Solutions to Introduction to Counting

Dennis Chen, Abrar Fiaz

CPV

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§ 1 Dennis Chen

The Committee of MAST needs to split 3 distinct jobs among 8 students. If each student can do as many jobs as they please, how many ways are there to assign the jobs?

§ 1.1 Solution

We have three choices: who to do job A, job B, and job C. Each of these choices have eight options, so the answer is $8^3 = 512$.

§ 2 Dennis Chen

There are 52 postal abbreviations for the 50 states of America, D.C. and Puerto Rico. If we choose a two-letter "word" at random (such as AA), what is the probability that we choose one of the 52 postal abbreviations?

§ 2.1 Solution

There are a total of 26^2 combinations, of which 52 satisfy the given condition. So the probability is $\frac{52}{26^2} = \frac{1}{13}$.

§ 3 Dennis Chen

Say Jim is getting an ice cream cone. He can get either 1, 2, or 3 scoops. For each scoop, he can get 3 different flavors. How many different ice creams can he get? (Order of the scoops matter!)

§ 3.1 Solution

If he gets one scoop, he has 3^1 choices. If he gets two, he has 3^2. If he gets three, he has 3^3. Adding it all up, he has $3 + 9 + 27 = 39$ choices.

§ 4 Dennis Chen

Find the amount of terms in $\{5, 7, 9 \cdots 39\}$. (This is an arithmetic sequence.)

§ 4.1 Solution

Subtract 3 to get $\{2, 4, 6 \cdots 36\}$. Divide by 2 to get $\{1, 2, 3 \cdots 18\}$. So the answer is 18.

§ 5 Dennis Chen

Find the amount of terms in $\{3, 6, 12, 24, 48, 96\}$. (This is a geometric sequence.)

§ 5.1 Solution

Multiply by $\frac{2}{3}$ to get $\{2, 4, 8, 16, 32, 64\}$. Then take $\log_2 x$ to get $\{1, 2, 3, 4, 5, 6\}$. So the answer is 6.

§ 6 Well-known

How many ways can you arrange the letters in TARGET?

§ 6.1 Solution

There are 6 letters, so there are 6! ways to rearrange them. However, there are two of the same letter, so we overcount by a factor of 2! Thus the answer is $\frac{6!}{2!} = 360$.

§ 7 Well-known

How many ways are there to arrange the letters in MATHCOUNTS?

§ 7.1 Solution

There are 10 letters and T is repeated twice, so the answer is $\frac{10!}{2!} = 181440$.

§ 8 Well-known

How many positive integers less than 1000 only have even digits?

§ 8.1 Solution

Say we have a three-digit sequence that corresponds to a positive integer with only even digits. Then each digit has 5 choices, so we have 125 sequences. But note 000 does not correspond to a positive intger, so the answer is $5^3 - 1 = 124$.

§ 9 Dennis Chen

We want to choose two disjoint committees of 4 people from a class of 12. How many ways can we do this, if the committees are distinct?

§ 9.1 Solution

We first begin by choosing the 8 people that will be part of a committee. There are $\binom{12}{8}$ ways to do this. We then choose 4 people to make one committee (the other four go to the other committee). There are $\binom{8}{4}$ ways to do this. Thus our answer is $\binom{12}{8}\binom{8}{4} = 34650$.

§ 10 Dennis Chen

What if the committees are indistinguishable?

§ 10.1 Solution

We first choose 8 people to form our two committees. There are $\binom{12}{8}$ ways to do this. Then we permute them and account for overcounting. There are 8! ways to permute them, but we overcount by a factor of 4!4!2!. This is because any rearrangement of the first four people in our permutation is counted 4! times. The same holds for the last four people. Then, we count every ordering of the committee 2! times as there are two groups. (For example, $\{\{A, B, C, D\}, \{E, F, G, H\}\}$ would be counted twice as $(\{A, B, C, D\}, \{E, F, G, H\})$ and $(\{E, F, G, H\}, \{A, B, C, D\})$.) Thus our answer is $\binom{12}{8}\frac{8!}{4!4!2!} = 17325$.

