Radical Axes

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§ 1 Power of a Point

We first extend the power of a point theorem to a definition.

Definition 1 (Power of a Point) The power of a point P with respect to circle ω with center O and radius r as $OP^2 - r^2$. We will denote this as $\mathcal{P}(P, \omega) = OP^2 - r^2$.

This yields the following corollary.

Fact 1 (Square of Tangent Line) The power of a point is the square of the length of the tangent line.

§ 2 Radical Axes

Definition 2 (Radical Axis) The radical axis of a pair of circles ω_1, ω_2 as the locus of points such that $\mathcal{P}(P, \omega_1) = \mathcal{P}(P, \omega_2)$.

Theorem 1 (Radical Axis Theorem) The radical axis of a pair of circles is a line.

Proof: We use coordinates to prove this.

Without loss of generality, let the center of ω_1 be $(0, 0)$ and let the center of ω_2 be $(x_0, 0)$. Denote the radii of ω_1, ω_2 as r_1, r_2, respectively. If the coordinates of P are (x, y), then $(x^2 + y^2) - r_1^2 = ([x - x_0]^2 + y^2) - r_2^2$. Rearranging, this yields $-r_1^2 = -2x_0x + x_0^2 - r_2^2$. This is the equation of a line, as desired. ■

A corollary that arises: if ω_1, ω_2 intersect at X, Y, then the radical axis is XY. This is because $\mathcal{P}(X, \omega_1) = 0 = \mathcal{P}(X, \omega_2)$ and $\mathcal{P}(Y, \omega_1) = 0 = \mathcal{P}(Y, \omega_2)$. Since two points are needed to determine a line, the proof is done.

Theorem 2 (Radical Center Theorem) Consider three circles $\omega_1, \omega_2, \omega_3$. Then their pairwise radical axes concur.

Proof: Without loss of generality, let the radical axis of ω_1, ω_3 and the radical axis of ω_1, ω_2 intersect at P. Then notice that $\mathcal{P}(P, \omega_1) = \mathcal{P}(P, \omega_3)$ and $\mathcal{P}(P, \omega_1) = \mathcal{P}(P, \omega_2)$, so $\mathcal{P}(P, \omega_2) = \mathcal{P}(P, \omega_3)$, implying that P lies on the radical axis of ω_2, ω_3, as desired. ■

§ 3 Techniques

§ 3.1 Basic Techniques

Keeping this simple result in mind kills problems involving common chords and external tangents.

Fact 2 (Radical Axis Bisects External Tangent) Consider two circles ω_1, ω_2 that intersect at X, Y. Let one of their common external tangents intersect ω_1 at A and ω_2 at B. Then XY bisects AB.

Proof: Note that XY is the radical axis. Let XY intersect AB at M. Since M lies on the radical axis, $AM = BM$. ■

§ 3.2 Advanced Techniques

We introduce two powerful techniques - Linearity of Power and circles with radius 0.

Theorem 3 (Linearity of Power) The function $f(P) = \mathcal{P}(P, \omega_1) - \mathcal{P}(P, \omega_2)$ changes at a linear rate as P moves along a fixed line ℓ.

Proof: Without loss of generality, let ℓ be the x axis, let the equation of ω_1 be $(x-h_1)^2 + (y-k_1)^2 = r_1^2$, and let the equation of ω_2 be $(x-h_2)^2 + (y-k_2)^2 = r_2^2$.

Let the coordinates of P be $(x, 0)$. Then

$$f(P) = (r_1^2 - ((h_1 - x)^2 + k_1^2)) - (r_2^2 - ((h_2 - x)^2 + k_2^2))$$

$$f(P) = r_1^2 - r_2^2 - (x^2 - 2h_1x + h_1^2 + k_1^2) + (x^2 - 2h_2x + h_2^2 + k_2^2)$$

$$f(P) = r_1^2 - r_2^2 + h_2^2 - h_1^2 + k_2^2 - k_1^2 + x(2h_2 - 2h_1).$$

Since all variables except for x are constant, $f(P)$ varies linearly. ■

Here's a really silly example of it.

Example 1 (MAST Diagnostic 2021/12) In $\triangle ABC$, let the foot of B to AC be E and the foot of C to AB be F. Suppose that the circle through F centered at B is externally tangent to the circle through E centered at C at some point D. Let G be the midpoint of EF. Prove that DG is perpendicular to BC.

Solution: Let the circle centered at B be ω_1 and the circle centered at C be ω_2., and let $f(P) = \mathcal{P}(P, \omega_1) - \mathcal{P}(P, \omega_2)$. Then by Linearity of Power, we want to show that $f(F) + f(E) = 0$. Note that $f(F) = \mathcal{P}(F, \omega_1) - \mathcal{P}(F, \omega_2) = -\mathcal{P}(F, \omega_2) = -(FC^2 - CE^2) = -(BC^2 - BF^2 - CE^2) = BF^2 + CE^2 - BC^2$. Also note that $f(E) = \mathcal{P}(E, \omega_1) - \mathcal{P}(E, \omega_2) = \mathcal{P}(E, \omega_1) = EB^2 - BF^2 = BC^2 - CE^2 - BF^2$.

Thus by Linearity of Power, $f(G) = 0$, implying that G lies on the radical axis of ω_1, ω_2. Since the radical axis is the line through D perpendicular to BC, we are done.

This is all motivated by the fact that G should lie on the radical axis of ω_1, ω_2. The linearity of power solution comes quite naturally because of the right triangles created by the altitudes.

And here's an example of a problem solved using circles with radius 0.

Example 2 (Iran TST 2011/1) In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of BC and let D and E be the feet of the altitudes from C and B, respectively. Let K and L be the midpoints of ME and MD. If KL intersects the line through A parallel to BC in T, prove that $TA = TM$.

