Introduction to Series

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APV

1 Introduction

We discuss algebraic manipulations with the roots of polynomials to find symmetric sums and to solve polynomials.

2 Symmetric Sums of Roots

2.1 Vieta's Formulas

To motivate this section, we give Vieta's upfront.

Vieta's Formulas. Consider polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with roots $r_1, r_2, \ldots, r_n$. Then

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k} r_{i_1} r_{i_2} \ldots r_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}.$$

So what does this mean, and what's the right way to think about Vieta's? The answer is through factoring the polynomial and matching coefficients.

Example (Quadratic). Let the roots of $x^2 + 7x - 30$ be r_1, r_2. Find $r_1 + r_2$.

We're not going to just use Vieta's Theorem blindly here. Instead we'll try to motivate its discovery.

Solution. Note that $x^2 + 7x - 30 = (x - r_1)(x - r_2)$, **by definition**. Expanding gives $x^2 + 7x + 30 = x^2 - (r_1 + r_2)x + r_1 r_2$. Matching coefficients

Let's explicitly formalize this matching coefficients idea.

Matching Coefficients. Consider polynomials $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$. Then $f(x) = g(x)$ if and only if $a_i = b_i$ for all i.[1]

With this idea in mind we can now prove Vieta's formulas.

Proof (Vieta's Formulas). Expand $(x - r_1)(x - r_2) \ldots (x - r_n)$ and compare its coefficients with the coefficients of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

[1] We use $\cdots$ even though the polynomials are terminating because $a_k = 0$ for all $k > \deg f$. In other words, at some point it's just coefficients of 0 going forward.

2.2 Newton's Sums

Newton's Sums are about the sum $r_1^k + r_2^k + \cdots + r_n^k$, where $r_1, r_2, \ldots, r_k$ are the roots of some polynomial P. These sums are often called **power sums**.

Newton's Sums. Consider some degree n polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with roots $r_1, r_2, \ldots, r_n$. Then let $Z_k = r_1^k + r_2^k + \cdots + r_n^k$. Then

$$\sum_{i=0}^{n} a_i Z_{k-n+i} = a_n Z_k + a_{n-1} Z_{k-1} + \cdots + a_{n-(k-1)} Z_1 + a_0 Z_{k-n} = 0.$$

The proof is actually surprisingly obvious.

Proof. Note that for any root x of $P(x)$, $x^{k-n}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0) = 0 = a_n x^k + a_{n-1}x^{k-1} + \cdots + a_0 x^{k-n}$. Now just take $x = r_1, r_2, \ldots, r_n$ and sum all the equations.

Here's a direct example of Newton's Sums.

Example (AMC 12A 2019/17). Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = as_k + bs_{k-1} + cs_{k-2}$ for $k = 2, 3, \ldots$ What is $a + b + c$?

Solution. By Newton's Sums, $s_{k+1} - 5s_k + 8s_{k-1} - 13s_{k-2}$, or $s_{k+1} = 5s_k - 8s_{k-1} + 13s_{k-2}$. Thus $a + b + c = 5 - 8 + 13 = 10$.

2.3 Reciprocal Roots

Reciprocal Roots. If $f(x) = a_k x^k + a_{k-1}x^{k-1} \ldots + a_0$ has roots $r_1, r_2 \ldots r_k$, then $g(x) = a_0 x^k + a_1 x^{k-1} \ldots + a_k$ has roots $\frac{1}{r_1}, \frac{1}{r_2} \ldots \frac{1}{r_k}$.

Proof With Vieta's. Use Vieta's to show that the polynomial with roots $\frac{1}{r_1}, \frac{1}{r_2} \ldots \frac{1}{r_k}$ and leading coefficient a_0 is indeed $g(x)$.

