Introduction to Logic
Assignment 5

Problem 1

Suppose ϕ is the formula $\neg(r \leftrightarrow \neg(p \wedge \neg q))$.

1.1 Find a formula that is logically equivalent to ϕ but contains none of the operators: $\wedge, \rightarrow, \leftrightarrow$.

1.2 Find a formula that is logically equivalent to ϕ but contains none of the operators: $\neg, \wedge, \vee, \leftrightarrow$.

Problem 2

The NOR operator, often denoted by $\downarrow$, is a binary truth-functional logical operator described by the following truth table

A	B	$A \downarrow B$
F	**F**	**T**
F	**T**	**F**
T	**F**	**F**
T	**T**	**F**

Suppose ϕ is the formula $\neg((p \vee \neg q) \wedge r)$. Find a formula that is logically equivalent to ϕ but contains no operators other than $\downarrow$ (i.e. none of the following symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$ occurs).

Problem 3

Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

$$\phi = (p \wedge q) \rightarrow r$$
$$\psi = (\neg r) \rightarrow (q \rightarrow \neg p)$$

Problem 4

The XOR operator, often denoted by $\oplus$, is a binary truth-functional logical operator described by the following truth table

A	B	$A \oplus B$
F	**F**	**F**
F	**T**	**T**
T	**F**	**T**
T	**T**	**F**

Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

$$\phi = p \oplus \neg(q \oplus \neg(r \oplus \neg(s \oplus \neg p)))$$
$$\psi = (p \leftrightarrow r) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow s)$$

Problem 5

Let f be a Boolean function with 5 arguments such that $f(x_1, x_2, x_3, x_4, x_5) = 1$ when *exactly two* of the variables $x_1, ..., x_5$ are 1. For example, $f(1,1,0,0,0) = f(0,0,1,0,1) = 1$, but $f(0,0,0,0,0) = f(1,0,0,0,0) = f(0,1,1,0,1) = 0$.

Draw a reduced OBDD for the Boolean function f.

Problem 6

Suppose A, B, C, and D are the sets given by:

$$
\begin{aligned}
A &= \{-1, 0, 1\} \\
B &= \{-6, 1, 2, 7, 9\} \\
C &= \{x \in \mathbb{Z} \,|\, 0 \leq x < 20 \text{ and } x \text{ is odd}\} \\
D &= \{x \in \mathbb{Z} \,|\, x = y + z \text{ for some y and z in B}\} \\
E &= \{x^2 \in \mathbb{Z} \,|\, 2x \in B\}
\end{aligned}
$$

List all members in each of the following sets.

6.1 $A \cup B$

6.2 $B - C$

6.3 $\wp(A)$

6.4 D

6.5 $A \times E$

6.6 $\wp(\wp(A \cap B))$

Problem 1

Suppose ϕ is the formula $\neg(r \leftrightarrow \neg(p \wedge \neg q))$.

1.1 Find a formula that is logically equivalent to ϕ but contains none of the operators: $\wedge, \rightarrow, \leftrightarrow$.

1.2 Find a formula that is logically equivalent to ϕ but contains none of the operators: $\neg, \wedge, \vee, \leftrightarrow$.

1.1

$$\neg(r \leftrightarrow \neg(p \wedge \neg q)) \equiv \neg(r \leftrightarrow \neg p \vee q)) \qquad p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$\equiv \neg(r \to (\neg p \vee q) \wedge (\neg p \vee q) \to r) \qquad p \to q \equiv \neg p \vee q$$

$$\equiv \neg((\neg r \vee \neg p \vee q) \wedge (\neg(\neg p \vee q) \vee r))$$

$$\equiv \neg(\neg r \vee \neg p \vee q) \vee \neg(\neg(\neg p \vee q) \vee r) \ \#$$

1.2

$$\neg(r \leftrightarrow \neg(p \wedge \neg q)) \equiv \neg(r \leftrightarrow (\neg p \vee q)) \qquad \neg p \vee q \equiv p \to q$$

$$\equiv \neg(r \leftrightarrow (p \to q)) \qquad p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$\equiv \neg((r \to (p \to q)) \wedge ((p \to q) \to r))$$

$$\equiv \neg(r \to (\overset{\neg p}{p} \to q)) \vee \neg((p \to \overset{q}{q}) \to r) \qquad \neg p \vee q = p \to q$$

$$\equiv (r \to (p \to q)) \to \overset{\neg\phi}{\neg((p \to q) \to r)} \qquad \neg\phi \equiv \phi \to \bot$$

$$\equiv (r \to (p \to q)) \to (((p \to q) \to r) \to \bot) \ \#$$

Problem 2

The NOR operator, often denoted by $\downarrow$, is a binary truth-functional logical operator described by the following truth table

