Bases

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We examine problems either explicitly or implicitly using different base systems.

§ 1 What is a Base?

The numerical system we use is commonly known as base 10. For example, 1234 in base 10 is $1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0$. But there's nothing special about 10. We could choose to represent numbers like $1 \cdot 8^3 + 2 \cdot 8^2 + 3 \cdot 8^1 + 4 \cdot 8^0$, or really like $1 \cdot n^3 + 2 \cdot n^2 + 3 \cdot n^1 + 4 \cdot n^0$. Remember that the base must be greater than the value of the largest digit. (For example, 658_7 is absurd because $7 \leq 8$.)

Definition 1 (Base b) In general, the number $\overline{a_n a_{n-1} \dots a_1 a_0}_b = a_n \cdot b^n + a_{n-1} \cdot b^{n-1} + \dots + a_1 \cdot b^1 + a_0 \cdot b^0$. We say that this number is written in base b.

§ 2 Period of a Repeating Decimal

You may know that $\frac{1}{7} = 0.\overline{142857}$ has a period of 6 digits. But what if I asked you to find the period of, say, $\frac{1}{997}$? You probably wouldn't want to divide it by hand. So how would you find it?

Example 1 (Period of $\frac{1}{13}$) Find the period of the decimal expansion of $\frac{1}{13}$.

Solution: We use some number theory concepts and a little bit of clever algebra here.

Let the decimal expansion of $\frac{1}{13}$ be $0.\overline{a_1 a_2 \cdots a_n}$. Then note that $\frac{10^n}{13} = a_1 a_2 \cdots a_n.\overline{a_1 a_2 \cdots a_n}$. Thus $\frac{10^n - 1}{13} = a_1 a_2 \cdots a_n$, implying that $13 | 10^n - 1$, so we want to find the smallest n such that $13 | 10^n - 1$. Note that $13 | 10^6 - 1$, since $10^6 - 1 = 999 \cdot 1001$, and $13 | 1001$. (We can check the proper divisors of 6 and note none of them work.)

Sometimes expressing a recursively defined sequence in some nice base can yield useful information about the sequence. This is best explained through an example.

Example 2 (2014 HMMT Feb) We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\left\lfloor \frac{x}{2} \right\rfloor$, while pressing the second button replaces x with $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence.)

Solution: Write the number displayed in base 2; the first button deletes the rightmost digit while the second button adds the string 01 to the right of the number. Note that we thus may achieve any number that has no consecutive 1's in its binary representation; call a number with this property a *nice number*. Since $2014_{10} = 11111011110_2$, and because all such nice numbers that have 11 digits are at most 10101010101_2, the problem is equivalent to finding all nice numbers that have 11 or fewer digits. The last part is a combinatorics problem; by considering the last digit of each nice number, we may find by recursion that the number of nice numbers with n or fewer digits is F_{n+2} where F_i is the i-th Fibonacci number - we quickly find that our answer is $F_{13} = 233$.

§ 3 Problems

Minimum is [34 ✎]. Problems with the ◇ symbol are required.

> "A person is smart. People are dumb."
>
> Kaguya-sama

[1 ◇] **Problem 1 (AMC 10A 2021/11)** For which of the following integers b is the base-b number $2021_b - 221_b$ not divisible by 3?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 8

[2 ✎] **Problem 2** (SMT 2012) Find the sum of all integers $x \geq 3$ such that

$$201020112012_x$$

is divisible by $x - 1$.

[2 ✎] **Problem 3** (PAMO 2003/3) Does there exists a base in which the numbers of the form:

$$10101, 101010101, 1010101010101, \cdots$$

are all prime numbers?

[2 ✎] **Problem 4** (AIME I 2020/3) A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

[3 ✎] **Problem 5** (ARML 2000) For an integer k in base 10, let $z(k)$ be the number of zeros that appear in the binary representation of k. Let $S_n = \sum_{k=1}^{n} x(k)$. Find S_{256}.

[3 ✎] **Problem 6** (AIME 2008 II/4) There exist r unique nonnegative integers $n_1 > n_2 > \cdots > n_r$ and r unique integers a_k $(1 \leq k \leq r)$ with each a_k either 1 or -1 such that

$$a_1 3^{n_1} + a_2 3^{n_2} + \cdots + a_r 3^{n_r} = 2008.$$

Find $n_1 + n_2 + \cdots + n_r$.

[3 ✎] **Problem 7** (AIME I 2001/8) Call a positive integer N a *7-10 double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

[4 ✎] **Problem 8** (AMC 10A 2018/25) For a positive integer n and nonzero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b, and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a + b + c$ for which there are at least two values of n such that $C_n - B_n = A_n^2$?

[6 ✎] **Problem 9** (AHSME 1993/30) Given $0 \leq x_0 < 1$, let

$$x_n = \begin{cases} 2x_{n-1} & \text{if } 2x_{n-1} < 1 \\ 2x_{n-1} - 1 & \text{if } 2x_{n-1} \geq 1 \end{cases}$$

for all integers $n > 0$. For how many x_0 is it true that $x_0 = x_5$?

[6✏] **Problem 10** (e-dchen Mock MATHCOUNTS) Find the sum of all odd n such that $\frac{1}{n}$ expressed in base 8 is a repeating decimal with period 4.

[6✏] **Problem 11** (HMMT 2002) A sequence $s_0, s_1, \cdots$ is defined by $s_0 = s_1 = 1$ and the following relations: $s_{2n} = s_n, s_{4n+1} = s_{2n+1}$, and $s_{4n-1} = s_{2n-1} + s_{2n-1}^2/s_{n-1}$. What is the value of s_{1000}?

[6✏] **Problem 12** (AIME II 2014/15) For any integer $k \geq 1$, let $p(k)$ be the smallest prime witch does not divide k. Define the integer function $X(k)$ to be the product of all primes less than $p(k)$ if $p(k) > 2$, and $X(k) = 1$ if $p(k) = 2$. Let $\{x_n\}$ be the sequence defined by $x_0 = 1$ and $x_{n+1}X(x_n) = x_n p(x_n)$ for $n \geq 0$. Find the smallest positive integer t such that $x_t = 2090$.

[9✏] **Problem 13** (SMT 2018) A sequence is defined as follows. Given a term a_n, we define the next term a_{n+1} as

$$\begin{cases} \frac{a_n}{2} & 2|a_n \\ a_n - 1 & 2 \nmid a_n \end{cases}$$

The sequence terminates when $a_n = 1$. Let $P(x)$ be the number of terms in such a sequence with initial term x. For instance, $P(7) = 5$ because its corredsponding sequence is $7, 6, 3, 2, 1$. Evaluate $P(2^{2018} - 2018)$.

[13✏] **Problem 14** (AIME II 2000/14) Every positive integer k has a unique factorial base expansion $(f_1, f_2, f_3, \ldots, f_m)$, meaning that $k = 1! \cdot f_1 + 2! \cdot f_2 + 3! \cdot f_3 + \cdots + m! \cdot f_m$, where each f_i is an integer, $0 \leq f_i \leq i$, and $0 < f_m$. Given that $(f_1, f_2, f_3, \ldots, f_j)$ is the factorial base expansion of $16! - 32! + 48! - 64! + \cdots + 1968! - 1984! + 2000!$, find the value of $f_1 - f_2 + f_3 - f_4 + \cdots + (-1)^{j+1} f_j$.