Orders and Primitive Roots

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§ 1   Orders

We begin by reviewing some introductory theorems.

Theorem 1 (Fermat) $a^{p-1} \equiv 1 \pmod{p}$ whenever p is prime and $p \nmid a$.

Theorem 2 (Euler) $a^{\phi(n)} \equiv 1 \pmod{n}$ whenever $\gcd(a, n) = 1$.

(If you have forgotten the proofs for these theorems, try to reprove them as an exercise or refer to **NQU-Mod**.) Notice that both Fermat and Euler are *weak*.

Example 1 (Stronger Euler on 1000) Show that there exists some $x < 400$ such that

$$a^x \equiv 1 \pmod{1000}$$

for all a relatively prime to 1000.

Solution: We claim that $x = 100$ is a satisfactory value of x.
   Notice that by Euler,

$$a^{100} \equiv a^{100 \pmod{\phi(8)}} \equiv a^{100 \pmod 4} \equiv a^0 \equiv 1 \pmod 8$$

and that

$$a^{100} \equiv a^{\phi(125)} \equiv 1 \pmod{125}$$

whenever $\gcd(a, 1000) = 1$. Thus, by CRT,

$$a^{100} \equiv 1 \pmod{1000}$$

for all a relatively prime to 1000.
   Often, there is a number $x < \phi(n)$ such that $a^x \equiv 1 \pmod n$ for some a. In order to properly discuss this x, we define **orders.**

Definition 1 (Orders) Let a and n be two relatively prime integers with $n > 1$. Then $\mathrm{ord}_n(a)$ (the order of a mod n) is the smallest positive integer x such that $a^x \equiv 1 \pmod n$.

We immediately notice the following fact.

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Fact 1 For relatively prime integers a and m, $a^m \equiv 1 \pmod{n}$ if and only if $\mathrm{ord}_n(a) \mid m$.

Proof: Clearly, $a^m \equiv 1 \pmod{n}$ if $\mathrm{ord}_n(a) \mid m$ by definition, so the if condition holds.

Now, if $a^m \equiv 1 \pmod{n}$, we see that $m \geq \mathrm{ord}_n(a)$ by definition. Thus, from the division algorithm, there exist two integers q and r such that

$$m = q \cdot \mathrm{ord}_n(a) + r,$$

where $0 \leq r \leq \mathrm{ord}_n(a) - 1$ and $q > 0$. Since $a^{\mathrm{ord}_n(a)} \equiv 1 \pmod{n}$, we see $a^{-q \cdot \mathrm{ord}_n(a)} \equiv (1)^{-q} \equiv 1$.

Thus,

$$a^r \equiv a^{m - q \cdot \mathrm{ord}_n(a)} \equiv 1 \pmod{n},$$

so by the minimality of $\mathrm{ord}_n(a)$, r can't be positive (as $r \leq \mathrm{ord}_n(a) - 1$). Thus, we must have $r = 0$, meaning $\mathrm{ord}_n(a) \mid m$, completing the proof. ∎

This fact is one of the most useful facts relating to orders - it allows us to take the order of some random value and relate it to the overall modulus. In other words, it allows us to get **global** information from **local** information - something that is very powerful in many places. We will explore two of those examples.

Example 2 (Fermat's Christmas Theorem) Show that if a prime $p > 2$ can be written as the sum of two squares, we must have $p \equiv 1 \pmod{4}$.

Solution: Suppose that $p = x^2 + y^2$ for some positive integers x and y. Clearly, $p \nmid x$ and $p \nmid y$, as if p divided either x or y, we would have $x^2 + y^2 > p$.

Since $x^2 + y^2 = p$, we see $x^2 + y^2 \equiv 0 \pmod{p}$. Thus, $(xy^{-1})^2 \equiv -1 \pmod{p}$, and (by squaring), $(xy^{-1})^4 \equiv 1 \pmod{p}$. Thus, $\mathrm{ord}_p(xy^{-1}) \mid 4$.

Notice that if $\mathrm{ord}_p(xy^{-1}) \mid 2$, then we would have $(xy^{-1})^2 \equiv 1 \pmod{p}$, but as we showed earlier, $(xy^{-1})^2 \equiv -1 \pmod{p}$. Since $p > 2$, this is clearly absurd, so we must have $\mathrm{ord}_p(xy^{-1}) \nmid 2$. Since the only factor of 4 that doesn't divide 2 is 4, we must have $\mathrm{ord}_p(xy^{-1}) = 4$.

