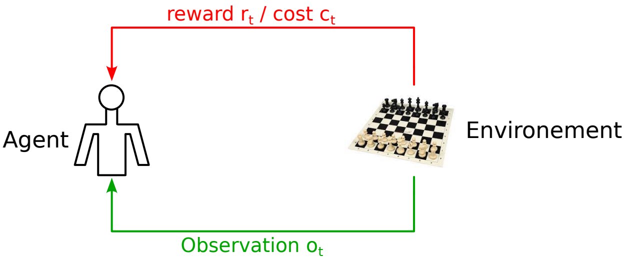
Agent 

 Environement

An agent wants to learn an environment's behavior, in this case, a game of chess

Sequential Decision Making

Suppose the following scenario



The environment reports a reward r_t or cost c_t to the agent as well as an observation o_t.

Sequential Decision Making

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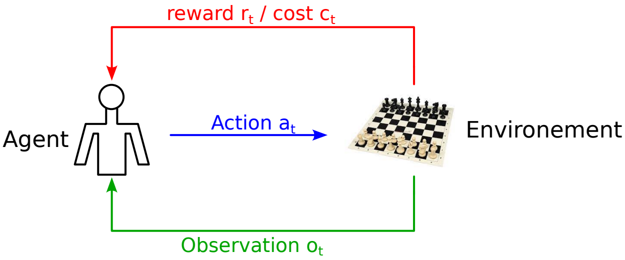

The agent then takes a decision given the information provided previously and applies the action a_t.

Sequential Decision Making

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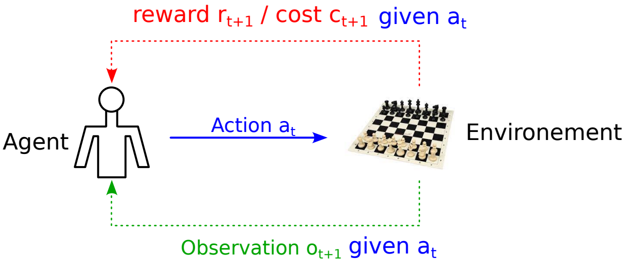

The environment then reports, given action a_t, an observation o_{t+1} and a reward r_{t+1} or cost c_{t+1}.

Sequential Decision Making

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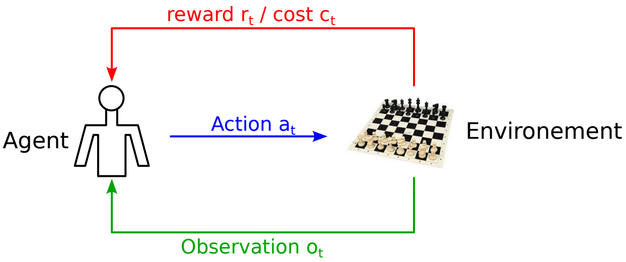
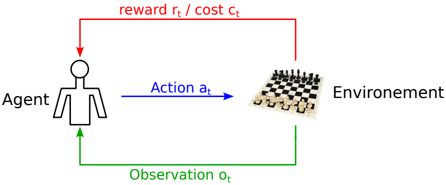


The goal of the agent is to choose the best actions that maximizes or minimizes the reward or cost.

History and State



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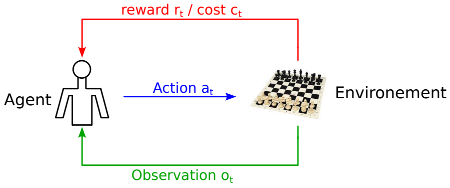

- History: sequence of past observation, actions and rewards up to the agent's decision time t

$$h_t = (o_1, r_1, a_1), (o_2, r_2, a_2), \ldots, (o_t, r_t)$$

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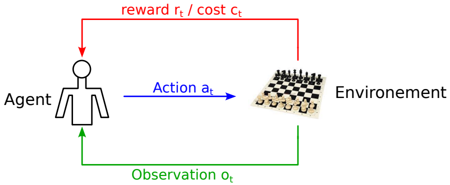

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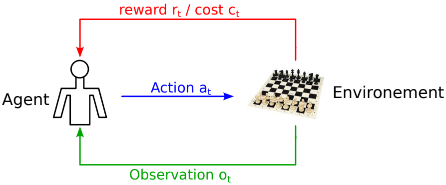

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$$h_t = (o_1, r_1, a_1), (o_2, r_2, a_2), \ldots, (o_t, r_t)$$

- Agent takes decisions given the history.
- A state is produced given the history, function of history: $s_t = f(h_t)$.

