The MAST Compendium

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Dedication

To the students who made this possible,
thank you.

Foreword

Preface

This book is a collection of the materials that I've used in my program, which I have fortuitously named MAST. The first name I came up with was Mid-AIME Training Program (MATP). I did not initially intend on coming up with a funny acronym for my program, but after seeing the letters "MAT" appear in succession, I knew that I could potentially score some extra points by making my acronym happen to be a word. Thus "Mid-AIME Self-Training" (MAST) was born. I wanted to make the base of the program wider and give it more potential to spread without rebranding, and the current name felt lacking somehow, so "Mid-AIME" became "Math Advancement (by)." And thus the final name "Math Advancement by Self-Training" was born.

MAST is not the progenitor of these materials. In the fall of 2017, during my middle school years, I ran a program for a small number of my classmates with the abbreviation MPP, though I presently cannot recall what the letters stand for. Looking back, it is a miracle that the program happened in the first place – indeed, it is odd that my classmates got their parents to agree to let them stay in a run-down garage for two hours a week – but I was not satisfied with the shape it had taken. As high school started in the fall of 2019, few of my classmates returned to the program during the third year as they expected high school to keep them busy, and the gap was being filled up by younger students. During the March of 2020, a variety of factors pushed and emboldened me to start MAST: I had almost been guaranteed to qualify for the upcoming USAJMO (it had not been cancelled at the time, and though in retrospect I was embarrassingly close to missing the qualification, I was unaware that it would be so hard to qualify this year), and I believed that the dwindling participation in MPP which so dispirited me would be fixed if only I had more dedicated students. These materials and this program had already been fundamentally transformed once, and not for the better. The "spirit" of the old program was no longer worth preserving, and with that, I made the decision to start the program and advertise it on the Art of Problem Solving website.

Initially I was extremely apprehensive of how the program was going to turn out. I did not envision, for better of for worse, that the program would garner as much interest as it had. The rolling application model I had worked to the program's favor, keeping the growth constant – quite impressive given the way most programs fizzled out after a few weeks. I quickly realized through my growing inability to reply to emails that keeping track of everything manually was impossible, so I put the program on hiatus and had Amol Rama design a website which could do this for me. This turned out to be a major improvement for keeping track of things, particularly in comparison to my habitual failure to respond to emails without having to be reminded that they exist. I thank all my students who took the pains to apply to my program and stuck with it despite the inconveniences that I may have put them through.

In parallel with the growth of the student base was the formation of the staff team. A few students noticed early on, when the direction of MAST was still very much up in the air, that they would be better suited on the staff team and were thus invited to join it. Still others sent me emails or messages asking if they could help somehow with the program. At the time growing the staff team so fast seemed like quite a gamble, but in retrospect the atmosphere I had built around the program made it so that all who asked to help out were qualified to do so (and consequently, nearly every staff request was accepted). I am very glad that I ended up making the right call here, and I hope the staff are also glad that they decided to participate in this program. My gratitude goes it to the staff team, for without them I could not have taken on as many students as I can now, and without them I would not have found my footing or have a clear vision for my program.

As the program continues to grow and my busywork does as well, my efforts to keep up with the work I need to do have been in vain. I think the most appropriate remark to end the preface with must be "Don't worry! The microphone won't catch anything," which I said in response to the general hubbub around my house, unaware that this had ironically been caught on the microphone. Though many of my overconfident assertions have not held true and embarrassing mistakes have been made (and will continue to be made), I thank my students for staying through the program and recommending it to their friends despite this, the staff team for supporting me through my many incompetencies, and you, the reader, for making this compendium worth compiling.

How to Use

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Logarithms

Chapter 1

1.1 Theory

We commonly have functions of the form $f(x) = x^n$, and to find $f^{-1}(x)$, we just take the nth root of both sides to get $\sqrt[n]{x} = f^{-1}(x)$. But how would we find the inverse of a function like $f(x) = n^x$? To do this, we create an inverse function known as a logarithm, where $n^{\log_n x} = x$.

Here are two examples to get you up to speed.

Example 1.1.1
Find $\log_2 8$.

Solution: Notice that $2^{\log_2 8} = 8 = 2^3$ by the definition of log, so $\log_2 8 = 3$.

Example 1.1.2
Simplify $\frac{\log_5 x}{\log_{25} x}$.

Solution: Let $25^a = x$. Then notice $5^{2a} = x$. Substituting yields $\frac{2a}{a} = 2$.

Here's a motivating exercise for what's going to come next.

Exercise 1.1.3
Evaluate $\log_2 16 + \log_2 32$, and then evaluate $\log_2 16 \cdot 32$.

1.1.1 Fundamental Rules

The fundamental two rules of logarithms are the addition and subtraction rules. Notice that addition outside becomes multiplication inside (and inversely, subtraction becomes division inside). This is because of the way exponents behave: $x^{a+b} = x^a \cdot x^b$.

Theorem 1.1.4 — Logarithm Addition
For positive a, b, c, $\log_a b + \log_a c = \log_a bc$.

Proof: Notice that $a^{\log_a b+\log_a c} = a^{\log_a b} \cdot a^{\log_a c} = bc = a^{\log_a bc}$. Since the bases are the same, it follows the exponents are the same. ■

Theorem 1.1.5 — Logarithm Subtraction
For positive a, b, c, $\log_a b - \log_a c = \log_a \frac{b}{c}$.

Proof: This is a repeat of logarithm addition. Notice that $a^{\log_a b-\log_a c} = \frac{a^{\log_a b}}{a^{\log_a c}} = \frac{b}{c} = a^{\log_a \frac{b}{c}}$. ■

Notice that we're exploiting the properties of logarithms, and we're expressing everything without using logarithms as soon as possible. This trend will continue in AIME problems; once the logs have been removed, there's not much underneath to solve.

1.1.2 Base Change

The base change rule allows you to express all logarithms in the same base; this is an extremely powerful, even if it doesn't look like much.

Theorem 1.1.6 — Base Change
For positive a, b, c, $\frac{\log_a b}{\log_a c} = \log_c b$.

Proof: Have $x = \log_a b$, $y = \log_a c$, and $z = \log_c b$. Notice that $a^x = b, a^y = c, c^z = b$. Then $(a^y)^z = a^x$, implying $yz = x$ or $\frac{x}{y} = z$. ■

We present the so-called logarithm chain rule as an exercise. (It's pretty useless and is only being presented as a check-up.)

Exercise 1.1.7 — Logarithm Chain Rule
For positive a, b, c, d, $(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$.

1.2 Examples

Here are some examples of AIME logarithm problems. I want to re-iterate the following with these two problems: usually, **interpreting the log condition is the entire problem**.

Example 1.2.1 — AIME II 2009/2
Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

Solution: We notice that $a^{(\log_3 7)^2} = (a^{\log_3 7})^{\log_3 7}$. Similar expressions hold for b, c.

We then substitute $a^{\log_3 7} = 27$ as defined in the problem statement, and we do the same for b, c. This becomes $27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25} = 3^{3\log_3 7} + 7^{2\log_7 11} + 11^{\frac{1}{2}\log_{11} 25} = 7^3 + 11^2 + 25^{\frac{1}{2}}$. This simplifies to 469, which is our answer.

Example 1.2.2 — AIME I 2011/9
Suppose x is in the interval $[0, \pi/2]$ and $\log_{24 \sin x}(24 \cos x) = \frac{3}{2}$. Find $24 \cot^2 x$.

Solution: We can rewrite this as $(24\sin x)^3 = (24\cos x)^2$, which implies $24\sin^3 x = \cos^2 x = 1 - \sin^2 x$. Thus we want to find the positive root of $24\sin^3 x + \sin^2 x - 1 = 0$. Using the Rational Root Theorem (aka guessing), we see that $\frac{1}{3}$ is a root. Thus $\cot x = 2\sqrt{2}$ and our answer is 192.

1.3 Problems

Minimum is [32 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "I just long for a world in which ordinary things are done in an ordinary way."
>
> Psycho-Pass

[1 ♟] **(AIME II 2020/3)** The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

[2 ♟] **(AIME 1986/8)** Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1000000. What is the integer nearest to S?

[2 ♟] **(AIME I 2020/2)** There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

[2 ♟] **(AIME I 2007/7)** Let $N = \sum\limits_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor)$.
Find the remainder when N is divided by 1000. ($\lfloor k \rfloor$ is the greatest integer less than or equal to k, and $\lceil k \rceil$ is the least integer greater than or equal to k.)

[3 ♟] **(SMT 2020)** If a is the only real number that satisfies $\log_{2020} a = 202020 - a$ and b is the only real number that satisfies $2020^b = 202020 - b$, what is the value of $a+b$?

[3 ♟] **(AIME II 2013/2)** Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a+b$.

[3 ♞] **(AIME II 2010/5)** Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

[3 ♟] **(AIME I 2006/9)** The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

[4 ♟] **(HMMT 2020)** Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

[4 ♟] **(AIME II 2007/12)** The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that $\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \le \log_3\left(\sum_{n=0}^{7} x_n\right) \le 57$, find $\log_3(x_{14})$.

