Solutions to Factoring a Polynomial

Dennis Chen

AQU

Contents

§ 1 MATHCOUNTS State 2020/Target/6

What is the value of $\sqrt{111,111,111 \cdot 1,000,000,011 + 4}$?

§ 1.1 Solution

Note that

$$\sqrt{111,111,111 \cdot 1,000,000,011 + 4} =$$
$$\sqrt{333,333,333 \cdot 333,333,337 + 4} =$$
$$\sqrt{(333,333,335 - 2)(333,333,335 + 2) + 4} =$$
$$333,333,335.$$

§ 2 Unsourced

Find $\frac{1999^3-1000^3-999^3}{1999\cdot 1000\cdot 999}$.

§ 2.1 Solution

Note $1999^3-1000^3-999^3-3\cdot 1999\cdot(-1000)\cdot(-999)=0$, since $1999-1000-999=0$. Thus $\frac{1999^3-1000^3-999^3}{3\cdot 1999\cdot 1000\cdot 999}=\frac{3\cdot 1999\cdot 1000\cdot 999}{1999\cdot 1000\cdot 999}=3$.

§ 3 PAMO 2003/3

Does there exists a base in which the numbers of the form:

$$10101, 101010101, 1010101010101, \cdots$$

are all prime numbers?

§ 3.1 Solution

No. Note that the first number is $b^4 + b^2 + 1 = (b^2 - b + 1)(b^2 + b + 1)$.

§ 4 Dennis Chen

Find all constants r such that $a - r | ar^2 + ar - 17a + 15$.

§ 4.1 Solution

Substitute $a = r$. The remainder is $r^3 + r^2 - 17r + 15 = (r+5)(r-1)(r-3)$. Thus the roots are $r = -5, 1, 3$.

§ 5 AIME 1985/3

Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$, where $i^2 = -1$.

§ 5.1 Solution

This implies that we want the imaginary term of $(a+bi)^3$ to be 107. Note that the imaginary part of $(a+bi)^3$ is $a^2bi - b^3i$, so $3a^2b - b^3 = b(3a^2 - b^2) = 107$. Since 107 is prime, we must have $b = 1$ or $b = 107$. We check that only the first case works since $107^2 + 1 \equiv 2 \pmod 3$, so $a = \sqrt{\frac{107+1^2}{3}} = 6$. Then note the real part of $(6+i)^3$ is $6^3 - 3 \cdot 6 = 198$.

§ 6 AMC 10B 2020/22

What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

§ 6.1 Solution

Let $2^{50} = x$. We want to find the remainder of $4x^4 + 202$ divded by $2x^2 + 2x + 1$. Long division gives 201.

§ 7 AHSME 1969/34

Find the remainder when x^{100} is divided by $x^2 - 3x + 2$.

§ 7.1 Solution

Note $x^2 - 3x + 2 = (x-1)(x-2)$. By Remainder Theorem,

$$x^{100} \equiv 1 \equiv x(2^{100} - 1) + (-2^{100} + 2) \pmod{x-1}$$

$$x^{100} \equiv 2^{100} \equiv x(2^{100} - 1) + (-2^{100} + 2) \pmod{x-2}$$

so

$$x^{100} \equiv x(2^{100} - 1) + (-2^{100} + 2) \pmod{x^2 - 3x + 2}$$

where the final step is motivated by wanting to manipulate the constants of the first two modular congruences so that they are identical.

§ 8 e-dchen Mock MATHCOUNTS

For any ordered pair of integers (a, b) such that $a, b \notin \{1, 2 \ldots 8\}$, $a \neq b$, and the remainder of

$$f(x) = (x-1)(x-2)(x-3)\ldots(x-8)$$

when divided by $x - a$ and $x - b$ are the same, find $a + b$.

§ 8.1 Solution

Notice that $f(a) = f(b)$. Without loss of generality, let $a \geq b$. Then notice that for $|f(a)| = |f(b)|$, we desire $|a - 8| = |b - 1|$. Since we cannot have $a, b \leq 1$ and $a \geq b$, we have $a > 8$. (This all arises from our problem conditions.) Then $|a - 8| = a - 8$. But by similar reasoning, we have $b < 1$, so $|b - 1| = 1 - b$. This yields $a - 8 = 1 - b \rightarrow a = 9 - b$, implying $a + b = 9 - b + b = 9$.

§ 9 AIME 1991/1

Find $x^2 + y^2$ if x and y are positive integers such that

$$xy + x + y = 71$$

$$x^2y + xy^2 = 880.$$

§ 9.1 Solution

Let $a = x+y$ and note that from the first equation, $xy = 71-a$. So $x^2y+xy^2 = xy(x+y) = (71-a)a = 880$. Since exactly one of $a, 71-a$ are even and $880 = 16 \cdot 55$, we must either have $a = 16$ or $a = 55$. The former must be correct since $xy \geq \max(x, y)$. Then note $x^2 + y^2 = (x+y)^2 - 2xy = 16^2 - 2 \cdot 55 = 146$.

§ 10 AIME I 2015/3

There is a prime number p such that $16p+1$ is the cube of a positive integer. Find p.

§ 10.1 Solution

Let $16p+1=a^3$. Then $16p=(a-1)(a^2+a+1)$. Note that a^2+a+1 is always odd since $a(a+1)$ is always even, so $a-1=16$. (We can check that $p=2$ doesn't work.) Thus $a=17$ and $p=\frac{17^3-1}{16}=17^2+17+1=307$.

