Fake Algebra

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AQU

Thanks to Valentio Iverson for many of the problems in this handout.

Sometimes we have algebraic identities or problems that suggest a geometric structure. Examples of such will be in the list of what to look for and demonstrated in the problem set.

§ 1 What to Look For

Here's a list of identities that suggest something geometric.

1. Stewart's Theorem - $man + dad = bmb + cnc$.
   - In particular, the Appolonius Theorem - if x is the length of the median through A, then $x = \sqrt{\frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}}$.
   - Also of note, $\sqrt{ab - xy}$ - if $\triangle ABC$ has angle bisector AD, and we label $AB = a$, $AC = b$, $BD = x$, $CD = y$, then $AD = \sqrt{ab - xy}$.
2. Sine Area Formula
   - $[ABC] = \frac{1}{2}ab\sin\theta$. This can be used in many places.
3. Heron's Formula
   - $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$. Be on the lookout for suspicious factorizations like $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.
4. Trig Identities
   - Know the angle addition formulas; $\sin(x+y) = \sin x \cos y + \cos x \sin y$.
   - By Pythagorean Identities, anything of the form $1 \pm x^2$, **particularly in the denominator**, suggest trig substitutions.
   - $x + \frac{1}{x}$ suggests the following; if $x = \tan\frac{\alpha}{2}$, then $\sin\alpha = \frac{2}{x+1/x}$.
5. Law of Cosines - Look for certain proportions or tell-tale signs of "sort of symmetrical but not quite" of the form of $x^2 + y^2 + axy$.

§ 1.1 Tangent Angle Addition

Tangent angle addition is closely related with complex numbers.

Theorem 1 (Tangent Addition in the Complex Plane) Given reals a, b,

$$\tan(\arctan a + \arctan b) = \frac{\operatorname{Im}\left((1+ai)(1+bi)\right)}{\operatorname{Re}\left((1+ai)(1+bi)\right)}.$$

This is just another way to state the tangent addition formula, so why is it so powerful? It is because of the following corollary.

Corollary 1 Given reals $a_1, a_2, \ldots, a_n$,

$$\tan\left(\sum_{k=1}^{n} \arctan a_k\right) = \frac{\operatorname{Im}\left(\prod_{k=1}^{n}(1+a_k i)\right)}{\operatorname{Re}\left(\prod_{k=1}^{n}(1+a_k i)\right)}.$$

We did not prove the two-variable case before for two reasons: one, it follows easily after expanding $(1+ai)(1+bi)$, and secondly, the general proof is more informative.

Proof: We use the first half of the mantra from complex numbers: **angles add**. For $1 \le k \le n$, define $z_k = 1 + a_k i$. Then note that

$$\sum_{k=1}^{n} \arg z_k = \arg\left(\prod_{k=1}^{n} z_k\right)$$

by said mantra. Now note $\arg z_k = \arctan a_k$ and $z_k = 1 + a_k i$ by definition; this gives us the very obvious equation

$$\left(\sum_{k=1}^{n} \arctan a_k\right) = (\arg\left(\prod_{k=1}^{n}(1+a_k i)\right).$$

Taking the tangent of both sides gives

$$\tan\left(\sum_{k=1}^{n} \arctan a_k\right) = \frac{\operatorname{Im}\left(\prod_{k=1}^{n}(1+a_k i)\right)}{\operatorname{Re}\left(\prod_{k=1}^{n}(1+a_k i)\right)},$$

as desired. ■

§ 2 Examples

The problems in this unit fall into two categories - geometric and trigonometric.

§ 2.1 Geometric

Geometric problems are fake algebra problems that can be expressed geometrically. One such famous class of problems is the "implicit Law of Cosines."

