Solutions to Modular Arithmetic

MAST

NQU

Contents

1 Unsourced

Find the inverse of $2 \pmod{p}$ for odd prime p in terms of p.

1.1 Solution

Note that $2 \cdot \left(\frac{p+1}{2}\right) \equiv 1 \pmod{p}$ for an odd prime p, so the answer is $\boldsymbol{\frac{p+1}{2}}$.

2 Unsourced

Find the remainder of 97! when divided by 101.

2.1 Solution

By Wilson's Theorem, since 101 is prime, $100! \equiv -1 \pmod{101}$. Then,

$$\begin{aligned} 97! \cdot 98 \cdot 99 \cdot 100 &\equiv -1 \\ 97! \cdot (-3) \cdot (-2) \cdot (-1) &\equiv -1 \\ 97! \cdot (-3) \cdot (-2) &\equiv 1 \\ 97! \cdot 6 &\equiv 1 \pmod{101}, \end{aligned}$$

and note that $6 \cdot 17 \equiv 102 \equiv 1 \pmod{101}$, so 17 is the inverse of $6 \pmod{101}$. Thus,

$$\begin{aligned} 97! \cdot 6 &\equiv 1 \\ 97! &\equiv 17 \pmod{101}, \end{aligned}$$

so the answer is **17**.

3 Unsourced

Find the remainder of $(p-2)!$ when divided by p, provided that p is prime.

3.1 Solution

By Wilson's Theorem, $(p-1)! \equiv -1 \pmod{p}$. Then,

$$\begin{aligned} (p-2)! \cdot (p-1) &\equiv -1 \\ (p-2)! \cdot (-1) &\equiv -1 \\ (p-2)! &\equiv 1 \pmod{p}, \end{aligned}$$

so the answer is **1**.

4 AMC 12A 2003/18

Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

4.1 Solution

Let $n = \overline{abcde}$ for digits a, b, c, d, e with $a \neq 0$. Note that $q = \overline{abc}$ and $r = \overline{de}$. Thus,

$$\begin{aligned} q + r &= \overline{abc} + \overline{de} \\ &= 100a + 10b + c + 10d + e \\ &= 100a + 10(b + d) + (c + e) \\ &\equiv a - b + c - d + e \pmod{11}. \end{aligned}$$

Now, we require $a - b + c - d + e \equiv 0 \pmod{11}$. By the divisibility rule for 11, this is equivalent to n being divisible by 11. The smallest 5-digit multiple of 11 is $11 \cdot 910 = 10010$, while the largest 5-digit multiple of 11 is $11 \cdot 9090 = 99990$, so the answer is $9090 - 910 + 1 = \mathbf{8181}$.

5 MAST Diagnostic 2020

How many integer values of $1 \leq x \leq 100$ makes $x^2 + 8x + 5$ divisible by 10?

5.1 Solution

Note that $x^2+8x+5 = (x+4)^2 - 11$. So, we are trying to find values of $1 \leq x \leq 100$ such that $(x+4)^2 - 11 \equiv 0 \pmod{10}$. Let $a = x+4$. Now, we are trying to find values of $5 \leq a \leq 104$ such that

$$\begin{aligned} a^2 - 11 &\equiv 0 \\ a^2 &\equiv 11 \\ a^2 &\equiv 1 \pmod{10}, \end{aligned}$$

which amounts to a having a units digit of 1 or 9. There are 10 values of $5 \leq a \leq 104$ such that a has a units digit of 1, and there are 10 values of a in the same interval with units digit 9. Thus, the answer is $10+10 = \mathbf{20}$.

6 1001 Problems in Number Theory

For which positive integers n is it true that $1 + 2 + \cdots + n \mid 1 \cdot 2 \cdot \cdots \cdot n$?

6.1 Solution

Suppose that n is odd. Since $n \mid n$ and $\frac{n+1}{2} \mid (n-1)!$, we have $\frac{n(n+1)}{2} \mid n!$. Suppose that n is even. We have the following:

1. Since n is relatively prime to $n+1$, $\frac{n}{2}$ is relatively prime to $n+1$.
2. $\frac{n}{2} \mid n!$.

Case 1. $n+1$ is prime (in particular, n is an odd prime in this case since we assumed that n is even).

Since $n+1 \nmid n!$, $\frac{n(n+1)}{2} \nmid n!$.

Case 2. $n+1$ is not a prime.

Since $n+1 \mid n!$, by (1) and (2), we have $\frac{n(n+1)}{2} \mid n!$.

Thus, the answer is n such that $n+1$ is not an odd prime.

7 Unsourced

What is the residue of $\frac{1}{1\cdot 2} \cdot \frac{1}{2\cdot 3} \cdot \cdots \cdot \frac{1}{11\cdot 12} \pmod{13}$?

