

Logarithms

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APV

§ 1 Theory

We commonly have functions of the form $f(x) = x^n$, and to find $f^{-1}(x)$, we just take the n th root of both sides to get $\sqrt[n]{x} = f^{-1}(x)$. But how would we find the inverse of a function like $f(x) = n^x$? To do this, we create an inverse function known as a logarithm, where $n^{\log_n x} = x$.

Here are two examples to get you up to speed.

Example 1 Find $\log_2 8$.

Solution: Notice that $2^{\log_2 8} = 8 = 2^3$ by the definition of log, so $\log_2 8 = 3$.

Example 2 Simplify $\frac{\log_5 x}{\log_{25} x}$.

Solution: Let $25^a = x$. Then notice $5^{2a} = x$. Substituting yields $\frac{2a}{a} = 2$.

Here's a motivating exercise for what's going to come next.

Exercise 1 Evaluate $\log_2 16 + \log_2 32$, and then evaluate $\log_2 16 \cdot 32$.

§ 1.1 Fundamental Rules

The fundamental two rules of logarithms are the addition and subtraction rules. Notice that addition outside becomes multiplication inside (and inversely, subtraction becomes division inside). This is because of the way exponents behave: $x^{a+b} = x^a \cdot x^b$.

Theorem 1 (Logarithm Addition) For positive a, b, c , $\log_a b + \log_a c = \log_a bc$.

Proof: Notice that $a^{\log_a b + \log_a c} = a^{\log_a b} \cdot a^{\log_a c} = bc = a^{\log_a bc}$. Since the bases are the same, it follows the exponents are the same. ■

Theorem 2 (Logarithm Subtraction) For positive a, b, c , $\log_a b - \log_a c = \log_a \frac{b}{c}$.

Proof: This is a repeat of logarithm addition. Notice that $a^{\log_a b - \log_a c} = \frac{a^{\log_a b}}{a^{\log_a c}} = \frac{b}{c} = a^{\log_a \frac{b}{c}}$. ■

Notice that we're exploiting the properties of logarithms, and we're expressing everything without using logarithms as soon as possible. This trend will continue in AIME problems; once the logs have been removed, there's not much underneath to solve.

§ 1.2 Base Change

The base change rule allows you to express all logarithms in the same base; this is an extremely powerful, even if it doesn't look like much.

Theorem 3 (Base Change) For positive a, b, c , $\frac{\log_a b}{\log_a c} = \log_c b$.

Proof: Have $x = \log_a b$, $y = \log_a c$, and $z = \log_c b$. Notice that $a^x = b$, $a^y = c$, $c^z = b$. Then $(a^y)^z = a^x$, implying $yz = x$ or $\frac{x}{y} = z$. ■

We present the so-called logarithm chain rule as an exercise. (It's pretty useless and is only being presented as a check-up.)

Exercise 2 (Logarithm Chain Rule) For positive a, b, c, d , $(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$.

§ 2 Examples

Here are some examples of AIME logarithm problems. I want to re-iterate the following with these two problems: usually, **interpreting the log condition is the entire problem**.

Example 3 (AIME II 2009/2) Suppose that a, b , and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$


Solution: We notice that $a^{(\log_3 7)^2} = (a^{\log_3 7})^{\log_3 7}$. Similar expressions hold for b, c .

We then substitute $a^{\log_3 7} = 27$ as defined in the problem statement, and we do the same for b, c . This becomes $27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25} = 3^{3 \log_3 7} + 7^{2 \log_7 11} + 11^{\frac{1}{2} \log_{11} 25} = 7^3 + 11^2 + 25^{\frac{1}{2}}$. This simplifies to 469, which is our answer.

Example 4 (AIME I 2011/9) Suppose x is in the interval $[0, \pi/2]$ and $\log_{24 \sin x}(24 \cos x) = \frac{3}{2}$. Find $24 \cot^2 x$.

Solution: We can rewrite this as $(24 \sin x)^3 = (24 \cos x)^2$, which implies $24 \sin^3 x = \cos^2 x = 1 - \sin^2 x$. Thus we want to find the positive root of $24 \sin^3 x + \sin^2 x - 1 = 0$. Using the Rational Root Theorem (aka guessing), we see that $\frac{1}{3}$ is a root. Thus $\cot x = 2\sqrt{2}$ and our answer is 192.

§ 3 Problems

Minimum is [32]. Problems with the  symbol are required.

“I just long for a world in which ordinary things are done in an ordinary way.”

Psycho-Pass

[1] **Problem 1** (AIME II 2020/3) The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[2] **Problem 2** (AIME 1986/8) Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1000000. What is the integer nearest to S ?

[2] **Problem 3** (AIME I 2020/2) There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[2] **Problem 4** (AIME I 2007/7) Let $N = \sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor)$.

Find the remainder when N is divided by 1000. ($\lfloor k \rfloor$ is the greatest integer less than or equal to k , and $\lceil k \rceil$ is the least integer greater than or equal to k .)

[3] **Problem 5** (SMT 2020) If a is the only real number that satisfies $\log_{2020} a = 202020 - a$ and b is the only real number that satisfies $2020^b = 202020 - b$, what is the value of $a + b$?

[3] **Problem 6** (AIME II 2013/2) Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

[3] **Problem 7** (AIME II 2010/5) Positive numbers x , y , and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

[3] **Problem 8** (AIME I 2006/9) The sequence a_1, a_2, \dots is geometric with $a_1 = a$ and common ratio r , where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \dots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r) .

[4] **Problem 9** (HMMT 2020) Let $a = 256$. Find the unique real number $x > a^2$ such that


$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$


[4] **Problem 10** (AIME II 2007/12) The increasing geometric sequence x_0, x_1, x_2, \dots consists entirely of integral powers of 3. Given that $\sum_{n=0}^7 \log_3(x_n) = 308$ and $56 \leq \log_3 \left(\sum_{n=0}^7 x_n \right) \leq 57$, find $\log_3(x_{14})$.


[4] **Problem 11** (AIME I 2009/7) The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1} - a_n)} - 1 = \frac{1}{n + \frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k .

[4] **Problem 12** (AIME I 2010/14) For each positive integer n , let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x .


[6]  **Problem 13 (AIME I 2005/8)** The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

[6]  **Problem 14** (AIME I 2013/8) The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

[7]  **Problem 15** (hARMLess Mock ARML 2019/10) Compute the sum of all positive integers that can be expressed in the form

$$\log_b(404!) - \log_b(c),$$

where b and c are positive integers such that $b > 1$ and $b + c$ is odd.

[9]  **Problem 16** (AIME I 2012/9) Let x , y , and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p + q$.