Proof Without Vieta's. Alternatively, consider $f(\frac{1}{x})$. $f(\frac{1}{x})$ has reciprocal roots so $x^k(\frac{1}{x^k}a_k + \frac{1}{x^{k-1}}a_{k-1} \ldots + a_0 = a_0x^k + a_1x^{k-1} \ldots + a_k) = (x - \frac{1}{r_[1})(x - \frac{1}{r_2}) \ldots (x - \frac{1}{r_k})$

After doing this, you can find symmetric sums of $\frac{1}{r_1} \ldots \frac{1}{r_k}$ using Vieta's and other polynomial techniques. In many cases like computing $\frac{1}{r_1} \cdots + \frac{1}{r_k}$ or $\sum \frac{1}{r_i r_j}$, it's simply easier to combine under a common denominator and apply Vieta's on $f(x)$. However, this trick is very useful for sums like $\frac{1}{r_1^2} + \frac{1}{r_2^2} \cdots + \frac{1}{r_k^2}$ where the numerator isn't very easy to work with using the original roots. In this example, once you switch to the reciprocal roots, it simply becomes Newton's Sums. This often trivializes certain problems.

One interesting corollary of this is that for polynomials with palindromic coefficents, that is $a_i = a_{k-i}$, r is a root of $f(x)$ if and only if $\frac{1}{r}$ is a root.

Example (AMC 12B 2021/16). Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3 + ax^2 + bx + c$, where $1 < a < b < c$. What is $g(1)$ in terms of a, b, and c?

Solution. Using the previous theorem, $cx^3 + bx^2 + ax + 1$ has the reciprocal roots of $f(x)$. Then, we divide by c to get a leading coefficient of 1 so $g(x) = \frac{cx^3+bx^2+ax+1}{c}$. Hence, $g(1) = \frac{a+b+c+1}{c}$.

2.4 Derivatives Trick

The following section requires pre-requisite knowledge of the Chain Rule and Quotient Rule from calculus. This trick mainly appears on college competitions, and those preparing primarily for the AMCs and AIME should not feel the need to know this.

Derivatives Trick. Let $f(x) = (x - r_1)(x - r_2)\dots(x - r_k)$. Then,

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{k} \frac{1}{x - r_i}$$

Proof. We take the natural log of both sides.

$$\ln(f(x)) = \sum_{i=1}^{k} \ln(x - r_i)$$

We take the derivative of both sides using the chain rule.

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{k} \frac{1}{x - r_i}$$

We can extend this further by taking the derivative again as much as we need to do to find $\sum_{i=1}^{k} \frac{1}{(x-r_k)^n}$ for n in general.

Derivatives Trick Extended.

$$\frac{f''(x)f(x) - f'(x)^2}{f(x)^2} = -\sum_{i=1}^{k} \frac{1}{(x - r_i)^2}$$

Proof. Use the Quotient Rule.

3 Solving Polynomials

3.1 Substitutions

The big idea is that you make substitutions to simplify a seemingly very complicated equation such as a quartic or higher degree polynomial. Then, you work backwards to find the solutions in the original variable. In this section, we'll present some common substitutions.

Note that a common substitution is $y = x + \frac{1}{x}$, particularly for polynomials which are palindromic.

Example (HMMT February 2014). Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

Solution. We divide by r^2 to get $r^2 + \frac{16}{r^2} + k(r + \frac{4}{r}) + 1 = 0$. Then subsitute $t = r + \frac{4}{r}$ to get

$$\begin{aligned} t^2 - 8 + kt + 1 &= 0 \\ t^2 + kt - 7 &= 0 \\ k^2 - 28 &= 0 \\ k &= \pm 2\sqrt{7}. \end{aligned}$$

When you have a factorization of the form $(x + a)(x + b)(x + c)(x + d)$ where $a + d = b + c = k$, group together $(x + a)(x + d) = x^2 + kx + ad$ and $(x + b)(x + c) = x^2 + kx + bc$. Then substituting $y = x^2 + kx$, or in some cases $y = x^2 + kx + \frac{ad+bc}{2}$ to take advantage of difference of squares, will help

This applies more generally in that you should try to group together terms to produce polynomials that only differ in the constant term.

Example (AMC 10A 2019/19). What is the least possible value of

$$(x + 1)(x + 2)(x + 3)(x + 4) + 2019$$

where x is a real number?