§ 11 AMC 12 2001/16

A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

§ 11.1 Solution

Since the sock must be put on the shoe, we can consider each sock and its corresponding shoe to be identical. Thus the answer is identical to the amount of ways to permute $AABBCC \cdots HH$, which is $\frac{16!}{2^8}$.

§ 12 2019 AMC 10A #17

A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

§ 12.1 Solution

Quick casework by leaving off one red, one green and one blue gives $\frac{8!}{3!\cdot 4!}+\frac{8!}{2!\cdot 2!\cdot 4!}+\frac{8!}{2!\cdot 3!\cdot 3!}=1260$.

§ 12.2 Clever Bijection

Note that we can put the last cube on top, and this will not generate more towers since there is only one way to put the last cube on top. So the answer is $\frac{9!}{4!3!2!}=2160$.

§ 13 MATHCOUNTS State 2020

Iris is playing a game that has a 5×5 gameboard like the one shown. The goal is to get her game piece from the square labeled $\star$ to the square labeled $\circ$ using a series of moves any positive integer number of squares up or any positive integer number of squares to the right. Note that moving two squares up in a single move is different than moving two squares up in two moves. How many unique sequences of moves can Iris make to get her game piece from $\star$ to $\circ$?

§ 13.1 Solution

Okay, one way you can do this is to just manually find the number of ways to go from one square to the next.[1] I am showing another way.

Notice that you need to do move 4 units right and 4 units up to reach that square in any ways. You do have the choice how you want to select ways for reaching 4 units. You can partition 4 into these ways:

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+1+1 \\ &= 2+2 \\ &= 1+1+1+1 \end{aligned}$$

Let's denote 1 unit right as R_1 and up as U_1 and similarly for others. So there are 5 kind of ways to reach 4 units right $\{R_1R_1R_1R_1, R_1R_3, R_2R_1R_1, R_4, R_2R_2\}$ and its permutations and same for 4 units up.

Now if you want to do the standard MISSISSIPPI method, you will need to pair up R's and U's and calculate the number of ways for each of them and then sum up everything.

We have two sets $\{R_1R_1R_1R_1, R_1R_3, R_2R_1R_1, R_4, R_2R_2\}$ and $\{U_1U_1U_1U_1, U_1U_3, U_2U_1U_1, U_4, U_2U_2\}$. We have the strings

$R_1R_1R_1R_1U_1U_1U_1U_1 \to 70$	$R_1R_3U_1U_1U_1U_1 \to 30$	$R_2R_1R_1U_1U_1U_1U_1 \to 105$	$R_4U_1U_1U_1U_1 \to 5$	$R_2R_1R_1U_4 \to 12$
$R_1R_1R_1R_1U_1U_3 \to 30$	$R_1R_3U_1U_3 \to 24$	$R_2R_1R_1U_1U_3 \to 60$	$R_4U_1U_3 \to 6$	$R_2R_2U_1U_3 \to 12$
$R_1R_1R_1R_1U_2U_1U_1 \to 105$	$R_1R_3U_2U_1U_1 \to 60$	$R_2R_1R_1U_2U_1U_1 \to 180$	$R_4U_2U_1U_1 \to 12$	$R_2R_2U_2U_1U_1 \to 30$
$R_1R_1R_1R_1U_4 \to 5$	$R_1R_3U_4 \to 6$	$R_2R_2U_1U_1U_1U_1 \to 15$	$R_4U_4 \to 2$	$R_2R_2U_4 \to 3$
$R_1R_1R_1R_1U_2U_2 \to 15$	$R_1R_3U_2U_2 \to 12$	$R_2R_1R_1U_2U_2 \to 30$	$R_4U_2U_2 \to 3$	$R_2R_2U_2U_2 \to 6$

That was a lot of cases. Summing up everything, we get 838 ways.[2]

[1] Try to approach this from backwards.
[2] Well, I am quite sure there must be a better way.

§ 14 Falling numbers

How many 4 digit falling numbers are there? (A falling number is a number whose last digit is strictly smaller than its second-to last digit, and so on, such as 4321.

