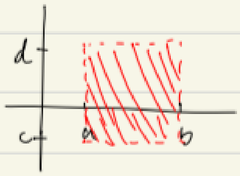
▸ Review Fubini's Theorem

<u>Theorem</u> Fubini's thm on rect. domain)

$f: \mathbb{R}^2 \to \mathbb{R}$ is cont. in $[a,b] \times [c,d]$



$$\iint_{\mathbb{R}} f(x,y)\, dx\,dy = \int_a^b \int_c^d f(x,y)\, dy\,dx$$
$$= \int_c^d \int_a^b f(x,y)\, dx\,dy$$

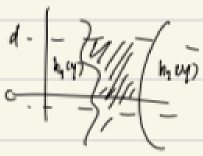
<u>Theorem</u> Fubini's Thm on non-react. domain)

If $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is cont. in D, then $\cdot \{(x,y) \in \mathbb{R} \mid x \in [a,b],\ y \in [g_1(x), g_2(x)]$ with $g_1 g_2$ cont on $[a,b]$, then

$$\iint_D f(x,y)\, dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\,dx$$

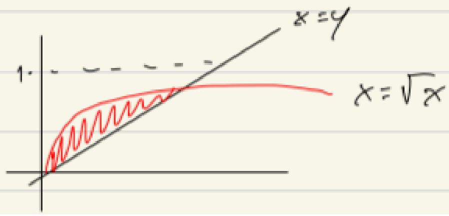
if $D = \{(x,y) \in \mathbb{R}^2 \mid x \in [h_1(y), h_2(y)],\ y \in \mathbb{R}^2 \mid x \in h_1(y), h_2(y),\ y = [c,d]\}$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\,dy$$



Ex. Find intregral of $f(x,y) = x^2 + y^2$ on $D = \{(x,y) \in \mathbb{R}^2 \mid y \le x \le \sqrt{y},\ 0 \le y \le 1\}$

1) $x = y$
2) $x = \sqrt{y}$



By the Fubini's Thm

$$I = \iint_D f(x,y)\, dx\,dy$$
$$= \int_c^d \int_{h_1 y}^{h_2 y} f(x,y)\, dx\,dy$$
$$= \int_0^1 \int_y^{\sqrt{y}} (x^2+y^2)\, dx\,dy$$
$$= \int_0^1 \frac{x^3}{3} + xy^2 \Big|_y^{\sqrt{y}}$$
$$= \int_0^1 \frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{3} - y^3\ dy$$
$$= \ldots$$

Expectation of g(x)

Proof Assume $X \geq 0$ and $g(x) \geq 0$ for the proof.

$X = X \cdot \underbrace{1_A(x)}_{x \geq 0} + X \cdot \underbrace{1_{A^c}(x)}_{x<0}$, $A = \{x \in S \mid x(\cdot) \geq 0\}$

$g = g(x)\, 1_B(x) + g(x)\, 1_{B^c}(x)$

$E[g(x)] = \int_0^\infty P(g(x) > y)\, dy$ * by previous proposition

$= \int_0^\infty \left(\int_{\{x; g(x) > y\}} f(x) dx \right) dy$

By the Fubini's Thm,

$E[g(x)] = \int_0^\infty \left(\int_0^{g(x)} dy \right) f(x)\, dx$

$= \int_0^\infty g(x) f(x)\, dx$

Proposition

Let X be a r.v. with pdf $f(x)$. Then $E(aX+b) = aE[X] + b$, $a, b \in \mathbb{R}$

Proof

$E(ax+b) = \int_{\mathbb{R}} (ax+b) f(x)\, dx$

$= a \int_{\mathbb{R}} x f(x)\, dx + b \int_{\mathbb{R}} f(x)\, dx$

$= aE(x) + b$

Normal Distribution
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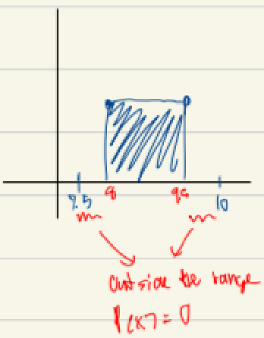
Lemma

1) If $X = \mu + \sigma Z$ with $Z \sim N(0,1)$ then X is normal distribution
2) If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, then $a \geq 0$. $Y \sim N(a\mu + b, a^2\sigma^2)$