Solution. We group together $(x + 1)(x + 4) = x^2 + 5x + 4$ and $(x + 2)(x + 3) = x^2 + 5x + 6$. We make the substitution $y = x^2 + 5x + 5$. Then, the expression is $(y - 1)(y + 1) + 2019 = y^2 + 2018 \geq 2018$ with equality if $y = 0$. We use the quadratic formula and see that Δ[a] > 0, so $y = 0$ is actually possible. Then, the minimum is **2018**.

[a] This denotes the discriminant of the quadratic.

3.2 Symmetry

Basically, we use some kind of symmetry in the function to find solutions. This can just be that the function is symmetric about $x = c$. Note that if we make a substitution, we can use the symmetry of the substitution, even if the polynomial in terms of the substitution isn't actually symmetric.

Example (HMMT February 2014). Find the sum of all real numbers x such that $5x^4+10x^3+10x^2+5x-11 = 0$.

Solution. First, note that we cannot just use Vieta's because it specifies **real** roots. Now, to make the equation more symmetric, we rewrite this as

$$5x^4 + 10x^3 + 10x^2 + 5x = 11.$$

The left hand, which we will call $f(x)$, side factors to

$$\begin{aligned} 5(x(x^3+1) + 2x^2(x+1)) &= 5(x+1)(x(x^2-x+1) + 2x^2) \\ &= 5(x+1)(x^3+x^2+x) \\ &= 5x(x+1)(x^2+x+1) \\ &= 5(x^2+x)(x^2+x+1) \\ &= 5((x^2+x)^2 + (x^2+x)) \end{aligned}$$

which is a polynomial in $x^2 + x$. Since $x^2 + x$ is symmetric about $\frac{-1}{2}$, $f(x)$ is also symmetric about $\frac{-1}{2}$. This implies that if $f(x) = -11$, then $f(-1-x) = 11$. So, the real solution x's come in pairs that sum to -1 (note that $\frac{-1}{2}$ isn't a solution). Now we need to find the number of these pairs. We subtitute in $x^2 + x = k$ to get

$$5k^2 + 5k = 11 \implies 5k^2 + 5k - 11$$

which has solutions $\frac{-5\pm7\sqrt{5}}{10}$. Now, note that $5k^2 + 5k = 5(k+\frac{1}{2})^2 - \frac{5}{4}$ so it has a minimum of $\frac{-5}{4}$. Then, $\frac{-5-7\sqrt{5}}{10}$ is less than this minimum but $\frac{-5+7\sqrt{5}}{10}$ is greater than this minimum. So, there is only one pair, yielding an answer of $\mathbf{-1}$.

3.3 Roots of Unity

Root of Unity. A complex number ω is a nth root of unity if $\omega^n = 1$.

For example, i is a 2nd root of unity. In the complex plane, the nth roots of unity will look like a regular n-gon centered around the origin.

Primitive Root of Unity. A complex number ω is a nth primitive root of unity if the smallest positive integer k such that $\omega^k = 1$ is n.

Cis. $\operatorname{cis}\theta = \cos\theta + \sin\theta$

The motivation behind this definition is that $\operatorname{cis}\theta$ is the point in the complex plane such that when you connect it with the origin, it forms an angle with measure θ with the x-axis.

De Moivre's. $\operatorname{cis}\alpha + \beta = (\operatorname{cis}\alpha)(\operatorname{cis}\beta)$

Proof. Use the cosine addition and sine addition formulas.

Formula for nth Root of Unity. An nth root of unity can be written in the form $\operatorname{cis}\frac{2k\pi}{n}$. for $0 \le k < n$.

Example (AMC 12B 2017/12). What is the sum of the roots of $z^{12} = 64$ that have a positive real part?

Solution. Substitute in $x = z\sqrt{2}$. Then, $x^{12} = 1$. Using the previous theorem, $x = \operatorname{cis}\frac{2k\pi}{12} = \operatorname{cis}\frac{k\pi}{6} = \cos\frac{k\pi}{6} + \sin\frac{k\pi}{6}$. The real part is $\cos\frac{k\pi}{6} > 0$ which happens for $k = 0, 1, 2, 10, 11$.