7.1 Solution

Since 13 is prime, by Wilson's theorem, 12! is the inverse of 12!. Then,

$$\begin{aligned}\prod_{i=1}^{11} \frac{1}{i(i+1)} &\equiv \frac{1}{11! \cdot 12!} \\ &\equiv \frac{1}{11!} \cdot \frac{1}{12!} \\ &\equiv \frac{12}{12!} \cdot 12! \\ &\equiv 12 \pmod{13},\end{aligned}$$

so the answer is **12**.

8 AMC 10A 2020/18

Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $0, 1, 2, 3$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

8.1 Solution

We require ad and bc to be of opposite parity. We apply complementary counting. The total number of ordered pairs (a, b, c, d) without the parity restriction is $4 \cdot 4 \cdot 4 \cdot 4 = 256$. If ad and bc are of the same parity, then there are 12 choices for (a, d) and 12 choices for (b, d) (at least one variable in each term has to be even). Then, there are $12 \cdot 12 = 144$ ordered pairs (a, b, c, d) such that ad and bc have the same parity. Thus, the answer is $256 - 144 = \mathbf{112}$.

9 AMC 10B 2018/16

Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

9.1 Solution

By Fermat's Little Theorem,

$$\sum_{i=1}^{2018} a_i^3 \equiv \sum_{i=1}^{2018} a_i \equiv 2018^{2018} \equiv (-1)^{2018} \equiv 1 \pmod{3}.$$

Since $a_i \equiv a_i^3 \pmod{2}$,

$$\sum_{i=1}^{2018} a_i^3 \equiv \sum_{i=1}^{2018} a_i \equiv 2018^{2018} \equiv 0 \pmod{2}.$$

Thus,

$$\sum_{i=1}^{2018} a_i^3 \equiv 4 \pmod{6},$$

so the answer is **4**.

10 PUMaC 2018

Find the number of positive integers $n < 2018$ such that $25^n + 9^n$ is divisible by 13.

10.1 Solution

Note that

$$\begin{aligned} 0 &\equiv 25^n + 9^n \\ &\equiv (-1)^n + (-4)^n \\ &\equiv 1^n + 4^n \\ -1 &\equiv 4^n \pmod{13}. \end{aligned}$$

Since $k = 3$ is the smallest possible positive integer such that $4^k \equiv -1 \pmod{13}$, we know that $n = 3m$ for odd positive integers m. Thus, $n \in \{3, 9, 15, \ldots, 2013\}$, so the number of possible values of n is **336**.

11 Unsourced

Prove $\phi(n)$ is composite for $n \geq 7$.

11.1 Solution

12 AMC 10B 2019/14

The base-ten representation for $19!$ is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

12.1 Solution

Since $1000 \mid 19!$, $H=0$. Note that $0 \le M+T \le 18$ and $-9 \le M-T \le 9$. First, by the divisibility rule for 9, $33+M+T \equiv 0 \pmod 9$, so $M+T \in \{3,12\}$. Next, by the divisibility rule for 11, $7+M-T \equiv 0 \pmod{11}$, so $M-T \in \{-4,4\}$. Since $M+T$ and $M-T$ must have the same parity, $M+T$ must be even, so $M+T=12$. Thus, $T+M+H = 12+0 = \mathbf{12}$.

13 Unsourced

Find the remainder of $5^{31} + 5^{17} + 1$ when divided by 31.

13.1 Solution

By Fermat's Little Theorem, $5^{31} \equiv 5 \pmod{31}$. Then,

$$\begin{aligned} 5^{31} + 5^{17} + 1 &\equiv 5 + 5^{17} + 1 \\ &\equiv 5 + (5^3)^5 \cdot 5^2 + 1 \\ &\equiv 5 + (1)^5 \cdot 5^2 + 1 \\ &\equiv 5 + 25 + 1 \\ &\equiv 0 \pmod{31}, \end{aligned}$$

so the answer is **0**.

14 OMO 15-16 Spring/9

Let $f(n) = 1 \times 3 \times 5 \times \cdots \times (2n-1)$. Compute the remainder when $f(1) + f(2) + f(3) + \cdots + f(2016)$ is divided by 100.

14.1 Solution

15 Unsourced

Prove that the equation $x^2 + y^2 + z^2 = x + y + z + 1$ has no solutions over the rationals.

15.1 Solution

16 MAST Diagnostic 2021

Find the remainder of $(1^3)(1^3+2^3)(1^3+2^3+3^3)\dots(1^3+2^3+3^3\cdots+99^3)$ when divided by 101.

16.1 Solution

Note that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Then, the given expression is equivalent to

$$\left(\frac{1(2)}{2}\cdot\frac{2(3)}{2}\cdots\frac{99(100)}{2}\right)^2 = \left(\frac{99!\cdot 100!}{2^{99}}\right)^2.$$

By Wilson's Theorem, $100! \equiv -1 \pmod{101}$. Additionally,

$$\begin{aligned} -1 &\equiv 100! \\ -1 &\equiv 99!\cdot 100 \\ -1 &\equiv 99!\cdot -1 \\ 1 &\equiv 99! \pmod{101}. \end{aligned}$$

By Fermat's Little Theorem, $2^{100} \equiv 1 \pmod{101}$, so

$$\begin{aligned} \frac{1}{2^{99}} &\equiv \frac{2}{2^{100}} \\ &\equiv 2 \pmod{101}. \end{aligned}$$

Thus,

$$\left(\frac{99!\cdot 100!}{2^{99}}\right)^2 \equiv (1\cdot -1\cdot 2)^2 \equiv 4 \pmod{101},$$

so the answer is **4**.