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  - Function of history $s_t^a = f(h)$.
- A process changes in a:
  - Deterministic way: given an action, one single observation and reward/cost are generated (ex: robotics)
  - Stochastic way: given an action, many possible observations and rewards/costs can be generated (ex: card game)

Components of Sequential Decision Process

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Agents the following components:

- Policy π: agent's behavior (actions) function
- Value function: Given a policy that the agent follows, how good is each state and/or action over a long run.
- Model: The agent's understanding of the environment i.e. how the environment responds to a given action provided by the agent.

Policy

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  - γ controls which is more important: immediate or future reward/cost.

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- A transition model predicts the new agent state:

$$P_{ss'}^{a} = P(s_{t+1} = s' | s_t = s, a_t = a)$$

- A reward model predicts the next immediate reward:

$$R_s^a = R(s,a) = \mathbb{E}[r_{t+1} | s_t = s, a_t = a]$$

Types of Agents

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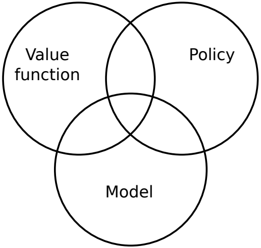

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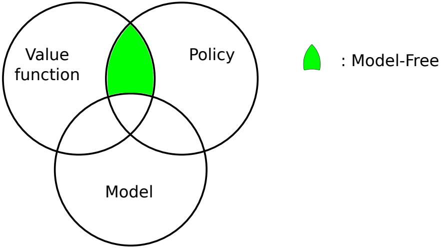

A model-free agent will optimize the policy without trying to learn the environment rules and state information.

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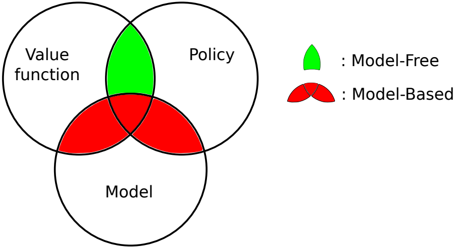

A model-based agent will optimize the policy by learning the model's behavior and state information.

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  - Exploration: discover more possibilities to find more information about the environment's behavior to avoid doing a mistake in the future.

Exploitation/Exploitation Trade Off

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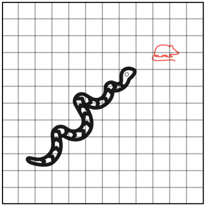
Agent
Simulator

Consider an agent being a computer bot.

Exploitation/Exploitation Trade Off

Agent Simulator



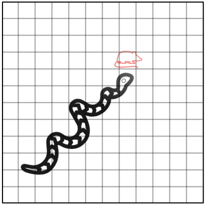
Snake Game

The agent, at time t, is trying to win a snake game by understanding the environment's behavior.

Exploitation/Exploitation Trade Off

Hmm instead of going up I will try to go right this time for future use

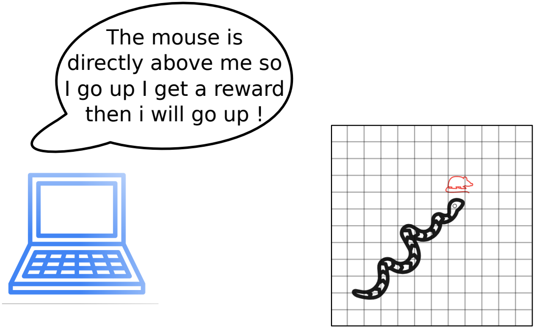
Agent Simulator



Snake Game

The agent when exploring is trying to do an action to explore.

Exploitation/Exploitation Trade Off



Agent Simulator

Snake Game

The agent when exploiting is trying to take a decision that maximizes the reward.

