ICMA350 Probability
Sample Quiz 2

Lecture 9 Lecture 10

Problem 1: Let X be a continuous random variable that follows a uniform distribution between 0 and 10.

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(a) Find the probability density function (PDF) of X.

(b) Calculate $P(4 < X < 7)$.

Lecture 3

(c) Determine the expected value and variance of X.

Problem 2: Let a continuous random variable Y have the following piecewise probability density function (PDF):

$$f_Y(y) = \begin{cases} \frac{y-2}{9} & \text{for } 2 \leq y < 5, \\ \frac{8-y}{9} & \text{for } 5 \leq y \leq 8, \\ 0 & \text{otherwise.} \end{cases}$$

?

(a) Derive the cumulative distribution function (CDF) of Y. Sketch the graph of the CDF.

(b) Calculate the probability that Y is less than 6 given that Y is greater than 3.

Problem 3: Let Z be an exponential random variable with rate parameter $\lambda = 3$.

(a) Write down the PDF and CDF of Z. Aon Pg 17

(b) Compute $P(Z > 2)$.

(c) Determine the expected value and standard deviation of Z.

Problem 4: Let X and Y be two independent random variables where $X \sim \text{Exp}(\lambda = 2)$ and $Y \sim \text{Exp}(\lambda = 4)$.

Lecture 16

(a) Find the joint PDF of X and Y.

(b) Compute $P(X < Y)$. Lecture 15

(c) Determine the expected value of $X + Y$.

Lecture 4

Problem 5: Consider two continuous random variables U and V with joint PDF given by $f(u, v) = ke^{-2u-3v}$, for $u > 0$ and $v > 0$.

(a) Find the value of k to make $f(x, y)$ a valid PDF.

(b) Verify if U and V are independent.

(c) Find the marginal PDFs of U and V. ?

(d) Calculate $P(U < 1, V < 2)$.

Lecture 15

Problem 6: A joint distribution of X and Y is given by the PDF $f(x, y) = kx(1 - y)$, for $0 < x < 1$ and $0 < y < 1$.

(a) Find the value of k to make $f(x, y)$ a valid PDF.

(b) Find the marginal PDFs of X and Y.

(c) Find $E(X|Y < 0.5)$.

(d) Are X and Y independent? Justify your answer.

Bonus Problem

Consider a random point (X, Y) that is uniformly distributed inside the circle centered at the origin with radius 2.

(a) Find the joint PDF of X and Y.

(b) Calculate the probability that the point (X, Y) falls within 1 unit distance from the origin.

(c) Determine the expected distance from the origin for the randomly chosen point.

Sample Quiz 2 Solutions

Solution to Problem 1:

(a) The PDF of X is $f(x) = \dfrac{1}{10-0} = \dfrac{1}{10}$ for $0 \leq x \leq 10$.

(b) $P(4 < X < 7) = \displaystyle\int_4^7 \frac{1}{10}dx = \frac{7-4}{10} = \frac{3}{10}$.

(c) Expected value: $E(X) = \dfrac{10+0}{2} = 5$

Variance: $Var(X) = \dfrac{(10-0)^2}{12} = \dfrac{100}{12} = \dfrac{25}{3}$.

Solution to Problem 2: **Problem 2**

Let a continuous random variable Y have the following piecewise probability density function (PDF):

$$f_Y(y) = \begin{cases} \frac{y-2}{9} & \text{for } 2 \leq y < 5, \\ \frac{8-y}{9} & \text{for } 5 \leq y \leq 8, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Derive the cumulative distribution function (CDF) of Y. Sketch the graph of the CDF.