[4 ♟] **(AIME I 2009/7)** The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} - 1 = \frac{1}{n+\frac{2}{3}}$ for $n \ge 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

[4 ♟] **(AIME I 2010/14)** For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \le 300$.
Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

[6 ♞] **(AIME I 2005/8)** The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

[6 ♟] **(AIME I 2013/8)** The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

[7 ♟] **(hARMLess Mock ARML 2019/10)** Compute the sum of all positive integers that can be expressed in the form

$$\log_b(404!) - \log_b(c),$$

where b and c are positive integers such that $b > 1$ and $b + c$ is odd.

[9 ♟] **(AIME I 2012/9)** Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p + q$.

Sequences in the AIME

Chapter 2

There are two types of problems in this unit. The first type is sequence problems: given some sequence, find a certain thing out about said sequence. Then there are restriction problems: with some restriction, how many ways can something be done? The former is solved with clever algebraic manipulation and is quite standard. The latter requires recursions.

2.1 Combinatorial Sequences

Example 2.1.1 — AIME I 2006/11

A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

- Any cube may be the bottom cube in the tower.
- The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

Solution: Let a_k be the amount of ways to make a tower of height k.

Note that $a_k = 3a_{k-1}$ for $k \geq 3$, as we can put the last cube above the cube with edge length $k-1$, above the cube with edge length $k-2$, or as the bottom cube in the tower. It is obvious that $a_2 = 2$, so $a_k = 2 \cdot 3^{k-2}$. Thus, $a_6 = 2 \cdot 3^6 = 1458$, so the answer is 458.

That just required a little bit of clever thinking. Let's take a look at a more involved example.

Example 2.1.2 — AMC 12B 2019/23

How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

Solution: Let a_n be the number of sequences of length n that start and end with 0, and have no two consecutive 0s and no three consecutive 1s.

Each sequence starts with 010 or 0110, so $a_n = a_{n-2} + a_{n-3}$. Since $a_1 = 1$, $a_2 = 0$, and $a_3 = 1$,

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
a_n	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37	49	65

Thus the answer is $a_{19} = 65$.

These types of problems are a two step process:

- Find a recursion that characterizes the number of ways to do something.
- Extract the answer.

Generally the first step is much harder than the second. But usually neither step is very hard if you have experience doing recursion problems.

2.2 Algebraic Sequences

Some sequences *feel* bashable because they have an organized structure, and you just have to figure out what it is. The only real choice that you have to make is whether to bash directly in terms of the constants or in terms of the varaible.

Example 2.2.1 — AMC 10A 2019/15
A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p+q$?

Solution: We calculate a few terms to get a feel for the problem. Note that $a_3 = \frac{3}{11}$, $a_4 = \frac{3}{15}$, and $a_5 = \frac{3}{19}$. This leads us to believe $a_n = \frac{3}{4n-1}$, and checking that this holds for a_1 is enough to convince us that $a_{2019} = \frac{3}{4 \cdot 2019 - 1} = \frac{3}{8075}$, so the answer is 8078.

In fact we can verify with induction that $a_n = \frac{3}{4n-1}$. The base case a_1 is already done, and

$$a_{n+1} = \frac{a_{n-1} \cdot a_n}{2a_{n-1} - a_n} = \frac{\frac{3}{4n-1} \cdot \frac{3}{4n-5}}{2 \cdot \frac{3}{4n-5} - \frac{3}{4n-1}} = \frac{9}{12n+9} = \frac{3}{4n+3}$$

finishes the inductive step.

This is a problem where you bash directly with the given numbers, because you feel the given numbers might be special in some way.

Example 2.2.2 — AIME II 2020/6
Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3$. Then t_{2020} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution: The fractions make the function *feel* periodic. So we write everything in terms of t_1 and t_2, because it is likely to stay periodic regardless of the values of t_1 and t_2. Note

$$t_3 = \frac{5t_2 + 1}{25t_1}$$

$$t_4 = \frac{5 \cdot \frac{5t_2+1}{25t_1} + 1}{25t_2} = \frac{5t_1 + 5t_2 + 1}{125t_1 t_2}$$

$$t_5 = \frac{5 \cdot \frac{5t_1+5t_2+1}{125t_1t_2} + 1}{25 \cdot \frac{5t_2+1}{25t_1}} = \frac{5t_1 + 5t_2 + 1 + 25t_1t_2}{125t_2^2 + 25t_2} = \frac{(5t_1+1)(5t_2+1)}{25t_2(5t_2+1)} = \frac{5t_1+1}{25t_2}$$

$$t_6 = \frac{5 \cdot \frac{5t_1+1}{25t_2} + 1}{25 \cdot \frac{5t_1+5t_2+1}{125t_1t_2}} = \frac{5t_1^2 + 5t_1t_2 + t_1}{5t_1 + 5t_2 + 1} = t_1.$$

So we see that this sequence repeats with a period of 5. Thus $t_{2020} = t_5 = \frac{5 \cdot 20+1}{5 \cdot 21} = \frac{101}{125}$, so the answer is 226.

2.3 Problems

[2 ♟] **(AMC 12A 2007/25)** Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

[2 ♟] **(AIME I 2001/14)** A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

[3 ♟] **(MAST Diagnostic 2020)** A secret spy organization needs to spread some secret knowledge to all of its members. In the beginning, only 1 member is *informed*. Every informed spy will call an uninformed spy such that every informed spy is calling a different uninformed spy. After being called, an uninformed spy becomes informed. The call takes 1 minute, but since the spies are running low on time, they call the next spy directly afterward. However, to avoid being caught, after the third call an informed spy makes, the spy stops calling. How many minutes will it take for every spy to be informed, provided that the organization has 600 spies?

[3 ♟] **(HMMT 2019)** Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1 b_1 = 20$, $a_2 b_2 = 19$, and $a_3 b_3 = 14$. Find the greatest possible value of $a_4 b_4$.

[3 ♟] **(AIME II 2016/9)** The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k.

[4 ♟] **(AMC 12A 2002/21)** Consider the sequence of numbers: $4, 7, 1, 8, 9, 7, 6, \ldots$ For $n > 2$, the n-th term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. What is the smallest value of n for which $S_n > 10,000$?

[4 ♟] **(AIME 1984/7)** The function f is defined on the set of integers and satisfies

$$f(n) = \begin{cases} n - 3 & \text{if } n \geq 1000 \\ f(f(n+5)) & \text{if } n < 1000 \end{cases}.$$

Find $f(84)$.

[4 ♟] **(AMC 12A 2001/25)** Consider sequences of positive real numbers of the form $x, 2000, y, \ldots$ in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

[4 ♞] **(PUMaC 2017)** Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $a_n = 11a_{n-1} - n$ for all $n > 1$. The smallest possible value of a_1 can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

[6 ♞] **(OMO 2019)** Susan is presented with six boxes $B_1, \ldots, B_6$, each of which is initially empty, and two identical coins of denomination 2^k for each $k = 0, \ldots, 5$. Compute the number of ways for Susan to place the coins in the boxes such that each box B_k contains coins of total value 2^k.

[6 ♟] **(MATHCOUNTS State 2020)** Hank builds an increasing sequence of positive integers as follows: The first term is 1 and the second term is 2. Each subsequent term is the smallest positive integer that does not form a three-term arithmetic sequence with any previous terms of the sequence. The first five terms of Hank's sequence are $1, 2, 4, 5, 10$. How many of the first 729 positive integers are terms in Hank's sequence?

[9 ♟] **(AMC 12B 2019/22)** Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \leq 4 + \frac{1}{2^{20}}.$$

Find $\lfloor \log_3 m \rfloor$.

[13 ♟] **(IMO 2014/1)** Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

[13 ♟] **(rd123 AIME 2020/12)** Define a sequence of positive integers with 6 terms, $(a_1, a_2, a_3, a_4, a_5, a_6)$, to be *crazy* if:

- $a_1 = 2016$
- $a_6 = 1$
- a_{k+1} is a proper divisor of a_k for $k \in \{1, 2, 3, 4, 5\}$.

Compute the last three digits of the number of crazy sequences.

Introduction to Counting

Chapter 3

We discuss independent choices, basic bijections, and combinations and permutations.

3.1 Independent Choices

We examine independent choices, and how many ways you can make all of these choices.

Definition 3.1.1 — Independent Choices
Two choices C_1 and C_2 are independent if and only if the outcome of C_1 does not affect the outcome of C_2, and vice versa.

This can be interpreted combinatorially and probabilisitcally.

Example 3.1.2 — Combinatoric Independent Choices
If there are three types of pizza you can buy and you want to buy two pizzas, the first type of pizza you buy does not influence your choices for the second type of pizza you buy. The order of the choices doesn't matter either.

Commonly you'll see this referred to as "choosing without restriction," which means that the choices are independent by definition.

Example 3.1.3 — Probabilistic Indepdendent Choices
If you flip a coin twice, the two flips are independent since the first flip does not affect the second.

Theorem 3.1.4 — Independent Choices
If you have **independent** choices $C_1, C_2 \ldots C_n$ with $o_1, o_2 \ldots o_n$ options respectively, the amount of distinct ways to make all of these choices is $o_1 \cdot o_2 \cdot \cdots \cdot o_n$.

Let's say you want to know how many sets of clothes you can wear, where a set of clothes has a shirt and pants. How many total options do we have considering a certain amount of individual choices with a certain amount of options? Not everyone will use the same terminology, so let's define a choice and an option.

Definition 3.1.5
A **choice** is the reason an option matters (we have x options for this choice), and an **option** is an option for a choice. For clarity we present an example.

