States

Dennis Chen

CQU

Independent probability problems are very easy. Dependent probability problems are not so easy. We introduce some techniques to solve dependent probability problems by considering the possible states of the problem.

§ 1 Probability

States are hard to explain without examples, so we will start with them first before explaining the theory.

Example 1 You are flipping a fair coin. What is the probability you get three heads in a row before you get two tails?

This one isn't hard (and probably doesn't even need states). Thus we present an example that doesn't need states to make states easier to grasp.

Solution 1: We start out with two attempts to flip three heads in a row, which succeeds with probability $\frac{1}{8}$. If we fail, that means we've flipped tails and lose one of our two attempts.

The amount of ways to succeed are with one or two attempts. With one attempt, the probability is $\frac{1}{8}$ and with two, the probability is $\frac{7}{8} \cdot \frac{1}{8}$. Thus the total probability is $\frac{15}{64}$.

Now let's do this with states!

Solution 2: Let P_n be the probability of success with n tails flipped. We want to find P_0. Notice that $P_0 = \frac{1}{8} \cdot 1 + \frac{7}{8}P_1$, as we have a $\frac{1}{8}$ chance of going to the success case. By states, $P_1 = \frac{1}{8} \cdot 1 + \frac{7}{8} \cdot 0$.

The rest of the solution is just arithmetic.

Just because we don't succeed if we flip tails at P_0 doesn't mean we failed. You don't *fail* until there's literally no way to succeed.[1] Flipping tails at P_0 only sends us to P_1.

States are hard and can get algebraically involved even if done right. The unofficial mantra is *keep it simple, stupid*.[2]

Theorem 1 (Probability in States) Consider the probability of some event occurring at an initial state S. Let the probability that you are sent from S to S_i be x_i, and let the probability of success at S_i be P_i. Then the probability of success at S is $\sum_{i=1}^{\infty} P_i x_i$.

A couple of reminders:

1. Remember that $\sum_{i=1}^{\infty} x_i = 1$.
2. If S_i doesn't exist or can't happen, then set x_i as 0.

[1] Which occurs when we flip tails at P_1.

[2] I've seen people try to define states of $P_{(H,T)}$, where H is the amount of heads flipped in a row and T is the amount of tails flipped.

3. For success and failure cases, set P_i as 1 and 0, respectively.
4. Most importantly, **this theorem applies to all** S_i. Since the success and failure states will eventually appear, you can use this to set a system of equations and solve for the probability of success at S.
5. It's okay to complementary count. The success and failure states are arbitrary. One man's success is another's failure.

§ 2 Expected Value

Let's talk about expected value with a simple question.

Example 2 On average, how many times do you have to flip a coin until it lands heads?

The first (and most naive) method one could think of is creating a summation.[3] This summation would look like $\sum_{n=1}^{\infty} \frac{n}{2^n}$. You could evaluate the summation to get 2.

We should notice that 2 is the reciprocal of $\frac{1}{2}$. Does the same hold for other probabilities? How many times must we roll a die before we expect to land on 1?[4]

To do this, let's use the idea of states. If we flip heads on our first go, then great! We're done! But if we flip tails, **we're back where we started.** Let's try to encode this into states.

Solution: Let E be the expected amount of flips to land heads. Then notice $E = (\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot E) + 1$. Solving, $E = 2$.

Let's decode what the stuff in the parentheses means. If we flip heads, which happens with a $\frac{1}{2}$ chance, the expected amount of flips to get heads is 0 (since we already flipped heads). And if we flip tails, which happens with $\frac{1}{2}$ chance, we're back where we started, so the expected amount of flips to get heads is still E. And of course, no matter what happens, we will always have used up a step, so we must add 1 to the expected value.

Can we generalize? Yes. The theorem below shall do so.

Theorem 2 (Expected Tries for Single Event) If you have an event that succeeds with probability p every step, the expected amount of steps it takes to succeed once is $\frac{1}{p}$.

Proof: Let E be the expected amount of steps it takes. Then $E = (p \cdot 0 + (1-p) \cdot E) + 1$. Solving, $E = \frac{1}{p}$. ■

And how many steps does it take to, say, succeed twice?

Theorem 3 (Linearity of Expectation) If you have an event that succeeds with probability p every step, the expected amount of steps it takes to succeed n times is $\frac{n}{p}$.

Proof: We induct. This is proven to be true for the base case. Now assume it is true for n. Let $E[n]$ denote the expected amount of steps to succeed n times. Then notice $E[n+1] = (p \cdot E[n] + (1-p) \cdot E[n+1]) + 1$. Substitute in $\frac{n}{p}$ for $E[n]$ and you get $E[n+1] = (pE[n] + (1-p) \cdot E[n+1]) + 1$. Solving, $E[n+1] = \frac{n+1}{p}$, as desired. ■

We've answered the obvious questions. Now let's show the power of states with something a bit harder: *consecutive* successes.