When we sum these roots, the imaginary parts cancel out and we get $1 + 2 \cdot \frac{\sqrt{3}2}{+}2 \cdot \frac{1}{2} = 2 + \sqrt{3}$.

3.4 Polynomial Interpolation

Fundamental Theorem of Algebra. If a polynomial $f(x)$ has degree $d > 0$, then it has d complex roots.

Equal Polynomials. If two polynomials $f(x)$ has degree d and $g(x)$ has a degree less than or equal to d and $f(x) = g(x)$ for $x_1, x_2 \dots x_{d+1}$, then $f(x) = g(x)$.

Proof. Consider the polynomial $f(x) - g(x)$ which has degree at most d. Then, $f(x) - g(x)$ has $d+1$ roots $x_1, x_2 \dots x_{d+1}$. Because there are more roots than the degree d, by the Fundamental Theorem of Algebra, this implies that $f(x) - g(x)$ is the zero polynomial and that $f(x) - g(x) = 0 \implies f(x) = g(x)$.

Lagrangian Interpolation. If a polynomial $f(x)$ of degree d passes through $(x_1, y_1), (x_2, y_2), \dots (x_{d+1}, y_{d+1})$, then

$$\begin{aligned} f(x) &= \frac{y_1(x-x_2)(x-x_3)\dots(x-x_{d+1})}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_{d+1})} \\ &+ \frac{y_2(x-x_1)(x-x_3)\dots(x-x_{d+1})}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_{d+1})} \\ &\dots \\ &+ \frac{y_{d+1}(x-x_1)(x-x_2)\dots(x-x_d)}{(x_{d+1}-x_1)(x_{d+2}-x_2)\dots(x_{d+1}-x_d)} \end{aligned}$$

This looks very intimidating but it's basically a very hamfisted way of getting a polynomial that goes through every point. The idea is that we can force it to be equal by strategically making all but one of the terms go to zero at one of the points.

Proof. If we represent the right hand side as $g(x)$, then we can clearly see that $g(x) = f(x)$ for $x = x_1, x_2 \dots x_{d+1}$ as most of the terms except the one we want cancel out. By the previous theorem, $f(x) = g(x)$.

Example (Mandelbrot). There is a unique polynomial $P(x)$ of the form

$$P(x) = 7x^7 + c_1x^6 + c_2x^5 + \cdots + c_6x + c_7$$

such that $P(1) = 1, P(2) = 2, \dots,$ and $P(7) = 7$. Find $P(0)$.

Solution. Note that $P(x) - x$ has the 7 roots $1, 2 \dots 7$. By the Fundamental Theorem of Algebra, it can have no other roots so $P(x) - x = a(x-1)(x-2)\cdots(x-7)$. We are given that the leading coefficient of $P(x)$ is 7 so $a = 7$. Then, $P(0) - 0 = -7! \cdot 7 = \mathbf{-35280}$.

3.5 Finite Differences

We are essentially using the idea of a derivative from calculus discretely. That is, instead of having differences be continuous, they're integers. If you haven't learned calculus, don't worry about the calculus analogies here; they're not actually needed to know finite differences.

Note that the following definition is commonly called the **backwards finite difference**.

Finite Difference.

$$\Delta_0(x) = f(x)$$

$$\Delta_n(x) = \Delta_{n-1}(x) - \Delta_{n-1}(x-1)$$

Constant Nth Finite Differences. For a n-degree polynomial $f(x)$, $\Delta_n(x)$ is constant for all x. This is equivalent to $\Delta_{n+1}(x) = 0$ for all x.

Proof. We prove by induction $\Delta_k(x)$ is a $n-k$ degree polynomial with leading coefficient $\frac{n!}{(n-k)!}$.

You can think of it as the same as $f^n(x)$ being constant for n-degree polynomials in calculus.