Markov Decision Process (MDP)

MDP and Markov Property

Definition: Markov State

A state s_t is Markov if and only if:

$$P(s_{t+1}|s_t) = P(s_{t+1}|s_t, s_{t-1}, \ldots, s_1)$$

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a $\longrightarrow$ b
means b depends on a

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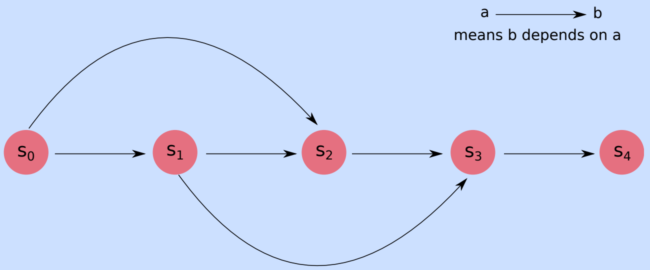

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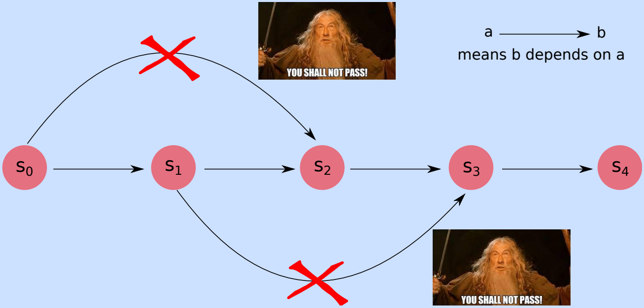

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- State transition matrix P where only the rows sum up to 1.

$$P = [P_{ss'}] \begin{bmatrix} P_{s_1 s_1} & P_{s_1 s_2} & \dots & P_{s_1 s_N} \\ P_{s_2 s_1} & P_{s_2 s_2} & \dots & P_{s_2 s_N} \\ \dots & \dots & \dots & \dots \\ P_{s_N s_1} & P_{s_N s_2} & \dots & P_{s_N s_N} \end{bmatrix}$$

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- Rows sum up to 1 but columns do not (proof next slide).

MDP and Markov Property

proof:

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$$\text{Sum of row } r = \sum_{i=1}^{N} P_{s_r s_i}(s_{t+1} = s_i | s_t = s_r)$$

MDP and Markov Property

proof:

- Rows sum up to 1:

$$\text{Sum of row } r = \sum_{i=1}^{N} P_{s_r s_i}(s_{t+1} = s_i | s_t = s_r)$$
$$= \sum_{i=1}^{N} \frac{P_{s_r s_i}(s_{t+1} = s_i, s_t = s_r)}{P(s_t = s_r)} \quad \textit{Bias Rule}$$

MDP and Markov Property

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&= \frac{\sum_{i=1}^{N} P_{s_r s_i}(s_{t+1}=s_i, s_t=s_r)}{P(s_t=s_r)} \\
&= \frac{P(s_t=s_r)}{P(s_t=s_r)} \quad \textit{Marginal Distribution} \\
&= 1
\end{aligned}
$$

MDP and Markov Property

proof:

- Columns do not sum up to 1:

$$\text{Sum of column } c = \sum_{i=1}^{N} P_{s_i s_c}(s_{t+1} = s_c | s_t = s_i)$$

$$= \sum_{i=1}^{N} \frac{P_{s_i s_c}(s_{t+1} = s_c, s_t = s_i)}{P(s_t = s_i)} \quad \textit{Bias Rule}$$

Nothing can be done here as the sum includes both numerator and denominator.

Markov Process

Markov Process

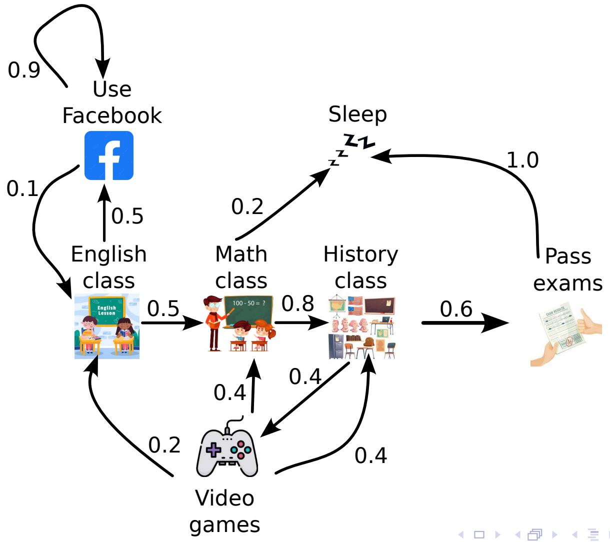
Definition: Markov Process or Markov Chain

A Markov Process or Chain is a tuple $< S, P >$ where:

- S is a finite set of states
- P is a state transition probability matrix

Markov Process

Example: Student Markov Chain



Markov Reward Process (MRP)

