Modern American Computational Geo

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1 Introduction

This is a unit about triangle centers, configurations, and nontrivial geometry skills(e.g. angle chasing) actually used in computational problems. Recently, AIME geo has featured more of these kinds of problems, especially the in the last five. In particular, the "ideas" of this unit won't really be in the lecture notes; rather, this unit will focus on the problem set. These problems are referred to as "American", both in the sense that they have configurational elements and nontrivial angle chasing, much like "American Geo" in the olympiad sense, and also in the sense that recent American computational problems really do look like this.
Computational problems will *still use computational methods,* despite the bulk of the problem being some olympiad-style angle chase. Thus, it is *vital* that you remember all your standard tools(e.g. ratio lemma, law of sines/cosines, Stewart's theorem, Ptolemy, Heron's, etc), as there will rarely be problems where you will have to do no significant algebra/length chasing at all.
Nevertheless, be prepared for some difficult geometry, bordering on olympiad-level. In fact, this unit will contain some olympiad problems.

2 Properties of Triangle Centers

Fact 5. Let I be the incenter of $\triangle ABC$ and let M be the arc midpoint of BC. Then M is the circumcenter of (BIC).

Proof. $\angle BIM = \frac{\angle B}{2} + \frac{\angle A}{2} = \angle IBC + \angle CAM = \angle IBC + \angle CBM = \angle IBM$, similarly for C.

This is one of the ubiquitous basic results in geometry that leads contestants to be embarrassed if overlooked. Make sure to have a thorough understanding of why it is true. As a corollary, the $A-$excenter also lies on (BIC).

Example (Orthocenter/Incenter Duality). Let H_A be the foot of the altitude from A, similarly for B and C. then the orthocenter H is the incenter of $H_AH_BH_C$. Similarly, if I_A is the $A-$excenter, then I is the orthocenter of $I_AI_BI_C$.

Proof. For one direction, note that (A, H_B, H_C, H) cyclic. For the other direction, simply note that I_AI is perpendicular to I_BI_C.

Next, some computational-style results, primarily just to make computations with triangle centers easier.

Circumradius formula. Let $[ABC]$ be the area of triangle $\triangle ABC$, and let a, b, c be side lengths. Then $R = \frac{abc}{4[ABC]}$.

Proof. Combine extended law of sines($R = \frac{a}{2 \sin A}$) and trig area formula($[ABC] = \frac{1}{2} bc \sin A$), where A denotes angle $\angle BAC$.

This particularly useful to find lengths of known chords on the circumcircle.

3 Other related points

Orthocenter reflection. Let H be the orthocenter. Then the reflection of H over BC, and the reflection of H over the midpoint of BC both lie on (ABC).

Proof. Simply use the fact that $\angle BHC = 180 - \angle A$.

Note that this means $H_A H \cdot H_A A = H_A B \cdot H_A C$.

Other properties of feet of the altitudes. Let H_B, H_C be the feet of the altitudes from B, C, respectively. Then BCH_BH_C is cyclic, or $\triangle AH_BH_C \sim \triangle ABC$.

This can be used with incircles/excircles effectively.

4 Problems

Minimum is [50 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "I'll swear when I want to"
>
> Evan Chen

[3 ♟] Problem 1 (AMC 12A 2019/25) Let $\triangle A_0B_0C_0$ be a triangle whose angle measures are exactly 59.999°, 60°, and 60.001°. For each positive integer n define A_n to be the foot of the altitude from A_{n-1} to line $B_{n-1}C_{n-1}$. Likewise, define B_n to be the foot of the altitude from B_{n-1} to line $A_{n-1}C_{n-1}$, and C_n to be the foot of the altitude from C_{n-1} to line $A_{n-1}B_{n-1}$. What is the least positive integer n for which $\triangle A_nB_nC_n$ is obtuse?

[3 ♟] Problem 2 (Purple Comet HS 2020/26) In $\triangle\ ABC$, $\angle A = 52^o$ and $\angle B = 57^o$. One circle passes through the points B, C, and the incenter of $\triangle\ ABC$, and a second circle passes through the points A, C, and the circumcenter of $\triangle\ ABC$. Find the degree measure of the acute angle at which the two circles intersect.

[4 ♟] Problem 3 (JMO 2019/4) Let ABC be a triangle with $\angle ABC$ obtuse. The A-excircle is a circle in the exterior of $\triangle ABC$ that is tangent to side BC of the triangle and tangent to the extensions of the other two sides. Let E, F be the feet of the altitudes from B and C to lines AC and AB, respectively. Can line EF be tangent to the A-excircle?