More specifically, we have

Formula for Nth Finite Difference. If the leading coefficient of a n-degree polynomial $f(x)$ is a, then $\Delta_n(x) = n! \cdot a$.

Proof. We prove by induction $\Delta_k(x)$ is a $n-k$ degree polynomial with leading coefficient $\frac{n!}{(n-k)!}$.

Again, this is analogous to using the Power Rule in Calculus repeatedly.

Example (HMMT February 2015 Team Reworded). The complex numbers x, y, z satisfy

$$xyz = -4$$

$$(x+1)(y+1)(z+1) = 7$$

$$(x+2)(y+2)(z+2) = -3$$

Find the value of $(x+3)(y+3)(z+3)$.

Solution. Consider $f(k) = (k-x)(k-y)(k-z)$. Then, $f(0) = -xyz = 4$, $f(-1) = -(x+1)(y+1)(z+1) = -7$, and $f(-2) = -(x+2)(y+2)(z+2) = 3$.

We know the leading coefficient of $f(k)$ is 1 so $\Delta_3(x) = 6$. We apply finite differences to get $f(-3) = 28$. Then, $(x+3)(y+3)(z+3) = -f(-3) = \boxed{-28}$

3.6 Root Rules

This is an assorted collection of rules and theorems about the roots of a polynomials. These don't come up often as the whole problem but may be useful as intermediate steps.

Descartes' Rule of Signs. The number of sign changes in the coefficients of a polynomial $f(x)$ is the maximum possible number of positive zeros. Also, the number of sign changes in the coefficients of polynomial $f(-x)$ from is the maximum possible number of negative zeros.

Example. Using Descartes' Rule of Signs, what is the maximum number of positive real solutions to $x^4 - x^3 + x^2 + 1$?

Solution. The coefficients change sign two times, 1 to -1 and -1 to 1. So, **2**.

Rational Root Theorem. A rational root of the polynomial $f(x) = a_k x^k + a_{k-1}x^{k-1} \ldots + a_0$ is in the form $\frac{p}{q}$ where p, q are relatively prime integers such that $p|a_0$ and $q|a_k$.

Conjugate Root Theorem. If a polynomial $f(x)$ with real coefficients has a complex root $a + bi$, then the complex conjugate $a - bi$ is also a root.

Example. Find the roots of the polynomial $x^4 - 14x^3 + 71x^2 - 136x + 58$ given that $5 - 2i$ is a root.

Solution. By Conjugate Root Theorem, $5+2i$ is also a root. Then, $(x-5-2i)(x-5+2i) = x^2 - 10x + 29$. We apply long division and the Quadratic Formula on $x^2 - 4x + 2$ to find the other roots of $2 \pm \sqrt{2}$.

Radical Conjugate Root Theorem. If a polynomial $f(x)$ with rational coefficients has a root of the form $a + b\sqrt{c}$, then $a - b\sqrt{c}$ is also a root.

4 Problems

Minimum is [60 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "It is the time you have wasted for your rose that makes your rose so important."
>
> The Little Prince

[1 ♟] Problem 1 (AMC 12B 2019/8)
Let $f(x) = x^2(1-x)^2$. What is the value of the sum

$$f\left(\frac{1}{2019}\right) - f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) - f\left(\frac{4}{2019}\right) + \cdots + f\left(\frac{2017}{2019}\right) - f\left(\frac{2018}{2019}\right)?$$

[2 ♟] Problem 2 (BMT 2015) Let r, s, and t be the three roots of the equation $8x^3 + 1001x + 2008 = 0$. Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

[3 ♟] Problem 3 (PuMAC 2019) Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x)$, $Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

[3 ♟] Problem 4 (PHS HMMT TST 2020) Let a, b, c be the distinct real roots of $x^3 + 2x + 5$. Find $(8-a^3)(8-b^3)(8-c^3)$.