Example 1 (Implicit Law of Cosines) Given

$$x^2 + xy + y^2 = a^2$$
$$y^2 + yz + z^2 = b^2$$
$$z^2 + zx + x^2 = c^2$$

for constants a, b, c, find the value of

$$xy + yz + xz.$$

Solution: Consider $\triangle ABC$ with point P in its interior satisfying

$$\angle APB = \angle BPC = \angle CPA = 120^\circ.$$

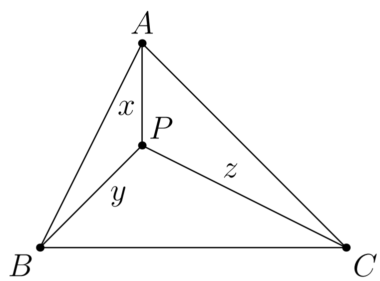
Then let $PA = x, PB = y$, and $PC = z$. By the Law of Cosines,

$$BC^2 = x^2 + xy + y^2 = a^2$$
$$CA^2 = y^2 + yz + z^2 = b^2$$
$$AB^2 = z^2 + zx + x^2 = c^2,$$

so the side lengths of $\triangle ABC$ are a, b, c. Now note that by the Sine Area Formula,

$$[ABC] = [PBC] + [PCA] + [PAB] = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} yz + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} zx + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} xy,$$

so the requested sum is $\frac{4}{\sqrt{3}}[ABC]$, where the specific values a, b, c can be used to determine the area. (Generally a, b, c will be contrived to give an easily computable area.)



The coefficients of xy, yz, zx need not be 1; they need only correspond to cosines of angles that add up to 360°.[1]

[1] To be more exact, $-\frac{1}{2}$ times the coefficients should correspond to cosines of angles that add to 360°.

§ 2.2 Trigonometric

Trigonometric problems are algebra problems that can be expressed trigonometrically. They are not "fake" algebra, despite the name of the unit.

Example 2 (HMMT Feb. Guts 2012/18) Let x and y be positive real numbers such that $x^2 + y^2 = 1$ and $(3x - 4x^3)(3y - 4y^3) = -\frac{1}{2}$. Compute $x + y$.

Solution: Let $x = \sin\alpha$ and $y = \cos\alpha = \sin(90^\circ - \alpha)$. Note that

$$(3x-4x^3)(3y-4y^3) = (4x^3-3x)(4y^3-3y) = \cos(3\alpha)\cos(3(90^\circ-\alpha)) = -\cos(3\alpha)\sin(3\alpha) = -\frac{1}{2}\sin(6\alpha) = -\frac{1}{2},$$

implying that $\alpha = 15^\circ$, so

$$x + y = \sin 15^\circ + \cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{6}}{2}.$$

Example 3 (CNCM R1/5) Positive reals $a, b, c \le 1$ satisfy $\frac{a+b+c-abc}{1-ab-bc-ca} = 1$. Find the minimum value of

$$\left(\frac{a+b}{1-ab} + \frac{b+c}{1-bc} + \frac{c+a}{1-ca}\right)^2.$$

Solution: Note that $\frac{a+b}{1-ab}$ looks suspiciously similar to $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.[2] This motivates substituting $a = \tan\alpha$, $b = \tan\beta$, and $c = \tan\gamma$. Now note that we want to maximize

$$|\tan(\alpha+\beta) + \tan(\beta+\gamma) + \tan(\gamma+\alpha),|$$

under the conditions that $0 < \alpha, \beta, \gamma \le \frac{\pi}{4}$ and $\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha} = 1$. But also note that

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha},$$

so $\tan(\alpha+\beta+\gamma) = 1$, implying $\alpha+\gamma+\beta = \frac{\pi}{4}$.

Now say $\alpha+\beta = x, \beta+\gamma = y, \gamma+\alpha = z$. Then we want to minimize $\tan x + \tan y + \tan z$, with the condition that $x + y + z = \frac{\pi}{2}$ and $0 < x, y, z \le \frac{\pi}{4}$. By Jensen's, $\tan x + \tan y + \tan z \ge 3\tan\left(\frac{x+y+z}{3}\right) = 3\tan\frac{\pi}{6} = \sqrt{3}$. So the answer is $(\sqrt{3})^2 = 3$.

The motivation for trying to expand $\tan(\alpha+\beta+\gamma)$ is that it seems likely to work, and nothing else seems workable. At this point the minimum is guessable.

[2] Alternatively, just recall the identity with the complex numbers representation of tangent angle addition.

§ 3 Problems

Minimum is [40✐]. Problems with the ♢ symbol are required.

> "Will there ever come a day when all my sins are forgiven?"
>
> My Home Hero

[2✐] **Problem 1** If $a < b < c < a + b$, order $\frac{b^2+c^2-a^2}{bc}, \frac{c^2+a^2-b^2}{ca}, \frac{a^2+b^2-c^2}{ab}$ in ascending order.

