Complex Numbers

Dennis Chen

GQV

We discuss geometric interpretations of complex numbers.

§ 1 Triangle Centers

We can describe triangle centers with complex coordinates. The most obvious one is the centroid.

Theorem 1 (Midpoint) The midpoint of a and b is $\frac{a+b}{2}$.

Proof: Convert to Cartesian Coordinates. ■

Theorem 2 (Centroid) The centroid of a, b, c is $\frac{a+b+c}{3}$.

For the rest of the centers, (ABC) is the unit circle **centered at the origin.** (In other words, $O = 0$.)

Theorem 3 (Circumcenter) The circumcenter is 0.

Proof: Because I said so. ■

Theorem 4 (Orthocenter) The orthocenter is $a + b + c$.

Proof: Note that $OH = 3OG$ due to the Euler Line. Since $O = 0$ and $G = \frac{1}{3}(a+b+c)$, $H = a+b+c$. ■

Remember that addition of complex numbers is a translation, and multiplication of complex numbers is a spiral similarity (a rotation and a dilation about the same point) around the origin. This means that given some conditions, we can equate them to other (more manageable) conditions pretty easily.

Example 1 (AMC 12B 2019/25) Let $ABCD$ be a convex quadrilateral with $BC = 2$ and $CD = 6$. Suppose that the centroids of $\triangle ABC, \triangle BCD$, and $\triangle ACD$ form the vertices of an equilateral triangle. What is the maximum possible value of $ABCD$?

Solution: We claim that $\triangle DAB$ is equilateral. To prove this, let the vertices have complex coordinates a, b, c, d. Then the centroids are $\frac{a+b+c}{3}, \frac{b+c+d}{3}, \frac{a+c+d}{3}$. The fraction is annoying, so we multiply by 3. So $a+b+c, b+c+d, a+c+d$ form equilateral triangles. Then subtract $a+b+c+d$ and we see that $-d, -a, -b$ form equilateral triangles. Multiplying by -1, we see that d, a, b form an equilateral triangle, implying that $\triangle DAB$ is equilateral.

Let $BCD = \theta$. Then

$$[ABCD] = [ABD] + [BCD] = \frac{\sqrt{3}(\sqrt{2^2 + 6^2 - 24\cos\theta})^2}{4} + \frac{1}{2} \cdot 2 \cdot 6 \cdot \sin\theta$$

$$[ABCD] = \sqrt{3}(10 - 6\cos\theta) + 6\sin\theta = 10\sqrt{3} + 6(\sin\theta - \sqrt{3}\cos\theta).$$

Since

$$10\sqrt{3} + 6(\sin(180 - \theta) + \sqrt{3}\cos(180 - \theta)) \leq 10\sqrt{3} + 6\sqrt{(1^2 + \sqrt{3}^2)},$$

our answer is $10\sqrt{3} + 12$.

§ 2 Complex Criterion

We introduce the perpendicularity, collinearity, concyclic, and equilateral triangle criterion in complex numbers.

Theorem 5 (Perpendicular Condition) For points A, B, C, D, $AB \perp CD$ if and only if $\frac{d-c}{b-a}$ is a purely imaginary number.

Proof: This implies the argument of $\frac{d-c}{b-a}$ is $\pm\frac{\pi}{2}$. ■

Theorem 6 (Collinear Condition) Points A, B, C are collinear if and only if $\frac{c-a}{c-b}$ is real.

Proof: This implies that the argument of $\frac{c-a}{c-b}$ is 0 or π. ■

Theorem 7 (Concyclic Condition) The complex number z is concyclic with z_1, z_2, z_3 if and only if $\frac{z_3-z_1}{z_2-z_1} \cdot \frac{z-z_2}{z-z_3}$ is real.

Proof: All angles are directed.

This is the same as claiming the argument of this product is 0 or π.

The argument of $\frac{z_3-z_1}{z_2-z_1}$ is $\measuredangle z_2 z_1 z_3$ and the argument of $\frac{z-z_2}{z-z_3}$ is $\measuredangle z_3 z z_2$. For the points to be concyclic, either $\measuredangle z_2 z_1 z_3 + \measuredangle z_3 z z_2 = 0$ or $\measuredangle z_2 z_1 z_3 + \measuredangle z_3 z z_2 = \pi$, as desired. ■

Here's a direct example of a problem using this condition.

Example 2 (AIME I 2017/10) Let $z_1 = 18 + 83i$, $z_2 = 18 + 39i$, and $z_3 = 78 + 99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3-z_1}{z_2-z_1} \cdot \frac{z-z_2}{z-z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

Solution: This implies z lies on the circumcircle of $\triangle z_1 z_2 z_3$. To maximize the imaginary part, the real part must be the same as the circumcenter.

We can now ignore complex numbers and use Cartesian Coordinates.

We want to find the x coordinate of the circumcenter of $(18, 83)$, $(18, 39)$, $(78, 99)$. The y coordinate is $\frac{83+39}{2} = 61$, so the circumcenter must satisfy $(x - 18)^2 + (61 - 39)^2 = (x - 78)^2 + (99 - 61)^2$, implying $x = 56$, which is our answer.

