ICMA350 Probability
Quiz 2 (10%)

Monday, 18 March 2024

This quiz, worth 40 points (total possible 50 points, capped at 40), constitutes 10% of the total term grade. Calculators are permitted; cheat sheets are not.

Problem 1 (10 points): Let X be uniformly distributed between 3 and 9.

(a) State the PDF and CDF of X.

(b) Calculate $P(X > 7)$.

(c) Compute the mean and variance of X.

Problem 2 (10 points): An exponential random variable Z has a rate parameter $\lambda = 2$.

(a) Derive the PDF and CDF of Z.

(b) Find $P(Z < 1)$.

(c) Determine $E(Z)$ and $Var(Z)$.

Problem 3 (10 points): If U and V are independent and uniformly distributed on $[0, 2]$,

(a) Write the joint PDF of U and V.

(b) Compute $P(U < 1, V < 1)$.

(c) Compute $P(U - V < 1)$.

Problem 4 (10 points): Let k be a positive real number. Consider random variables X and Y with the following joint PDF:

$$f(x, y) = \begin{cases} k(6x + 4y) & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of k to make $f(x, y)$ a valid joint PDF.

(b) Find the marginal PDFs of X and Y.

(c) Compute $E(X|Y = 0.5)$.

(d) Are X and Y independent? Justify your answer.

Bonus Problem (10 points):

A dart is thrown at a square target with sides of length 2 units, such that its position (X, Y) relative to the center $(0, 0)$ is uniformly distributed over the square.

(a) Describe the joint PDF of X and Y.

(b) What is the probability that the dart lands within 1 unit of the center?

(c) If the dart lands in the upper right quadrant $(X > 0, Y > 0)$, what is the expected value of X?

ICMA350 Probability: Quiz 2 Solutions

Solution to Problem 1:

(a) The PDF of X is $f(x) = \frac{1}{9-3} = \frac{1}{6}$ for $3 \leq x \leq 9$, and the CDF is $F(x) = \frac{x-3}{6}$ for $3 \leq x \leq 9$.

(b) $P(X > 7) = 1 - F(7) = 1 - \frac{7-3}{6} = \frac{1}{3}$.

(c) The mean of X is $E(X) = \frac{3+9}{2} = 6$, and the variance is $Var(X) = \frac{(9-3)^2}{12} = 3$.

Solution to Problem 2:

(a) The PDF of Z is $f(z) = 2e^{-2z}$ for $z \geq 0$, and the CDF is $F(z) = 1 - e^{-2z}$.

(b) $P(Z < 1) = F(1) = 1 - e^{-2} \approx 0.8647$.

(c) The expected value of Z is $E(Z) = \frac{1}{2}$, and the variance is $Var(Z) = \frac{1}{2^2} = \frac{1}{4}$.

Solution to Problem 3:

(a) Since U and V are independent and uniformly distributed on $[0, 2]$, their individual PDFs are $f_U(u) = \frac{1}{2}$ for $0 \leq u \leq 2$ and $f_V(v) = \frac{1}{2}$ for $0 \leq v \leq 2$, respectively. Therefore, the joint PDF $f_{U,V}(u, v)$ is given by:

$$f_{U,V}(u, v) = f_U(u) \times f_V(v) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, \quad \text{for } 0 \leq u, v \leq 2.$$

(b) Compute $P(U < 1, V < 1)$:

Using the independence of U and V, the probability that $U < 1$ and $V < 1$ is the product of their individual probabilities:

$$P(U < 1, V < 1) = P(U < 1) \times P(V < 1) = \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = \frac{1}{4}.$$

(c) Compute $P(U - V < 1)$:

To compute this probability, we consider the total area where $U - V < 1$ in the uv-plane. This condition can be visualized as a region in the square $[0, 2] \times [0, 2]$. However, for uniform distributions U and V on $[0, 2]$, we calculate this probability directly:

The condition $U - V < 1$ can be split into two regions based on the values of U and V:

1. When $V \leq 1$, U can range from 0 to $V + 1$. 2. When $V > 1$, U ranges from 0 to 2 since U cannot exceed its maximum value of 2.

Therefore, we calculate the probability as follows:

$$P(U - V < 1) = \int_0^1 \int_0^{v+1} \frac{1}{4}\, du\, dv + \int_1^2 \int_0^2 \frac{1}{4}\, du\, dv.$$

For the first integral, where $0 \leq V \leq 1$:

$$\int_0^1 \int_0^{v+1} \frac{1}{4}\, du\, dv = \int_0^1 \left[\frac{u}{4}\right]_0^{v+1} dv = \int_0^1 \frac{v+1}{4}\, dv = \left[\frac{v^2}{8} + \frac{v}{4}\right]_0^1 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