[2✐] **Problem 2** Prove that the A and B angle bisectors of a triangle are equal in length if and only if $BC = CA$.

[3✐] **Problem 3** (AIME 1986/2) Evaluate the product $(\sqrt{5}+\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})$.

[3✐] **Problem 4** Let x and y be real numbers such that $(x-5)^2 + (y-5)^2 = 18$. Determine the maximum value of $\frac{y}{x}$.

[3✐] **Problem 5** Let a, b, c be positive reals. Prove that $\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$.

[3✐] **Problem 6** Minimze $\sqrt{x^2 - 3x + 3} + \sqrt{y^2 - 3y + 3} + \sqrt{x^2 - \sqrt{3}xy + y^2}$ over the reals.

[3✐] **Problem 7** Prove that for reals $a, b \geq 1$,

$$\sqrt{a^2-1} + \sqrt{b^2-1} \leq ab.$$

[3✐] **Problem 8** What value of x maximizes $(21 + x)(1 + x)(x - 1)(21 - x)$, if x must be positive?

[4✐] **Problem 9** (TrinMaC 2020/19) Compute

$$\sum_{n=0}^{\infty} \cos^{-1}\left(\frac{\sqrt{n(n+1)(n+2)(n+3)}+1}{(n+1)(n+2)}\right).$$

[4♢] **Problem 10** Let a, b, c, d be real numbers such that $a^2 - b^2 - c^2 + d^2 = ad + bc$ and $a^2 + b^2 - c^2 - d^2 = 0$. Determine the value of $\frac{ab+cd}{ad+bc}$.

[4♢] **Problem 11 (AIME II 2006/15)** Given that x, y, and z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

[4✏] **Problem 12** Consider sequence a_n with $a_1 = \sqrt{3}$ and $a_n a_{n-1}^2 + 2a_{n-1} - a_n = 0$ for $n \geq 2$. Find a_{1000}.

[6✏] **Problem 13** (AIME 1991/15) For positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers whose sum is 17. There is a unique positive integer n for which S_n is also an integer. Find this n.

[6✏] **Problem 14** If x, y, z are positive numbers such that

$$x^2 + xy + \frac{1}{3}y^2 = 25$$

$$\frac{1}{3}y^2 + z^2 = 9$$

$$z^2 + zx + x^2 = 16,$$

find $xy + 2yz + 3zx$.

[9✏] **Problem 15** (HMMT 2014) Given a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$
$$b^2 + bc + c^2 = -2$$
$$c^2 + ca + a^2 = 1,$$

compute $(ab + bc + ca)^2$. (Here, $i = \sqrt{-1}$.)

[9✏] **Problem 16** Find all triples (x, y, z) such that $xy + yz + zx = 1$ and $5(x + \frac{1}{x}) = 12(y + \frac{1}{y}) = 13(z + \frac{1}{z})$.

[9✏] **Problem 17** (rd123/tworigami Mock AIME 2020/13) If a, b, c, d are positive real numbers such that

$$\begin{aligned} ab + cd &= 90, \\ ad + bc &= 108, \\ ac + bd &= 120, \\ a^2 + b^2 &= c^2 + d^2, \end{aligned}$$

and $a + b + c + d = \sqrt{n}$ for some integer n, find n.

[13✏] **Problem 18** (PUMaC 2018) Let a, b, c be nonzero reals such that $\frac{1}{abc} + \frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. The maximum possible value of

$$\frac{4}{a^2+1} + \frac{4}{b^2+1} + \frac{7}{c^2+1}$$

is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

[13✏] **Problem 19** (2018 Mock AIME, by TheUltimate123) Let a,b,c,d be positive real numbers such that

$$195 = a^2 + b^2 = c^2 + d^2 = \frac{13(ac+bd)^2}{13b^2 - 10bc + 13c^2} = \frac{5(ad+bc)^2}{5a^2 - 8ac + 5c^2}$$

Then $a + b + c + d$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

[13✏] **Problem 20** (Mildorf AIME) Let Ω denote the value of the sum

$$\sum_{k=1}^{40} \cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right).$$

The value of $\tan(\Omega)$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

[13 💎] **Problem 21 (IMO 2001/6)** Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.