Uniform r.v. 43

Ex let X be the time to failure of a computer. Assume that $X \sim \mu([8, 10])$. Find $P(7.5 < X < 9.5)$



$$P(7.5 < X < 9.5) = P(8 \leq X \leq 9.5)$$
$$= \int_8^{9.5} \frac{1}{10-8}\, dx$$
$$= \frac{1}{2} x \Big|_8^{9.5}$$
$$= 0.75$$

IDK wtf is this 44

$$f_X(x)\Delta x \approx P(x \leq X \leq x + \Delta x)$$
$$= F_X(x + \Delta x) - F_X(x)$$
$$f_X(x) = \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}$$
$$f_X(x) = \lim_{\Delta x \to 0} \left(\frac{F_X(x+\Delta x) - F_X(x)}{\Delta x} \right)$$
$$= \frac{d}{dx} F_X(x)$$

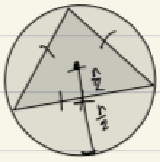
* $F_X(x)$ is pointwise differentiable.

<u>Theorem</u> Change of variable in one dimension

$Y = g(X)$ where X, Y are r.v.'s and g is differentiable and strictly increasing (or decreasing) (g is bijective) then $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$, $x = g^{-1}(y)$

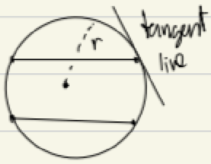
Bertrand's product

Experiment: Draw a random chord of a circle with center o and radius r.
Compute P(chord > the side of inscribed equilibrium triangle)



Method 1: chord is determined by its distance D to center o.
D = distance from the chord to the center o
E = "event that chord is longer than the equilateral triangle.
$P(E) = P(D \le \frac{r}{2}) = \frac{1}{2}$.

Method 2: Chord is parametized by θ.



θ = angle between chord and tangent
$\theta \sim U[0, 90^\circ]$
$P(E) = P(60^\circ < \theta < 90^\circ)$

Ex Chi-square PDF
if $X \sim N(0,1)$, then $Y = X^2$ is called chi-square r.v (df = 1), $f_Y(y) = ?$

Sol^n $X \sim N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$
$\varphi(x) = f_X(x)$

$\Phi(x) = \int_{-\infty}^{x} f_X(t)\, dt$

$= \int_{-\infty}^{x} \varphi(t)\, dt$

$F_Y(y) = P(X^2 \le y)$
$= P(-\sqrt{y} \le X \le \sqrt{y})$
$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$
$= \Phi(\sqrt{y}) - (1 - \Phi(\sqrt{y}))$
$= 2\Phi(\sqrt{y}) - 1$
$\frac{d}{dy} F_Y(y) = 2 \frac{d}{dy} \Phi(\sqrt{y}) - 0$
$= 2\varphi(\sqrt{y}) \cdot \frac{1}{2} y^{-\frac{1}{2}}$ $\frac{d}{dy}\sqrt{y}$

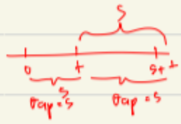
$f_Y(y) = f_X(\sqrt{y})\, y^{-\frac{1}{2}}$, $y \ge 0$

Remark: The chi square r.v. with df = n is $X_n^2 = X_1^2 + X_2^2 + \cdots + X_n^2$ where X_i's are independent and identically distributed^n with $N \sim (0,1)$

Memoryless Property

Not depend on the past

Def A r.v. X has a memoryless property if $P(X > s+t \mid X > t) = P(X > s)$



If $X \sim Exp(\lambda)$ then X has a memoryless property.

Theorem

If X is a continuous r.v. with memoryless property, then X is exponentially distributed.

Proof Let $F_X(x) = P(X \le x)$. We have $P(X > s+t \mid X > t) \overset{\text{memoryless}}{=} P(X > s)$

Let $G_X(x) = 1 - F_X(x)$.

We will show that $G_X(s+t) = G_X(s)G_X(t)$ —— (*)

Since $F_X(x)$ is cont., $G_X(x)$ is cont. From (*), $G_X(x)$ must be an exponential function.

Theorem • example of memoryless on discrete r.v.

Let $X \sim Geo(p)$ then X has a memoryless property.

TTTT_ _ TH

$P(\text{success}) = p \quad P(X=k) = (1-p)^{k-1}p$

$P(\text{failure}) = 1-p \quad P(X \le k) = \sum_{i=1}^{k} (1-p)^{i-1} p$

$$= \frac{p(1-(1-p)^k)}{1-(1-p)} \quad \text{Geometric series} \quad = 1-(1-p)^k$$

Def

Let X, Y be two continuous r.v.

