Homework 6

March 2024

Task 3: Quick Sort Recurrence

(i)

$$f(n) = (n+1) + \frac{2}{n}(f(n-1) + f(n-2) + ... + f(1))$$
$$f(n-1) = n + \frac{2}{n-1}(f(n-2) + f(n-3) + ... + f(1))$$

Multiplying the first equation by n and the second equation by n - 1, we have

$$nf(n) = n(n+1) + 2(f(n-1) + f(n-2) + ... + f(1)) \tag{1}$$
$$(n-1)f(n-1) = n(n-1) + 2(f(n-2) + f(n-3) + ... + f(1)) \tag{2}$$

Subtracting equation (2) from equation (1), we have

$$nf(n) - (n-1)f(n-1) = 2n + 2(f(n-1))$$
$$nf(n) = 2n + (n+1)f(n-1)$$

In other words,

n · f (n) = 2n + (n + 1)f(n 1)

(ii) Let g(n) = $\frac{f(n)}{n+1}$. Can you write what you have in terms of the function g?
(Hint: divide equation (3) by n(n + 1))

$$nf(n) = 2n + (n+1)f(n-1)$$

Dividing equation (3) by n(n + 1), we have

$$\frac{nf(n)}{n(n+1)} = \frac{2n}{n(n+1)} + \frac{(n+1)f(n-1)}{n(n+1)}$$
$$\frac{f(n)}{n+1} = \frac{2}{n+1} + \frac{f(n-1)}{n}$$

Since we let g(n) = $\frac{f(n)}{n+1}$, g(n) = $\frac{2}{n+1}$ + g(n - 1).

(iii) Your task in this step is to find a closed form for g.
First, let's unravel g(n). By the definition of g(n),

$$
\begin{aligned}
g(n) &= \frac{2}{n+1} + g(n-1) \\
g(n) &= \frac{2}{n+1} + \frac{2}{n} + g(n-2) \\
g(n) &= \frac{2}{n+1} + \frac{2}{n} + \frac{2}{n-1} + g(n-3)
\end{aligned}
$$

If we keep expanding the recurrence, g(0) = $\frac{f(0)}{1}$ = 0, we'll get

$$
\begin{aligned}
g(n) &= \frac{2}{n+1} + \frac{2}{n} + \frac{2}{n-1} + ... + \frac{2}{3} + 1 \\
g(n) &= 2\left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n+1}\right)
\end{aligned}
$$

To find g(n), you multiply every term of the (n + 1)-th Harmonic number sequence by 2, and then you subtract 2 from the sum. This can be written as 2 · (H_{n+1} - 1).

(iv) Now that you can express g(n) in terms of some expression involving Harmonic numbers, you can proceed to derive a closed form for f(n). Finally use the following fact to conclude that f(n) is O(n * ln(n)):

Fact: $H_n \leq$ 1 + ln(n), where ln denotes the natural logarithm.

If we define g(n) as $\frac{f(n)}{n+1}$, we can multiply both sides by n + 1, giving us f(n) = (n + 1) · g(n). Therefore,

$$
\begin{aligned}
f(n) &= (n+1)g(n) \\
f(n) &= (n+1)(2H_{n+1} - 2)
\end{aligned}
$$

We can use the fact: $H_n \leq 1 + ln(n)$, which implies $H_{n+1} \leq 1 + ln(n+1)$, by substituting H_{n+1} with this inequality, we can then derive the closed form expression of f(n).

$$
\begin{aligned}
f(n) &= (n+1)(2(1 + ln(n+1)) - 2) \\
f(n) &= 2(n+1)ln(n+1)
\end{aligned}
$$

To conclude that f(n) ∈ O(n · ln(n)), we will verify the following limit:

$\lim_{n\to\infty} \frac{f(n)}{nln(n)} < \infty$

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{f(n)}{nln(n)} \\
&= \lim_{n\to\infty} \frac{2(n+1)ln(n+1)}{nln(n)} \\
&= \lim_{n\to\infty} \frac{2(n+1)}{n} \lim_{n\to\infty} \frac{ln(n+1)}{ln(n)} \\
&= 21 \\
L &= 2 < \infty
\end{aligned}
$$

Therefore, f(n) ∈ O(n · ln(n)).