A	B	$A \downarrow B$	$\equiv \neg(A \vee B)$	$\equiv \neg A \wedge \neg B$
F	**F**	**T**	T	T
F	**T**	**F**	F	F
T	**F**	**F**	F	F
T	**T**	**F**	F	F

Suppose ϕ is the formula $\neg((p \vee \neg q) \wedge r)$. Find a formula that is logically equivalent to ϕ but contains no operators other than $\downarrow$ (i.e. none of the following symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$ occurs).

$A \downarrow A \equiv \neg A$

A	$A \downarrow A$
T	F
F	T

$A \wedge B = (A \downarrow A) \downarrow (B \downarrow B)$

$\neg B \equiv B \downarrow B$
$\neg A \equiv A \downarrow A$

A	B	$A \downarrow B$	$\neg A$	$\neg B$	$\neg A \downarrow \neg B$	$\equiv (A \downarrow A) \downarrow (B \downarrow B)$	$\equiv A \wedge B$
T	T	T	F	F	T	T	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	F	T	T	F	F	F

$A \vee B = (A \downarrow B) \downarrow (A \downarrow B)$

$\neg A \equiv A \downarrow A$

A	B	$A \downarrow B$	$\neg(A \downarrow B)$	$\equiv (A \downarrow B) \downarrow (A \downarrow B)$	$\equiv A \vee B$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	T	T

$\neg A \equiv A \downarrow A$

$A \wedge B \equiv (A \downarrow A) \downarrow (B \downarrow B)$

$A \vee B \equiv (A \downarrow B) \downarrow (A \downarrow B)$

$A \downarrow B \equiv \neg(A \vee B) \equiv \neg A \wedge \neg B$

1st Solution

$$\neg((p \vee \neg q) \wedge r) \equiv \neg(p \vee \neg q) \vee \neg r$$ $\neg A \equiv A \downarrow A$

$$\equiv \overset{\neg(A \vee B)}{\neg(p \vee (q \downarrow q))} \vee (r \downarrow r)$$ $\neg(A \vee B) \equiv A \downarrow B$

$$\equiv \overset{A}{(p \downarrow (q \downarrow q))} \vee \overset{B}{(r \downarrow r)}$$ $A \vee B \equiv (A \downarrow B) \downarrow (A \downarrow B)$

$$\equiv ((p \downarrow (q \downarrow q)) \downarrow (r \downarrow r)) \downarrow ((p \downarrow (q \downarrow q)) \downarrow (r \downarrow r))$$ #

2nd Solution

$$\neg((p \vee \neg q) \wedge r) \equiv \neg(p \vee \neg q) \vee \neg r$$

$$\equiv (\neg p \wedge q) \vee \neg r$$ $\neg A \equiv A \downarrow A$

$$\equiv (\overset{A}{(p \downarrow p)} \wedge \overset{B}{q}) \vee (r \downarrow r)$$ $A \wedge B = (A \downarrow A) \downarrow (B \downarrow B)$

$$\equiv \overset{A}{[((p \downarrow p) \downarrow (p \downarrow p)) \downarrow (q \downarrow q)]} \vee \overset{B}{(r \downarrow r)}$$ $A \vee B \equiv (A \downarrow B) \downarrow (A \downarrow B)$

$$\equiv \Big[[((p \downarrow p) \downarrow (p \downarrow p)) \downarrow (q \downarrow q)] \downarrow (r \downarrow r)\Big] \downarrow \Big[[((p \downarrow p) \downarrow (p \downarrow p)) \downarrow (q \downarrow q)] \downarrow (r \downarrow r)\Big]$$ #