[3] In fact, this can be useful. At the very least, it's an interesting way to relate certain summations with expected value.
[4] The answer is unsurprisingly 6.

Example 3 (Two Heads Up) How many times must you flip a coin before you expect it to land heads twice in a row?

The naive answer is 4 (and it is the wrong answer!) and the reasoning for it being wrong is simple. The expected amount of flips should not be the same as having it land heads twice (without having to be consecutive). We use states to do this instead.

Solution: Let E_n be the amount of times you expect to flip a coin for it to land heads n times in a row. Then $E_2 = (\frac{1}{2} \cdot E_1 + \frac{1}{2} \cdot E_2) + 1$ and $E_1 = (\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot E_2) + 1$. Simplifying both of these equations, $E_2 = E_1 + 2$ and $2E_1 = E_2 + 2$. Substituting E_2 into the second equation yields $2E_1 = E_1 + 2 + 2 = E_1 + 4$, so $E_1 = 4$. Substituting that into the first equation yields $E_2 = 4 + 2 = 6$, so the expected amount of flips to get two heads in a row is 6.

The generalization for expected value is very similar to probability (just add a 1 to the end).

Theorem 4 (Expected Value in States) Consider the probability of some event occurring at an initial state S. Let the probability that you are sent from S to S_i be x_i, and let the expected amount of steps at S_i be E_i. Then the expected amount of steps to success at S is $\sum\limits_{i=1}^{\infty} E_i x_i + 1$.

A couple of reminders:

1. Remember that $\sum\limits_{i=1}^{\infty} x_i = 1$.
2. If S_i doesn't exist or can't happen, then set x_i as 0.
3. For the success case, set E_i as 0.
4. Most importantly, **this theorem applies to all** S_i. Since the success state will eventually appear, you can use this to set a system of equations and solve for the expected value until success at S.
5. There is only one success condition. You cannot complementary count.

§ 3 Various Examples

We present more examples to solidify the reader's understanding of states.

Example 4 (BMT Discrete 2019/1) A fair coin is repeatedly flipped until 2019 consecutive coin flips are the same. Compute the probability that the first and last flips of the coin come up differently.

Solution: We complementary count because that makes the initial state less annoying.

Say without loss of generality the first flip comes up heads. Notice that if a run of heads breaks, we have 1 tail and thus need to flip 2018 more tails in a row, and vice versa. Therefore, let p_1 represent the probability of success when we have a run of one head at the end and let p_2 represent the probability of success when we have a run of one tail at the end, and note that

$$p_1 = \frac{1}{2^{2018}} + (1 - \frac{1}{2^{2018}})p_2$$

$$p_2 = (1 - \frac{1}{2^{2018}})p_1,$$

implying that

$$p_1 = \frac{1}{2^{2018}} + (1 - \frac{1}{2^{2018}})^2 p_1$$

$$p_1(2-\frac{1}{2^{2018}})(\frac{1}{2^{2018}}) = \frac{1}{2^{2018}}$$
$$p_1 = \frac{2^{2018}}{2^{2019}-1}.$$

So the answer is $\frac{2^{2018}-1}{2^{2019}-1}$.

Example 5 Consider a number line with integers $0, 1, 2 \ldots n$. Every second, a particle initially at the origin randomly moves to an adjacent integer. (If it is at 0, it goes to 1 all the time.) In terms of n, find the expected amount of time for the particle to reach n.

Solution: We claim the answer is n^2.

Let $E(k)$ denote the expected amount of seconds required to move from k to n. Note that $E(0) = E(1)+1$, $E(n) = 0$, and for all other k, $E(k) = \frac{1}{2}(E(k-1) + E(k+1)) + 1$.

Then we have the system of equations

$$E(0) = E(1) + 1$$
$$E(1) = \frac{1}{2}(E(0) + E(2)) + 1$$
$$E(2) = \frac{1}{2}(E(1) + E(3)) + 1$$
$$\cdots$$
$$E(n-1) = \frac{1}{2}(E(n-2) + E(n)) + 1$$
$$E(n) = 0.$$

We claim that $E(k) = E(k+1) + (2k+1)$ for all suitable k. We prove this by induction with the base case of $k = 0$. Then $E(k+1) = \frac{1}{2}(E(k) + E(k+2)) + 1 = \frac{E(k+1)}{2} + \frac{2k+1}{2} + \frac{E(k+2)}{2} + 1$, implying $E(k+2) = E(k+1) + 2(k+1) + 1$, as desired.

§ 4 Freedom

Similar to the Freedom section of **CQV-Perspectives**, here you are trying to find out *what the states are*. We begin with a well-known example.