[3 ♟] Problem 5 (AMC 10A 2017/24) For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

[3 ♞] Problem 6 (AMC 10A 2019/24) Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

[3 ♟] Problem 7 (Canada) If a, b, c are roots of $a^3 - a - 1 = 0$, compute

$$\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c}$$

[3 ♟] Problem 8 (William Dai) If r_1, r_2, r_3 and r_4 are the roots of $x^4 + 5x^3 + 3x^2 + 2x + 1$, find $\frac{1}{r_1^3} + \frac{1}{r_2^3} + \frac{1}{r_3^3} + \frac{1}{r_4^3}$.

[3 ♟] Problem 9 (AIME 1993/5) Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x-n)$. What is the coefficient of x in $P_{20}(x)$?

[4 ♟] Problem 10 (BMT 2019)

Let r_1, r_2, r_3 be the (possibly complex) roots of the polynomial $x^3 + ax^2 + bx + \frac{4}{3}$. How many pairs of integers a, b exist such that $r_1^3 + r_2^3 + r_3^3 = 0$?

[4 ♞] Problem 11 (AMC 12B 2017/23) The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2,4)$, $B(3,9)$, and $C(4,16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

[4 ♟] Problem 12 (HMMT February 2013) Let z be a non-real complex number with $z^{23} = 1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}.$$

[4 ♟] Problem 13 (FARML 2007/T9) For fixed numbers x, y, z, let $p(n) = x^n + y^n + z^n$. If $p(2) = 2$, $p(4) = \frac{3}{2}$, and $p(6) = \frac{29}{24}$, compute $p(8)$.

[4 ♟] Problem 14 (David's Problem Stash) Let a, b, and c be nonzero real numbers such that $a + b + c = 0$ and

$$28(a^4 + b^4 + c^4) = a^7 + b^7 + c^7.$$

Find $a^3 + b^3 + c^3$.

[4 ♟] Problem 15 Give all unordered pairs of (x, y) where x and y are complex numbers satisfying:

$$x + y = 3$$

$$x^5 + y^5 = 33$$

[4 ♞] Problem 16 (HMMT 2008) The equation $x^3 - 9x^2 + 8x + 2 = 0$ has three real roots p, q, r. Find $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$.

[4 ♟] Problem 17 (NanoMath Fall Meet 2020) If $x + y = 6$ and $x^3 + y^3 = 108$, find $x^5 + y^5$.

[4 ♟] Problem 18 (2019 AIME I) Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[6 ♟] Problem 19 (AoPS Forums)

Given x_1, x_2, x_3, x_4 are the roots of $P(x) = 2x^4 - 5x + 1$, find the value of $\sum_{i=1}^{4} \frac{1}{(1-x_i)^3}$.

[6 ♟] Problem 20 (AIME I 2014/14)

Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4$$

There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

[6 ♟] Problem 21 (SLKK AIME 2020) Let a, b, and c be the three distinct solutions to $x^3 - 4x^2 + 5x + 1 = 0$. Find

$$(a^3 + b^3)(a^3 + c^3)(b^3 + c^3).$$

[6 ♟] Problem 22 (AIME I 2019/10) For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x - z_1)^3 (x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_k \right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[9 ♟] **Problem 23 (HMMT Feburary 2017)** A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $\lfloor 2017P(2017) \rfloor$

[9 ♟] **Problem 24 (AIME 1990/15)** Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$ax + by = 3,$$

$$ax^2 + by^2 = 7,$$

$$ax^3 + by^3 = 16,$$

$$ax^4 + by^4 = 42.$$

[12 ♟] **Problem 25 (HMMT February 2020)** Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

> **Remark:** The following problem is not explicitly algebra but it does use several polynomial techniques. It's very, very hard!

[21 ♟] **Problem 26 (SLKK AIME 2020)** Let $p = 991$ be a prime. Let S be the set of all lattice points (x, y), with $1 \le x, y \le p - 1$. On each point (x, y) in S, Olivia writes the number $x^2 + y^2$. Let $f(x, y)$ denote the product of the numbers written on all points in S that share at least one coordinate with (x, y). Find the remainder when

$$\sum_{i=1}^{p-2} \sum_{j=1}^{p-2} f(i, j)$$

is divided by p.