Now, from Fermat, $(xy^{-1})^{p-1} \equiv 1 \pmod{p}$. Thus, $\mathrm{ord}_p(xy^{-1}) = 4 \mid p - 1$, so $p \equiv 1 \pmod{4}$.

Observe how we did not *ever* try to find x, or y, or xy^{-1}. We only tried to find $\mathrm{ord}_p(xy^{-1})$. The idea of finding orders instead of variables is quite useful.

Oftentimes, this idea works, but sometimes, we need to use another idea − exploiting minimality.

Example 3 (China TST 2006 Quiz) Find all positive integers a and n such that

$$\frac{(a+1)^n - a^n}{n}$$

is an integer.

Solution: Assume for the sake of contradiction that $n > 1$. Let p be the smallest prime that divides n (p exists as $n > 1$). Since $\frac{(a+1)^n - a^n}{n} \in \mathbb{Z}$, we must have $(a+1)^n \equiv a^n \pmod{p}$. Thus (since a is clearly not a multiple of p), $((a+1)a^{-1})^n \equiv 1 \pmod{p}$, so $\mathrm{ord}_p((a+1)a^{-1}) \mid n$.

Observe that from Fermat, $\mathrm{ord}_p((a+1)a^{-1}) \mid p - 1$. Thus, $\mathrm{ord}_p((a+1)a^{-1}) \mid \gcd(p-1, n)$. Notice that if $\gcd(p-1, n) > 1$, then there is some prime $q < p$ that divides n, contradicting minimality of p. Thus, we must have $\gcd(p-1, n) = 1$, so $\mathrm{ord}_p((a+1)a^{-1}) = 1$.

Thus,

$$(a+1)a^{-1} \equiv 1 \pmod{p} \iff a+1 \equiv a \pmod{p} \iff 1 \equiv 0 \pmod{p},$$

a contradiction.

Thus, $n = 1$. Substituting that into the original expression, we see that

$$\frac{(a+1)^n - a^n}{n} = \frac{a + 1 - a}{1} = 1,$$

so $\frac{(a+1)^n - a^n}{n}$ is an integer whenever $\boldsymbol{n = 1.}$

There are two main takeaways from this problem that apply to most order problems.

✦ The modulus is the most important: Like in the previous example, notice that the method of solving this problem was "Find n" - not "find a" (even with primitive roots, the idea is to pick a instead of finding it) It's almost always a better idea to restrict the modulus than restrict the equivalent numbers in problems like these

✦ Minimality arguments: Notice how important the fact that p was the smallest prime factor of n was. Without it, the problem would be much more difficult.

§2 Primitive Roots

We know that we always have $\text{ord}_n(a) \leq \phi(n)$, but can we ever achieve the maximum? In other words, does there exist a value a for a certain n such that $\text{ord}_n(a) = \phi(n)$? What properties might this a have? In order to properly discuss these numbers, we define a **primitive root**.

Definition 2 (Primitive Root) Let a and n be two positive integers. a is called a primitive root modulo n if and only if $\text{ord}_n(a) = \phi(n)$.

Before discussing the applications of primitive roots, we prove that they always exist modulo p, where p is prime.

Definition 3 (Polynomial Ring) Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients.

Theorem 3 (Lagrange) Let $f(x) \in \mathbb{Z}[x]$ such that not all coefficients of f are multiples of some prime p. Then the equation

$$f(x) \equiv 0 \pmod{p}$$

has at most $\deg f$ incongruent solutions $(\text{mod } p)$.

Proof: We proceed with induction on $\deg f$.

Consider when $\deg f = 0$. Then, by definition, $f(x) = c$, where $p \nmid c$. Thus, the equation $f(x) \equiv 0$ $(\text{mod } p)$ has no solutions, so the claim holds in the base case.

Now, assume the claim holds for all polynomials of degree m for some $m \in \mathbb{N}$. We will show it holds for all polynomials of degree $m + 1$.

Consider some polynomial $f(x) \in \mathbb{Z}[x]$ with degree $m + 1$. If $f(x) \equiv 0$ $(\text{mod } p)$ has no solutions, the claim holds. Otherwise, assume that there exists some constant a such that $f(a) \equiv 0$ $(\text{mod } p)$. From the definition of modular arithmetic, there exists some integer q such that $f(a) - pq = 0$. From the remainder theorem, this means $x - a \mid f(x) - pq$.