Example 3.1.6
If you have 5 shirts and 7 pairs of pants, how many suits, consisting of a shirt and a pair of pants, can you wear?

Solution: We have two **choices**: what shirt to wear, and what pants to wear. For the shirts, we have 5 **options**, and for the pants, we have 7 **options**.

3.2 Basic Bijections

It's well-known that there are n terms in the sequence $\{1,2,3\cdots n\}$. We can use this to count the amount of terms in an arithmetic sequence, geometric sequence, etc. This is because no matter how we transform the individual terms, the size of each set stays the same.

Theorem 3.2.1 — Terms in an Arithmetic Sequence
In an arithmetic sequence with beginning term a, ending term z, and with common difference d, there are $\frac{z-a}{d}+1$ terms.

Proof: Let the set be denoted as S. Applying $f(x) = x + (d - a)$ to all members of S yields another set $S_{f(x)}$. Notice that the members of $S_{f(x)}$ are $\{d, 2d, 3d \ldots d + z - a\}$. Then applying $g(x) = \frac{x}{d}$ to all members of $S_{f(x)}$ yields another set $S_{g(x)}$, whose members are $\{1, 2, 3 \ldots \frac{z-a}{d}+1\}$. Thus, S has $\frac{z-a}{d}+1$ members, as desired. ∎

Theorem 3.2.2 — Terms in a Geometric Sequence
In a geometric sequence with beginning term a, ending term z, and common ratio r, there are $\log_r(\frac{z}{a})+1$ terms.

Proof: Let the set be denoted as S. Applying $f(x) = \frac{rx}{a}$ to all members of S yields another set $S_{f(x)}$ with members $r, r^2, r^3 \ldots r\frac{z}{a}$. Then applying $g(x) = \log_r x$ to all members of $S_{f(x)}$ yields another set $S_{g(x)}$, whose members are $\{1, 2, 3 \ldots \log_r(\frac{z}{a})+1\}$. Thus, S has $\log_r(\frac{z}{a})+1$ members, as desired. ∎

Even though the notation for each theorem and its proof are different, they contain the same idea. (This is evident due to the nearly-identical structure.) Keep in mind that it is fine to forget the theorems if you remember the main idea - a clever use of bijections to get the set $\{1, 2, 3 \ldots n\}$.

3.3 Permutations and Combinations

When we discuss Combinations and Permutations, there are two standard examples to refer to.

Example 3.3.1 — Permutations
Out of a group of n distinguishable objects, how many ways are there to line k of them up (order matters)?

Example 3.3.2 — Combinations
Out of a group of n people, how many ways are there to choose k?

Theorem 3.3.3 — Permutations
Let $P(n,r)$ denote the amount of ways to permute n objects in a line of length r. If $n \geq r$, then $P(n,r) = \frac{n!}{(n-r)!}$. If $n < r$, then $P(n,r) = 0$.

Proof: For the first object in line, we have n choices as to what it is. For the next, we have $n-1$ choices, and so on, until we have $n-r+1$ choices for the last one. Notice that this gives us a total of $n(n-1)(n-2)\dots(n-r+1)$ ways to line them up. If $n \geq r$, then this expression is equal to $\frac{n!}{(n-r)!}$. However, if $n < r$, then the expression is equal to $n(n-1)(n-2)\cdots(n-n)(n-(n+1))\cdots(n-r) = 0$ (as $n-n=0$). ■

Theorem 3.3.4 — Combinations
Let $\binom{n}{r}$ denote the amount of ways to choose r objects out of n. Then for all $n \geq r$, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, an for all $n < r$, $\binom{n}{r} = 0$.

Proof: The result is obvious for $n < r$.

If $n \geq r$, then notice that $P(n,r) = \frac{n!}{(n-r)!}$. But notice that when we count combinations, **order doesn't matter**. As we've counted each combination $r!$ times (due to the $r!$ possible permutations), $\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$, as desired. ■

In competition math, combinations is used more commonly than permutations simply because the idea of combinations is more interesting. (This can be seen with the study of combinatorial identities, which nearly exclusively use combinations.)

3.3.1 Permuting with Restrictions

Sometimes we'll have to permute with restrictions. In this case, **take care of restrictions first**.

Example 3.3.5
How many 4 digit numbers have distinct digits?

Solution: There are 9 choices for the first digit, 9 choices for the second, 8 for the third, and 7 for the fourth. So the answer is $9 \cdot 9 \cdot 8 \cdot 7 = 4536$.

If we tried to take care of anything other than the first digit first, we'd have to deal with annoying casework.

Example 3.3.6
Find the number of ways to put 8 rooks on a chessboard such that

- no two rooks are attacking each other, and
- no rook is in one of the four corners of the chessboard.

Solution: Note that we are putting a rook in each row such that no two rooks are in the same column, and the rooks in the first and eigth row are restricted. There are 6 ways to choose the column of the rook in the first row and 5 ways to choose the column of the rook in the second row. For the other 6 rooks, there are 6 available columns and no restrictions, so there are 6! ways to arrange them. Thus the answer is $6 \cdot 5 \cdot 6! = 21600$.

3.4 Block Walking

Commonly in MATHCOUNTS, you'll see a problem where you have something on a grid and you want to move it somewhere else, typically with restrictions. This type of problem is known as a block walking problem.

Example 3.4.1 — MATHCOUNTS 2020

A checker starts at square 4 of the checkerboard shown here. At any time, it can move to any diagonally adjacent square below its current position. How many possible ways are there for the checker to move from square 4 to square 32?

Solution: Fill in the grid with the number of ways to get to each number. Notice that the number of ways to get to a number is the sum of the number of ways to get to the two numbers diagonally above it.

3.5 Problems

Minimum is [30 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "Then what is living to you?"
> "Hmm... Perhaps redemption."
>
> Oyasumi, Punpun

[1 ♟] The Committee of MAST needs to split 3 unique jobs among 8 students. If each student can do as many jobs as they please, how many ways are there to assign the jobs?

[1 ♟] There are 52 postal abbreviations for the 50 states of America, D.C. and Puerto Rico. If we choose a two-letter "word" at random (such as AA), what is the probability that we choose one of the 52 postal abbreviations?

[2 ♟] Jim is getting an ice cream cone. He can get either 1, 2, or 3 scoops. For each scoop, he can get 3 different flavors. How many different ice creams can he get? (Order of the scoops matter!)

[1 ♟] Find the amount of terms in the set $\{5, 7, 9 \cdots 39\}$. (This is an arithmetic sequence.)

[1 ♟] Find the amount of terms in the set $\{3, 6, 12, 24, 48, 96\}$. (This is a geometric sequence.)

[1 ♟] How many ways can you arrange the letters in TARGET?

[1 ♟] How many ways are there to arrange the letters in MATHCOUNTS?

[3 ♟] How many positive integers less than 1000 only have even digits?

[3 ♟] We want to choose two disjoint committees of 4 people from a class of 12. How many ways can we do this, if the committees are distinct?

[2 ♟] What if the committees are indistinguishable?

[4 ♟] **(AMC 12 2001/16)** A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

[4 ♟] **(AMC 10A 2019/17)** A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

[4 ♞] **(AMC 10A 2021/25)** How many ways are there to place 3 indistinguishable red chips, 3 indistinguishable blue chips, and 3 indistinguishable green chips in the squares of a 3×3 grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally.

[4 ♟] **(MATHCOUNTS State 2020)** Iris is playing a game that has a 5×5 gameboard like the one shown. The goal is to get her game piece from the square labeled $\star$ to the square labeled $\circ$ using a series of moves any positive integer number of squares up or any positive integer number of squares to the right. Note that moving two squares up in a single move is different than moving two squares up in two moves. How many unique sequences of moves can Iris make to get her game piece from $\star$ to $\circ$?

[6 ♞] How many 4 digit falling numbers are there? (A falling number is a number whose last digit is strictly smaller than its second-to last digit, and so on, such as 4321.)

[6 ♟] **(AMC 10A 2021/21)** In how many ways can the sequence $1, 2, 3, 4, 5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

[6 ♟] (HMMT Feb. Guts 2012/20) Let n be the maximum numbers of bishops that can be placed on the squares of a 6×6 chessboard such that no two bishops are attacking each other. Find $n + k$.[1]

[9 ♟] (AIME 1990/8) In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:

- The marksman first chooses a column from which a target is to be broken.
- The marksman must then break the lowest remaining target in the chosen column.

If the rules are followed, in how many different orders can the eight targets be broken?

[13 ♟] (AIME I 2010/7) Define an ordered triple (A, B, C) of sets to be minimally intersecting if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \varnothing$. For example, $(\{1,2\}, \{2,3\}, \{1,3,4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1,2,3,4,5,6,7\}$. Find the remainder when N is divided by 1000.
Note: $|S|$ represents the number of elements in the set S.

[1] Rules of chess

Perspectives

Chapter 4

Before we begin, we establish some notation that will make it much easier (and faster!) to write intersections and unions.

$A \cap B$ denotes the intersection of A and B (the set of elements that is in both sets A and B). $A \cup B$ denotes the union of A and B (the set of elements in set A, set B, or both sets.) $|A|$ denotes the size of set A, and $|A \cap B|$ denotes the size of the intersection of A and B.

