Logarithms

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APV

§ 1 Theory

We commonly have functions of the form $f(x) = x^n$, and to find $f^{-1}(x)$, we just take the nth root of both sides to get $\sqrt[n]{x} = f^{-1}(x)$. But how would we find the inverse of a function like $f(x) = n^x$? To do this, we create an inverse function known as a logarithm, where $n^{\log_n x} = x$.

Here are two examples to get you up to speed.

Example 1 Find $\log_2 8$.

Solution: Notice that $2^{\log_2 8} = 8 = 2^3$ by the definition of log, so $\log_2 8 = \mathbf{3}$.

Example 2 Simplify $\frac{\log_5 x}{\log_{25} x}$.

Solution: Let $25^a = x$. Then notice $5^{2a} = x$. Substituting yields $\frac{2a}{a} = \mathbf{2}$.

Here's a motivating exercise for what's going to come next.

Exercise 1 Evaluate $\log_2 16 + \log_2 32$, and then evaluate $\log_2 16 \cdot 32$.

§ 1.1 Fundamental Rules

The fundamental two rules of logarithms are the addition and subtraction rules. Notice that addition outside becomes multiplication inside, and similarly, subtraction outside becomes division inside. This is a consequence of the way exponents behave: $x^{a+b} = x^a \cdot x^b$.

Theorem 1 (Logarithm Addition) Given positive reals a, b, c with $a > 1$, $\log_a b + \log_a c = \log_a bc$.

Proof: Notice that $a^{\log_a b + \log_a c} = a^{\log_a b} \cdot a^{\log_a c} = bc = a^{\log_a bc}$. Since the bases are the same, it follows the exponents are the same. ■

Theorem 2 (Logarithm Subtraction) Given positive reals a, b, c with $a > 1$, $\log_a b - \log_a c = \log_a \frac{b}{c}$.

Proof: This is a repeat of logarithm addition. Notice that $a^{\log_a b - \log_a c} = \frac{a^{\log_a b}}{a^{\log_a c}} = \frac{b}{c} = a^{\log_a \frac{b}{c}}$. ■

Notice that we're exploiting the properties of logarithms, and we're expressing everything without using logarithms as soon as possible. This trend will continue in AIME problems; once the logs have been removed, there's not much underneath to solve.

§ 1.2 Base Change

The base change rule allows you to express all logarithms in the same base; this is extremely powerful, even if it doesn't look like much.

Theorem 3 (Base Change) Given positive reals a, b, c with $a > 1$, $\frac{\log_a b}{\log_a c} = \log_c b$.

Proof: Have $x = \log_a b$, $y = \log_a c$, and $z = \log_c b$. Notice that $a^x = b, a^y = c, c^z = b$. Then $(a^y)^z = a^x$, implying $yz = x$ or $\frac{x}{y} = z$. ■

This is one of the fundamental manipulation techniques for logarithmic manipulations. This is very convenient because it can be used to make all the logarithms share an arbitrary common base. Specifically,

$$\log_b a = \frac{\log a}{\log b}.$$

Example 3 (AMC 12B 2021/9) What is the value of

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

Solution: First we convert everything to base 10 with the base change theorem. Note that our expression is equivalent to

$$\frac{\log 80 \log 40 - \log 160 \log 20}{(\log 2)^2}.$$

Here is where things get a little tricky. Note that $\log 160 = \log 80 + \log 2$ and $\log 20 = \log 40 - \log 2$, so our expression becomes

$$\begin{aligned}\frac{\log 80 \log 40 - (\log 80 + \log 2)(\log 40 - \log 2)}{(\log 2)^2} &= \frac{\log 2(\log 80 - \log 40) + (\log 2)^2}{(\log 2)^2} \\ &= \frac{\log 2(\log 2) + (\log 2)^2}{(\log 2)^2} \\ &= \mathbf{2}.\end{aligned}$$

The motivation for expressing $\log 160$ as $\log 80 + \log 2$ is twofold. The general reason is that we want things to simplify with an SFFT-esque expression, and the reason we use $\log 2$ specifically is because it is in the denominator.

