Special Functions

math1

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There are multiple types of function-related problems that could appear on the AIME. Some functions will be defined within the problem, while others will be in terms of *special functions* such as floor, ceiling, or absolute value functions. Here, we explore these kinds of problems and how to approach them. We begin with some helpful terminology.

Monotonically Increasing/Decreasing. We say that a function f defined over the reals is **monotonically increasing** if, for all reals a, b with $a < b$, $f(a) \leq f(b)$. Similarly, f is **monotonically decreasing** if, for all reals a, b with $a < b$, $f(a) \geq f(b)$.

Strictly Increasing/Decreasing. We say that a function f defined over the reals is **strictly increasing** if, for all reals a, b with $a < b$, $f(a) < f(b)$. Similarly, f is **strictly decreasing** if, for all reals a, b with $a < b$, $f(a) > f(b)$.

1 Floors and Ceilings

Let's begin with some definitions.

Floor Function. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

Similarly, we define the ceiling function.

Ceiling Function. Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x.

It is also useful to define the fractional part of a number. There are actually conflicting definitions for this, mainly arising from the issues with negative numbers, but we will use the below definition throughout this handout. Most math competitions also use the below definition, and they usually explicitly define the floor, ceiling, and fractional part functions within the problem itself.

Fractional Part. Let $\{x\} = x - \lfloor x \rfloor$.

To see these functions in action, we present some examples.

Example. We have

$$\lfloor 5 \rfloor = 5, \ \lfloor 3.4 \rfloor = 3, \ \lfloor -2.6 \rfloor = -3,$$

and

$$\lceil 5 \rceil = 5, \ \lceil 3.4 \rceil = 4, \ \lceil -2.6 \rceil = -2,$$

and

$$\{5\} = 0, \ \{3.4\} = 0.4, \ \{-2.6\} = 0.4.$$

Floor and ceiling function problems come in various flavors, so it's quite difficult to group them under one umbrella. However, there are a few general heuristics we can keep in mind; these will be covered in each subsection of this section.

1.1 Analyzing near-integer inputs

One useful technique is to look at what happens to the function *near* integer inputs. The reason for this is the following: floor and ceiling functions usually "jump" (or, more formally, are discontinuous) right before integer inputs, so analyzing what happens right before or after that jump is a useful way to better understand the behavior of a function.

Example (math1's Computer Science Teacher). Determine whether the following statement is true for all positive real x:

$$\lfloor \log_2(x) + 1 \rfloor = \lceil \log_2(x+1) \rceil.$$

Solution. Let k be a positive integer. Suppose that $0 < \varepsilon < \min(1, k - \log_2(2^k - 1))$. Note that $2^k < 2^{k-\varepsilon} + 1$, so we consider

$$\lfloor \log_2(2^{k-\varepsilon}) + 1 \rfloor = \lfloor k - \varepsilon + 1 \rfloor = k = \lceil \log_2(2^k) \rceil < \lceil \log_2(2^{k-\varepsilon} + 1) \rceil,$$

which is a counter-example to the given statement.

Remark: The brevity of the solution is deceiving; finding a range for ε requires some care. The first step is to convince yourself that the answer is more likely false than true; adding 1 on the inside of the logarithm on the RHS is bizarre, and it feels like we can find a counter-example at edge cases.

To continue, we analyze how these functions behave when we look at inputs near "jumps". Those jumps occur on the LHS at powers of 2, while on the RHS they occur at numbers 1 less than a power of 2. Perhaps we can find a good counter-example between those two; this motivates $x = 2^{k-\varepsilon}$.

To prove that there is a counter-example, we would like $k < \lceil \log_2(2^{k-\varepsilon} + 1) \rceil$. Since $k = \log_2(2^k)$, it would be great if we could get $2^k < 2^{k-\varepsilon} + 1$. We can solve for such values of ε:

$$\begin{aligned} 2^k &< 2^{k-\varepsilon} + 1 \\ 2^k - 1 &< 2^{k-\varepsilon} \\ \log_2(2^k - 1) &< k - \varepsilon \\ \varepsilon &< k - \log_2(2^k - 1). \end{aligned}$$

The rest follows from the solution.

Example (AIME 1985/10). How many of the first 1000 positive integers can be expressed in the form

$$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor,$$

where x is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z?

Solution. We try to understand this function by looking at inputs where the function jumps. There are coefficients inside the floor function, so we look at inputs near and at $\frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}$, and 1.

