Radical Axes

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§ 1 Power of a Point

We first extend the power of a point theorem to a definition.

Definition 1 (Power of a Point) The power of a point P with respect to circle ω with center O and radius r as $OP^2 - r^2$. We will denote this as $\mathcal{P}(P, \omega) = OP^2 - r^2$.

This yields the following corollary.

Fact 1 (Square of Tangent Line) The power of a point is the square of the length of the tangent line.

§ 2 Radical Axes

Definition 2 (Radical Axis) The radical axis of a pair of circles ω_1, ω_2 as the locus of points such that $\mathcal{P}(P, \omega_1) = \mathcal{P}(P, \omega_2)$.

Theorem 1 (Radical Axis Theorem) The radical axis of a pair of circles is a line.

Proof: We use coordinates to prove this.

Without loss of generality, let the center of ω_1 be $(0, 0)$ and let the center of ω_2 be $(x_0, 0)$. Denote the radii of ω_1, ω_2 as r_1, r_2, respectively. If the coordinates of P are (x, y), then $(x^2 + y^2) - r_1^2 = ([x - x_0]^2 + y^2) - r_2^2$. Rearranging, this yields $-r_1^2 = -2x_0x + x_0^2 - r_2^2$. This is the equation of a line, as desired. ■

A corollary that arises: if ω_1, ω_2 intersect at X, Y, then the radical axis is XY. This is because $\mathcal{P}(X, \omega_1) = 0 = \mathcal{P}(X, \omega_2)$ and $\mathcal{P}(Y, \omega_1) = 0 = \mathcal{P}(Y, \omega_2)$. Since two points are needed to determine a line, the proof is done.

Theorem 2 (Radical Center Theorem) Consider three circles $\omega_1, \omega_2, \omega_3$. Then their pairwise radical axes either concur or are all parallel.

Proof: Without loss of generality, let the radical axis of ω_1, ω_3 and the radical axis of ω_1, ω_2 intersect at P.[a] Then notice that $\mathcal{P}(P, \omega_1) = \mathcal{P}(P, \omega_3)$ and $\mathcal{P}(P, \omega_1) = \mathcal{P}(P, \omega_2)$, so $\mathcal{P}(P, \omega_2) = \mathcal{P}(P, \omega_3)$, implying that P lies on the radical axis of ω_2, ω_3, as desired. ■

[a] If they do not intersect, then all three radical axes are parallel; this is because the proof works regardless of which pair of radical axes you intersect as it is symmetric.

§ 3 Techniques

§ 3.1 Basic Techniques

Keeping this simple result in mind kills problems involving common chords and external tangents.

Fact 2 (Radical Axis Bisects External Tangent) Consider two circles ω_1, ω_2 that intersect at X, Y. Let one of their common external tangents intersect ω_1 at A and ω_2 at B. Then XY bisects AB.

Proof: Note that XY is the radical axis. Let XY intersect AB at M. Since M lies on the radical axis, $AM = BM$. ■

§ 3.2 Advanced Techniques

We introduce two powerful techniques - Linearity of Power and circles with radius 0.

Theorem 3 (Linearity of Power) The function $f(P) = \mathcal{P}(P, \omega_1) - \mathcal{P}(P, \omega_2)$ changes at a linear rate as P moves along a fixed line ℓ.

Proof: Without loss of generality, let ℓ be the x axis, let the equation of ω_1 be $(x-h_1)^2 + (y-k_1)^2 = r_1^2$, and let the equation of ω_2 be $(x-h_2)^2 + (y-k_2)^2 = r_2^2$.

Let the coordinates of P be $(x, 0)$. Then

$$f(P) = (r_1^2 - ((h_1 - x)^2 + k_1^2)) - (r_2^2 - ((h_2 - x)^2 + k_2^2))$$

$$f(P) = r_1^2 - r_2^2 - (x^2 - 2h_1x + h_1^2 + k_1^2) + (x^2 - 2h_2x + h_2^2 + k_2^2)$$

$$f(P) = r_1^2 - r_2^2 + h_2^2 - h_1^2 + k_2^2 - k_1^2 + x(2h_2 - 2h_1).$$

Since all variables except for x are constant, $f(P)$ varies linearly. ■

Here's a really silly example of it.