4.1 The Principle of Inclusion-Exclusion

Perhaps you've heard of "Venn Diagram" problems; if you haven't, here's an example problem.

Example 4.1.1 — Two Sets
20 students are taking Spanish and 30 students are taking French. If everyone takes at least one language and there are 45 total students, how many students are **only** taking Spanish?

Solution: Let the amount of students taking Spanish and French be x. Then note that there are $20 - x$ students taking only Spanish, $30 - x$ students only taking French. We can add the students in all three of these groups to find our total sum. Since there are only 45 students, $20 - x + 30 - x + x = 45 \rightarrow 50 - x = 45 \rightarrow x = 5$.

The Principle of Inclusion-Exclusion is about splitting the students into groups depending on the exact **number** of classes they take (usually the exact classes they take are irrelevant), and making sure you count each student exactly once. We note that if we count everybody for each time they are in Spanish and each time they are in French, then we will "double-count" (count twice) the people in both Spanish and French. This means that we must subtract the people in both Spanish and French. Fortunately, this overcounting/undercounting behavior is actually quite predictable.

Theorem 4.1.2 — The Two-Set Case
For sets A_1, A_2,
$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Proof: Notice that you count the elements in **exactly** one set once, but you count the elements in two sets twice. Thus, we must subtract the elements in both sets to account for this overcounting. ■

Let's take a look at the case of 3 people.

Theorem 4.1.3 — The Three-Set Case
For sets A_1, A_2, A_3,
$$|A_1 \cup A_2 \cup A_3| =$$
$$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|.$$

Proof: We note that if we count A_1, A_2, A_3 once, then we count everything in exactly two sets twice. Thus we subtract $|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1|$. This means our current value is $|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1|$, but we have counted every person in $|A_1| \cap |A_2| \cap |A_3|$ 0 times. So we have to add $|A_1 \cap A_2 \cap A_3|$, giving us our final value as

$$|A_1 \cup A_2 \cup A_3| =$$

$$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|.$$

■

Keeping in mind how many times we count each number, we can generalize PIE. Let's say we have X sets. Then we add the amount of terms in the individual sets, subtract the terms in 2 sets, add the amount of terms in 3 sets, and so on. Note that this means the amount of terms in **at least** that many sets, not exactly. The general rule is we add the amount of terms in K sets if K is odd and we subtract the amount of terms in K sets if K is even. (See the top of the section for a more formalized statement and proof.)

For example, with four sets A_1, A_2, A_3, A_4, we have

$$A_1 \cup A_2 \cup A_3 \cup A_4 = \sum_{i=1}^{4} A_i - \sum_{\text{sym}} A_1 \cap A_2 + \sum_{\text{sym}} A_1 \cap A_2 \cap A_3 - A_1 \cap A_2 \cap A_3 \cap A_4.$$

As an exercise, do this with 5 sets. (Do this only if you feel like it; it isn't very important.)

You may be noticing a pattern here; we are "adding" intersections of an odd number of sets and "subtracting" intersections of an even number of sets. The natural question to ask is, "Does this hold in general, and why?" The answer to the first question is the general Principle of Inclusion-Exclusion, and the second is a Perspectives-style argument.

Theorem 4.1.4 — The Principle of Inclusion-Exclusion

Given sets $A_1, A_2, \ldots, A_n$,

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\text{sym}} |\bigcap_{j=1}^{i} A_j|.$$

We do the following problem to motivate the proof.

Example 4.1.5 — AIME 1983/13

For $\{1, 2, 3, \ldots, n\}$ and each of its non-empty subsets a unique alternating sum is defined as follows. Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. For example, the alternating sum for $\{1, 2, 3, 6, 9\}$ is $9 - 6 + 3 - 2 + 1 = 5$ and for $\{5\}$ it is simply 5. Find the sum of all such alternating sums for $n = 7$.

Solution: Note that any subset A not containing 7 can be matched with a subset B containing 7, and further note that $S(A) + S(B) = 7$. Since there are $2^6 = 64$ sets A (we can treat the empty set as having alternating sum 0), the answer is $64 \cdot 7 = 448$.

The reason behind the statement of PIE is that we need to ensure that each element is counted exactly once. Thus it logically follows that the simplest and fundamentally moral way to prove this is by proving each element is counted at most once, and the motivation behind our induction-style argument is the intuition we got from the last example.

Proof: We prove that each element is counted once.

Say that some element X is in k sets. Without loss of generality, these sets are $A_1, A_2, \ldots, A_k$.

We proceed by induction. This is obvious for $k = 1$.

If this is true for k, we prove this is true for $k+1$. For every set of sets not containing A_{k+1} with size i, there is a set of sets containing A_{k+1} with size $i+1$. In PIE, the sum of how many times these sets are counted is 0. There is also one additional set of sets $\{A_{k+1}\}$, so X is counted exactly once. ■

4.2 Clever Bijections

4.2.1 Stars and Bars

The answer to the question "how many ways can we give identical things to non-identical people?" Also known as sticks and stones or balls and urns. This is a clever trick often used in lower-level competition mathematics, and it's best thought of as a clever bijection.

Theorem 4.2.1 — Stars and Bars
The number of ways to distribute n indistinguishable items to k distinguishable people is $\binom{n+k-1}{k-1}$.

Proof: Let there be $k-1$ dividers and n items in a line. Then we distribute the items between each set of dividers (and to the left of the leftmost divider and to the right of the rightmost divider) to the people in that order. Note that there are $\binom{n+k-1}{k-1}$ ways to do this, and this corresponds directly to the number of ways to directly distribute the items to the people.

$$* * | * | * * * | *$$

Stars and Bars for $n = 7$ and $k = 4$.

■

We present a fairly straightforward application of Stars and Bars with no restrictions.

Example 4.2.2 — AMC 10A 2003/21
Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Solution: There are six stars and three bars, so the answer is $\binom{6+3-1}{3-1} = \binom{8}{2} = 28$.

You usually will have some sort of restrictions - most commonly, certain people must get a minimum of the distributed item. (This is evidenced by how hard it was for me to find an example for stars and bars with no restrictions.) In this case, allot the items "beforehand" and ignore the restrictions, while starting with less items than before you took care of the restrictions. This is all fairly abstract, so a concrete example will help.

Example 4.2.3 — AMC 8 2019/25
Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

Solution: We distribute 2 apples to each of the 3 people. So we have 18 apples left and no more restrictions, so the answer is $\binom{18+3-1}{3-1} = \binom{20}{2} = 190$.

There will be times when you need to do another clever bijection to make stars and bars easier. Some example include having a limit on how many items people can receive, and having a number of items very close to this limit - in this case, we can think about distributing "negative items" - that is, how far away each person is from receiving the maximum.

4.2.2 Picking Unordered Elements

Alternatively titled ascending numbers. We take a look at two generic examples that cover the section pretty well.

Example 4.2.4 — Ascending Numbers
An *ascending number* is a number whose digits increase from left to right. How many four digit ascending numbers are there?

Solution: We choose four distinct digits between 1 and 9, and the order is fixed. Thus, the answer is just $\binom{9}{4} = 126$.

Now what if the number is just non-decreasing and can stay the same?

Example 4.2.5 — Non-descending Numbers
A *nondescending number* is a number whose digits never decrease (but may stay the same) from left to right. How many four digit nondescending numbers are there?

Solution: Let the digits be a, b, c, d. Then we desire $1 \leq a \leq b \leq c \leq d \leq 9$. But note this is also equivalent to $1 \leq a < b + 1 < c + 2 < d + 3 \leq 12$. So we pick 4 distinct numbers and match them with $a, b + 1, c + 2, d + 3$. There are $\binom{9-1+4}{4} = \binom{12}{4}$ ways to do this, so the answer is 495.

This can also be done with stars and bars - if we let the "baskets" be the digits $1, 2, \ldots, 9$ and the "stars" be the 4 digits we choose. We also get the value of $\binom{9-1+4}{4} = 495$.

4.3 Combinatorial Identites

We take a look at some famous combinatorial identities like Hockey-Stick, Vandermonde, and the Binomial Theorem (and friends). We will look at the combinatorial proofs (aka bijections) when possible because algebra is boring and straightforward (and sometimes not possible).

We first start with the most boring one (which is done by algebra).

Theorem 4.3.1 — Shift 1
For positive integers n, k,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

Proof: Note that $\frac{n!}{k!(n-k)!} = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!}$. ■

This theorem does come into play as part of some harder problems, so it is good to be able to manipulate binomials this way. But there will probably never be a problem *based* on this theorem.

Theorem 4.3.2 — Hockey-Stick
For positive integers n, k,

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

Diagram from AoPS Wiki.

The Hockey-Stick identity is named such because it looks like a hockey stick in Pascal's Triangle.

Proof: Have a particle on the lattice grid starting at $(0,0)$, and allow it to either move 1 unit right or 1 unit up in each move. Then note that $\sum_{i=k}^{n} \binom{i}{k}$ is the sum of the number of ways to get to $(k,0),(k,1),\ldots,(k,n-k)$, and that $\binom{n+1}{k+1}$ is the number of ways to get to $(k+1,n-k)$. But note that to get to $(k+1,n-k)$, we go from a point (k,i) to a point $(k+1,i)$ and then go straight up, which there is always exactly one way to do once you get to (k,i). Thus the two values are equal.

Example for $n=7$ and $k=3$.