Problem 3

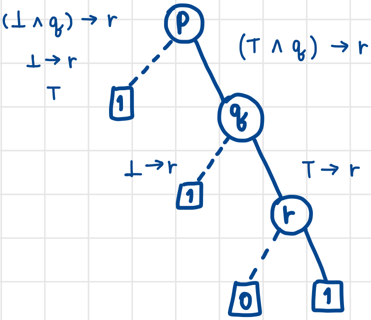
Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

$$\phi = (p \wedge q) \to r$$
$$\psi = (\neg r) \to (q \to \neg p)$$

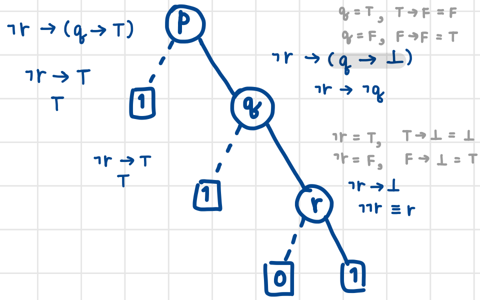
$\bot \to A = T$ $\quad A \to T = T$

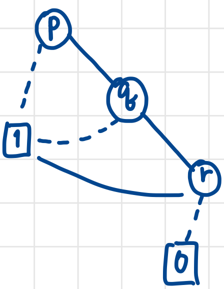
$T \to A = A$ $\quad A \to \bot = \neg A$

ϕ

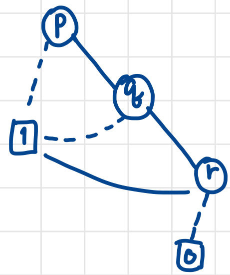


ψ





Logically equivalent



Problem 4

The XOR operator, often denoted by $\oplus$, is a binary truth-functional logical operator described by the following truth table

A	B	$A \leftrightarrow B$
F	F	T
F	T	F $= A$
T	F	F
T	T	T $= A$

A	B	$A \oplus B$
F	**F**	**F**
F	**T**	**T**
T	**F**	**T**
T	**T**	**F**

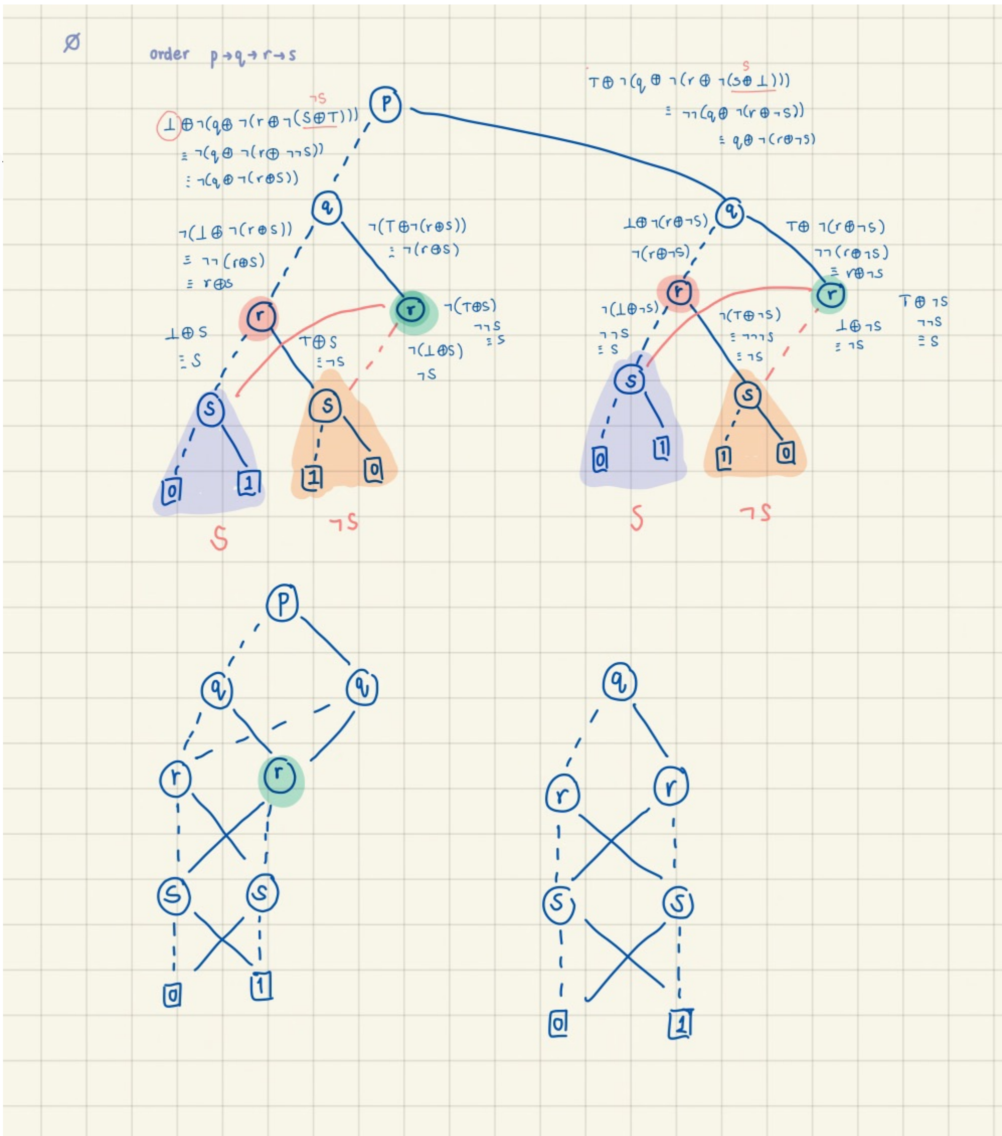
$\boxed{\begin{array}{l} A \oplus \top \equiv \top \oplus A \equiv \neg A \\ A \oplus \bot \equiv \bot \oplus A \equiv A \end{array}}$