Thus, there exists some $g(x) \in \mathbb{Z}[x]$ such that $f(x) = g(x) \cdot (x - a) + pq$. Thus, $f(x) \equiv g(x)(x - a)$ $(\text{mod } p)$, and since $\deg(x - a) = 1$, we have $\deg g = m$. Now, notice $g(x) \equiv 0$ $(\text{mod } p)$ has at most m solutions (inductive hypothesis) and $x - a \equiv 0$ $(\text{mod } p)$ has one solution. Thus, $f(x) \equiv 0$ $(\text{mod } p)$ has at most $m + 1$ solutions, completing the inductive step and finishing the proof. ∎

We can now show that primitive roots always exist modulo p where p is prime. In fact, we an prove something much stronger.

Theorem 4 (Number of Repeating Orders) Let p be a prime, and $d \mid p - 1$. Then there are exactly $\phi(d)$ elements with order d modulo p.

Proof: Consider the polynomial $x^d - 1$. Clearly, $x^{p-1} - 1 = (x^d - 1)\frac{x^{p-1}-1}{x^d-1}$. From the geometric series formula, $\frac{x^{p-1}-1}{x^d-1} \in \mathbb{Z}[x]$, and from Fermat, $x^{p-1} - 1 \equiv 0 \pmod{p}$ has $p - 1$ solutions.

Now, from Lagrange, $x^d - 1 \equiv 0 \pmod{p}$ has at most d non-congruent solutions $\pmod{p}$, and $\frac{x^{p-1}-1}{x^d-1} \equiv 0 \pmod{p}$ has at most $p - d$ non-congruent solutions $\pmod{p}$. Since $(x^d - 1)\frac{x^{p-1}-1}{x^d-1} \equiv 0$ $\pmod{p}$ has exactly p solutions $\pmod{p}$, $x^d - 1$ and $\frac{x^{p-1}-1}{x^d-1}$ must each respectively have exactly d and $p - d$ non-congruent solutions $\pmod{p}$.

Let $\Omega(q)$ be the number of prime factors of q counted with multiplicity, where $q \mid p - 1$. We will show by strong induction on $\Omega(q)$ that there are $\phi(q)$ non-congruent numbers which have order q modulo p.

If $\Omega(q) = 0$, $q = 1$. Clearly, there is only one number with order $\phi(1) = 1$ modulo p, proving the first base case.

If $\Omega(q) = 1$, q would be prime. Consider the number of solutions to $x^q - 1 \equiv 0 \pmod{p}$. From Fact 1, we know that $x^q - 1 \equiv 0 \pmod{p}$ if and only if $\text{ord}_p(x) \mid q$. Since q is prime, the number of solutions to $x^q - 1 \equiv 0 \pmod{p}$ is equal to the number of solutions to $\text{ord}_p(x) = 1$ plus the number of solutions to $\text{ord}_p(x) = q$. Since there is only one x such that $\text{ord}_p(x) = 1$ and $x^q - 1 \equiv 0 \pmod{p}$ has q solutions, there are $q - 1 = \phi(q)$ numbers with order q modulo p. Thus, the second base case is true.

Now, assume that for all $q \mid p - 1$ with $\Omega(q) \leq m$, $\text{ord}_p(x) = q$ has $\phi(q)$ solutions. We will show that for any $r \mid p - 1$ and $\Omega(r) = m + 1$, there are $\phi(r)$ solutions to $\text{ord}_p(x) = r$.

Let the proper divisors of r be $1, r_1, r_2, \ldots, r_n$. Consider the number of solutions to $x^r - 1 \equiv 0 \pmod{p}$. We know that the number of solutions to $x^r - 1 \equiv 0 \pmod{p}$ is equal to the number of solutions to $\text{ord}_p(x) = 1$ plus the number of solutions to $\text{ord}_p(x) = r_1$, ..., plus the number of solutions to $\text{ord}_p(x) = r$.

Clearly, $\Omega(r_i) \leq m$ for all $1 \leq i \leq n$. By the inductive hypothesis, there are $\phi(r_i)$ solutions to $\text{ord}_p(x) = r_i$ for all $1 \leq i \leq n$. From Fact 2, it follows that there are $\phi(r)$ solutions to $\text{ord}_p(x) = r$, completing the inductive step and finishing the proof. ∎

It turns out that primitive roots exist mod n if and only if n is either $2, 4, p^k$, or $2p^k$, where p is an odd prime and k is a positive integer. This will turn out to be very useful.

Proof: Let g^m and g^n be two distinct elements in $\{g, g^2, g^3, \ldots, g^{p-1}\}$. Notice that $g^m \not\equiv g^n \pmod{p}$, as if $g^m \equiv g^n \pmod{p}$, then we would have $p - 1 \mid m - n$. Thus, all the elements in $\{g, g^2, g^3, \ldots, g^{p-1}\}$ are distinct modulo p.