■

Theorem 4.3.3 — Vandermonde

For positive integers m,n,k,

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}=\binom{m+n}{k}.$$

Proof: Note that this is the same as picking a committee of k people from $m+n$ people, since for every committee, there is some arbitrary number i such that we pick i from the group of m and the rest of the $k-i$ from the group of n.

■

4.4 Freedom

As you might recall from **CPV-Intro**, the number of choices between independent events is multiplicative.[1] However, sometimes it isn't clear what the independent events are, or if there even are any at all. The goal of this section is to develop the intuition of when and how counting problems hide their independent choices.

Example 4.4.1 — Coins

Linus is flipping a fair coin n times. In terms of n, how many ways can he end his nth flip with an even number of heads?

Solution: Note that regardless of what he gets on his first $n-1$ flips, there is exactly 1 choice for the final flip based on the current parity. If the current number of heads is odd, then the last flip must be heads, and if the current number of heads is even, then the last flip must be tails.

As an exercise, find the number of ways Linus can flip his coin to end with an odd number of heads. Why is it the same even when a change of perspectives (when n is even, and the argument "even heads is equivalent to odd tails, and heads are no different from tails" doesn't hold) is not possible?

Example 4.4.2

Farmer John has N cows of heights $a_1,\ldots,a_N$. His barn has N stalls with max height limits $b_1,\ldots,b_N$ (so for example, if $b_5=17$, then a cow of height at most 17 can reside in stall 5). In how many distinct ways can Farmer John arrange his cows so that each cow is in a different stall, and so that the height limit is satisfied for every stall?

Devise an $O(N^2)$ algorithm to determine the answer.

Solution: This is just glorified counting with restrictions. Note the cow with the tallest height has the most restrictions, the cow with the second tallest height has the second most restrictions, so on. So it is only natural to place the cows in order of height.

Sort $a_1,a_2,\ldots,a_N$ such that $a_1>a_2>\cdots>a_N$. Then say cow a_i fits into k_i stalls. Note that a_i can be placed into

[1] As an example, if you want to pick a piece of paper out of 5 pieces and a pencil out of 10 pencils, there are $5\cdot 10$ total choices.

$k_i - (i-1)$ stalls, because the previous $i-1$ cows are in stalls that the ith cow can fit in. This is because those $i-1$ cows are taller, so by definition they must be in a stall tall enough to fit the ith cow.

Thus the answer is just

$$\prod_{i=0}(k_i - (i-1)).$$

Since each of the N k_i can be determined in $O(N)$ time, this algorithm is $O(N^2)$.

This idea of "do whatever for the first $n-k$ moves and meet restrictions in the last k moves" is often considered difficult and thus not much extra stuff is added. So if you can get used to it, it's free points for you.[2] These problems also have the added bonus of being very pleasant to solve – something atypical of most AIME combo.

To finish off, here's an example of a harder freedom problem.

Example 4.4.3 — HMMT Feb. Guts 2011/10

In how many ways can one fill a 4×4 grid with a 0 or 1 in each square such that the sum of the entries in each row, column, and long diagonal is even?

Solution: Surprisingly, this problem is completely independent. The free squares are denoted with an F below:

F	F	D	D
F	F	F	D
F	F	F	D
D	D	D	D

The proof this works is left to the reader.

4.4.1 Binomial Sums

We start with the most obvious result.

Theorem 4.4.4 — Binomial Theorem

For a positive number n,

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Proof: Note that $(1+1)^n = \sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Combinatorially, there are 2 choices for each of the n terms in the expansion; this leads to 2^n terms, each with value 1. ∎

Closely related is the following theorem, which is obvious when n is odd but not so much when n is even.

Example 4.4.5 — Even Binomial Theorem

For any $n \geq 1$, $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = \frac{2^n}{2}$.

Solution: This is the number of ways to flip n coins such that an even of them land heads.

Finally, we end with a useful identity that is proven with an *algebraic* change of perspectives, rather than a combinatorial one.

[2] You only need to look at AMC 10A 2020/23 and AMC 10B 2020/23, which are on different versions of this handout, to get an idea of *how inflated* this sense of difficulty is. Both are in the last 5 of the AMCs and are only worth three points each – something rare even in an R unit.

Example 4.4.6 — Binomial with Coefficient

For any $n \geq 1$, $\sum\limits_{i=0}^{n} i\binom{n}{i} = \frac{n2^n}{2}$.

Solution: A combo problem is always easy when algebra saves the day.

Note $2\sum\limits_{i=0}^{n} i\binom{n}{i} = \sum\limits_{i=0}^{n} i\binom{n}{i} + \sum\limits_{i=0}^{n} i\binom{n}{n-i} = \frac{n2^n}{2} = n\sum\limits_{i=0}^{n}\binom{n}{i} = n2^n$. Dividing by 2 yields the desired result.

For those of you who know what the Roots of Unity Filter is, this is a very primitive form of it.

4.5 Problems

Minimum is [40 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "I won't ever allow you to do anything to abandon these possibilities and choose death!"
>
> Fullmetal Alchemist: Brotherhood

[2 ♟] **(AMC 8 2011/6)** In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

[2 ♟] How many integers from 1 to 100 (inclusive) are multiples of 2 or 3?

[3 ♟] **(AMC 10A 2018/11)** When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

[2 ♟] There are 3 distinct six-sided dice, one red, white, and blue. How many ways can the sum of the 15 faces showing on the three die equal 56, if each die orientation is only considered unique if the sum of its faces that are showing are unique?

[2 ♟] **(AMC 10B 2017/13)** There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

[3 ♞] **(AMC 10B 2020/23)** Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1)$, and $D(1,-1)$. Consider the following four transformations:

- L, a rotation of 90° counterclockwise around the origin;
- R, a rotation of 90° clockwise around the origin;
- H, a reflection across the x-axis; and
- V, a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.

[3 ♞] **(AIME I 2020/7)** A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

[3 ♟] We have 7 balls each of different colors (red, orange, yellow, green, blue, indigo, violet) and 3 boxes each of different shapes (tetrahedron, cube, dodecahedron). How many ways are there to place these 7 balls into the 3 boxes such that each box contains at least 1 ball?

[3 ♟] **(AIME II 2009/6)** Let m be the number of five-element subsets that can be chosen from the set of the first 14 natural numbers so that at least two of the five numbers are consecutive. Find the remainder when m is divided by 1000.

[4 ♟] **(AIME II 2002/9)** Let $\mathcal{S}$ be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (Disjoint sets are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

[4 ♟] **(Mildorf AIME)** Let N denote the number of 7 digit positive integers have the property that their digits are in increasing order. Determine the remainder obtained when N is divided by 1000. (Repeated digits are allowed.)

[4 ♟] **(AIME I 2020/9)** Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random with replacement from the set S and labeled a_1, a_2, and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

[6 ♟] **(AIME II 2013/9)** A 7×1 board is completely covered by $m \times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Let N be the number of tilings of the 7×1 board in which all three colors are used at least once. For example, a 1×1 red tile followed by a 2×1 green tile, a 1×1 green tile, a 2×1 blue tile, and a 1×1 green tile is a valid tiling. Note that if the 2×1 blue tile is replaced by two 1×1 blue tiles, this results in a different tiling. Find the remainder when N is divided by 1000.

[6 ♞] **(AIME I 2015/12)** Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

[9 ♟] **(AIME 1986/13)** In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTTHHTHTTTHHTTH of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

[9 ♟] **(CMIMC 2018)** Compute the number of rearrangements $a_1, a_2, \ldots, a_{2018}$ of the sequence $1, 2, \ldots, 2018$ such that $a_k > k$ for exactly one value of k.

Basics of Geometry

Chapter 5

This is geometry lite - just similarity and angle chasing. Things that you should keep in mind include similarity/congruence criterion, collinearity/concurrency angle conditions, parallel line angle conditions, and the fact that a tangent is perpendicular to the radius.

5.1 Theory

Here are two theorems to keep in mind.

Theorem 5.1.1 — Inscribed Angle
Let A, B be points on a circle with center O.
If C is a point on major arc AB, then $\angle ACB = \frac{\angle AOB}{2}$.
If C is a point on minor arc AB, then $\angle ACB = 180° - \frac{\angle AOB}{2}$.

Proof: Let D be the antipode of C. Then $\angle ACD = \frac{180° - \angle AOC}{2} = \frac{\angle AOD}{2}$. Thus addition or subtraction, depending on whether O is inside acute angle $\angle ACB$, of $\angle ACD$ and $\angle BCD$ will yield the result. ■

Theorem 5.1.2 — Tangent Angle
Consider circle ω with center O and points A, B on ω. Let ℓ be the tangent to ω through B and let θ be the acute angle between AB and ℓ. Then $\theta = \frac{\angle AOB}{2}$.

Proof: Let B' be the antipode of B. Then note that $\theta = 90° - \angle ABB' = \frac{180° - \angle AOB'}{2} = \frac{\angle AOB}{2}$. ■

A corollary of this theorem is that if C is some point on $\widehat{AB}$, then $\theta = \angle ACB$.
With the Inscribed Angle Theorem in mind, try to prove these two theorems.

Theorem 5.1.3 — Angle of Secants/Tangents
Let lines AX and BY intersect at P such that A, X, P and B, Y, P are collinear in that order. Then $\angle APB = \frac{\angle AOB - \angle XOY}{2}$.