The CDF, denoted by $F_Y(y)$, is calculated by integrating the PDF:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt$$

We'll integrate piecewise over the intervals where the PDF is defined:

* **For $y < 2$:**

$$F_Y(y) = 0$$

* **For $2 \leq y < 5$:**

$$F_Y(y) = \int_2^y \frac{t-2}{9}dt = \frac{(y-2)^2}{18}$$

* **For $5 \leq y \leq 8$:**

$$F_Y(y) = \frac{1}{2} + \int_5^y \frac{8-t}{9}dt = \frac{-y^2+16y-46}{18}$$

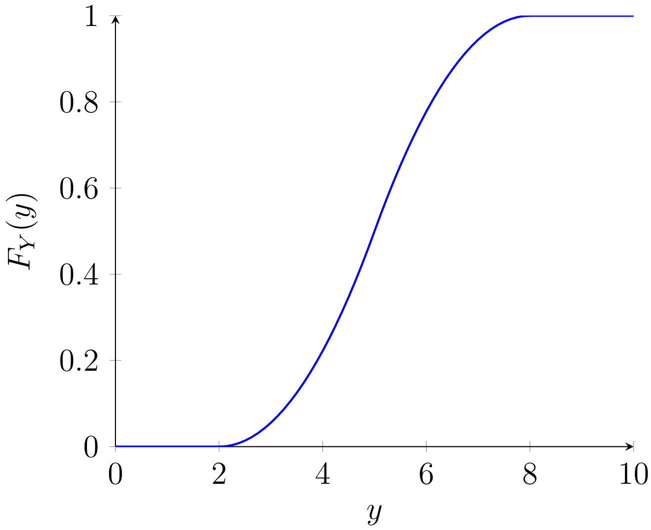
* **For $y > 8$:**

$$F_Y(y) = 1$$

Therefore, the CDF is:

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 2, \\ \dfrac{(y-2)^2}{18} & \text{for } 2 \leq y < 5, \\ \dfrac{-y^2+16y-46}{18} & \text{for } 5 \leq y \leq 8, \\ 1 & \text{for } y > 8. \end{cases}$$

(Graph of the CDF:



(b) Calculate the probability that Y is less than 6 given that Y is greater than 3.
We'll use conditional probability:

$$P(Y < 6 \mid Y > 3) = \frac{P(3 < Y < 6)}{P(Y > 3)}$$

Using the CDF:
* $P(3 < Y < 6) = F_Y(6) - F_Y(3) = \frac{13}{18}$
* $P(Y > 3) = 1 - F_Y(3) = 1 - \frac{1}{18} = \frac{17}{18}$
Therefore:

$$P(Y < 6 \mid Y > 3) = \frac{13/18}{17/18} = \frac{13}{17}$$

Solution to Problem 3:

(a) PDF of Z is $f(z) = 3e^{-3z}$ for $z \geq 0$, and CDF is $F(z) = 1 - e^{-3z}$.

(b) $P(Z > 2) = 1 - F(2) = e^{-6}$.

(c) Expected value $E(Z) = \frac{1}{3}$ and standard deviation $\sigma_Z = \frac{1}{3}$.

Solution to Problem 4:

(a) Since X and Y are independent, the joint PDF is $f(x, y) = f_X(x)f_Y(y) = 2e^{-2x} \cdot 4e^{-4y}$ for $x, y \geq 0$.

(b) $P(X < Y) = \int_0^\infty \int_0^y 2e^{-2x} \cdot 4e^{-4y} dxdy = \frac{1}{3}$.

(c) $E(X + Y) = E(X) + E(Y) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Solution to Problem 5:

(a) To find k, ensure the total integral over the support equals 1: $\int_0^\infty \int_0^\infty ke^{-2u-3v} dudv = 1$, solving gives $k = 6$.

(b) To check independence, compare the product of marginal PDFs with the joint PDF.

(c) Marginal PDFs: $f_U(u) = \displaystyle\int_0^\infty 6e^{-2u-3v}dv = 2e^{-2u}$, $f_V(v) = \int_0^\infty 6e^{-2u-3v}du = 3e^{-3v}$.

(d) $P(U < 1, V < 2) = \int_0^1 \int_0^2 6e^{-2u-3v}dvdu$.