Let $f(x) = \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor$. For a very small value of $0 < \varepsilon < 0.001$, we have

$$f\left(\frac{1}{8} - \varepsilon\right) = \left\lfloor \frac{1}{4} - 2\varepsilon \right\rfloor + \left\lfloor \frac{1}{2} - 4\varepsilon \right\rfloor + \left\lfloor \frac{3}{4} - 6\varepsilon \right\rfloor + \lfloor 1 - 8\varepsilon \rfloor = 0.$$

Now, what happens exactly at the jumping point?

$$f\left(\frac{1}{8}\right) = \left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{3}{4} \right\rfloor + \lfloor 1 \rfloor = 1.$$

Nothing is too special here; the $x = \frac{1}{6} - \varepsilon, \frac{1}{6}$ case is similar in that f increases by 1. However, this gives us a hint: if there is an input where the function somehow skips over integers, we will be able to count how many integers were skipped to solve the problem. Indeed,

$$f\left(\frac{1}{4} - \varepsilon\right) = 2 \quad \text{and} \quad f\left(\frac{1}{4}\right) = 4.$$

Note that we skipped over an integer, 3. Since f is monotonically increasing, f can *never* equal 3. Additionally,

$$f\left(\frac{1}{2} - \varepsilon\right) = 6 \quad \text{and} \quad f\left(\frac{1}{2}\right) = 10,$$
$$f(1 - \varepsilon) = 16 \quad \text{and} \quad f(1) = 20.$$

In both of these cases, we have skipped over 3 integers.

To generalize this phenomenon, notice that when x has a denominator of 4, f skips over 1 integer, and when x has a denominator of 2 or is an integer, f skips over 3 integers. Now, all we have to do is count the number of integers skipped and subtract that number from 1000.

Note that the largest x with denominator 4 *unsimplified* such that $f(x) \le 1000$ is $x = \frac{200}{4}$. In particular, we have 100 inputs that, when simplified, have a denominator of 4, and we have 100 inputs that, when simplified, have a denominator of 2 or are integers. Thus, the answer is

$$1000 - 1 \cdot 100 - 3 \cdot 100,$$

which is **600**.

Remark: It is important to understand why the function doesn't skip over integers at any other type of input (and would definitely be required on an olympiad), so we leave this as an exercise.

2 Absolute Value

Sometimes, the easier absolute value problems simply require you to get rid of the absolute values.

Example (AIME 1983/2). Let $f(x) = |x - p| + |x - 15| + |x - p - 15|$, where $0 < p < 15$. Determine the minimum value taken by $f(x)$ for x in the interval $p \le x \le 15$.

Solution. Note that $|x - p| = x - p$, $|x - 15| = 15 - x$, and $|x - p - 15| = 15 - (x - p) = 15 - x + p$. Thus, $f(x) = 30 - x$ for $x \in [p, 15]$; in particular, the minimum over this interval is $f(15) = $ **15**.

More involved absolute value problems generally can be placed in two categories: casework and graphing. We present examples of both.

2.1 Casework

2.2 Graphing

Graphing is a useful tool that can be applied far more extensively than just in absolute value problems. However, these applications are too detailed to cover here and will possibly be covered in a separate unit. Here, we will explore how they apply specifically to absolute value problems.

3 Problem-Defined Functions

This section does not involve any theory; instead, the key concept is simply to **experiment** to better understand the given function.

Example (2021 AMC 10A/18). Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Furthermore, suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\dfrac{17}{32}$ **(B)** $\dfrac{11}{16}$ **(C)** $\dfrac{7}{9}$ **(D)** $\dfrac{7}{6}$ **(E)** $\dfrac{25}{11}$

Solution. Let's apply some wishful thinking. We want the function to be negative, but everything seems positive in the given conditions. So, can we somehow get a negative sign? Indeed, we can in the following way:

$$f(a \cdot b) - f(b) = f(a).$$

This is promising, since we have control over a. In particular, let's make b an arbitrary prime p so we only have to worry about a:

$$f(a \cdot p) - p = f(a).$$

Let's get rid of $a \cdot p$ now; let $a = \frac{k}{p}$ for some positive integer k:

$$f(k) - p = f\left(\frac{k}{p} \cdot p\right) - p = f\left(\frac{k}{p}\right).$$

Aha! If $f(k) < p$, then $f\left(\frac{k}{p}\right)$ is negative. We now look towards the answer choices to see if there are any prime denominators, and indeed, $\frac{25}{11}$ is one of the answer choices. Letting $k = 25$ and $p = 11$, we have

$$f\left(\frac{25}{11}\right) = f(25) - 11 = f(5) + f(5) - 11 = 5 + 5 - 11 = -1 < 0,$$

as required. So, the answer is $\boldsymbol{\frac{25}{11}}$.

Remark: Here, we used the classic themes of **wishful thinking** and **plugging in convenient values to make things disappear**. The latter is likely not new to those who have experience solving functional equations.

4 Problems

> "Do. Or do not. There is no try."
>
> Yoda

[4 ♟] Problem 1 (AIME 1984/12) A function f is defined for all real numbers and satisfies

$$f(2+x) = f(2-x) \qquad \text{and} \qquad f(7+x) = f(7-x)$$

for all real x. If $x = 0$ is a root of $f(x) = 0$, what is the least number of roots $f(x) = 0$ must have in the interval $-1000 \le x \le 1000$?

[6 ♟] Problem 2 (AIME I 2021/8) Find the number of integers c such that the equation

$$\left||20|x| - x^2| - c\right| = 21$$

has 12 distinct real solutions.

[6 ♟] Problem 3 (AMC 12A 2020/25) The number $a = \frac{p}{q}$, where p and q are relatively prime positive integers, has the property that the sum of all real numbers x satisfying

$$\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$$

is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is $p + q$?