For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

1)

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 \end{array}\right] \begin{array}{l} \\ \rightarrow R_2 - 2R_1 \\ \rightarrow R_3 - R_1 \end{array}$$

free variable

$b_2 - 2b_1 = 0$

$b_2 = 2b_1$

$b_3 - b_1 = 0$

$b_3 = b_1$

Find the reduced R for each of these (block) matrices:

2)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $2R_1$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\frac{1}{3}R_2$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_1 - 3R_2$

$B = \begin{bmatrix} A & A \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \end{bmatrix}$ $\frac{1}{A}R_1$

$C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$

$= \begin{bmatrix} A & A \\ 0 & -A \end{bmatrix}$ $R_2 - R_1$

$= \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ $-R_2$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\frac{1}{A}R_1$, $\frac{1}{A}R_2$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $R_1 - R_2$

Find the complete solution (also called the *general solution*) to

3

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3-2 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1-1 \end{array}\right] \quad R_2 - R_3$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \tfrac{1}{2}R_2$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1-0.5 \\ 0 & 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 - R_2$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \downarrow & \uparrow & \downarrow \\ \text{pivot} & \text{free variable} & \text{pivot} & \text{free variable} \end{array} \end{array}$$

pivot : x , z
free : y, t

Partial Sol

$Ax = b$
free variable = 0

$y, t = 0$

$x + 3y + 0z + 0t = 0.5$

$x = 0.5$

$z + 2t = 0.5$

$z = 0.5$

Special Sol.

$Ax = 0$
free variable = 1

$y = 1$
$t = 0$

$x + 3y + 0z + 0t = 0.5$
$x + 3 = 0.5$
$x = -2.5$

$z + 2t = 0.5$
$z = 0.5$

$t = 1$
$y = 0$

$x + 3y = 0.5$
$x = 0.5$

$z + 2t = 0.5$
$z = -2.5$

4.

Find a basis for each of the four subspaces associated with A:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$R_2 - R_1 \quad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 \end{array}$$

REF

$$R_3 - R_2 \quad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 - (R_2 - R_1) \end{array}$$

บน 1 → 0

RREF

$$R_1 - 2R_2 \quad \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - 2(R_2 - R_1) \\ R_2 - R_1 \\ R_3 - (R_2 - R_1) \end{array} \Bigg\} \begin{array}{l} 3R_1 - 2R_2 \\ \cancel{R_1 - 2R_2 + 2R_1} \\ R_2 - R_1 \\ R_1 - R_2 + R_3 \end{array}$$

pivot pivot

Basis $C(A)$:
Column Space

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{row of pivot}$$

pivot pivot

Basis of row $= C(A^T)$:
Row space

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Null space

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot pivot

pivot

$x_2 + 2x_3 - 2x_5$

$x_2 = -2x_3 + 2x_5$

pivot

$x_4 + 2x_5 = 0$

$x_4 = -2x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{bmatrix} x_1 \\ -2x_3 + 2x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Left null space

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot

$R_1 - 2(R_2 - R_1)$

$R_2 - R_1$

$R_3 - (R_2 - R_1)$

$3R_1 - 2R_2$

$\cancel{R_1 - 2R_2 + 2R_1}$

$R_2 - R_1$

$R_1 - R_2 + R_3$

$R_1 - R_2 + R_3$

$(1)R_1 - (1)R_2 + (1)R_3$

Left null space $= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Check

$$A = \begin{array}{cc} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} & \begin{array}{c} \times 1 \\ \times -1 \\ \times 1 \end{array} \end{array}$$

$$= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ -0 & -1 & -2 & -4 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Project $\boldsymbol{b}$ onto the column space of A by solving $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$:

5 $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$.

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be perpendicular to the columns of A.

$p = A(A^TA)^{-1} A^T \cdot b$

$$p = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} \right)^{-1} \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1+1+0 & 1+1+0 \\ 1+1+0 & 1+1+1 \end{pmatrix}^{-1} \begin{pmatrix} 4+4+0 \\ 4+4+6 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{6-4}\right) \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{2}\right) \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

$$= \underset{3\times 3}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 8 \\ 14 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12-14 \\ -8+14 \end{bmatrix}$$

$$= \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} -2 \\ 6 \end{bmatrix}}$$

$$= \underset{3\times 1}{\begin{bmatrix} -2+6 \\ -2+6 \\ 6 \end{bmatrix}} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$e = b - p$

$$= \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\hat{x} = 2\times 1$

$= (A^TA)(A^Tb)$

$A^TA(x) = A^TB$

$2x_1 + 2x_2 = 8$

$2x_1 + 3x_2 = 14$

$x = (A^TA)^{-1} A^TB$ ✓

$\hat{x} = (A^Tb)(A^TA)^{-1}$

$$A^TA = \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^Tb = \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 8 \\ 14 \end{bmatrix}}$$

$$\underset{2\times 2}{\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 8 \\ 14 \end{bmatrix}}$$