We say that X and Y are jointly distributed. if for each $B \subseteq \mathbb{R}^2$,

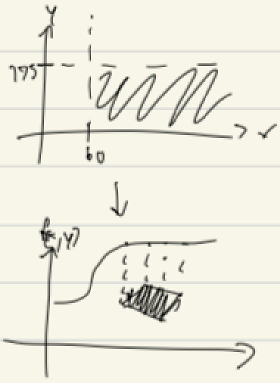
$P[(X,Y) \in B] = \iint_{(x,y) \in B} f_{x,y}(x,y)\, dx\, dy$ for some function $f_{x,y}(x,y)$. We call $f_{x,y}$ the joint probability mass function of X, Y.

Ex// What is the prob. that a person weights over 60 kg and is under 175 cm. ?

Sol^n X = weight a person (kg)

Y = Height of the same person (cm)

$$P(\underbrace{60 < X < \infty,\ 0 \leq Y < 175}_{B\ =\ (60, +\infty) \times (0, 175)})$$



$$P((X,Y) \in B) = \iint_B f_{x,y}(x,y)\, dx\, dy$$

$$= \int_0^{175} \int_{60}^{\infty} f_{x,y}(x,y)\, dx\, dy$$

If discrete, $P_x(x) = \sum_y P_{x,y}(x,y)$

$P_y(y) = \sum_x P_{x,y}(x,y)$

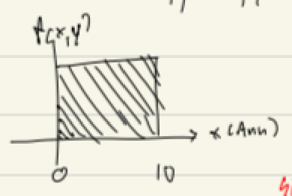
Likewise, for cont. r.v. X, Y,

$$f_x(x) = \int_{-\infty}^{+\infty} f_{x,y}(x,y)\, dy$$

$$f_y(x) = \int_{-\infty}^{+\infty} f_{x,y}(x,y)\, dx$$

Ex// Ann and Bob both pick a number between 0 and 10, according to a count uniform dist.

What is $f_{x,y}(x,y)$



$P(0 \leq X \leq 10,\ 0 \leq Y \leq 10) = 1$

Thus, $\int_0^{10}\int_0^{10} \underbrace{f_{x,y}(x,y)}_{c}\, dx\, dy = 1$

Since, it is uniform

$c \int_0^{10}\int_0^{10} dx\, dy = 1$

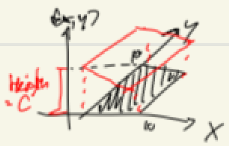
$c \int_0^{10} x \Big|_{x=0}^{10} dy = 1$

$10c \int_0^{10} dy = 1$

$10c\ y \Big|_0^{10} = 1$

$(10)(c)(10) = 1$

$c = \frac{1}{100}$



$$\therefore f_{x,y}(x,y) = \begin{cases} \frac{1}{100} ; & 0 \leq x \leq 10,\ 0 \leq y \leq 10 \\ 0 ; & \text{otherwise} \end{cases}$$

Conditional PDF

Let X be a cont. r.v and B is an event $f_X(x)$ is the prob density f^n (PDF)

$$P(X \in B) = \int_B f_X(x)\,dx$$

let A be an event with $P(X \in A) > 0$

Def Conditional PDF of X given A is $P(X \in B \mid X \in A) = \int_B f_{X|A}(x)\,dx$ where

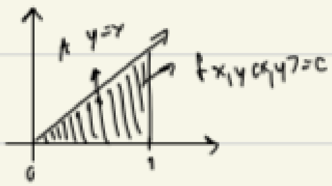
$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(X \in A)} & ; x \in A \\ 0 & ; \text{otherwise} \end{cases}$$

Remark
$$\int_B f_{X|A}(x)\,dx = \int \frac{f_X(x)}{P(X \in A)}\,dx$$
$$= \int_{A \cap B} \frac{f_X(x)}{P(A)}\,dx$$
$$= \frac{1}{P(A)} \int_{A \cap B} f_X(x)\,dx$$
$$= \frac{1}{P(A)} \cdot P(A \cap B)$$
$$= \frac{P(A \cap B)}{P(A)}$$
$$= P(B \mid A)$$

Conditional PDF of a r.v X on r.v Y

Def is defined by $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$ → Joint function → density function

$$f_{X,Y}(x,y) = \begin{cases} c & ; 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$



$$1 = \iint_A f_{X,Y}(x,y)\,dy\,dx$$
$$= \int_0^1 \int_0^x c\,dy\,dx$$
$$= \int_0^1 cx\,dx$$
$$= \frac{cx^2}{2}\Big|_{x=0}^{1}$$
$$= c \cdot \frac{1^2}{2}$$
$$1 = \frac{c}{2} \Rightarrow c = 2$$

