Solutions to Basics of Geometry

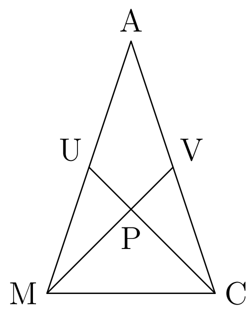
Dennis Chen

GPV1

Contents

§ 1 AMC 10A 2020/12

Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



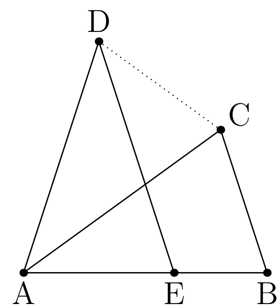
§ 1.1 Solution

Because the centroid splits the median in a 2 to 1 ratio, $PM = PC = 8$. Also note that the height from P to MC is $\frac{1}{3}$ of the height from A to MC, so $[AMC] = 3[PMC]$. Also note that $[PMC] = \frac{8 \cdot 8}{2} = 32$, so $[AMC] = 96$.

§ 2 Brazil 3rd Phase Level 2 2004/1

In the figure, ABC and DAE are isosceles triangles ($AB = AC = AD = DE$) and the angles BAC and ADE have measures $36°$.

1. Using geometric properties, calculate the measure of angle $\angle EDC$.
2. Knowing that $BC = 2$, calculate the length of segment DC.
3. Calculate the length of segment AC.



§ 2.1 Solution

The main idea is expressing angles in terms of differences of other angles.

1. Note that $\angle DAC = \angle DAE - \angle CAB = 72° - 36° = 36°$, and also note that $\angle EDC = \angle ADC - \angle ADE = 72° - 36° = 36°$.
2. Note that $\angle DAC = \angle CAB$ from our progress on the first part, so $DC = BC = 2$.
3. By the Law of Sines, $\frac{AC}{DC} = \frac{AC}{2} = \frac{\sin 72°}{\sin 36°} = \frac{2 \sin 36° \cos 36°}{\sin 36°} = 2 \cos 36°$, so $AC = 4 \cos 36° = \sqrt{5} + 1$.

§ 3 Unsourced

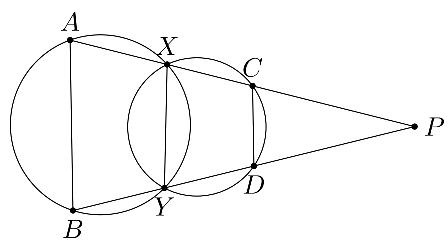
Let circles ω_1 and ω_2 intersect at X, Y. Let line ℓ_1 passing through X intersect ω_1 at A and ω_2 at C, and let line ℓ_2 passing through Y intersect ω_1 at B and ω_2 at D. If ℓ_1 intersects ℓ_2 at P, prove that $\triangle PAB \sim \triangle PCD$.

§ 3.1 Solution

We only prove it for the configuration below. We can use directed angles or casework to take care of all configurations.

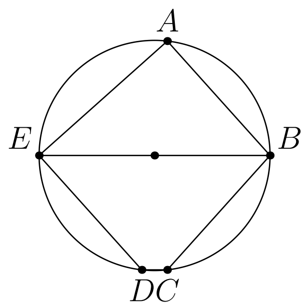
We want to prove that $\angle PAB = \angle PCD$. Note that

$$\angle PCD = \angle 180^\circ - \angle XCD = \angle XYD = 180^\circ - \angle XYB = \angle XAB = \angle PAB.$$



§ 4 AMC 10B 2011/17

In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



§ 4.1 Solution

Note that $\angle EAB = 90°$, as AB is a diameter. Thus $\angle AEB = 40°$ and $\angle ABE = 50°$. Since AB and DE are parallel, the measures of the arcs must be the same, so $\angle BED = 50°$. By the cyclic quadrilateral criterion, $\angle BCD = 180° - \angle BED = 130°$.

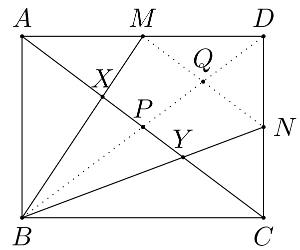
Note that the fact that DC is parallel to EB is irrelevant, other than determining the configuration.

§ 5 Dennis Chen

Consider rectangle $ABCD$ with $AB = 6$, $BC = 8$. Let M be the midpoint of AD and let N be the midpoint of CD. Let BM and BN intersect AC at X and Y respectively. Find XY.