§ 14.1 Solution

You can choose 4 digits in $\binom{10}{4}$ ways and for every selection of 4 numbers, there is going to be 1 way to order them in a strictly decreasing way. Hence, the answer is 210.

§ 15 AMC 10A 2021/21

In how many ways can the sequence $1, 2, 3, 4, 5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

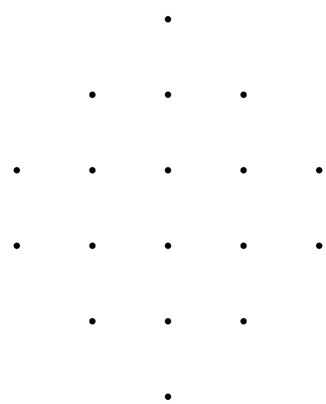
§ 15.1 Solution

§ 16 HMMT Feb. Guts 2012/20

Let n be the maximum numbers of bishops that can be placed on the squares of a 6×6 chessboard such that no two bishops are attacking each other. Find $n + k$.[3]

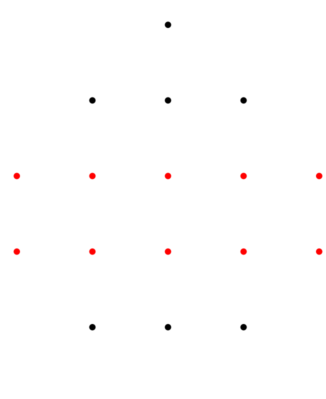
§ 16.1 Solution

We can color the chessboard black and white; note that black squares and white squares are independent and symmetric. We only care about black squares. To make everything clearer, rotate the chessboard by 45° to get the following diagram.



Each black dot represents a black square.

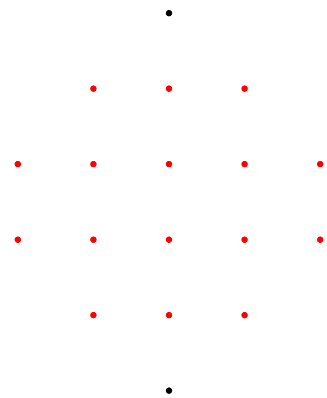
Notice that each column can have at most one bishop. Since 5 bishops is possible (this is incredibly easy to verify), we now have to construct all possible arrangements of 5 bishops. Note that the outer two columns must have bishops, and there are 2 ways to choose the bishop for the left outer column which locks the position of the bishop on the right outer column. Now we can delete the middle two rows.



Red dots cannot have a bishop on them, because the outer two columns already have bishops.

By similar reasoning to above, there are two ways to choose the bishops in the remaining outer rows. So the final row is just left with two dots, so there's 2 more choices there. Thus, there are $2^3 = 8$ total ways to arrange the black bishops.

[3] Rules of chess



Red dots cannot have a bishop on them, because all columns except the center already have bishops.

Now remember that there are also white bishops. So $n = 2 \cdot 5$ and $k = 8^2$, and $n + k = 2 \cdot 5 + 8^2 = 10 + 64 = 74$.

§ 17 AIME 1990/8

In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:

- ✦ The marksman first chooses a column from which a target is to be broken.
- ✦ The marksman must then break the lowest remaining target in the chosen column.

If the rules are followed, in how many different orders can the eight targets be broken?

§ 17.1 Solution

We can treat each column's targets as indistinguishable as they correspond to exactly one order of breaking, so the problem is the same as permuting $AAABBBCC$. Thus the answer is $\frac{8!}{3!3!2!} = 560$.

§ 18 AIME I 2010/7

Define an ordered triple (A, B, C) of sets to be minimally intersecting if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Find the remainder when N is divided by 1000.

Note: $|S|$ represents the number of elements in the set S.

§ 18.1 Solution

We first pick the common elements. There are $7 \cdot 6 \cdot 5$ ways to do this. Then we have 4 terms left, each with 4 options. We can either put it in set A, B, C, or we can put it in no set. So the amount of ways to do this is $7 \cdot 6 \cdot 5 \cdot 4^4 = 53760$, so the answer is 760.