$2x_1 + 2x_2 = 8$

$2x_1 + 3x_2 = 14$

$-x_2 = 6$

$x_2 = 6$

$x_1 = -2$

$$p = A\hat{x} = \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} -2 \\ 6 \end{bmatrix}}$$

$$p = \underset{3\times 1}{\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}}$$

$e = b - p$

$$= \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

⑥ The average of the four times is $\hat{t} = \frac{1}{4}(0+1+3+4) = 2$. The average of the four b's is $\hat{b} = \frac{1}{4}(0+8+8+20) = 9$.
Verify that the best line goes through the center point $(\hat{t}, \hat{b}) = (2, 9)$.

t =	0	1	3	4	2
b =	0	8	8	20	9

best fit pass (2,9)

$y = mx + c$

$b = mt + (1)c$

$$\underset{B}{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}} = \underset{\substack{A \\ 4\times 2}}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}} \underset{\substack{\hat{x} \\ 2\times 1}}{\begin{bmatrix} c \\ m \end{bmatrix}}$$

$\hat{x} = \begin{bmatrix} c \\ m \end{bmatrix}$

$\hat{x} = (A^TA)^{-1}(A^Tb)$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 8+8+20 \\ 0+8+24+80 \end{bmatrix}$$

$$= \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

A^TA

$$= \underset{2\times 4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}} \underset{4\times 2}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}$$

$$= \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$$

$(A^TA)^{-1}$

$$= \frac{1}{104-64}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13}{20} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

$$= \underset{2\times 2}{\begin{bmatrix} \frac{13}{20} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 36 \\ 112 \end{bmatrix}}$$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{117}{5} + \frac{-112}{5} \\ \frac{-36}{5} + \frac{112}{10} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \longrightarrow = \begin{bmatrix} C \\ M \end{bmatrix}$$

so $C = 1$
$M = 4$

from $b = mt + c$

$b = 4(t) + 1$

from

Verify that the best line goes through the center point $(\widehat{t}, \widehat{b}) = (2, 9)$.

$9 = 4(2) + 1$

$9 = 9$

Find $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ (orthonormal) as combinations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (independent columns). Then write A as QR:

7

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \quad q_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\vec{v}_1 = q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = q_2 - \left(\frac{q_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{\overset{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}{\cancel{\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{v}_3 = q_3 - \left(\frac{q_3 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 - \left[\frac{q_3 \cdot v_2}{v_2 \cdot v_2}\right] \cdot v_2$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\overset{\begin{bmatrix} 0 \\ 0 \\ 18 \end{bmatrix}}{\cancel{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}}}{\begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}} \cdot \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix}$$

$$Q^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 1 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 1/5 & 0 \end{array}\right] \quad \begin{array}{l} R_2/3 \\ R_3/5 \end{array}$$

$$R = \begin{array}{c} Q^{-1} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/3 \\ 0 & 1/5 & 0 \end{bmatrix} \end{array} \begin{array}{c} A \\ \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} \end{array} = \begin{array}{c} R \\ \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$QR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

1.

For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2.

Find the reduced R for each of these (block) matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

3.

Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

4.

Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5.

Project $\boldsymbol{b}$ onto the column space of A by solving $A^{\mathrm{T}} A \widehat{\boldsymbol{x}} = A^{\mathrm{T}} \boldsymbol{b}$ and $\boldsymbol{p} = A \widehat{\boldsymbol{x}}$:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be perpendicular to the columns of A.

6.

The average of the four times is $\widehat{t} = \frac{1}{4}(0+1+3+4) = 2$. The average of the four b's is $\widehat{b} = \frac{1}{4}(0+8+8+20) = 9$.

Verify that the best line goes through the center point $(\widehat{t}, \widehat{b}) = (2, 9)$.

7.

Find $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ (orthonormal) as combinations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (independent columns). Then write A as QR:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$