Theorem 8 (Equilateral Triangles) Complex numbers a, b, c form an equilateral triangle if and only if $a^2 + b^2 + c^2 = ab + bc + ca$.

Proof: We prove this for complex numbers $0, b - a, c - a$. Note

$$(b-a)^2 + (c-a)^2 = (b-a)(c-a) \Leftrightarrow a^2 + b^2 + c^2 = ab + bc + ca.$$

Then let $b - a = x$ and $c - a = y$.

Then note $x^2 + y^2 = xy$ implies $x = \operatorname{cis}(\pm 60°)y$. ■

§ 3 Vectors

Vectors can be used similarly to complex numbers. They have a few unique uses that are more convenient than complex numbers. Here's an obvious (but useful) theorem.

Theorem 9 (Polygon) Given points $A_1, A_2, \ldots, A_n$,

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_nA_1} = 0.$$

Example 3 (IMO 2005/1) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Solution: Note that

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2B_1} + \overrightarrow{B_1B_2} + \overrightarrow{B_2C_1} + \overrightarrow{C_1C_2} + \overrightarrow{C_2A_1} = 0.$$

Since $\overrightarrow{A_1A_2}$, $\overrightarrow{B_1B_2}$, and $\overrightarrow{C_1C_2}$ make angles of 120° with each other (they are parallel to sides of an equilateral triangle),

$$\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 0.$$

This implies that

$$\overrightarrow{A_2B_1} + \overrightarrow{B_2C_1} + \overrightarrow{C_2A_1} = 0,$$

which implies that they form an equilateral triangle. Thus $\triangle A_1A_2B_1 \cong \triangle B_1B_2C_1 \cong \triangle C_1C_2A_1$. Thus $\triangle A_1B_1C_1$ is equilateral and the lines concur in the center of the triangle.

§ 4 Problems

Minimum is [40✎]. Problems with the ◇ symbol are required.

> "It is weakness that brought us this fear. It is weakness that made us strong."
>
> My Home Hero

[2✎] **Problem 1** Consider $\triangle ABC$ with circumcenter O, orthocenter H, and centroid G. Prove that any one of the four imply the other three:

1. $O = H$
2. $H = G$
3. $G = O$
4. $\triangle ABC$ is equilateral.

[2✎] **Problem 2** Consider convex non-self intersecting quadrilateral $ABCD$, and let the midpoints of AB, BC, CD, DA be P, Q, R, S.

1. Prove that $PQRS$ is a parallelogram.
2. Prove that $PQRS$ is a rhombus if and only if $AC = BD$.

[3✎] **Problem 3** (AIME II 2005/9) For how many positive integers n less than or equal to 1000 is $(\sin t + i\cos t)^n = \sin nt + i\cos nt$ true for all real t?

[3✎] **Problem 4** (AIME I 2020/8) A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates $60°$ counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to the point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[3✎] **Problem 5** (Napoleon's Theorem) Let equilateral triangles $\triangle ABR$, $\triangle BCP$, and $\triangle CAQ$ be constructed externally from $\triangle ABC$. Prove their centers form an equilateral triangle.

[3◇] **Problem 6 (AMC 12B 2020/23)** How many integers $n \geq 2$ are there such that whenever $z_1, z_2, ..., z_n$ are complex numbers such that

$$|z_1| = |z_2| = ... = |z_n| = 1 \text{ and } z_1 + z_2 + ... + z_n = 0,$$

then the numbers $z_1, z_2, ..., z_n$ are equally spaced on the unit circle in the complex plane?

[4✎] **Problem 7** (AMC 12A 2019/21) Let

$$z = \frac{1+i}{\sqrt{2}}.$$

What is

$$\left(z^{1^2} + z^{2^2} + z^{3^2} + \cdots + z^{12^2}\right) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \frac{1}{z^{3^2}} + \cdots + \frac{1}{z^{12^2}}\right)?$$

[4] **Problem 8 (AIME II 2012/6)** Let $z = a + bi$ be the complex number with $|z| = 5$ and $b > 0$ such that the distance between $(1 + 2i)z^3$ and z^5 is maximized, and let $z^4 = c + di$. Find $c + d$.

[4] **Problem 9** (AIME 1994/8) The points $(0, 0)$, $(a, 11)$, and $(b, 37)$ are the vertices of an equilateral triangle. Find the value of ab.

[4] **Problem 10** (EGMO 2013/1) The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled.

[6] **Problem 11 (CMIMC Algebra 2016/6)** For some complex number ω with $|\omega| = 2016$, there is some real $\lambda > 1$ such that ω, ω^2, and $\lambda\omega$ form an equilateral triangle in the complex plane. Then, λ can be written in the form $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers and b is squarefree. Compute $\sqrt{a+b+c}$.

[6] **Problem 12** (AIME I 2019/12) Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

[6] **Problem 13** (AIME II 2014/10) Let z be a complex number with $|z| = 2014$. Let P be the polygon in the complex plane whose vertices are z and every w such that $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$. Then the area enclosed by P can be written in the form $n\sqrt{3}$, where n is an integer. Find the remainder when n is divided by 1000.

[9] **Problem 14 (AIME II 2012/14)** Complex numbers a, b and c are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b, and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

[13] **Problem 15** (AIME I 2017/15) The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.