Solutions to Logarithms

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AQU

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§ 1 AIME II 2020/3

The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

§ 1.1 Solution

Note that $\log_2 3^{\frac{20}{x}} = \log_2 3^{\frac{2020}{x+3}}$, implying

$$\begin{aligned}\frac{20}{x} &= \frac{2020}{x+3}\\ 20x+60 &= 2020x\\ 60 &= 2000x\\ x &= \frac{3}{100}.\end{aligned}$$

Thus the answer is $3+100=103$.

§ 2 AIME 1986/8

Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1000000. What is the integer nearest to S?

§ 2.1 Solution

The log addition rule implies that $10^6 \cdot 10^S$ is the product of all of the divisors of 10^6. Since $10^6 = 2^6 \cdot 5^6$, 10^6 has $7 \cdot 7 = 49$ divisors, so $10^6 \cdot 10^S = (10^6)^{\frac{49}{2}} = 10^{147}$, implying $S = 141$.

§ 3 AIME I 2020/2

There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

§ 3.1 Solution

Let $x = 2^a$. This implies that $\frac{\frac{1}{3}(a+1)}{\frac{1}{2}a} = \frac{\frac{1}{2}a}{a}$, or $\frac{a+1}{3} = \frac{a}{4}$, or $\frac{1}{3} = \frac{-a}{12}$. Thus $a = -4$ and $x = 2^{-4} = \frac{1}{16}$, so the answer is $1 + 16 = 17$.

§ 4 AIME I 2007/7

Let $N = \sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor)$.

Find the remainder when N is divided by 1000. ($\lfloor k \rfloor$ is the greatest integer less than or equal to k, and $\lceil k \rceil$ is the least integer greater than or equal to k.)

§ 4.1 Solution

Note that $\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor = 1$ unless k is a power of 2. Then

$$N = \sum_{k=1}^{1000} k - \sum_{i=1}^{9} 2^i = 500 \cdot 1001 - 2^{10} + 1$$

$$N \equiv 500 - 24 + 1 \equiv 477 \pmod{1000}.$$

§ 5 SMT Algebra 2020/1

If a is the only real number that satisfies $\log_{2020} a = 202020 - a$ and b is the only real number that satisfies $2020^b = 202020 - b$, what is the value of $a + b$?

§ 5.1 Solution

Note that $b = \log_{2020} a$. Then the first equation implies

$$\log_{2020} a + a = 202020,$$

or

$$a + b = 202020.$$

§ 6 AIME II 2013/2

Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

§ 6.1 Solution

Note that this implies

$$\begin{aligned}\log_{2^a}(\log_{2^b}(2^{1000})) &= 1\\ \log_{2^b} 2^{1000} &= 2^a\\ \frac{1000}{b} &= 2^a.\end{aligned}$$

Thus the possible pairs are $(1, 500), (2, 250), (3, 125)$. So the answer is $(1+500)+(2+250)+(3+125) = 881$.

§ 7 AIME II 2010/5

Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

§ 7.1 Solution

Let $x = 10^a$, $y = 10^b$, and $z = 10^c$. Then $a + b + c = 81$ and $a(b + c) + bc = ab + bc + ca = 468$. Then

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{(a + b + c)^2 - 2(ab + bc + ca)} = \sqrt{81^2 - 2 \cdot 468} = 75.$$

§ 8 AIME I 2006/9

The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

§ 8.1 Solution

This implies

$$a^{12}b^{66} = 8^{2006}$$

$$a^2b^{11} = 2^{1003}.$$

If $a = 2^x$ and $b = 2^y$, then

$$2x + 11y = 1003.$$

Note that $1003 = 2 + 11 \cdot 91$, so the possible values of b are $0, 2, \cdots 90$, giving us 46 possible pairs.

§ 9 HMMT Februrary Algebra and Number Theory 2020/3

Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

§ 9.1 Solution

Let $\log_a x = k$ and $\log_a k = m$.

Note that

$$\log_a m = \log_{a^2} \log_{a^2}(\frac{1}{2}k) = \log_{a^2}(\frac{m}{2} - \frac{1}{16}),$$

implying $m^2 = \frac{m}{2} - \frac{1}{16}$, or $m = \frac{1}{4}$.

Since $\log_{256} k = \frac{1}{4}$, then $k = 4$, and since $\log_{256} x = 4$, then $x = 256^4 = 2^{32}$.