$A \leftrightarrow \top \equiv \top \leftrightarrow A \equiv A$

$A \leftrightarrow \bot \equiv \bot \leftrightarrow A \equiv \neg A$

Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

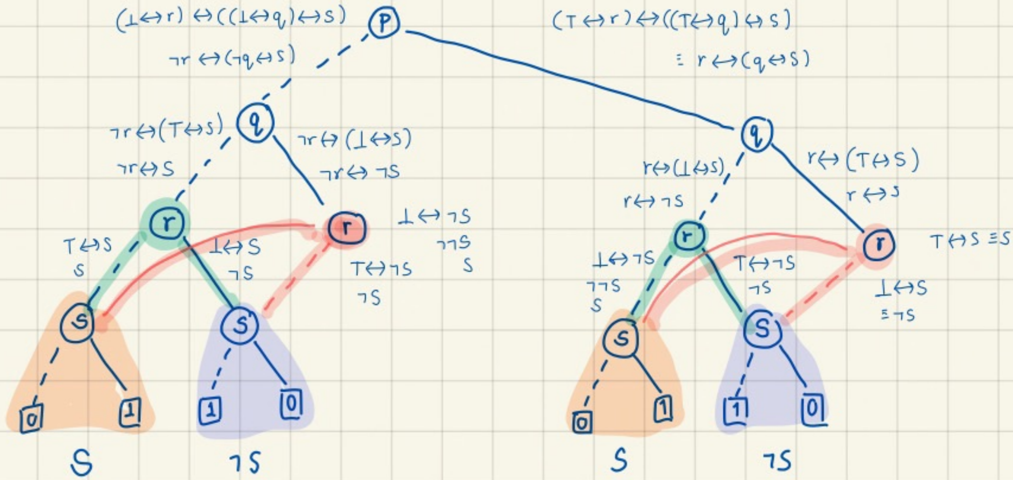
$$\phi = p \oplus \neg(q \oplus \neg(r \oplus \neg(s \oplus \neg p)))$$
$$\psi = (p \leftrightarrow r) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow s)$$

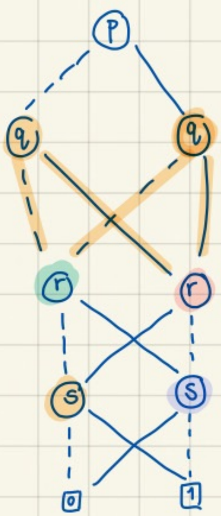


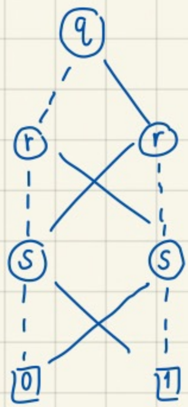
$A \leftrightarrow \top \equiv \top \leftrightarrow A \equiv A$