§ 5.1 Solution

Let BD intersect AC and MN at P and Q, respectively. Note that $\triangle BXY \sim \triangle BMN$, and also note that the similarity taking $\triangle BXY$ to $\triangle BMN$ also takes P to Q. Since $BP = \frac{2}{3}BQ$, $XY = \frac{2}{3}MN = \frac{10}{3}$.

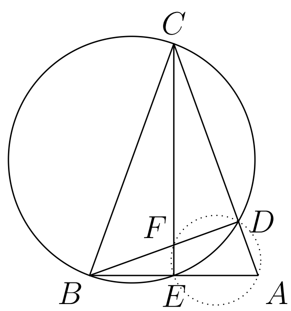


§ 6 AMC 10A 2019/13

Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40^\circ$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

§ 6.1 Solution

Notice that $\angle BFC = \angle DFE$. Now note that $\angle BEC = \angle BDC = 90^\circ$, so CE and BD are altitudes. Now note that $\angle FEA = \angle FDA = 90^\circ$, so $FEAD$ is also cyclic. Thus $\angle DFE = 180^\circ - \angle DAE = 110^\circ$.



§ 7 Miquel's Theorem

Consider $\triangle ABC$ with D on BC, E on CA, and F on AB. Prove that (AEF), (BFD), and (CDE) concur.

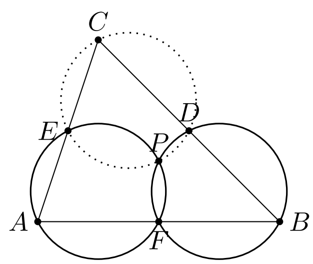
§ 7.1 Solution

We only prove the case where D, E, F are on segments BC, CA, AB. The other configurations can be proved with directed angles.

Let (AEF) and (BFD) intersect at P. We then prove that P lies on (CED). Note that

$$\angle DPE = 360^\circ - \angle EPF - \angle FPD = 360^\circ - (180^\circ - \angle A) - (180^\circ - \angle B) = \angle A + \angle B.$$

Since $\angle A + \angle DPE = 180^\circ$, $(CDPE)$ is cyclic.

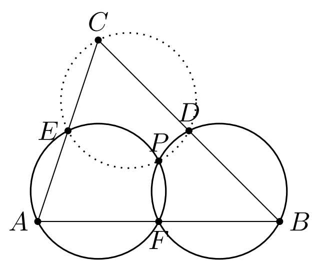


§ 8 Unsourced

Consider $\triangle ABC$ with D on segment BC, E on segment CA, and F on segment AB. Let the circumcircles of $\triangle FBD$ and $\triangle DCE$ intersect at $P \neq D$. If $\angle A = 50^\circ, \angle B = 35^\circ$, find $\angle DPE$.

§ 8.1 Solution

By Miquel's Theorem, $(CDPE)$ is cyclic, so $\angle DPE = 180^\circ - \angle C = 85^\circ$.



§ 9 AIME II 2018/4

In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \dfrac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

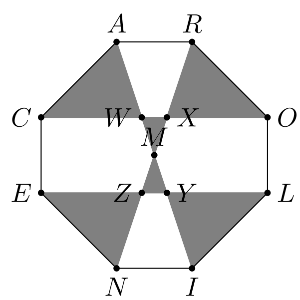
§ 9.1 Solution

Say that the center of $CAROLINE$ is M, and let MA and MR intersect CO at W and X, and let MI and MN intersect LE at Y and Z.

By similar triangles, $WX = \frac{AR}{3} = \frac{1}{3}$, so $[MWX] = \frac{1}{2} \cdot (\frac{1}{3} \cdot \frac{1}{2}) = \frac{1}{12}$. This implies that $CW = \frac{CO - WX}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$, so $[CAW] = \frac{1}{2} \cdot (\frac{4}{3} \cdot 1) = \frac{2}{3}$. Now note

$$[CORNELIA] = 4[CAW] + 2[WMX] = 4 \cdot \frac{2}{3} + 2 \cdot \frac{1}{12} = \frac{17}{6},$$

so the answer is $17 + 6 = 23$.

