MIDTERM 2010

2.1 (5 marks) Let p, q, r, s, and t be the following propositions:

p : The patient has a fever.

q : Rashes can be found on the patient's body.

r : Anti-allergic drugs are ineffective in treating the rashes on the patient's body.

s : The patient has contracted the Dengue virus.

t : The patient has developed an allergy.

Write the following propositions using p, q, r, s, t and logical connectives.

(a) Rashes can be found on the patient's body and anti-allergic drugs are ineffective in treating the rashes. $q \wedge r$

(b) If anti-allergic drugs are effective in treating the rashes on the patient's body, then the patient has not contracted the Dengue virus. $\neg r \rightarrow \neg s$

(c) The patient who has contracted the Dengue virus must have a fever. $s \rightarrow p$

(d) If rashes can be found on the patient's body, then the patient has either developed an allergy or contracted a Dengue virus but not both. $q \rightarrow ((\neg t \wedge s) \vee (t \wedge \neg s))$

(e) Having rashes on the body is a necessary symptom for the patient to have contracted the Dengue virus. $s \rightarrow q$

Keywords

equivalent

a and b $\equiv$ $a \wedge b$

a or b $\equiv$ $a \vee b$

if a then b $\equiv$ $a \rightarrow b$

p must have q $\equiv$ $p \rightarrow q$

either p or q, but not both $\equiv$ $p \oplus q$

$\equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

p is necessary for q $\equiv$ $q \rightarrow p$

p is sufficient for q $\equiv$ $p \rightarrow q$

Not both p and q $\equiv$ $\neg(p \wedge q)$

Neither p nor q $\equiv$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Either p or q $\equiv$ p or q $\equiv p \vee q$

(a) "Sweden and Norway will both not adopt the Euro."

$s =$ Sweden will adopt the Euro.
$n =$ Norway will adopt the Euro.

Ans. $\neg s \wedge \neg n$

(b) "Sweden and Norway will not both adopt the Euro."

$s =$ Sweden will adopt the Euro.
$n =$ Norway will adopt the Euro.

Ans. Typically, this sentence is interpreted as

"It is not the case that Sweden and Norway will both adopt the Euro",

which translates to $\neg(s \wedge n)$ and is equivalent to

Logical Consequence
Natural deduction

There are sets of formulas $\Gamma \{\psi_1, \ldots, \psi_n\}$.
From these premises, we will derive ϕ.

$$\Gamma \left\{ \begin{array}{c} \psi_1 \\ \vdots \\ \psi_n \end{array} \right\} \text{By using natural deduction rules.}$$

$$\frac{\quad\quad\quad}{\phi} \quad \leftarrow \text{if we can derive } \phi \text{ from these formulas}$$

we can say that $\Gamma \underset{\text{derives}}{\vdash} \phi$

$\wedge$ Rules

- $\wedge I$ (Introduction)

$p \,.\, q \qquad p \wedge q \quad \wedge I$

- $\wedge E$ (Elimination)

$p \wedge q \qquad p \qquad q \quad \wedge E$

$\vee$ Rules

- $\vee I$ (Introduction)

$p \qquad p \vee q \quad \vee I$

$q \qquad p \vee q \quad \vee I$

- $\vee E$ (Elimination)

As additional premises

$$\frac{\boxed{p \vee q} \quad \boxed{\begin{array}{c} \text{Assume } p \\ \vdots \\ \hline x \end{array}} \quad \boxed{\begin{array}{c} \text{Assume } q \\ \vdots \\ \hline x \end{array}}}{x}$$

$\rightarrow I$

$p \rightarrow q$

$\uparrow$ infering

$\rightarrow E$ (arrow Elimination) Modus Ponen

$$\begin{array}{lll} ① & p \rightarrow q & \\ ② & p & \\ \hline ③ & q & \rightarrow E \; ①, ② \end{array}$$

Modus Tollen (MT)

$$\begin{array}{lll} ① & p \rightarrow q & \\ ② & \neg q & \\ \hline & \neg p & MT \; ①, ② \end{array}$$

¬ Rules

¬¬E

$$\frac{①\ \neg\neg p}{p}\ \neg\neg E$$

¬¬I

$$\frac{①\ p}{\neg\neg p}\ \neg\neg I$$

Proof By Contradiction

① $\neg p$ ⋮ if we can derive
② $\perp$ ⋮

we may infer ⇒ p PBC ①,②

Contrapositive Proof

① $\neg p$
⋮
$\neg q$

$q \rightarrow p$

Disjunctive Syllogism

$$\frac{①\ p \vee q \quad ②\ \neg p}{q}\ \text{DS ①,②}$$

$$\frac{①\ p \vee q \quad ②\ \neg q}{p}\ \text{DS ①,②}$$

If $\Gamma \vdash \phi$, $\Gamma \models \phi$

ϕ is a logical consequence of Γ

2.5 [1]If Cain married his sister, his marriage was incestuous. [2]If he did not marry his sister, then Adam and Eve were not the progenitors of the entire human race. It follows that [3]if Adam and Eve were the progenitors of the whole human race, then Cain's marriage was incestuous.