For the second integral, where $1 < V \leq 2$:

$$\int_1^2 \int_0^2 \frac{1}{4}\, du\, dv = \int_1^2 \left[\frac{u}{4}\right]_0^2 dv = \int_1^2 \frac{1}{2}\, dv = \left[\frac{v}{2}\right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}.$$

Combining these results, the total probability is:

$$P(U - V < 1) = \frac{3}{8} + \frac{1}{2} = \frac{3}{8} + \frac{4}{8} = \frac{7}{8}.$$

Solution to Problem 4:

(a) Determine the value of k to make $f(x, y)$ a valid joint PDF:

The joint PDF $f(x, y) = k(6x + 4y)$ must integrate to 1 over the domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$:

$$\int_0^1 \int_0^1 k(6x + 4y)\, dy\, dx = 5k.$$

For $f(x, y)$ to be a valid PDF, this integral must equal 1, thus:

$$5k = 1 \Rightarrow k = \frac{1}{5}.$$

(b) Find the marginal PDFs of X and Y:

The marginal PDF of X, $f_X(x)$, is found by integrating the joint PDF over y:

$$f_X(x) = \int_0^1 \frac{1}{5}(6x + 4y)\, dy = \frac{2}{5} + \frac{6x}{5}, \quad 0 \leq x \leq 1.$$

Similarly, the marginal PDF of Y, $f_Y(y)$, is found by integrating the joint PDF over x:

$$f_Y(y) = \int_0^1 \frac{1}{5}(6x + 4y)\, dx = \frac{3}{5} + \frac{4y}{5}, \quad 0 \leq y \leq 1.$$

(c) Compute $E(X|Y = 0.5)$:

First, we need to find the conditional density function $f_{X|Y}(x|Y = 0.5)$. Given the joint PDF $f(x, y) = \frac{1}{5}(6x+4y)$ and the value of Y fixed at 0.5, the conditional PDF of X given $Y = 0.5$ is obtained by normalizing the joint PDF with respect to X, considering $y = 0.5$:

$$f_{X|Y}(x|Y = 0.5) = \frac{f(x, 0.5)}{f_Y(0.5)} = \frac{\frac{1}{5}(6x + 2)}{\frac{3}{5} + \frac{4\times0.5}{5}} = \frac{6x + 2}{5}.$$

Note that the denominator $f_Y(0.5)$ is the value of Y's marginal PDF at $y = 0.5$, which we calculated in part (b). Now, we can compute the conditional expectation $E(X|Y = 0.5)$ as follows:

$$E(X|Y = 0.5) = \int_0^1 x \cdot f_{X|Y}(x|Y = 0.5)\, dx = \int_0^1 x \cdot \frac{6x + 2}{5}\, dx.$$

Evaluating this integral gives:

$$E(X|Y = 0.5) = \frac{1}{5}\left[\int_0^1 6x^2\, dx + \int_0^1 2x\, dx\right] = \frac{1}{5}\left[\frac{6x^3}{3}\Big|_0^1 + x^2\Big|_0^1\right] = \frac{1}{5}[2 + 1] = \frac{3}{5}.$$

Thus, the expected value of X given that $Y = 0.5$ is $\frac{3}{5}$.

(d) Are X and Y independent? Justify your answer:

To check if X and Y are independent, we compare the product of the marginal PDFs with the joint PDF:

$$\frac{1}{5}(6x + 4y) \stackrel{?}{=} \left(\frac{2}{5} + \frac{6x}{5}\right)\left(\frac{3}{5} + \frac{4y}{5}\right).$$

Since the left-hand side (joint PDF) does not equal the right-hand side (product of marginal PDFs) for all values of x and y, X and Y are not independent.

Bonus Problem Solution:

(a) Describe the joint PDF of X and Y:

Since the dart's position (X, Y) is uniformly distributed over the square with sides of length 2 units, centered at the origin, the range of X and Y is from -1 to 1. The joint PDF $f_{X,Y}(x, y)$ for a uniform distribution over this square is constant:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4} & \text{if } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) What is the probability that the dart lands within 1 unit of the center?

The event that the dart lands within 1 unit of the center corresponds to the dart landing inside a circle of radius 1 centered at the origin. The area of this circle is $\pi \times 1^2 = \pi$. Since the total area of the square is $2 \times 2 = 4$, the probability that the dart lands within this circle is:

$$P(\text{Dart lands within 1 unit of center}) = \frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi}{4}.$$

(c) If the dart lands in the upper right quadrant ($X > 0, Y > 0$), what is the expected value of X?

In the upper right quadrant, both X and Y range from 0 to 1. Since X is uniformly distributed in this range (given that the dart lands in this quadrant), the expected value of X, denoted as $E(X|X > 0, Y > 0)$, is the midpoint of the interval $[0, 1]$:

$$E(X|X > 0, Y > 0) = \frac{0 + 1}{2} = \frac{1}{2}.$$

This reflects the symmetry and uniform distribution of X within the upper right quadrant.