$A \leftrightarrow \bot \equiv \bot \leftrightarrow A \equiv \neg A$

ψ order $p \to q \to r \to s$





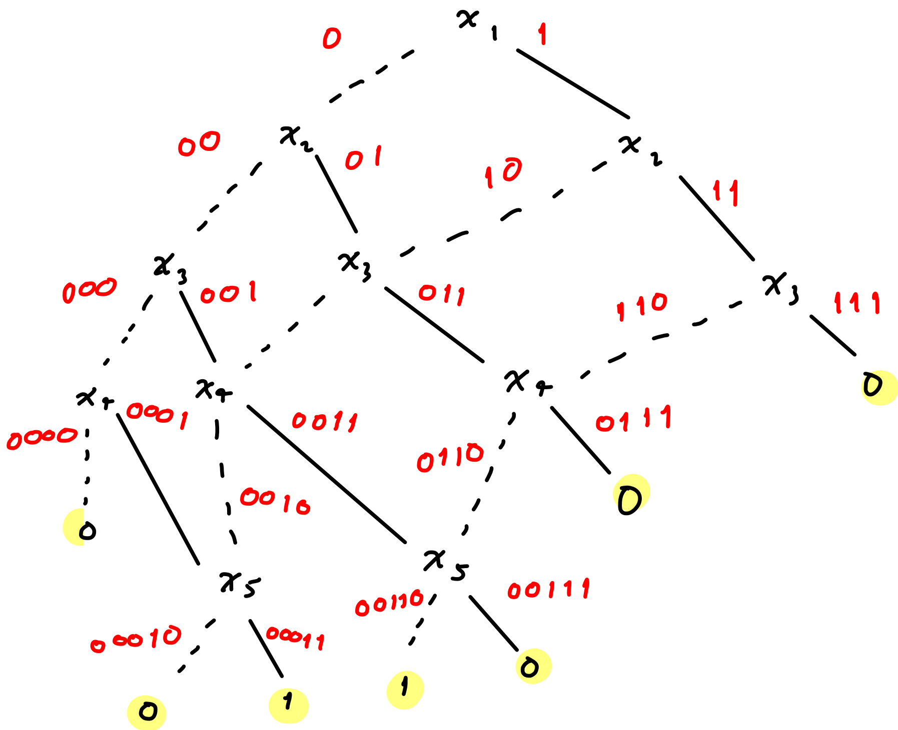


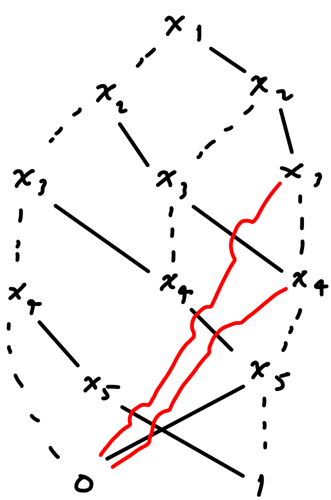
$\varnothing$ and ψ are logically equivalent.

Problem 5

Let f be a Boolean function with 5 arguments such that $f(x_1, x_2, x_3, x_4, x_5) = 1$ when *exactly two* of the variables $x_1, ..., x_5$ are 1. For example, $f(1, 1, 0, 0, 0) = f(0, 0, 1, 0, 1) = 1$, but $f(0, 0, 0, 0, 0) = f(1, 0, 0, 0, 0) = f(0, 1, 1, 0, 1) = 0$.

Draw a reduced OBDD for the Boolean function f.

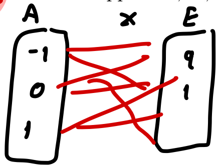




Problem 6

4.5

Suppose A, B, C, and D are the sets given by:



$$A = \{-1, 0, 1\}$$
$$B = \{-6, 1, 2, 7, 9\}$$
$$C = \{x \in \mathbb{Z} \,|\, 0 \leq x < 20 \text{ and } x \text{ is odd}\}$$
$$D = \{x \in \mathbb{Z} \,|\, x = y + z \text{ for some y and z in B}\}$$
$$E = \{x^2 \in \mathbb{Z} \,|\, 2x \in B\}$$

4.4

$\boxed{x+y=y+x}$

x	y	x+y
-6	-6	-12
-6	1	-5
-6	2	-4
-6	7	1
-6	9	3
1	1	2
~~1~~	~~2~~	~~3~~
1	7	8
1	9	10
2	2	4
2	7	9
2	9	11
7	7	14
7	9	16
9	9	18

4.1 $A \cup B$ $\{-1, 0, 1, -6, 2, 7, 9\}$

4.2 $B - C$ $\{-6, 2\}$

4.3 $\wp(A)$ $\{\{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}, \emptyset\}$

4.4 D $\{-12, -5, -4, 1, 2, 3, 4, 8, 9, 10, 11, 14, 16, 18\}$

4.5 $A \times E$ $\{(-1, 9), (0, 9), (1, 9), (-1, 1), (0, 1), (1, 1)\}$

4.6 $\wp(\wp(A \cap B))$ $A \cap B = \{1\}$

$\wp(A \cap B) = \{\emptyset, \{1\}\}$

$\wp(\wp(A \cap B)) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$