We present the so-called logarithm chain rule as an exercise. (It's pretty useless and is only being presented as a check-up.)

Exercise 2 (Logarithm Chain Rule) Given positive reals a, b, c, d with $a > 1$ and $c > 1$, $\log_a b \log_c d = \log_a d \log_c b$.

§ 2 Examples

Here are some examples of AIME logarithm problems. I want to re-iterate the following with these two problems: usually, **interpreting the log condition is the entire problem**.

Example 4 (AIME II 2009/2) Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

Solution: We notice that $a^{(\log_3 7)^2} = (a^{\log_3 7})^{\log_3 7}$. Similar expressions hold for b, c.

We then substitute $a^{\log_3 7} = 27$ as defined in the problem statement, and we do the same for b, c. This becomes $27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25} = 3^{3\log_3 7} + 7^{2\log_7 11} + 11^{\frac{1}{2}\log_{11} 25} = 7^3 + 11^2 + 25^{\frac{1}{2}}$. This simplifies to **469**, which is our answer.

Example 5 (AIME I 2011/9) Suppose x is in the interval $[0, \pi/2]$ and $\log_{24\sin x}(24\cos x) = \frac{3}{2}$. Find $24\cot^2 x$.

Solution: We can rewrite this as $(24\sin x)^3 = (24\cos x)^2$, which implies $24\sin^3 x = \cos^2 x = 1 - \sin^2 x$. Thus we want to find the positive root of $24\sin^3 x + \sin^2 x - 1 = 0$. Using the Rational Root Theorem (aka guessing), we see that $\frac{1}{3}$ is a root. Thus $\cot x = 2\sqrt{2}$ and our answer is **192.**

§ 3 Problems

Minimum is [32✎]. Problems with the ◇ symbol are required.

> "I just long for a world in which ordinary things are done in an ordinary way."
>
> Psycho-Pass

[1✎] **Problem 1** (AIME II 2020/3) The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

[2✎] **Problem 2** (AIME 1986/8) Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1000000. What is the integer nearest to S?

[2✎] **Problem 3** (AIME I 2020/2) There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

[2✎] **Problem 4** (AIME I 2007/7) Let $N = \sum\limits_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor)$.
Find the remainder when N is divided by 1000. ($\lfloor k \rfloor$ is the greatest integer less than or equal to k, and $\lceil k \rceil$ is the least integer greater than or equal to k.)

[3✎] **Problem 5** (SMT 2020) If a is the only real number that satisfies $\log_{2020} a = 202020 - a$ and b is the only real number that satisfies $2020^b = 202020 - b$, what is the value of $a+b$?

[3✎] **Problem 6** (AIME II 2013/2) Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a+b$.

[3◇] **Problem 7 (AIME II 2010/5)** Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

[3✎] **Problem 8** (AIME I 2006/9) The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

[4✎] **Problem 9** (HMMT 2020) Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

[4✎] **Problem 10** (AIME II 2007/12) The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that $\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \le \log_3\left(\sum_{n=0}^{7} x_n\right) \le 57$, find $\log_3(x_{14})$.

[4✎] **Problem 11** (AIME I 2009/7) The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} - 1 = \frac{1}{n+\frac{2}{3}}$ for $n \ge 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

[4✎] **Problem 12** (AIME I 2010/14) For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \le 300$.
Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

[6 💎] **Problem 13 (AIME I 2005/8)** The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

[6 ✎] **Problem 14** (AIME I 2013/8) The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

[6 ✎] **Problem 15** (hARMLess Mock ARML 2019/10) Compute the sum of all positive integers that can be expressed in the form

$$\log_b(404!) - \log_b(c),$$

where b and c are positive integers such that $b > 1$ and $b + c$ is odd.

[9 ✎] **Problem 16** (AIME I 2012/9) Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p + q$.