§ 11 AIME 1987/14

Compute

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}.$$

§ 11.1 Solution

Note that by Sophie Germain, $n^4+4\cdot 3^4 = (n^2+2\cdot 3^2-2\cdot 3\cdot n)(n^2+2\cdot 3^2+2\cdot 3\cdot n) = (n^2-6n+18)(n^2+6n+18) = ((n-3)^2+9)((n+3)^2+9)$. So the fraction is equivalent to

$$\prod_{i=0}^{4} \frac{(10+i)^4}{(4+i)^4} =$$

$$\prod_{i=0}^{4} \frac{((7+12i)^2+9)((13+12i)^2+9)}{((1+12i)^2+9)((7+12i)^2+9)}$$

which telescopes to $\frac{(61^2+9)}{(1^2+9)} = \frac{3730}{10} = 373$.

§ 12 Dennis Chen

Consider cubic $p(x)$ such that $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 0$. Find $p(5)$.

§ 12.1 Solution

Let $Q(x) = P(x) - x$. Then note $Q(1) = Q(2) = Q(3) = 0$ and $Q(4) = -4$, so $Q(x) = c(x-1)(x-2)(x-3)$. Also note that $Q(4) = 6c = -4$, so $c = -\frac{2}{3}$. Thus $P(5) = Q(5) + 5 = -\frac{2}{3} \cdot 4 \cdot 3 \cdot 2 + 5 = -11$.

§ 13 JMC 10 2020/22

What is the remainder of $17^7 + 17^2 + 1$ when divided by 307^2?

§ 13.1 Solution

Note $307 = 17^2+17+1$. Let $17 = x$. Then we want to find the remainder of x^7+x^2+1 divided by $(x^2+x+1)^2$. Note that $x^2 + x + 1 = \frac{x^3-1}{x-1}$, so $\frac{x^7+x^2+1}{x^2+x+1} = (x-1)\frac{(x^7-x)}{x^3-1} + 1 = x(x-1)(x^3+1) + 1$. The remainder when dividing by $\frac{x^3-1}{x-1}$ again is $x(x-1)(2) + 1 = 2x^2 - 2x + 1 = -4x - 1$.

Thus the remainder of $\frac{17^7+17^2+1}{307}$ divided by 307 is $307 - 4 \cdot 17 - 1 = 238$, so the remainder of 17^7+17^2+1 divided by 307^2 is $238 \cdot 307 = 73066$.

§ 14 AIME I 2013/5

The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c}$, where a, b, and c are positive integers. Find $a+b+c$.

§ 14.1 Solution

This implies $9x^3 = x^3 + 3x^3 + 3x + 1 = (x+1)^3$, or $\sqrt[3]{9}x = x + 1$. Thus $x(\sqrt[3]{9} - 1) = 1$, implying

$$x = \frac{1}{\sqrt[3]{9}-1} = \frac{\sqrt[3]{9}^2 + \sqrt[3]{9} + 1}{(\sqrt[3]{9}-1)(\sqrt[3]{9}^2 + \sqrt[3]{9} + 1)} = \frac{\sqrt[3]{81} + \sqrt[3]{9} + 1}{8},$$

so the answer is $81 + 9 + 8 = 98$.

§ 15 AIME 1998/13

Find a if a and b are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.

§ 15.1 Solution

Note the roots of $x^2 - x - 1 = 0$ are $x = \frac{1 \pm \sqrt{5}}{2}$. Then note that $x^{16}(ax + b) = -1$ for both of these values of x. Let $(\frac{1+\sqrt{5}}{2})^{16} = x + y\sqrt{5}$ for rational x, y, and note that $(\frac{1-\sqrt{5}}{2})^{16} = x - y\sqrt{5}$. Then we solve the system of equations

$$(x + y\sqrt{5})(\frac{a + 2b}{2} + \frac{a\sqrt{5}}{2}) = -1$$

$$(x - y\sqrt{5})(\frac{a + 2b}{2} - \frac{a\sqrt{5}}{2}) = -1.$$

We note this is secretly equivalent to just solving the first one. The irrational term being 0 implies that $ya + 2b + xa = 0$, or $a = -\frac{2b}{x+y}$, and the rational term being -1 implies that $xa + 2xb + 5ya = -2$, or $xb - b + \frac{4yb}{x+y} = -1$, or $b = \frac{-1}{x-1+\frac{4y}{x+y}}$. All that is left to do is to painstakingly bash out $(\frac{1+\sqrt{5}}{2})^{16} = \frac{2207}{2} + \frac{987\sqrt{5}}{2}$, which gives us $b = -1597$ and $a = 987$. Thus the answer is 987.

§ 16 AIME II 2000/13

The equation $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ has exactly two real roots, one of which is $\frac{m+\sqrt{n}}{r}$, where m, n and r are integers, m and r are relatively prime, and $r > 0$. Find $m + n + r$.

§ 16.1 Solution

Note the equation implies

$$2(1000x^6 - 1) + x(100x^4 + 10x^2 + 1) =$$
$$2((10x^2)^3 - 1^3) + x(100x^4 + 10x^2 + 1) =$$
$$2(10x^2 - 1)(100x^4 + 10x^2 + 1) + x(100x^4 + 10x^2 + 1) =$$
$$(20x^2 + x - 2)(100x^4 + 10x^2 + 1).$$

Note that $100x^4 + x^2 + 1 \geq 1 > 0$ by the Trivial Inequality. So we find the larger root of $20x^2 + x - 2$ by the Quadratic Formula, which is $\frac{-1+\sqrt{161}}{40}$. Thus the answer is $-1 + 161 + 40 = 200$.