Solution: We claim that the line through A parallel to BC, MD, and ME are tangent to (ADE). (This is known as the Three Tangent Lemma.)

Proof: Let H be the orthocenter of $\triangle ABC$. Note that AH is the diameter of (ADE) as $\angle ADH = \angle AEH = 90°$.

Since AH is perpendicular to BC and the line through A is parallel to BC, it is a tangent.

To show ME is tangent, we show $\angle DEM = \angle DAE = \angle A$. Notice that $MB = MC = MD = ME$, since M is the center of $(BCDE)$.

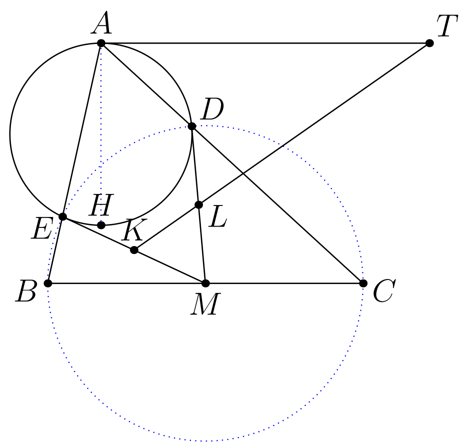
Notice that

$$\angle DEM = \angle DEB + \angle BEM = \angle DEB + \angle EBM = \angle DEB + \angle EBC$$

$$\angle DEB + \angle EBC = 90° - \angle C + 90° - \angle B = \angle A,$$

proving that ME is tangent to (AEF) as desired. ■

Thus KL is the radical axis of (ADE) and the circle centered at M with radius 0. Since T lies on the radical axis and TA is tangent to (ADE), $TA = TM$.



§ 4 Problems

Minimum is [32✐]. Problems with the ♦ symbol are required.

[1✐] **Problem 1** If circle ω with center O has radius 3 and $OP = 5$, find $\mathcal{P}(P, \omega)$.

[1✐] **Problem 2** Consider two externally tangent circles ω_1, ω_2. Let them have common external tangents AC, BD such that A, B are on ω_1 and C, D are on ω_2. Let AC intersect BD at P, and let the common internal tangent intersect AC and BD at X and Y. If $\frac{[PCD]}{[PAB]} = \frac{1}{25}$, find $\frac{[PCD]}{[PXY]}$.

[1✐] **Problem 3** (Mandelbrot 2012) Let A and B be points on the lines $y = 3$ and $y = 12$, respectively. There are two circles passing through A and B that are also tangent to the x axis, say at P and Q. Suppose that $PQ = 2012$. Find AB.

[2✐] **Problem 4** (HMMT 2020/T3) Let ABC be a triangle inscribed in a circle ω and ℓ be the tangent to ω at A. The line through B parallel to AC meets ℓ at P, and the line through C parallel to AB meets ℓ at Q. The circumcircles of ABP and ACQ meet at $S \neq A$. Show that AS bisects BC.

[2✐] **Problem 5** (Geometry Bee 2019) Circles O_1 and O_2 are constructed with O_1 having radius of 2, O_2 having radius of 4, and O_2 passing through the point O_1. Lines ℓ_1 and ℓ_2 are drawn so they are tangent to both O_1 and O_2. Let O_1 and O_2 intersect at points P and Q. Segment $\overline{EF}$ is drawn through P and Q such that E lies on ℓ_1 and F lies on ℓ_2. What is the length of $\overline{EF}$?

[3♦] **Problem 6 (USAJMO 2012/1)** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

[3✐] **Problem 7** (MAST Diagnostic 2020) Consider $\triangle ABC$, and let the feet of the B and C altitudes of the triangle be X, Y. Let XY intersect BC at P. Then prove that the circumcircles of $\triangle PBY$ and $\triangle PCX$ concur with AP.

[4✐] **Problem 8** (GOTEEM 1) Let ABC be a scalene triangle. The incircle of $\triangle ABC$ is tangent to sides BC, CA, AB at D, E, and F, respectively. Let G be a point on the incircle of $\triangle ABC$ such that $\angle AGD = 90°$. If lines DG and EF intersect at P, prove that AP is parallel to BC.

[4✐] **Problem 9** (USAMO 2009/1) Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

[4✐] **Problem 10** (IMO 1995/1) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

[6✐] **Problem 11** Consider scalene $\triangle ABC$ with incenter I. Let the A excircle of $\triangle ABC$ intersect the circumcircle of $\triangle ABC$ at X, Y. Let XY intersect BC at Z. Then choose M, N on the A excircle of $\triangle ABC$ such that ZM, ZN are tangent to the A excircle of $\triangle ABC$. Prove I, M, N are collinear.

[6✐] **Problem 12** (AIME II 2010/15) In triangle ABC, $AC = 13$, $BC = 14$, and $AB = 15$. Points M and D lie on AC with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on AB with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio $\frac{BQ}{CQ}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.

[6 ✐] **Problem 13** (AIME I 2016/15) Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

[9 ✐] **Problem 14** (PUMaC 2017) Triangle ABC has incenter I. The line through I perpendicular to AI meets the circumcircle of ABC at points P and Q, where P and B are on the same side of AI. Let X be the point such that $PX \parallel CI$ and $QX \parallel BI$. Show that PB, QC, and IX intersect at a common point.

[13 ✐] **Problem 15** (USAMTS 2018) Acute scalene triangle $\triangle ABC$ has circumcenter O and orthocenter H. Points X and Y, distinct from B and C, lie on the circumcircle of $\triangle ABC$ such that $\angle BXH = \angle CYH = 90°$. Show that if lines XY, AH, and BC are concurrent, then $\overline{OH}$ is parallel to $\overline{BC}$.