1. $s \rightarrow i$ } premises
2. $\neg s \rightarrow \neg p$
3. $p \rightarrow i$ conclusion

1.	$s \rightarrow i$	premises
2.	$\neg s \rightarrow \neg p$	
3.	p	assumption
4.	$\neg\neg p$	$\neg\neg I$ 3
5.	$\neg\neg s$	MT 2,4
6.	s	$\neg\neg E$ 5
7.	i	MP 1,6

$$\frac{\neg s \rightarrow \neg p \quad \neg(\neg p)}{\neg(\neg s)}$$

8. $p \rightarrow i$ $\rightarrow I$ 3-7

1. $s \to (c \vee m)$
2 $\neg(i \vee c)$ premises
3. $\neg(f \vee m)$

4. s assumption
5. $c \vee m$ MP 1, 4

6. c assumption
7. $i \vee c$ $\vee$i 6
8. $\bot$ $\neg$E 2, 7

m assumption
$f \vee m$ $\vee$i 6
$\bot$ $\neg$E 3, 7

9. $\bot$ $\vee$E 5, 6-8

10. $\neg s$ $\neg$I 4-9

$\{p\} \vdash p \wedge (q \vee p)$

1 p premise
2 $q \vee p$ $\vee$I 1
3 $p \wedge (q \vee p)$ $\wedge$I 2, 3

$(p \to (q \to r)) \vdash (q \to (p \to r))$

1. $p \to (q \to r)$
2. p Assume
3. $q \to r$ MP 1, 2

4. q Assume
5. r MP 3, 4

6. $p \to r$ $\to$I 2, 5
7. $q \to (p \to r)$ $\to$I 4, 6

$q \to (p \to r)$

Suppose ϕ is the formula $p \leftrightarrow (q \wedge \neg r)$.

p	q	r	¬r	q∧¬r	p≡q∧¬r	
T	T	T	F	F	F	
T	T	F	T	T	T	$p \wedge q \wedge \neg r$
T	F	T	F	F	F	
T	F	F	T	F	F	
F	T	T	F	F	T	$\neg p \wedge q \wedge r$
F	T	F	T	T	F	
F	F	T	F	F	T	$\neg p \wedge \neg q \neg r$
F	F	F	T	F	T	$\neg p \wedge \neg q \neg r$

DNF: $(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$

Suppose ϕ is the formula $p \leftrightarrow (q \wedge \neg r)$.

p	q	r	¬r	q∧¬r	p≡q∧¬r	
$\neg p$ T	$\neg q$ T	$\neg r$ T	F	F	F	$\neg p \vee \neg q \vee \neg r$
T	T	F	T	T	T	
$\neg p$ T	q F	$\neg r$ T	F	F	F	$\neg p \vee q \vee \neg r$
$\neg p$ T	q F	r F	T	F	F	$\neg p \vee q \vee r$
F	T	T	F	F	T	
p F	$\neg q$ T	r F	T	T	F	$p \vee \neg q \vee r$
F	F	T	F	F	T	
F	F	F	T	F	T	

CNF: $(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r)$

$$r \vee \neg s \vee t$$
$$p \vee \neg r \vee \neg s$$
$$\neg t$$
$$\neg p \vee \neg q \vee t$$
$$s \vee t$$
$$\neg p \vee q$$

$B_p = \{ p \vee \neg r \vee \neg s,\ \neg p \vee \neg q \vee t,\ \neg p \vee q \}$

$\neg q \vee \neg r \vee \neg s \vee t,\ q \vee \neg r \vee \neg s$

$B_q = \{ \}$

$B_r = \{ r \vee \neg s \vee t \}$

$B_s = \{ s \vee t \}$

$B_t = \{ \neg t \}$

$B_q = \{ \neg q \vee \neg r \vee \neg s \vee t,\ q \vee \neg r \vee \neg s \}$

$\neg r \vee \neg s \vee t$

$B_r = \{ r \vee \neg s \vee t \}$

$B_s = \{ s \vee t \}$

$B_t = \{ \neg t \}$

$$B_r = \{ r \vee \neg s \vee t,\ \neg r \vee \neg s \vee t \}$$

$$\neg s \vee t$$

$$B_s = \{ s \vee t \}$$

$$B_t = \{ \neg t \}$$

$$B_s = \{ s \vee t,\ \neg s \vee t \}$$

$$t$$

$$B_t = \{ \neg t \}$$

$$B_t = \{ \neg t,\ t \}$$

$$\perp$$

$\therefore$ $\perp$ Not satisfiable

OBDDs

Ex. $\boxed{(p \to q) \to (p \to r)}$

Order $p \to q \to r$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$T \to p \equiv p$
$p \to T \equiv T$

$\bot \to p \equiv T$
$p \to \bot \equiv \neg p$

$(\bot \to q) \to (\bot \to r)$

$T \to T$

T

$(T \to q) \to (T \to r)$

$q \to r$

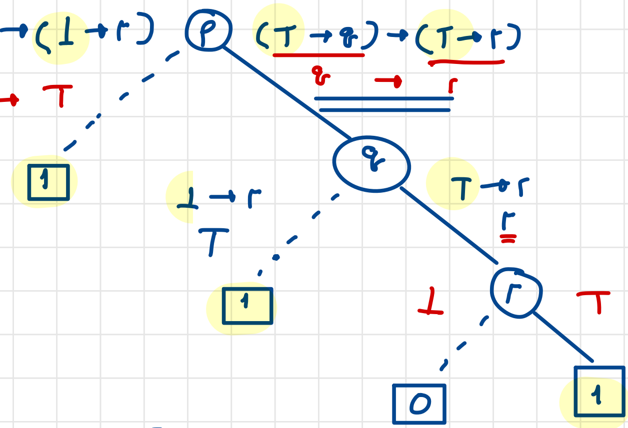
$\bot \to r$

T

$T \to r$

r

$\bot$ $\quad$ T



$T = 1 \quad F = 0$