Theorem 5.1.4 — Angle of Chords
Let chords AC, BD intersect at P. Then $\angle APB = \frac{\angle AOB + \angle COD}{2}$.

Here's a very important application of Inscribed Angle.

Theorem 5.1.5 — Cyclic Quadrilaterals
Any one of the three implies the other two:

1. Quadrilateral $ABCD$ is cyclic.
2. $\angle ABC + \angle ADC = 180°$.
3. $\angle BAC = \angle BDC$.

5.2 Examples

We present several examples of angle chasing problems, sorted by "flavor."

5.2.1 Computational Problems

This is a compilation of computational problems meant to serve as low-level examples for first-time readers. If this is your first time encountering the material, I strongly suggest you focus on this section.

Example 5.2.1 — AMC 10B 2011/18
Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

Solution: Note that $\angle CMD = \angle AMD = \angle AMD = \angle MDC$, implying that $CM = CD = 6$. Thus $\angle BMC = 30°$, implying that $\angle AMD = 75°$.

Example 5.2.2
Two circles ω_1, ω_2 intersect at P, Q. If a line intersects ω_1 at A, B and ω_2 at C, D such that A, B, C, D lie on the lie in that order, and P and Q lie on the same side of the line, compute the value of $\angle APC + \angle BQD$.

Solution: Without loss of generality, let P be closer to ℓ than Q. Note

$$\angle APC = 180 - \angle PAB - \angle BCP = \angle DCP - \angle PAB$$

$$\angle BQD = \angle BQP + \angle DQP.$$

Since $\angle PAB = \angle BDP$, the sum is $\angle DCP + \angle DQP = 180$.

5.2.2 Construct the Diagram

These problems are very simple; just construct the diagram and the problem will solve itself for you.

Example 5.2.3 — USA EGMO TST 2020/4
Let ABC be a triangle. Distinct points D, E, F lie on sides BC, AC, and AB, respectively, such that quadrilaterals $ABDE$ and $ACDF$ are cyclic. Line AD meets the circumcircle of $\triangle ABC$ again at P. Let Q denote the reflection of P across BC. Show that Q lies on the circumcircle of $\triangle AEF$.

Solution: Note that Q is the intersection of BE and CF, since $\angle EBD = \angle CAP = \angle CBP$ and $\angle FCB = \angle BAP = \angle BCP$. Now note that $\angle BQC = \angle BPC = 180^\circ - \angle A$.

The motivation is just drawing the diagram – as soon as you figure out that Q lies on BE and CF, the problem solves itself from there.

Here's a slightly harder example.

Example 5.2.4 — KJMO 2015/1
In an acute, scalene triangle $\triangle ABC$, let O be the circumcenter. Let M be the midpoint of AC. Let the perpendicular from A to BC be D. Let the circumcircle of $\triangle OAM$ hit DM at $P \neq M$. Prove that B, O, P are colinear.

Solution: Instead we show that the intersection of MD and BO, which we will call P', lies on (MAO). The central claim is that $PABD$ is cyclic.

Note $\angle PDA = \angle MDA = 90^\circ - \angle C$, and also note that $\angle PAD = \angle PAB - \angle DAB$. Note that $\angle PAB = \angle C$ since $\angle APB = 90^\circ$ and $\angle ABP = \angle ABO = 90^\circ - \angle C$ and $\angle BAD = 90^\circ - \angle B$. Thus $\angle PAD = \angle B + \angle C - 90^\circ$.

Now consider $\triangle PAD$. Note $\angle DPA = 180^\circ - (\angle PDA + \angle PAD) = 180^\circ - \angle B$. Thus $PABD$ is cyclic.

This implies that $\angle APO = \angle APB = \angle ADB = 90^\circ$. Since $\angle AMO = 90^\circ$ as well, we are done.

This final example demonstrates the power of wishful thinking.

Example 5.2.5 — ISL 2010/G1
Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Solution: We work in directed angles because there are plenty of configuration issues. (If you don't know what directed angles are, consult the chapter on them.)

Note that $AFPQ$ is cyclic, as

$$\measuredangle AFQ = \measuredangle BFD = \measuredangle ACB = \measuredangle APB = \measuredangle APQ.$$

Now note that

$$\measuredangle APQ = \measuredangle AFQ = \measuredangle BFD = \measuredangle ACB$$
$$\measuredangle PQA = \measuredangle PFA = \measuredangle EFA = \measuredangle ACB,$$

implying that $\angle APQ = \angle PQA$, or that $AP = AQ$.

I personally thought this problem was harder than the other two, especially since the cyclic quadrilateral had an asymmetric structure with respect to the whole diagram.[1] We're inclined to look for cyclic quadrilaterals involving

[1]This is explain by the entire diagram being asymmetric.

A, P, Q in some way because the problem is essentially equivalent to showing that $\angle AQP = \angle APQ$, and a little bit of experimentation shows that it's hard to show directly. The motivation for trying to prove F is the point on (APQ) is drawing in the circumcircles for both configurations, and noting that the second intersection point of them is F.

The rest of the motivation is quite straightforward – all you have to do afterwards is try to solve the problem with the assumption that $AFPQ$ is cyclic, and that part is fairly easy if you have any knowledge about the orthic triangle.

5.2.3 Tangent Angle Criterion

When tangent lines are given, you have to pay close attention the the tangent angle criterion.

Example 5.2.6 — British Math Olympiad Round 1 2000/1
Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

Solution: Note that $\angle RMQ = 180° - \angle RNQ = 180° - (\angle PNM + \angle QNM) = 180° - (\angle QPM + \angle PQM) = \angle PMQ$.

(This actually only takes care of the case where R is in between P and N. Can you show this is true for the other configuration as well?)

Let's expound on the motivation for this. We want to prove that PQ bisects $\angle PMN$, but it's quite hard to find the supplement of $\angle PMQ$ and $\angle RMQ$. This then motivates showing that $\angle PMQ = \angle RMQ$, because those angles seem more workable. We start by manipulating $\angle RMQ$ because it seems more unwieldy, and it feels like there are more ways to get to $\angle PMQ$ than $\angle RMQ$. (This part is personal preference, but a good rule of thumb is to try to manipulate the least independently defined points into the most independently defined points.[2])

The cyclic quadrilateral $RMNQ$ is the source of the only useful manipulation we can do with $\angle RMQ$, so we're pretty much forced into using it. Now looking at $\triangle PNQ$ as a whole motivates $\angle RNQ = 180° - (\angle PNM + \angle QNM)$, and at this point we want to start manipulating $\angle PMQ$. We're forced into doing $180° - (\angle QPM + \angle PQM) = \angle PMQ$, because tangent lines have lots of potential for angle chasing and it's the only place to go.

Now the rest of the problem will just come naturally by just trying things.

5.2.4 Orthocenter

Sometimes a problem will ask you to prove that $AH \perp BC$ for some point H not on BC. This is generally difficult to do directly, and one of the more elementary methods used is to show that H is the orthocenter of $\triangle ABC$, or $BH \perp CA$ and $CH \perp AB$.

This is obviously not always going to be true, so make sure that this actually seems true before you try too hard to prove it.

Example 5.2.7 — Swiss Math Olympiad 2007/4
Let ABC be an acute-angled triangle with $AB > AC$ and orthocenter H. Let D be the projection of A on BC. Let E be the reflection of C about D. The lines AE and BH intersect at point S. Let N be the midpoint of AE and let M be the midpoint of BH. Prove that MN is perpendicular to DS.

Solution: We claim S is the orthocenter of $\triangle DEM$. To do this, it suffices to show that $SN \perp DM$ and $SM \perp DN$. Let H' be the second intersection of AH with (ABC).

Note that $DM \parallel BH'$ by a homothety about H, $\angle MAE = \angle DAC = 90° - \angle C$, and $\angle AMB = \angle C$, proving $SN \perp DM$.

Now note that $DN \parallel AC$ by a homothety about E, proving $SM \perp DN$.

[2] A heuristic for the independence of a point is how much it would affect the diagram on GeoGebra if it was deleted.

5.3 Problems

Minimum is [40 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "How arrogant. The life of each human is worth one, that's it. Nothing more, nothing less."
>
> Fullmetal Alchemist: Brotherhood

[2 ♟] **(AMC 10A 2020/12)** Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?

[2 ♟] **(Brazil 2004)** In the figure, ABC and DAE are isosceles triangles ($AB = AC = AD = DE$) and the angles BAC and ADE have measures $36°$.

1. Using geometric properties, calculate the measure of angle $\angle EDC$.
2. Knowing that $BC = 2$, calculate the length of segment DC.
3. Calculate the length of segment AC.

[2 ♟] Let circles ω_1 and ω_2 intersect at X, Y. Let line ℓ_1 passing through X intersect ω_1 at A and ω_2 at C, and let line ℓ_2 passing through Y intersect ω_1 at B and ω_2 at D. If ℓ_1 intersects ℓ_2 at P, prove that $\triangle PAB \sim \triangle PCD$.

[2 ♟] **(AMC 10B 2011/17)** In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?

[3 ♟] **(Dennis Chen)** Consider rectangle $ABCD$ with $AB = 6$, $BC = 8$. Let M be the midpoint of AD and let N be the midpoint of CD. Let BM and BN intersect AC at X and Y respectively. Find XY.