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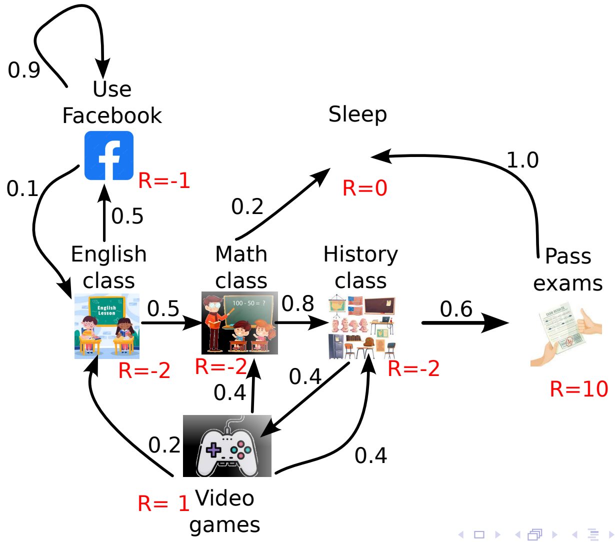
Definition: Markov Reward Process

A Markov Reward Process is a tuple $< S, P, R, \gamma >$ where:

- S is a finite set of states
- P is a state transition probability matrix
- R is a reward function
- γ is a discount factor

Markov Reward Process (MRP)

Example: Student MRP



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Definition: Return

The return G_t at time t of an MRP is the total discounted reward from t to T.

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$$

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Definition: State Value Function

As defined before, slide 10. Its the expected return starting from state s:

$$V(s) = \mathbb{E}[G_t | s_t = s]$$

Bellman Equation for MRP

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$$\begin{aligned} V(s) &= \mathbb{E}[r_{t+1} + \gamma V(s_{t+1}) | s_t = s] \text{ or,} \\ V(s) &= R_s + \gamma \sum_{s' \in S} P_{ss'} V(s') \end{aligned} \tag{1}$$

Bellman Equation for MRP

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The last step simply uses the rule that says $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$ and that

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- Better solution: Dynamic programming.

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Definition: Markov Decision Process

Its a tuple $< S, A, P, R, \gamma >$

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Model-Based Reinforcement Learning Methods

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Definition: Principle of Optimality

A policy $\pi(a|s)$ achieves the optimal value from state s, $V_\pi(s) = V^*(s)$ if and only if:

For any state s' reachable from s, π achieves the optimal value from state s',

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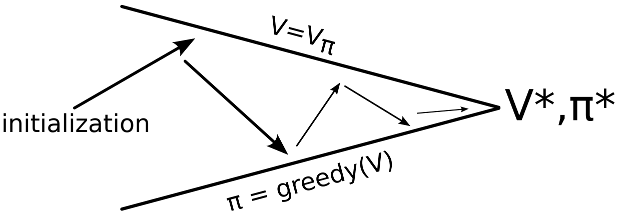

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- The last step consisting on choosing the policy of the optimal value function:

$$\pi(s) = arg \max_{a \in A}(R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a(s'))$$

Policy VS Value Iteration

Table: Comparing Policy and Value Iteration algorithms to solve MDP model-based problems.

Properties	**Policy Iteration**	**Value Iteration**
Initialization	Random policy	Random value function
Complexity	Very complex	Simple
Convergence	Guaranteed	Guaranteed
Number of iterations till convergence	Small number	High number
Speed	Fast	Slow

Model-Free Reinforcement Learning Methods

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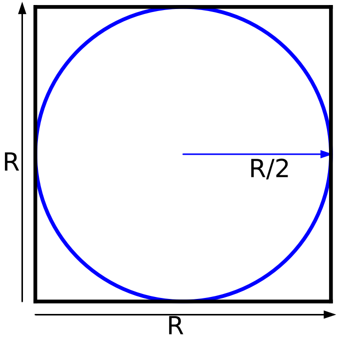

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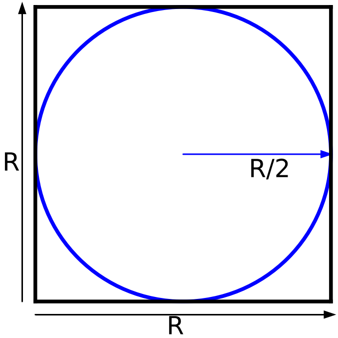

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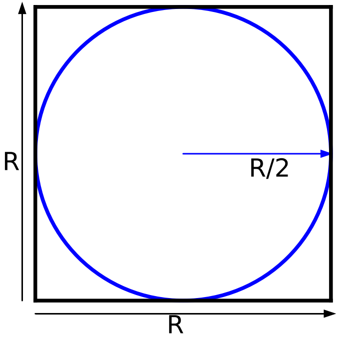

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- The area of the circle is: $A_{cr} = \frac{\pi R^2}{4}$

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To estimate the value of π, we can randomly choose points in the $2D$ grid, but guaranteed to be inside the square.