Thus, since there are $p - 1$ elements in $\{g, g^2, g^3, \ldots, g^{p-1}\}$ and only $p - 1$ non-zero residues modulo p, non-zero residues modulo p is equivalent to a certain element of the set $\{g, g^2, g^3, \ldots, g^{p-1}\} \pmod{p}$. Thus,

$$\{g, g^2, g^3, \ldots, g^{p-1}\} \equiv \{1, 2, 3, \ldots, p-1\} \pmod{p}.$$

∎

Primitive roots an often be used to convert questions dealing with the set $\{1, 2, 3, \ldots, p-1\}$ into ones which deal with the set $\{g, g^2, g^3, \ldots, g^{p-1}\}$ – a powerful exchange for many reasons.

They are also typically used when orders aren't powerful enough to solve a problem.

Example 4 (Primitive Root Problem) Find all positive two digit integers $\overline{ab}$ with $a \neq b$ such that $\overline{ab} \mid k^a - k^b$ for all integers k.

Solution: Let p be any prime that divides $\overline{ab}$, and let g be a primitive root modulo p.

Since we have $\overline{ab} \mid k^a - k^b$ for all integers k, we must have $p \mid \overline{ab} \mid g^a - g^b$, so $g^a \equiv g^b \pmod{p}$. Multiplying by g^{-b}, we get $g^{a-b} \equiv 1 \pmod{p}$, so since $\text{ord}_p(g) = p - 1$ (as x is a primitive root modulo p), we have $p - 1 \mid a - b$. Thus, since the maximum value of $|a - b|$ is 9 and $a - b \neq 0$, we see that

$$(p-1) + 1 \leq |a - b| + 1 \leq 10 \iff p \in \{2, 3, 5, 7\}.$$

Thus, the only primes that can divide $\overline{ab}$ when $a \neq b$ are $\{2, 3, 5, 7\}$, and if a prime p divides $\overline{ab}$, $p - 1 \mid a - b$. We proceed with casework

Case 1: $7 \mid \overline{ab}$.

If $7 \mid \overline{ab}$, then $6 \mid a - b$, so either $a = b + 6$ or $b = a + 6$. Thus, we must have $\overline{ab} \in \{17, 28, 39, 60, 71, 82, 93\}$, but since $7 \mid \overline{ab}$, we must have $\overline{ab} = 28$. Checking (with CRT and Euler), we see $\overline{ab} = 28$ works.

Case 2: $5 \mid \overline{ab}$ and $7 \nmid \overline{ab}$.

Clearly, we must have either $b = 0$ or $b = 5$. Since $5 \mid \overline{ab}$, we have $4 \mid a - b$, so we must have either $a = b + 4$, $a = b + 8$, $a = b - 4$, or $a = b - 8$. Thus, $\overline{ab} \in \{40, 80, 45, 85, 15\}$. We can't have $\overline{ab} = 45$ or 85, as if $\overline{ab} = 45$, then $3 \mid \overline{ab}$ but $2 \nmid a - b$, and if $\overline{ab} = 85$, then $17 \mid \overline{ab}$. Now, notice that if $\overline{ab} = 40$ or 80, then $k^a - 1 \neq 0 \pmod{8}$ whenever k is even, so we must have $\overline{ab} = 15$. Checking (with CRT and Euler), we see $\overline{ab} = 15$ works.

Case 3: $3 \mid \overline{ab}$ and $5 \nmid \overline{ab}$ and $7 \nmid \overline{ab}$.

Notice that we have $\overline{ab} = 3^p \cdot 2^q$, where $p > 1$. Thus, we have $\overline{ab} \in \{27, 81, 12, 18, 24, 36, 48, 54, 72, 96\}$. We can't have $\overline{ab} \in \{27, 81, 12, 18, 36, 54, 72, 96\}$, as then $3 \mid \overline{ab}$ but $2 \nmid a - b$. Thus, $\overline{ab} \in \{24, 48\}$. Now, notice that $\overline{ab} \neq 24$, since whenever $k \equiv 2 \pmod{8}$, $k^2 - k^4 \not\equiv 0 \pmod{8}$. Thus, we must have $\overline{ab} = 48$. Checking (with CRT and Euler), we see $\overline{ab} = 48$ works.

Case 4: 2 is the only prime that divides $\overline{ab}$.

Notice that we must have $\overline{ab} = 2^a$, where $a > 1$. Thus, $\overline{ab} \in \{16, 32, 64\}$, but notice that when this is true, $\overline{ab} \nmid 2^a - 2^b$. Thus, this case gives no solutions.