[3 ♟] **(AMC 10A 2019/13)** Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

[3 ♞] **(Miquel's Theorem)** Consider $\triangle ABC$ with D on BC, E on CA, and F on AB. Prove that (AEF), (BFD), and (CDE) concur.

[2 ♟] Consider $\triangle ABC$ with D on segment BC, E on segment CA, and F on segment AB. Let the circumcircles of $\triangle FBD$ and $\triangle DCE$ intersect at $P \neq D$. If $\angle A = 50°$, $\angle B = 35°$, find $\angle DPE$.

[3 ♟] **(AIME II 2018/4)** In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Let K be the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. Then $K = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

[4 ♞] **(Brazil 2007)** Let ABC be a triangle with circumcenter O. Let P be the intersection of straight lines BO and AC and ω be the circumcircle of triangle AOP. Suppose that $BO = AP$ and that the measure of the arc OP in ω, that does not contain A, is $40°$. Determine the measure of the angle $\angle OBC$.

[4 ♟] Consider square $ABCD$ and some point P outside $ABCD$ such that $\angle APB = 90°$. Prove that the angle bisector of $\angle APB$ also bisects the area of $ABCD$.

[4 ♞] **(AMC 10B 2018/12)** Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

[6 ♟] **(Formula of Unity 2018)** A point O is the center of an equilateral triangle ABC. A circle that passes through points A and O intersects the sides AB and AC at points M and N respectively. Prove that $AN = BM$.

[6 ♞] **(AMC 10A 2021/17)** Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. GIven that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

[6 ♟] **(Memorial Day Mock AMC 10 2018/21)** In the following diagram, $m\angle BAC = m\angle BFC = 40°$, $m\angle ABF = 80°$, and $m\angle FEB = 2m\angle DBE = 2m\angle FBE$. What is $m\angle ADB$?

[6 ♟] **(FARML 2012/6)** In triangle ABC, $AB = 7$, $AC = 8$, and $BC = 10$. D is on AC and E is on BC such that $\angle AEC = \angle BED = \angle B + \angle C$. Compute the length AD.

[9 ♟] **(USAJMO 2020/4)** Let $ABCD$ be a convex quadrilateral inscribed in a circle and satisfying $DA < AB = BC < CD$. Points E and F are chosen on sides CD and AB such that $BE \perp AC$ and $EF \parallel BC$. Prove that $FB = FD$.

[13 ♟] **(MAST Diagnostic 2020)** Consider $\triangle ABC$ with D on line BC. Let the circumcenters of $\triangle ABD$ and $\triangle ACD$ be M, N, respectively. Let the circumcircle of $\triangle MND$ intersect the circumcircle of $\triangle ACD$ again at $H \neq D$. Prove that A, M, H are collinear.

Prime Factorization

Chapter 6

These are problems where you want to look at the highest power of a prime that divides a number. This is the ν_p function.

6.1 Divisiblity

Here is the formal definition of divisibility.

Definition 6.1.1 — Divisibility
For integers a, b, we say a divides b if and only if there exists some integer c such that $ac = b$.
We denote this as $a \mid b$.

This implies the following three facts.

Fact 6.1.2 — Divisibility Results
Given integers a, b, c,

- If $a \mid b$ and $b \mid c$ then $a \mid c$. (This may be referred to as the "chain rule" of divisibility.)
- If $a \mid b$ then $a \mid bc$ for all integer c.
- If $a \mid b$ and $a \mid c$, then $a \mid b + c$ and $a \mid b - c$.

6.1.1 P-adic Valuation

P-adic valuation, or the ν_p function, asks for the largest power of p that divides an integer.

Definition 6.1.3 — P-adic Valuation
For a positive integer n, $\nu_p(n)$ is the largest integer that satisfies $p^{\nu_p(n)} \mid n$.

Remember that $\nu_p(n)$ is only defined for prime p.
This implies the following obvious but very useful fact.

Fact 6.1.4 — P-adic Inequality
If $a \mid b$, then for all primes p, $\nu_p(a) \leq \nu_p(b)$.

Proof: We proceed by contradiction. Say $\nu_p(a) > \nu_p(b)$. Then $p^{\nu_p(a)} \mid a \mid b$, implying that $p^{\nu_p(a)} \mid b$ by the chain rule of divisibility. But $\nu_p(b)$ is the largest power of p that divides b, contradiction. ■

6.1.2 GCD and LCM

Definition 6.1.5 — Greatest Common Divisor
We define $\gcd(a_1, a_2 \ldots a_n)$ as the largest positive integer such that

$$\gcd(a_1, a_2 \ldots a_n) \mid a_i$$

for all $1 \le i \le n$.

Definition 6.1.6 — Least Common Multiple
We define $\operatorname{lcm}(a_1, a_2 \ldots a_n)$ as the smallest **positive** integer such that

$$a_i \mid \operatorname{lcm}(a_1, a_2 \ldots a_n)$$

for all $1 \le i \le n$.

As an exercise, list the divisors of 0, the numbers that 0 divides, and find $\gcd(0, 8)$.

Now we take a look at the prime powers of gcd and lcm .

Fact 6.1.7 — P-adic Maximum and Minimum
Given integers $a_1, a_2, \ldots, a_n$,

- $v_p(\gcd(a_1, a_2, \ldots, a_n)) = \min(v_p(a_1), v_p(a_2), \ldots, v_p(a_n)))$.
- $v_p(\operatorname{lcm}(a_1, a_2, \ldots, a_n)) = \max(v_p(a_1), v_p(a_2), \ldots, v_p(a_n))$.

The proof is an obvious consequence of the P-adic Inequality.

6.2 Well-known Divisor Tricks

You should know everything here.

Theorem 6.2.1 — Fundamental Theorem of Arithmetic
Every number greater than 1 is either a prime or can be uniquely, up to order, expressed as a product of primes.

If you are curious about the proof, you may check out `https://gowers.wordpress.com/2011/11/18/proving-the-fundamental`

Theorem 6.2.2 — Number of Divisors
Say the prime factorization of n is $p_1^{q_1} \cdot p_2^{q_2} \cdot \cdots \cdot p_k^{q_k}$. Then n has $(p_1 + 1)(p_2 + 1) \cdots (p_k + 1)$ positive divisors.

Proof: This is a simple combinatorics problem.

Note that there are $q_i + 1$ numbers between 0 and q_i to pick from, and you choose the exponent of each prime p_i for the divisor. So in total there are $(q_1 + 1)(q_2 + 1) \cdots (q_k + 1)$ choices. ■

Theorem 6.2.3 — Sum of Divisors
Say the prime factorization of n is $p_1^{q_1} \cdot p_2^{q_2} \cdot \cdots \cdot p_k^{q_k}$. Then the sum of the factors of n is

$$\left(\frac{p_1^{q_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{q_1+1} - 1}{p_2 - 1}\right) \cdots \left(\frac{p_k^{q_k+1} - 1}{p_k - 1}\right) =$$

$$\left(1 + p_1 + p_1^2 + \cdots + p_1^{q_1}\right)\left(1 + p_2 + p_2^2 + \cdots + p_2^{q_2}\right) \cdots \left(1 + p_k + p_k^2 + \cdots + p_k^{q_k}\right).$$

The latter part of the proof is going to seem magical. Take some time to digest it.

Proof: By geometric series, $\frac{p_i^{q_i+1}-1}{p_i-1} = 1 + p_i + p_i^2 + \cdots + p_i^{q_i}$.

Notice that expanding the product gives you every possible combination of powers of $p_1, p_2, \ldots, p_k$. This means that summing all of these combinations together is equivalent to summing up all of the divisors of n. ■

For concreteness, we present a few examples.

Example 6.2.4
Find the prime factorization of 216.

Solution: The prime factorization is $2^3 \cdot 3^3$, and this is unique by the Fundamental Theorem of Arithmetic.

Example 6.2.5
Find the number of divisors of 216.

Solution: The prime factorization is $2^3 \cdot 3^3$, as we have established in the previous example. Now note that we can choose the power of 2 of the divisor in 4 ways, and we can choose the power of 3 of the divisor in 4 ways as well. Thus there are $4 \cdot 4 = 16$ total divisors.

Example 6.2.6
Find the sum of the divisors of 216.

Solution: The sum of the divisors is $(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2 + 3^3) = 15 \cdot 40 = 600$.

Expand the sum out to convince yourself that all divisors are characterized exactly once.

Here is a much harder example, motivated by an obvious fact.

Fact 6.2.7 — Odd Number of Divisors
A number has an odd number of positive divisors if and only if it is a perfect square.

The proof is just looking at the number of divisors formula. Alternatively, each divisor d gets paired off with $\frac{n}{d}$, except for $\sqrt{n}$.

Example 6.2.8 — 104 NT
Twenty bored students take turns walking down a hall that contains a row of closed lockers, numbered 1 to 20. The first student opens all the lockers; the second student closes all the lockers numbered 2, 4, 6, 8, 10, 12, 14, 16, 18, 20; the third student operates on the lockers numbered 3, 6, 9, 12, 15, 18: if a locker was closed, he opens it, and if a locker was open, he closes it; and so on. For the ith student, he works on the lockers numbered by multiples of i: if a locker was closed, he opens it, and if a locker was open, he closes it. What is the number of the lockers that remain open after all the students finish their walks?