Example 6 (Airplane Probability Problem) 100 passengers board an airplane with exactly 100 seats. Everyone has a ticket with an assigned seat number. However, the first passenger has lost their ticket and takes a random seat. Every subsequent passenger attempts to choose their own seat, but takes a random seat if theirs is taken. What is the probability the last passenger sits at his seat?

Solution: Starting with the first passenger, we see that the uncertainty ends either when the 1st or 100th seat is taken. If the 1st seat is taken, everyone else files in to their own seats, and if the 100th seat is taken, then the 100th passenger cannot sit there. The answer is then $\frac{1}{2}$.

However, freedom is best characterized by the following MAST Diagnostic problem.

Example 7 (MAST Diagnostic S1/C6) Andy the unicorn is on a number line from 1 to 2019. He starts on 1. Each step, he randomly and uniformly picks a number greater than the number he is currently on, and goes to it. He stops when he reaches 2019. What is the probability he is ever on 1984?

Solution: Say Andy is in state A if he is between 1 and 1983 inclusive and in state B if he is between 1984 and 2019 inclusive. The main claim is that **we only care about the move from A to B**, since it is the only move you can land on 1984 with. Since any of the numbers in B are equally likely to be chosen, the answer is $\frac{1}{2019-1984+1} = \frac{1}{36}$.

§ 5 Problems

Minimum is [50]. Problems with the symbol are required.

> "Isn't everybody sick to death of all this stuff?
> Can't we all stand up and say enough!"
>
> Death Note Musical

[1] **Problem 1** There are n people, each with a test. The teacher, who is lazy, randomly passes the tests back. What is the expected amount of people who will receive their own test back?

[1] **Problem 2** (MATHCOUNTS 2017) There are 100 chickens in a circle. Each chicken randomly and simultaneously pecks the chicken to its left or right. How many chickens are expected to not be pecked?

[2] **Problem 3** Consider a number line with a drunkard at 0, and two cops at -2019 and 1000. Each second, the drunkard will randomly move to an adjacent integer with equal probability. The cops must move to an adjacent integer of their choice every second as well, and the movements of the cops and drunkard happen simultaneously. If the goal of the cops is to occupy the same number as the drunkard, what is the expected amount of seconds it will take the cops to occupy the same space as the drunkard? Assume optimal movement from the cops.

[2] **Problem 4** (SMT 2020) A rook is on a chess board with 8 rows and 8 columns. The rows are numbered 1, 2, ..., 8 and the columns are lettered a, b, ..., h. The rook begins at $a1$ (the square in both row 1 and column a). Every minute, the rook randomly moves to a different square either in the same row or the same column. The rook continues to move until it arrives a square in either row 8 or column h. After infinite time, what is the probability the rook ends at $a8$?

[2] **Problem 5** (AIME II 2019/2) Lily pads $1, 2, 3, \ldots$ lie in a row on a pond. A frog makes a sequence of jumps starting on pad 1. From any pad k the frog jumps to either pad $k+1$ or pad $k+2$ chosen randomly with probability $\frac{1}{2}$ and independently of other jumps. The probability that the frog visits pad 7 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

[3] **Problem 6** Bob is flipping a fair coin and wants to get n heads in a row. In terms of n, how many times should he expect to flip his coin?

[3] **Problem 7** (AMC 12B 2019/19) Raashan, Sylvia, and Ted play the following game. Each starts with \$1. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives \$1 to that player. What is the probability that after the bell has rung 2019 times, each player will have \$1? (For example, Raashan and Ted may each decide to give \$1 to Sylvia, and Sylvia may decide to give her her dollar to Ted, at which point Raashan will have \$0, Sylvia will have \$2, and Ted will have \$1, and that is the end of the first round of play. In the second round Rashaan has no money to give, but Sylvia and Ted might choose each other to give their \$1 to, and the holdings will be the same at the end of the second round.)

[3] **Problem 8** (AMC 10B 2019/21) Debra flips a fair coin repeatedly, keeping track of how many heads and how many tails she has seen in total, until she gets either two heads in a row or two tails in a row, at which point she stops flipping. What is the probability that she gets two heads in a row but she sees a second tail before she sees a second head?

[3] **Problem 9** (AIME I 2019/5) A moving particle starts at the point $(4, 4)$ and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a, b), it moves at random to one of the points $(a-1, b)$, $(a, b-1)$, or $(a-1, b-1)$, each with probability $\frac{1}{3}$, independently of its previous moves.

The probability that it will hit the coordinate axes at $(0,0)$ is $\frac{m}{3^n}$, where m and n are positive integers. Find $m+n$.