Thus, the solution set is $\overline{ab} \in \{\mathbf{15, 28, 48}\}.$

Understand why primitive roots were used and how they were used. If we have freedom to pick the values of our variables, it is often fruitful to use primitive roots.

§ 3   Problems

For some of the problems presented, it may be useful to know the **Lifting The Exponent Lemma**. We will not prove the lemma here. (If you want a thorough treatment of LTE, see Raymond Feng's **NRU-Prime**.)

> **Theorem 5 (Lifting The Exponent)** Let $v_p(n)$ where p is prime be the number such that $p^{v_p(n)} \mid n$ and $p^{v_p(n)+1} \nmid n$.
>
> - If an odd prime $p \mid a - b$ but $p \nmid a$ and $p \nmid b$, we have
>
> $$v_p(a^n - b^n) = v_p(a - b) + v_p(n).$$
>
> - If an odd prime $p \mid a + b$ but $p \nmid a$ and $p \nmid b$, we have
>
> $$v_p(a^n + b^n) = v_p(a + b) + v_p(n)$$
>
>   if n is odd, and
>
> $$v_p(a^n + b^n) = 0$$
>
>   if n is even.
>
> - if $2 \mid x - y$ but $2 \nmid x$ and $2 \nmid y$, then whenever $2 \mid n$, we have
>
> $$v_2(x^n - y^n) = v_2(x - y) + v_2(n) + v_2(x + y) - 1.$$

Minimum is [37✏]. Problems with the ◈ symbol are required.

> "The Mafia is grievously wounded – but not mortally."
>
> Five Families

[1✏] **Problem 1** Show $n \nmid 2^n - 1$ for all $n > 1$. (This is actually a weaker form of Example 3.)

[1✏] **Problem 2** (PUMaC Div. A NT 2020/1) Compute the last two digits of $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}}$.

[2✏] **Problem 3** (AIME I 2019/14) Find the least odd prime factor of $2019^8 + 1$.

[3◈] **Problem 4 (China TST 1993/1)** For all primes $p \geq 3$ such that $p - 1 \nmid 120$, define

$$F(p) = \sum_{k=1}^{\frac{p-1}{2}} k^{120}$$

and $f(p) = \frac{1}{2} - \left\{ \frac{F(p)}{p} \right\}$, where $\{x\} = x - [x]$, find the value of $f(p)$.

[3✏] **Problem 5** (Euler) Prove that all factors of $2^{2^n} + 1$ are of the form $k \cdot 2^{n+1} + 1$.

[3✏] **Problem 6** (PUMaC Div. A NT 2016/7) Compute the number of positive integers n between 2017 and 2017^2 such that $n^n \equiv 1 \pmod{2017}$, given that 2017 is prime.

[4◈] **Problem 7** Suppose p is a prime such that there exists an integer q such $q^2 \equiv -3 \pmod{p}$. Find all solutions to $x^3 \equiv 1 \pmod{p}$ in terms of p and q.

[6✎] **Problem 8** (Taiwan TST Quiz 2014/3J/2) Alice and Bob play the following game. They alternate selecting distinct nonzero digits (from 1 to 9) until they have chosen seven such digits, and then consider the resulting seven-digit number by concatenating the digits in the order selected, with the seventh digit appearing last (i.e. $\overline{A_1 B_2 A_3 B_4 A_6 B_6 A_7}$). Alice wins if and only if the resulting number is the last seven decimal digits of some perfect seventh power. Please determine which player has the winning strategy.

[6✎] **Problem 9** (ISL 2006/N5) Find all integral solutions to $\frac{x^7-1}{x-1} = y^5 - 1$.

[6✎] **Problem 10** (DIME 2020/14) For a positive integer n not divisible by 211, let $f(n)$ denote the smallest positive integer k such that $n^k - 1$ is divisible by 211. Find the remainder when

$$\sum_{n=1}^{210} n f(n)$$

is divided by 211.

[9✎] **Problem 11** (IMO 1990/3, part of IMO 1999/4) Find all positive integers n such that $n^2 \mid 2^n + 1$.

[6✎] **Problem 12** (IMO 1999/4) Find all the pairs of positive integers (x, p) such that p is a prime, $x \le 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

[9✎] **Problem 13** (Weak Dirichlet) Prove that there are infinitely many primes $p \equiv 1 \pmod{k}$ for any positive integer k.

[9✎] **Problem 14** (ISL 2012/N6) Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

[13✎] **Problem 15** (ISL 2003/N7) The sequence a_0, a_1, a_2, ... is defined as follows:

$$a_0 = 2, \qquad a_{k+1} = 2a_k^2 - 1 \quad \text{for } k \ge 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.