Solution to Problem 6:

(a) Finding the value of k to make $f(x, y)$ a valid PDF:

The joint PDF $f(x, y) = kx(1 - y)$ must integrate to 1 over the domain $0 < x < 1$ and $0 < y < 1$:

$$\int_0^1 \int_0^1 kx(1 - y)\, dy\, dx = \frac{k}{4}.$$

For $f(x, y)$ to be a valid PDF, this integral must equal 1, thus:

$$\frac{k}{4} = 1 \Rightarrow k = 4.$$

(b) Finding the marginal PDFs of X and Y:

The marginal PDF of X, $f_X(x)$, is found by integrating the joint PDF over y:

$$f_X(x) = \int_0^1 kx(1 - y)\, dy = kx \left[y - \frac{y^2}{2} \right]_0^1 = 2x \quad \text{for } 0 < x < 1.$$

Similarly, the marginal PDF of Y, $f_Y(y)$, is found by integrating the joint PDF over x:

$$f_Y(y) = \int_0^1 kx(1 - y)\, dx = k(1 - y) \left[\frac{x^2}{2} \right]_0^1 = 2(1 - y) \quad \text{for } 0 < y < 1.$$

(c) Finding $E(X|Y < 0.5)$:

First, find the conditional PDF $f_{X|Y}(x|Y < 0.5)$: Since $Y < 0.5$, the conditional PDF is proportional to the joint PDF:

$$f_{X|Y}(x|Y < 0.5) = \frac{f(x, y)}{P(Y < 0.5)} = \frac{4x(1 - y)}{1/2} = 8x(1 - y) \quad \text{for } 0 < x < 1, 0 < y < 0.5.$$

Then, the conditional expectation is:

$$E(X|Y < 0.5) = \int_0^1 x \cdot 8x\, dx = 8 \int_0^1 x^2\, dx = 8 \left[\frac{x^3}{3} \right]_0^1 = \frac{8}{3}.$$

(d) Checking if X and Y are independent:

X and Y are independent if and only if $f(x, y) = f_X(x)f_Y(y)$ for all x, y.
We have $f_X(x) = 2x$, $f_Y(y) = 2(1-y)$, and $f(x, y) = 4x(1-y) = f_X(x)f_Y(y) = 4x(1-y)$ for all x, y. Therefore, X and Y are independent.

Solution to Bonus Problem:

Consider a random point (X, Y) that is uniformly distributed inside the circle centered at the origin with radius 2.

(a) Finding the joint PDF of X and Y:

Since the point (X, Y) is uniformly distributed inside the circle, the joint probability density function (PDF) is constant within the circle and zero outside. The area of the circle is 4π, so the joint PDF $f(x, y)$ for points inside the circle is given by:

$$f(x,y) = \begin{cases} \dfrac{1}{4\pi} & \text{if } x^2 + y^2 \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Calculating the probability that the point (X, Y) falls within 1 unit distance from the origin:

The probability that the point falls within a smaller circle of radius 1 centered at the origin is the ratio of the area of the smaller circle to the area of the larger circle. The area of the smaller circle is $\pi(1^2) = \pi$. Therefore, the probability is:

$$P(\sqrt{X^2 + Y^2} < 1) = \frac{\pi}{4\pi} = \frac{1}{4}.$$

(c) Determining the expected distance from the origin for the randomly chosen point:

The expected distance from the origin for a point uniformly distributed inside a circle of radius 2 is calculated using polar coordinates. The expected value $E[D]$ is given by:

$$E[D] = \int_0^{2\pi} \int_0^2 \frac{r^2}{\pi R^2} dr d\theta = \frac{1}{\pi R^2} \int_0^{2\pi} d\theta \int_0^2 r^2 dr$$

Substitute $R = 2$ into the equation:

$$E[D] = \frac{1}{4\pi} \cdot 2\pi \left[\frac{r^3}{3}\right]_0^2 = \frac{1}{4\pi} \cdot 2\pi \cdot \frac{8}{3} = \frac{4}{3}.$$

Therefore, the expected distance from the origin for a randomly chosen point inside the circle is $\frac{4}{3}$.