Ex. $X \sim Exp(\lambda)$. $f_X(x) = \lambda e^{-\lambda x}$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t)}{P(X>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

or we can do the following

$$f_{X|X>s}(x) = \begin{cases} \frac{\lambda e^{-\lambda x}}{e^{-\lambda s}} & x>s \\ 0 & ; \text{otherwise} \end{cases}$$

$$P(X > s+t \mid X > s) = \int_{s+t}^{\infty} f_{X|X>s} \, dx = \int_{s+t}^{\infty} \frac{\lambda e^{-\lambda x}}{e^{-\lambda s}} \, dx = \lambda \int_{s+t}^{\infty} e^{-\lambda(x-s)} \, dx = \lambda \int_{s+t}^{\infty} e^{-\lambda u} \, du = -e^{\lambda u} \Big|_{s+t}^{\infty} = e^{-\lambda t}$$

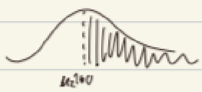
Ex. The height X of a randomly chosen Thai women can be approx by $X \sim N(\underbrace{160}_{\mu}, \underbrace{10^2}_{\sigma^2})$

A) Assume that a random women has height 155 cm. What is the prob that she has height at least 160 cm

Warm up

B) If we don't have the information "≥ 155 cm" what is the prob that a random woman has height at least 160 cm.

B) $$P\{X \geq 160\} = \int_{160}^{\infty} \frac{1}{\sqrt{2\pi}\, 10} e^{-\frac{(x-160)^2}{2(100)}} \, dx = \frac{1}{2}$$



$\frac{1}{2}$

A) $$P\{X \geq 160 \mid X \geq 155\} = \frac{P(X \geq 160)}{P(X \geq 155)} = 0.7231$$

use computer to compute

or by integration of $f_{X|A} \begin{cases} \frac{f_X(x)}{P(X \in A)} & ; x \geq 155 \\ 0 & ; \text{otherwise} \end{cases}$

What is $f_{X|Y}(x|y)$?

Sol $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx$$

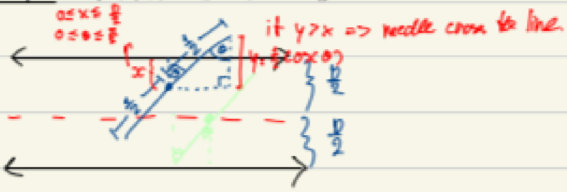
from condition $$= \int_{x=y}^{1} 2 \, dx = 2x \Big|_{x=y}^{1} = 2(1) - 2y = 2(1-y)$$

$$\therefore \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{2}{2(1-y)} & ; y \leq x \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Buffon's Needle problem

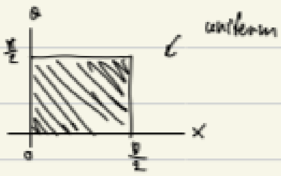
What is the probability that a needle will cross the line where the width between 2 lines is D and the length of a needle is L.

Sol^n Assume $L \le D$



Let X be the distance from the center of a needle to the closer line.

θ : the vertical angle * X and θ are independent



$$P(\text{a needle intersect a line}) = P(X \le \tfrac{L}{2}\cos\theta)$$
$$= P\left(\frac{X}{\cos\theta} \le \frac{L}{2}\right)$$

$$f_X(x) \begin{cases} \frac{1}{\frac{D}{2}} & ; 0 \le x \le \frac{D}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

$$f_\theta(y) = \begin{cases} \frac{1}{\frac{\pi}{2}} & ; 0 \le \theta \le \frac{\pi}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

Note that X, Y are independent then $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Thus, $P\left\{\frac{X}{\cos\theta} \le \frac{L}{2}\right\}$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{L}{2}\cos(y)} f_{X,\theta}(x,y)\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{L\cos(y)}{2}} f_X(x)\, f_\theta(y)\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{L\cos(y)}{2}} \frac{2}{D}\cdot\frac{2}{\pi}\,dx\,dy$$

$$= \frac{4}{\pi D}\int_0^{\frac{\pi}{2}}\int_0^{\frac{L\cos(y)}{2}} dx\,dy$$

$$= \frac{2}{\pi D}\int_0^{\frac{\pi}{2}} \cos(y)\,dy$$

$$= \frac{2}{\pi D}\left(\sin(\tfrac{\pi}{2}) - \sin(0)\right)$$

$$= \frac{2}{\pi D} L$$

$\therefore$ Probability of crossing $= \frac{2}{\pi}$

Proposition

Let X, Y be random variables then X and Y are independent if and only if

1) The joint CDF, F satisfies

$F(a,b) = F_X(a)\,F_Y(b)$ for all $a, b \in \mathbb{R}$.