[4 ♟] Problem 4 (AIME I 2016/6) In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, find IC.

[4 ♞] Problem 5 (AIME I 2011/4) In triangle ABC, $AB = 125$, $AC = 117$, and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.

[4 ♟] Problem 6 (PUMaC 2018 G3) Let $\triangle ABC$ satisfy $AB = 17$, $AC = \frac{70}{3}$ and $BC = 19$. Let I be the incenter of $\triangle ABC$ and E be the excenter of $\triangle ABC$ opposite A. (Note: this means that the circle tangent to ray AB beyond B, ray AC beyond C, and side BC is centered at E.) Suppose the circle with diameter IE intersects AB beyond B at D. Find BD.

[4 ♟] Problem 7 (OMO Spring 2020/15) Let ABC be a triangle with $AB = 20$ and $AC = 22$. Suppose its incircle touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively, and P is the foot of the perpendicular from D to $\overline{EF}$. If $\angle BPC = 90^\circ$, then compute BC^2.

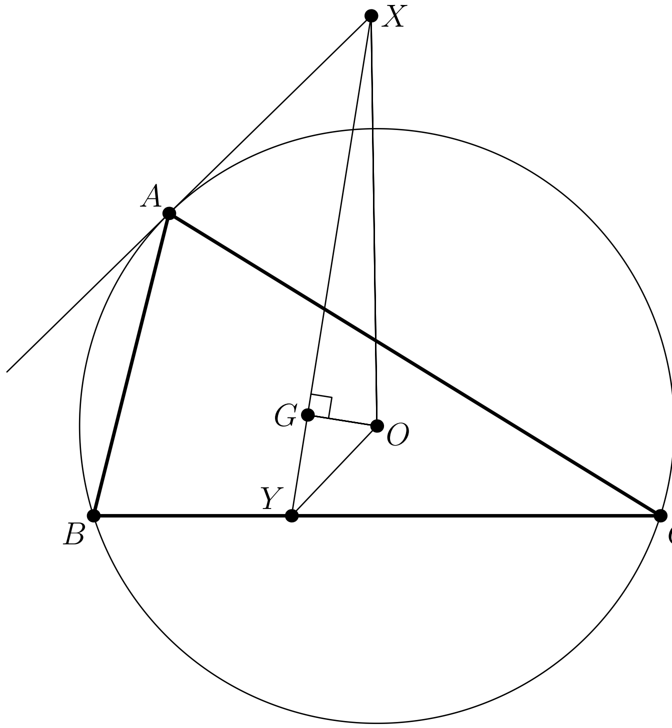
[4 ♟] Problem 8 (IMO 2006/1) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

[6 ♞] Problem 9 (HMMT 2019 G7) Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let H be the orthocenter of ABC. Find the radius of the circle with nonzero radius tangent to the circumcircles of AHB, BHC, CHA.

[6 ♟] Problem 10 (AIME II 2021/14) Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G. Let X be the intersection of the line tangent to the circumcircle of $\triangle ABC$ at A and the line perpendicular to GO at G. Let Y be the intersection of lines XG and BC. Given that the measures of $\angle ABC$, $\angle BCA$, and $\angle XOY$ are in the ratio $13 : 2 : 17$, the degree measure of $\angle BAC$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



[9 ♟] **Problem 11 (AIME II 2012/15)** Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side BC at D and circle ω at a second point E. Let γ be the circle with diameter DE. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[9 ♟] **Problem 12 (AIME II 2019/15)** In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

[9 ♟] **Problem 13 (AIME I 2020/13)** Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, $CA = 6$, find the area of $\triangle AEF$.

[13 ♟] **Problem 14 (AIME II 2020/15)** Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT = CT = 16$, $BC = 22$, and $TX^2 + TY^2 + XY^2 = 1143$. Find XY^2.

[13 ♟] **Problem 15 (ISL 2019 G2)** Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

[13 ♟] **Problem 16 (MOP 2019 HW)** Let $\triangle ABC$ be a triangle and let E and F be the feet of the altitudes from B and C. Assume line EF is tangent to the incircle of $\triangle ABC$. Let the excircle of triangle $\triangle ABC$ opposite the vertex A be tangent to BC at point A_1. Define points B_1 on AC and C_1 on AB analogously, using the excircles opposite B and C, respectively. Prove that points A, A_1, B_1, C_1 are concyclic.