§ 10 AIME II 2007/12

The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that $\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \le \log_3\left(\sum_{n=0}^{7} x_n\right) \le 57$, find $\log_3(x_{14})$.

§ 10.1 Solution

Note that

$$x_7 \le \sum_{i=0}^{7} x_i \le 3x_7,$$

which implies

$$\log_3 x_7 \le \log_3(\sum_{i=0}^{7} x_i) \le 1 + \log_3 x_7,$$

so $\log_3 x_7 = 56$. Let $x_7 = a$ and $\frac{x_7}{x_6} = r$. Then

$$\frac{a^8}{r^2 8} = \frac{3^{56\cdot 8}}{r^2 8} = 3^{308},$$

implying $r = 3^5$. Then note $x_1 4 = x_7 \cdot r^7 = 3^{56} \cdot 3^{5\cdot 7} = 3^{91}$, so the answer is 91.

§ 11 AIME I 2009/7

The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} - 1 = \frac{1}{n+\frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

§ 11.1 Solution

This implies

$$5^{a_{n+1}-a_n} = 1 + \frac{1}{n+\frac{2}{3}}$$

$$a_{n+1} - a_n = \log_5(1 + \frac{1}{n+\frac{2}{3}}) = \log_5(\frac{3n+5}{3n+2})$$

$$a_{n+1} - a_n = \log_5(3n+5) - \log_5(3n+2).$$

Note that $(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1) = a_n - a_1 = \log_5(3n+5) - 1$. So $3n+2$ must be a power of 5 greater than 5. Since $5^2 \equiv 1 \pmod 5$, 25 doesn't work. So

$$125 = 3n + 2$$

$$123 = 3n$$

$$41 = n.$$

§ 12 AIME I 2020/14

For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

§ 12.1 Solution

Note that $f(n)$ is monotonously increasing. The average value of each term should be roughly 3, so n is around 100. Since $f(109) = 300$ and $f(110) > 300$, 109 is the answer.

Comment: As far as I'm aware, there's no good way to do this problem.

§ 13 AIME I 2005/8

The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

§ 13.1 Solution

Let $y = 2^{111x}$. Then

$$\frac{1}{4}y^3 + 4y = 2y^2 + 1$$
$$y^3 - 8y^2 + 16y - 4 = 0$$

We want to find

$$\log_{2^{111}}(y_1y_2y_3) = \log_{2^{111}}(4) = \frac{2}{111},$$

so the answer is $2 + 111 = 113$.

§ 14 AIME I 2013/8

The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

§ 14.1 Solution

This implies $-1 \le \log_m(nx) \le 1$, or

$$\frac{1}{n} \le mx \le n$$

$$\frac{1}{mn} \le x \le \frac{n}{m}$$

So the domain has length $\frac{m^2-1}{mn} = \frac{1}{2013}$. So to minimize $m + n$, we minimize m. We must have $m|2013$ and $m > 1$, so the smallest possible m is $m = 3$. We plug this in and find $\frac{8}{3n} = \frac{1}{2013}$, implying $n = 5368$. So the minimum $m + n$ is 5371, and thus the answer is 371.

§ 15 AIME I 2012/9

Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p+q$.

§ 15.1 Solution

Note that this is the same as

$$\frac{\log(4y^2)}{\log(x)} = \frac{\log(16z^2)}{\log(2x)} = \frac{\log(8yz)}{\log(2x^4)}.$$

Since $\log(4y^2), \log(8yz), \log(16z^2)$ is a geometric series, so is $\log(x), \log(2x^4), \log(2x)$. Thus $2x^4 = \sqrt{x(2x)}$, implying $x = 2^{\frac{-1}{6}}$.

Then plugging the value of x into the first two equations yields

$$-6\log_2(2y) = \frac{6}{5}\log_2(4z),$$

implying

$$-5\log_2(y) - 5 = \log_2(z) + 2$$
$$-7 = \log_2(y^5z)$$
$$y^5z = \frac{1}{2^7}$$

So $xy^5z = \frac{1}{2^7 \cdot 2^{\frac{1}{6}}} = \frac{1}{2^{\frac{43}{6}}}$. Thus the answer is 49.

Comment: The answer is much easier to get if you let $2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) = 2$. Some meta-reasoning as to why this is okay: The problem never specifies what the three expressions are equal to, so it's either a fixed value or you can set it to anything you want. If it was the former, it'd be more likely that the problem would ask for the fixed value.