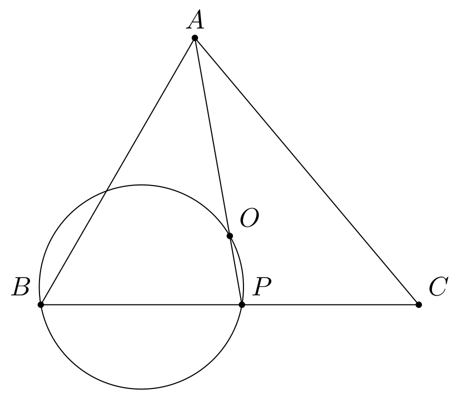


§ 10 Brazil 3rd Phase Level 2 2007/1

Let ABC be a triangle with circumcenter O. Let P be the intersection of straight lines BO and AC and ω be the circumcircle of triangle AOP. Suppose that $BO = AP$ and that the measure of the arc OP in ω, that does not contain A, is $40°$. Determine the measure of the angle $\angle OBC$.

§ 10.1 Solution

Note that $BO = AO = AP$, so $\triangle AOP$ is isosceles. Thus $\angle OAP = 20°$, implying $\angle AOC = 140°$, and $\angle AOP = \angle APO = 80°$, implying that $\angle AOB = 100°$. So $\angle OBC = 360° - 140° - 100° = 120°$, or $\angle OBC = 30°$.

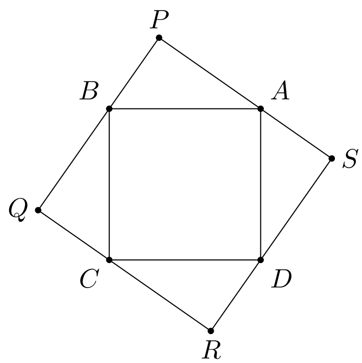


§ 11 Unsourced

Consider square $ABCD$ and some point P outside $ABCD$ such that $\angle APB = 90°$. Prove that the angle bisector of $\angle APB$ also bisects the area of $ABCD$.

§ 11.1 Solution

Let Q, R, S be the rotations of P about O by $90°, 180°, 270°$ counterclockwise. Note that PR is the angle bisector of $\angle APB$ and PR bisects the area of $[PQRS]$. Since the area we added to both halves of $ABCD$ is the same, PR also bisects $ABCD$.

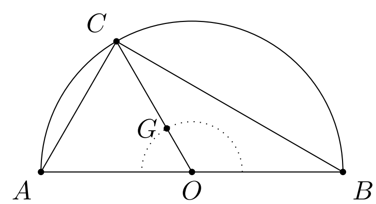


§ 12 AMC 10B 2018/12

Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

§ 12.1 Solution

Let O be the center of the circle and G be the centroid of $\triangle ABC$. Since O is also the midpoint of AB and thus lies on CG, we're motivated to make use of the $2:1$ ratio. CO always has length 12, so it follows that GO always has length 4. This means that the locus of G is a circle with center O and radius 4 by definition, so the area is $16\pi \approx 50$.

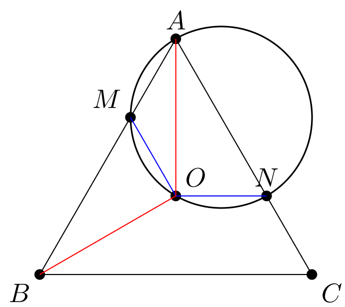


§ 13 Formula of Unity Qualifying Round Grade 11 2018/4

A point O is the center of an equilateral triangle ABC. A circle that passes through points A and O intersects the sides AB and AC at points M and N respectively. Prove that $AN = BM$.

§ 13.1 Solution

Note that $\angle MBO = 30^\circ = \angle NAO$, $\angle ANO = 180^\circ - \angle AMO = \angle BMO$, and $AO = BO$, so $\triangle BMO \cong \triangle ANO$. Thus $AN = BM$.



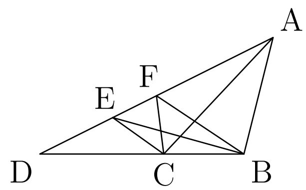
§ 14 AMC 10A 2021/17

Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. GIven that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

§ 14.1 Solution

§ 15 Memorial Day Mock AMC 10 2018/21

In the following diagram, $m\angle BAC = m\angle BFC = 40^\circ$, $m\angle ABF = 80^\circ$, and $m\angle FEB = 2m\angle DBE = 2m\angle FBE$. What is $m\angle ADB$?



§ 15.1 Solution

Since $\angle BAC = \angle BFC = 40^\circ$, $BAFC$ is cyclic. Now let $\angle EDB = x$. Since $\angle FEB = 2x$, $\angle EDB + \angle EBD = 2x$. This implies that $\angle EBD = x$, or that $ED = EB$.