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This means that $\pi = 4 * P$

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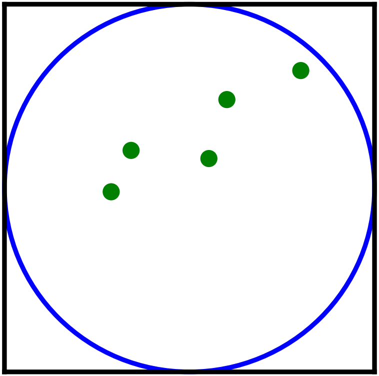
To estimate the probability P of a point being in the circle, with enough generated points we have:

$$P = \frac{\textit{number of points landing inside the circle}}{\textit{total number of generated points}}$$

Monte-Carlo Simulation

$n = 5$, $\pi = 4.0$



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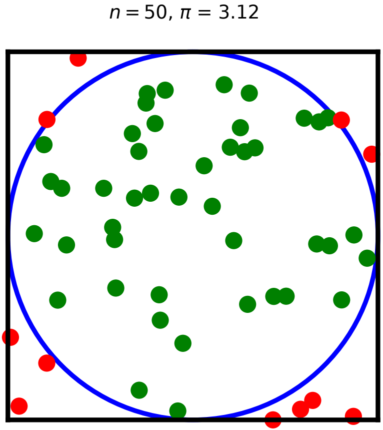

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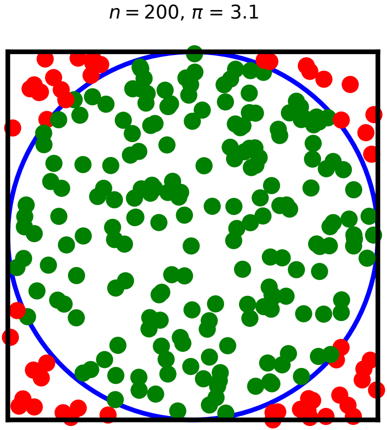

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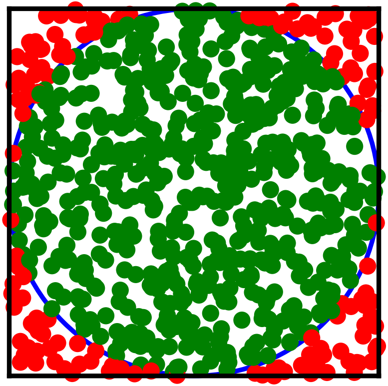
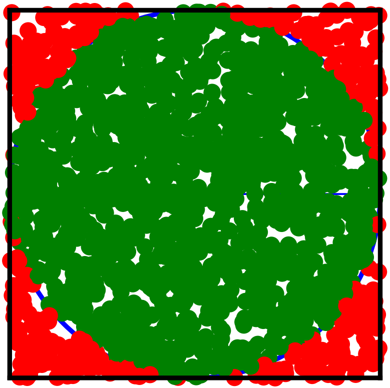
$n = 500$, $\pi = 3.264$



Monte-Carlo Simulation

$n = 1000$, $\pi = 3.22$



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$n = 5000$, $\pi = 3.1656$

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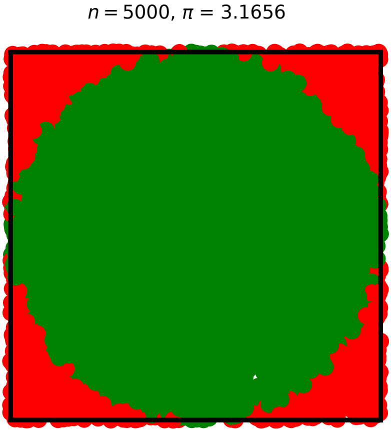

If we keep generating points we will converge to almost 3.14

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First-Visit MC Policy Evaluation