2) The joint pdf f satisfies $f(a,b) = f_X(a)\,f_Y(b)$. for all $a, b \in \mathbb{R}$

Proposition

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Ex// The joint density of X and Y is given by

a) $f(x,y) = \begin{cases} xe^{-(x+y)} & ; x>0, y>0 \\ 0 & ; \text{otherwise} \end{cases}$

Are X and Y independent?

b) If instead, $f(x,y)$ were given by

$f(x,y) = \begin{cases} 2 & ; 0<x<y, 0<y<1 \\ 0 & ; \text{otherwise} \end{cases}$

Would X and Y independent?

if we want x, integrate respect to y

a) $f_X(x) = \int_{-\infty}^{\infty} xe^{-(x+y)} \, dy$

$= \int_{0}^{\infty} xe^{-(x+y)} \, dy$

$= xe^{-x} \int_{0}^{\infty} e^{-y} \, dy$

$= xe^{-x} \left(-e^{-y} \Big|_0^{\infty}\right)$

$= xe^{-x} (0 + e^{-0})$

$= xe^{-x}$

$\int_{-\infty}^{\infty} xe^{-(x+y)} \, dx$

$= \int_{0}^{\infty} xe^{-(x+y)} \, dx$

$= e^{-y} \int_{0}^{\infty} xe^{-x} \, dx$

$f_{X,Y}(x,y) = xe^{-(x+y)}$

$f_X(x) f_Y(y)$

Thus, X and Y are independent

Chapter 7 properties of expectation

Indicator function

Def Let A, B be sets (events). We can define a rv. I_A and I_B by

$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

I_A is called an indicator function of A.

Ex// $I_A(0.1) = 1$

$I_A(-2) = 0 = I_A(1.5)$

Theorem

Let A and B be event then 1) $I_A = \underbrace{I_A \cdots I_A}_{k \text{ fixes}}$

$= I_A$ for any positive integer k

2) $I_{A^c} = 1 - I_A$

3) $I_{A \cap B} = I_A I_B$

4) $I_{A \cup B} = I_A + I_B - I_A I_B$

Theorem Let A be an event

$P(A) = E(I_A)$

Theorem Boole's Inequality

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le P(A_1) + P(A_2) + \ldots + P(A_i)$$

Ex We have a deck of n cards labeled 1 throug n. A card is a match if the card's position in the deck matches the card's label.

Let X be the number of matches. Find $E(X)$.

<u>Sol^n</u> $I_j = \begin{cases} 1 & \text{if the } j^{th} \text{ card is a match} \\ 0 & \text{if otherwise} \end{cases}$

$j : 1, 2, \ldots, n$

$X = \#$ matches $(0, 1, 2, \ldots, n)$

$X = I_1 + I_2 + \ldots + I_n$

I_2

$\boxed{1}\ \boxed{2} \to 1$

$\boxed{2}\ \boxed{1} \to 0$

$E[I_2] = \frac{1}{2} \cdot 1 + 0 \cdot \frac{1}{2} = \frac{1}{2}$

I_3

1 2 3

1 3 2

2 3 1

2 1 3

3 1 2

3 2 1

$E[I_3] = \frac{2}{6} \cdot 1 + 0 \cdot \frac{4}{6} = \frac{1}{3}$

$E[I_j] = \frac{1}{n}$

$$\begin{aligned} E[X] &= E[I_1 + I_2 + \ldots I_n] \\ &= \frac{1}{n} + \frac{1}{n} + \ldots \frac{1}{n} \\ &= \frac{n}{n} \\ &= 1 \end{aligned}$$

Proposition

X, Y r.v's g a function of 2 variables. $Z = g(X, Y)$, then $E(Z) = E(g(X,Y))$

$$= \sum_y \sum_x g(x,y)\, p(x,y)$$

for discrete r.v's

$$\begin{aligned} E(Z) &= E(g(x,y)) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy \end{aligned}$$

66 and after is
moment, take aom's