[4 💎] **Problem 10 (ART 2019/4)** Consider a number line with integers $-65, -64 \ldots 62, 63$. Every second, a particle at the origin randomly moves to an adjacent integer. Find the expected amount of seconds for the particle to reach either -65 or 63.

[4 ✎] **Problem 11** (AMC 10B 2019/18) Henry decides one morning to do a workout, and he walks $\frac{3}{4}$ of the way from his home to his gym. The gym is 2 kilometers away from Henry's home. At that point, he changes his mind and walks $\frac{3}{4}$ of the way from where he is back toward home. When he reaches that point, he changes his mind again and walks $\frac{3}{4}$ of the distance from there back toward the gym. If Henry keeps changing his mind when he has walked $\frac{3}{4}$ of the distance toward either the gym or home from the point where he last changed his mind, he will get very close to walking back and forth between a point A kilometers from home and a point B kilometers from home. What is $|A-B|$?

[4 ✎] **Problem 12** (TMC 2020 10B/18) Edwin has two chess pieces that he places both on the center square of a 5×5 chessboard. He sets a border one square wide on the edges of the chessboard, leaving a 3×3 area in the middle. In one move, each piece moves as follows:

- The white piece moves one square either vertically or horizontally and then two squares in a perpendicular direction.
- The black piece moves one square either vertically or horizontally.

Each piece moves repeatedly until it first lands on a square in the border, at which point it stops moving. If both pieces move randomly but always abide by their rules, what is the probability that the white and black pieces will end up on the same square after they each stop moving?

[4 ✎] **Problem 13** (AIME 1995/15) Let p be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Given that p can be written in the form m/n where m and n are relatively prime positive integers, find $m+n$.

[4 ✎] **Problem 14** (AMC 10A 2020/23) Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

[4 ✎] **Problem 15** (Mildorf) A single atom of Uranium rests at the origin. Each second, the particle has a $\frac{1}{4}$ chance of moving one unit in the negative x direction and a $\frac{1}{2}$ chance of moving in the positive x direction. If the particle reaches $(-3,0)$, it ignites a fission that will consume the earth. If it reaches $(7,0)$, it is harmlessly diffused. The probability that, eventually, the particle is safely contained can be expressed as $\frac{m}{n}$ for some relatively prime positive integers m and n. Determine the remainder obtained when $m+n$ is divided by 1000.

[6 ✎] **Problem 16** (AVHS 2017) Drake the toy snake is moving in the coordinate plane and he starts at the origin. Every second, if he is at (x,y), he either moves to $(x-1,y)$, $(x+1,y)$, $(x,y-1)$, or $(x,y+1)$. What is the expected amount of seconds it takes for his four most recent moves to draw out a unit square?

[6 ✎] **Problem 17** Arthur the arthropod sits at a vertex of a cube. Every minute he teleports to one of the three adjacent vertices, each having equal probability of being selected. After six minutes, what is the probability that he is back at the start?

[6✐] **Problem 18** (AIME I 2021/12) Let $A_1A_2A_3 \ldots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[6💎] **Problem 19 (HPMS MATHCOUNTS Tryouts)** 64 balls, labeled with the integers from 1 through 64, are placed in a bag. Balls are removed form the bag (without replacement) one by one until a ball with an odd number is removed. What is the probability that among the balls removed from the bag is the ball labeled 42? Express your answer as a common fraction.

[6✐] **Problem 20** (AIME I 2016/13) Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located at the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where $y = 0$, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

[9✐] **Problem 21** (Variation on MAST S1/C6) Andy the unicorn is on a number line from 1 to 2019. He starts on 1. Each step, he randomly picks a number greater than the number he is currently on, and the probabilities are distributed such that the probability of him going to $n + 1$ is half the probability of him going to n, where $n, n + 1$ are both integers greater than his current position and less than 2019.[5] He stops when he reaches 2019. What is the probability he is ever on 1984?

[13✐] **Problem 22** (CMIMC Team 2019/7) Suppose you start at 0, a friend starts at 6, and another friend starts at 8 on the number line. Every second, the leftmost person moves left with probability $\frac{1}{4}$, the middle person with probability $\frac{1}{3}$, and the rightmost person with probability $\frac{1}{2}$. If a person does not move left, they move right, and if two people are on the same spot, they are randomly assigned which one of the positions they are. Determine the expected time until you all meet in one point.

[13💎] **Problem 23 (NARML 2020/8)** The mad scientist Kyouma is traveling on a number line from 1 to 2020, subject to the following rules:

- He starts at 1.
- Each move, he randomly and uniformly picks a number greater than his current number to go to.
- If he reaches 2020, he is instantly teleported back to 1.
- There is a time machine on 199.
- A foreign government is waiting to ambush him on 1729.

What is the probability that he gets to the time machine before being ambushed?

[5] This means that $n \leq 2018$.