Algorithm 1 First-Visit Monte-Carlo Prediction

Input: Policy π, int E the number of episodes
Output: Value function V_π

Initialize #visits $N(s) = 0$ and Returns $G(s) = 0$ $\forall s \in S$
for $e = 1$ to E **do**
  Generate using π an episode $s_0, a_0, r_1, s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, a_T$
  $g \leftarrow 0$
  for $t = T - 1$ to $t = 0$ **do**
    $g \leftarrow g + r_{t+1}$
    if state s_t is not in the sequence $s_0, s_1, \ldots, s_{t-1}$ **then**
      $G(s_t) \leftarrow G(s_t) + g$
      $N(s_t) \leftarrow N(s_t) + 1$
    end if
  end for
end for
return $V(s) \leftarrow \dfrac{G(s)}{N(s)} \forall s \in S$

Every-Visit MC Policy Evaluation

Algorithm 2 Every-Visit Monte-Carlo Prediction

Input: Policy π, $int\ E$ the number of episodes
Output: Value function V_π

Initialize #visits $N(s) = 0$ and Returns $G(s) = 0\ \forall s \in S$
Initialize $S(s) = 0\ \forall s\ inS$
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  Generate using π an episode $s_0, a_0, r_1, s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, a_T$
  $g \leftarrow$ Return from time-step t to horizon T
  for $t = T - 1$ to $t = 0$ **do**
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  end for
end for
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- Update $V(s)$ incrementally after episode $s_0, a_0, r_1, s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T$

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$$\begin{aligned} V_n(s_t) &= \frac{1}{n}\sum_{i=1}^{n} G_{t,i} \\ &= \frac{1}{n}(G_{t,n} + \sum_{i=1}^{n-1} G_{t,i}) \end{aligned}$$

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- New Estimate = Old Estimate + Step $*$ (Target $-$ Old Estimate)

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- This is done incrementally while going through the episode.

Off Policy Control: Q-Learning

Off Policy Control: Q-Learning

Definition: State-Action Value function

The state-action value function $q_\pi(s,a)$ of an MDP is the expected Return for when starting in a state s, and taking action a following the policy π.

$$Q_\pi(s,a) = \mathbb{E}_\pi[G_t | s_t = s, a_t = a]$$
$$V_\pi(s) = \sum_{a \in A} \pi(a|s).Q(s,a)$$

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Bellman Expectation Equation for state-action value function:

$$Q_\pi(s, a) = \mathbb{E}_\pi[r_{t+1} + \gamma Q_\pi(s_{t+1}, a_{t+1}) | s_t = s, a_t = a] \tag{2}$$

Bellman Optimality Equation for state-action value function:

$$Q_\pi(s_t, a_t) = R_{s_t}^{a_t} + \gamma \max_{a \in A} Q_\pi(s_{t+1}, a) \tag{3}$$

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- Lets try to approximate the optimal state-action value function Q^* using the Bellman optimality equation.

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- For each tuple $s_t, a_t, r_{t+1}, s_{t+1}$ in a given episode, we apply the following:

$$Q(s_t, a_t) = Q(s_t, a_t) + \alpha(r_{t+1} + \gamma \max_{a \in A} Q(s_{t+1}, a) - Q(s_t, a_t))$$

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- α is the learning rate $\alpha \in [0, 1]$

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- Solution: $\epsilon - greedy$ approach ($\epsilon \in [0, 1]$):
  - With probability $1 - \epsilon$ choose the optimal action that maximizes the Q function
  - With probability ϵ choose a random action

The Q-Learning step-by-step Algorithm

Algorithm 3 Q-Learning Algorithm

Initialize $Q(s, a) \forall s \in S, a \in A$ randomly and $Q(terminal\_state, .) = 0$
for $e = 1$ to E **do**
  Initialize s
  while $s \neq terminal\_state$ **do**
    Choose a from s using $\epsilon - greedy$
    Take action a and observe s' and r
    $Q(s, a) \leftarrow Q(s, a) + \alpha[r + \gamma \max_{a' \in A} Q(s', a') - Q(s, a)]$
  end while
end for
return Q

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- Reinforcement Learning is a way of training based on a reward system
- RL is simply "the science of sequential decision making"
- RL consists on a smart Agent interacting with an Environment in order to learn how it works resulting in maximization of the reward
- Depending on the case scenario, a choice of algorithms should be done to solve such a problem