17 Wolstenholme's Theorem

Prove that for all prime $p \geq 5$, we have $p^2 \mid (p-1)!\left(\sum_{i=1}^{p-1} \frac{1}{i}\right)$.

17.1 Solution

18 AIME 1989/9

One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer such that $133^5 + 110^5 + 84^5 + 27^5 = n^5$. Find the value of n.

18.1 Solution

By working modulo 3, we find that $n^5 \equiv 0 \pmod 3$, so $3 \mid n$. By working modulo 10, we find that the $n^5 \equiv 4 \pmod{10}$, so $n \equiv 4 \pmod{10}$. Thus, $n \in \{144, 174, 204, \dots\}$. However, note that

$$174^5 \approx (1.3 \cdot 133)^5 > 3.2 \cdot 133^5 > 133^5 + 133^5 + 133^5 + (0.2 \cdot 133)^5 > n^5,$$

so 174 is too large. Thus, the answer is **144**.

19 USAMO 1979/1

Determine all non-negative integral solutions $(n_1, n_2, \dots, n_k)$, if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$

19.1 Solution

We claim that there are no solutions. Indeed, taking mod 16 on both sides,

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 \equiv 15 \pmod{16}.$$

However, since $n_i^4 \in \{0, 1\} \pmod{16}$ for $1 \le i \le 14$, this is impossible.

20 AIME II 2017/8

Find the number of positive integers n less than 2017 such that

$$1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \frac{n^4}{4!} + \frac{n^5}{5!} + \frac{n^6}{6!}$$

is an integer.

20.1 Solution

21 IMO 1970/4

Find all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets so that the product of the numbers in each subset is equal.

21.1 Solution

22 IMO 2005/4

Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

22.1 Solution

23 AIME I 2013/15

Let N be the number of ordered triples (A, B, C) of integers satisfying the conditions

- $0 \le A < B < C \le 99$,
- there exist integers a, b, and c, and prime p where $0 \le b < a < c < p$,
- p divides $A - a$, $B - b$, and $C - c$, and
- each ordered triple (A, B, C) and each ordered triple (b, a, c) form arithmetic sequences.

Find N.

23.1 Solution

24 USEMO 2019/4

Prove that for any prime p, there exists a positive integer n such that

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \equiv 2020 \pmod{p}.$$

24.1 Solution

25 Unsourced

The expansion of $\frac{1}{7}$ is $0.\overline{142857}$, which is a repeating decimal with a 6 digit long sequence. How many digits long is the expansion of $\frac{1}{13}$?

25.1 Solution

Long division gives $\frac{1}{13} = 0.\overline{076923}$, so the answer is **6**.

26 Unsourced

We define the cycle of a repeating fraction $\frac{m}{n}$ as the minimum number i such that $\frac{m}{n} = 0.\overline{a_1 a_2 a_3 \dots a_i}$. Find the cycle of $\frac{1}{23}$.

26.1 Solution

It suffices to find the smallest positive integer i such that $10^i \equiv 1 \pmod{23}$. By Fermat's Little theorem, $i = 22$ is a valid solution. To prove that this is the smallest such value, we need to check 10^1, 10^2, and 10^{11}. The first two clearly don't work. For the third case,

$$10^{11} \equiv (10^2)^5 \cdot 10 \equiv 8^5 \cdot 10 \equiv 64^2 \cdot 8 \cdot 10 \equiv (-5)^2 \cdot 8 \cdot 10 \equiv 2 \cdot 80 \equiv 160 \equiv -1 \pmod{23}.$$

Thus, 10^{11} does not work, so the answer is **22**.

27 AMC 10A 2019/18

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323\ldots_k$. What is k?

27.1 Solution

Note that

$$\frac{2k+3}{k^2-1} = \frac{7}{51}.$$

Simplifying, we have $102k + 160 = 7k^2$. By inspection, we test a few values by verifying units digits before plugging in the entire guess, and find that $k = \mathbf{16}$.

28 e-dchen Mock MATHCOUNTS

What is the sum of all odd n such that $\frac{1}{n}$ expressed in base 8 is a repeating decimal with period 4?

28.1 Solution

29 AMC 12A 2014/23

The fraction

$$\frac{1}{99^2} = 0.\overline{b_{n-1}b_{n-2}\ldots b_2b_1b_0},$$

where n is the length of the period of the repeating decimal expansion. What is the sum $b_0 + b_1 + \cdots + b_{n-1}$?

29.1 Solution

30 AMC 12B 2016/22

For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. Find n.

30.1 Solution