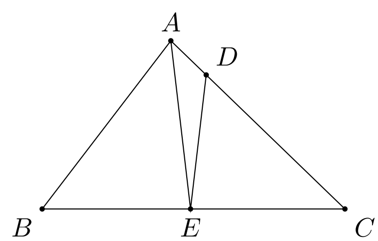
Now look at $\triangle ADB$. Note that $\angle ADB = x$, $\angle DAB = \angle FAC + \angle CAB = 2x + 40^\circ$, and $\angle ABD = \angle ABF + \angle FBC = 80^\circ + x$, so $x + 2x + 40^\circ + 80^\circ + x = 5x + 120^\circ = 180^\circ$, implying that $x = 12^\circ$.

§ 16 FARML 2012/6

In triangle ABC, $AB = 7$, $AC = 8$, and $BC = 10$. D is on AC and E is on BC such that $\angle AEC = \angle BED = \angle B + \angle C$. Compute the length AD.

§ 16.1 Solution

Angle chase to find $\triangle ABC \sim \triangle EDC \sim \triangle EBA$. So $BE = 7 \cdot \frac{7}{10} = \frac{49}{10}$, implying $CE = 10 - \frac{49}{10} = \frac{51}{10}$, and $CD = \frac{10}{8} \cdot \frac{51}{10} = \frac{51}{8}$, implying $AD = 8 - \frac{51}{8} = \frac{13}{8}$.



§ 17 USAJMO 2020/4

Let $ABCD$ be a convex quadrilateral inscribed in a circle and satisfying $DA < AB = BC < CD$. Points E and F are chosen on sides CD and AB such that $BE \perp AC$ and $EF \parallel BC$. Prove that $FB = FD$.

§ 17.1 Solution

We outline the solution, which is motivated from trying to construct circles.

1. Prove that $FB = FE$.

2. Prove that $(AFOED)$ is cyclic, where O is the circumcenter of $(ABCD)$.

To prove the first assertion, note that $\overline{BC} \parallel \overline{EF}$, $\angle FEB = \angle CEB$, and since $\triangle ABC$ is isosceles, $\angle ABE = \angle CBE$. Thus $\angle ABE = \angle FBE = \angle FEB$, so $FB = FE$.

To prove the second assertion, note that $\angle FAD = \angle BAD = 180^\circ - \angle BCD = 180^\circ - \angle FED$, so A lies on (FED). Now note that $\angle AOD = 2\angle ACD$ and

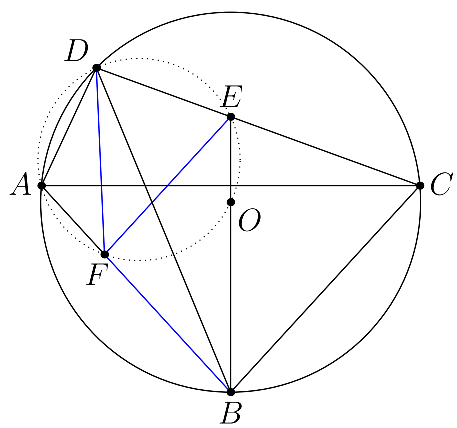
$$\angle AED = 180^\circ - \angle AEC = 180^\circ - 2\angle AEB = \angle 180^\circ - 2(90^\circ - \angle CAE) = 2\angle CAE = 2\angle ACD,$$

so O also lies on $(AFED)$.

To finish, note that $\angle BFE = 180^\circ - 2\angle FBE = 2\angle BCA$, implying that

$$\angle DFE = 180^\circ - \angle AFD - \angle BFE = 180^\circ - \angle AOD - 2\angle ACD =$$

$$180^\circ - 2\angle ACD - 2\angle BCA = 180^\circ - 2\angle BCD = 180^\circ - \angle BOD = \angle EOD = \angle EFD.$$



§ 18 MAST Diagnostic 2020

Consider $\triangle ABC$ with D on line BC. Let the circumcenters of $\triangle ABD$ and $\triangle ACD$ be M, N, respectively. Let the circumcircle of $\triangle MND$ intersect the circumcircle of $\triangle ACD$ again at $H \neq D$. Prove that A, M, H are collinear.

§ 18.1 Solution

It is obvious that $\triangle AMN \sim \triangle DMN$, and note that $\triangle AMN \sim \triangle ABC$ since $\angle AMN = \frac{1}{2}\angle AMD = \angle ABC$. Now note that $\measuredangle DNM = \measuredangle DHM$ and $\measuredangle AHD = \measuredangle ACD$ by cyclic quadrilaterals and $\measuredangle DNM = \measuredangle DCA$ by similar triangles, so $\measuredangle DHM = \measuredangle DHA$.