Example 1 (MAST Diagnostic 2021/12) In $\triangle ABC$, let the foot of B to AC be E and the foot of C to AB be F. Suppose that the circle through F centered at B is externally tangent to the circle through E centered at C at some point D. Let G be the midpoint of EF. Prove that DG is perpendicular to BC.

Solution: Let the circle centered at B be ω_1 and the circle centered at C be ω_2., and let $f(P) = \mathcal{P}(P, \omega_1) - \mathcal{P}(P, \omega_2)$. Then by Linearity of Power, we want to show that $f(F) + f(E) = 0$. Note that $f(F) = \mathcal{P}(F, \omega_1) - \mathcal{P}(F, \omega_2) = -\mathcal{P}(F, \omega_2) = -(FC^2 - CE^2) = -(BC^2 - BF^2 - CE^2) = BF^2 + CE^2 - BC^2$. Also note that $f(E) = \mathcal{P}(E, \omega_1) - \mathcal{P}(E, \omega_2) = \mathcal{P}(E, \omega_1) = EB^2 - BF^2 = BC^2 - CE^2 - BF^2$.

Thus by Linearity of Power, $f(G) = 0$, implying that G lies on the radical axis of ω_1, ω_2. Since the radical axis is the line through D perpendicular to BC, we are done.

This is all motivated by the fact that G should lie on the radical axis of ω_1, ω_2. The linearity of power solution comes quite naturally because of the right triangles created by the altitudes.

And here's an example of a problem solved using circles with radius 0.

Example 2 (Iran TST 2011/1) In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of BC and let D and E be the feet of the altitudes from C and B, respectively. Let K and L be the midpoints of ME and MD. If KL intersects the line through A parallel to BC in T, prove that $TA = TM$.

Solution: We claim that the line through A parallel to BC, MD, and ME are tangent to (ADE). (This is known as the Three Tangent Lemma.)

Proof: Let H be the orthocenter of $\triangle ABC$. Note that AH is the diameter of (ADE) as $\angle ADH = \angle AEH = 90°$.

Since AH is perpendicular to BC and the line through A is parallel to BC, it is a tangent.

To show ME is tangent, we show $\angle DEM = \angle DAE = \angle A$. Notice that $MB = MC = MD = ME$, since M is the center of $(BCDE)$.

Notice that

$$\angle DEM = \angle DEB + \angle BEM = \angle DEB + \angle EBM = \angle DEB + \angle EBC$$

$$\angle DEB + \angle EBC = 90° - \angle C + 90° - \angle B = \angle A,$$

proving that ME is tangent to (AEF) as desired. ■

Thus KL is the radical axis of (ADE) and the circle centered at M with radius 0. Since T lies on the radical axis and TA is tangent to (ADE), $TA = TM$.

§ 4 Problems

Minimum is [TBD✐]. Problems with the ♦ symbol are required.

[1✐] **Problem 1** Consider a point P with power 36 respective to a circle with center O. If $PO = 10$, find the radius of the circle.

[2✐] **Problem 2** Consider a coordinate plane with two circles tangent to the x axis at X, Y, respectively. If the circles intersect at P, Q, and $XY = 8$, is it possible for P to lie on $y = 3$ and Q to lie on $y = 12$?

[3✐] **Problem 3** (APMO 2020/1) Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

[3✐] **Problem 4** (EGMO 2019/4) Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

[4✐] **Problem 5** (IMO 2004/1) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

[6✐] **Problem 6** (ISL 2017/G1) Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

[6✐] **Problem 7** (Swiss TST 2018/3) Let ABC be a triangle with $\angle ABC \neq \angle BCA$. The inscribed circle k of the triangle ABC is tangent to the sides BC, CA, AB at points D, E, F respectively. The segment AD intersects k again at P. Let Q be the point of intersection of EF with the line perpendicular to AD passing through P. Let X, Y be the points of intersection of AQ with DE, DF respectively. Prove that A is the midpoint of the segment XY.

[7✐] **Problem 8** (ISL 2017 G5) Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.