Solution: Note that a locker is only open if it is interacted with an odd number of times, and the number of times a locker is interacted with is the number of divisors it has. Since perfect squares are the only integers with an odd number of divisors, the open lockers are just $1, 4, 9, 16$. Thus 4 lockers are open.

Example 6.2.9 — AIME I 2005/12
For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \le 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \le 2005$ with $S(n)$ even. Find $|a - b|$.

Solution: Note $\tau(n)$ is odd if and only if n is a perfect square, implying that $S(n)$ is odd if n is greater than an odd number of squares and even if n is greater than an even number of squares.

We can explicitly characterize this as

$$S(n)\begin{cases} \text{is odd if } 1^2 \le n < 2^2 \text{ or } 3^2 \le n < 4^2 \text{ or } 5^2 \le n < 6^2 \text{ or} \ldots \\ \text{is even if } 2^2 \le n < 3^2 \text{ or } 4^2 \le n < 5^2 \text{ or } 6^2 \le n < 7^2 \text{ or} \ldots \end{cases}.$$

Now we use the difference of squares formula to find a and b. Note that the largest square smaller than 2005 is $44^2 = 1936$, so

$$a - b = (-1^2 + 2^2 - 3^2 + 4^2 - \cdots - 43^2 + 44^2) - (-2^2 + 3^2 - 4^2 + 5^2 - \cdots + 42^2 - 43^2) - (2005 - 1936 + 1)$$

$$a - b = (1 + 2 + 3 + 4 + \cdots + 43 + 44) - (2 + 3 + 4 + \cdots + 43) - 70$$

$$a - b = 1 + 44 - 70 = -25.$$

Thus the answer is 25.

6.3 Assorted Examples

These are some examples of prime factorization analysis problems. Since this is a technique rather than a theorem, we will show, not tell.

Example 6.3.1 — Dennis' Mock AIME 2020/4
Find the number of ordered pairs of positive integers (a, b) such that $\gcd(a, b) = 20$ and $\operatorname{lcm}(a, b) = 19!$

Solution:

1. Note $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$, so $ab = 20!$
2. Let $\frac{a}{20} = x$ and $\frac{b}{20} = y$. Then note $\gcd(x, y) = 1$ and $\operatorname{lcm}(a, b) = \operatorname{lcm}(20x, 20y) = 20\operatorname{lcm}(x, y) = 20xy$. Thus $xy = \frac{19!}{20}$.
3. Note picking (x, y) uniquely determines (a, b).
4. Now note we have to either assign *all* of the powers of a prime to x or to y.
5. Check how many primes divide $\frac{19!}{20}$.

Example 6.3.2 — AIME 1987/7
Let $[r, s]$ denote the least common multiple of positive integers r and s. Find the number of ordered triples (a, b, c) of positive integers for which $[a, b] = 1000$, $[b, c] = 2000$, and $[c, a] = 2000$.

Solution:

1. Let $a = 2^a \cdot 5^x$, $b = 2^b \cdot 5^y$, $c = 2^c \cdot 5^z$.

2. Note that $\max(a, b) = 3$, $\max(b, c) = 4$, $\max(c, a) = 4$. This notably implies $c = 4$. (Why can't we have $a = 4$ or $b = 4$?)

3. Note that $\max(x, y) = \max(y, z) = \max(z, x) = 3$. How many of x, y, z can be less than 3 at a time? How many ways can we do this?

4. Multiply the number of ways to distribute the powers of 2 by the number of ways to distribute the powers of 5.

Here is a prime factorization analysis problem that doesn't have GCD or LCM in the problem statement.

Example 6.3.3 — ARML 2008
If n has 60 positive factors, compute the largest number of positive factors that n^2 could have.

Solution: Let the prime factorization of n be $p_1^{q_1} \cdot p_2^{q_2} \cdot \cdots \cdot p_k^{q_k}$. Then note

$$(p_1 + 1)(p_2 + 1)\cdots(p_k + 1) = 60.$$

We want to maximize

$$(2p_1 + 1)(2p_2 + 1)\cdots(2p_k + 1).$$

This occurs when $p_1 = p_2 = 1, p_3 = 2, p_4 = 4$. Thus our answer is $3 \cdot 3 \cdot 5 \cdot 9 = 405$.

6.4 Problems

Minimum is [45 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "I won't ask you to buy me curry bread anymore. Goodbye."
>
> Yugami-kun Has No Friends

[1 ♟] Answer the following:

- What are the divisors of 0?
- What integers does 0 divide?
- Find $\gcd(0, 8)$.
- Why isn't something like $\operatorname{lcm}(0, 8)$ defined?

[1 ♟] **(AMC 10A 2005/15)** Find the number of positive cubes that divide $3! \cdot 5! \cdot 7!$

[1 ♟] **(AMC 8 2013/10)** What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?

[2 ♞] **(PUMaC 2016)** What is the smallest positive integer n such that $2016n$ is a perfect cube?

[2 ♟] **(SMT 2018)** One of the six digits in the expression $435 \cdot 605$ can be changed so that the product is a perfect square N^2. Compute N.

[2 ♟] **(AIME I 2010/1)** Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[2 ♟] **(AMC 12B 2002/12)** For which integers n is $\dfrac{n}{20-n}$ the square of an integer?

[3 ♞] **(Scrabbler AMC 10)** Let n be the smallest positive integer with the property that $\operatorname{lcm}(n, 2020!) = 2021!$, where $\operatorname{lcm}(a, b)$ denotes the least common multiple of a and b. How many positive factors does n have?

[3 ♟] **(Switzerland Prelimiary Round 2018/N1)** Let $n \geq 2$ be a positive integer, and let $d_1, \cdots, d_r$ be all the positive divisors of n that are smaller that n. Determine all n for which

$$\operatorname{lcm}(d_1, \cdots, d_r) \neq n.$$

[3 ♞] **(AMC 12A 2016/22)** How many ordered triples (x, y, z) of positive integers satisfy $\operatorname{lcm}(x, y) = 72$, $\operatorname{lcm}(x, z) = 600$, and $\operatorname{lcm}(y, z) = 900$?

[4 ♟] **(CMC 10A 2021/22)** For a certain positive integer n, there are exactly 2021 ordered pairs of positive divisors (d_1, d_2) of n for which d_1 and d_2 are relatively prime. What is the sum of all possible values of the number of divisors of n?

[4 ♟] **(AHSME 1987/23)** If p is a prime and both roots of $x^2 + px - 444p = 0$ are integers, then what is p?

[4 ♟] **(HMMT 2018)** Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. What is A?

[6 ♟] **(AMC 12B 2007/24)** Find all pairs of positive integers (a, b) such that $\gcd(a, b) = 1$ and $\frac{a}{b} + \frac{14b}{9a}$ is an integer.

[6 ♟] **(AIME I 2020/10)** Let m and n be positive integers satisfying the conditions

- $\gcd(m + n, 210) = 1$,
- m^m is a multiple of n^n, and
- m is not a multiple of n.

Find the least possible value of $m + n$.

[6 ♟] **(ARML 2010)** Compute the smallest positive integer n such that n^n has at least $1,000,000$ positive divisors.

[6 ♟] **(AMC 10B 2018/23)** How many ordered pairs (a, b) of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \operatorname{lcm}(a, b) + 12 \cdot \gcd(a, b),$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b, and $\operatorname{lcm}(a, b)$ denotes their least common multiple?

[9 ♞] **(AMC 12A 2021/25)** Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

[9 ♟] **(AMC 10A 2018/22)** Let a, b, c, and d be positive integers such that $\gcd(a, b) = 24$, $\gcd(b, c) = 36$, $\gcd(c, d) = 54$, and $70 < \gcd(d, a) < 100$. Which of the following must be a divisor of a?

(A) 5 **(B)** 7 **(C)** 11 **(D)** 13 **(E)** 17

[9 ♟] **(AMC 12B 2010/25)** For every integer $n \geq 2$, let $\operatorname{pow}(n)$ be the largest power of the largest prime that divides n. For example $\operatorname{pow}(144) = \operatorname{pow}(2^4 \cdot 3^2) = 3^2$. What is the largest integer m such that 2010^m divides $\prod_{n=2}^{5300} \operatorname{pow}(n)$?

[13 ♟] **(ISL 2007/N2)** Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

[13 ♟] **(PUMaC 2016)** Let $k = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7^3 \cdot 53$. let S be the sum of $\frac{\gcd(m,n)}{\operatorname{lcm}(m,n)}$ over all ordered pairs of positive integers (m, n) where $mn = k$. If S can be written in simplest form as $\frac{r}{s}$, compute $r + s$.

Optimization and Basic Inequalities

Chapter 7

7.1 Optimization

In these types of problems, we usually have a list of numbers or actions and want to minimize of maximize some output. In many cases we try to perturb one of the properties or variables given in the problem, or set them to an extreme value and see what results we have. More often than not you will see such problems take the form of a word problem with a lengthy exposition. Sometimes the heart of the problem is simply solving some system of equations (usually that has multiple solutions) and providing the least value of some expression.

7.1.1 Examples

Example 7.1.1 — AIME 1983/2
Let $f(x) = |x-p| + |x-15| + |x-p-15|$, where $0 < p < 15$. Determine the minimum value taken by $f(x)$ for x in the interval $p \le x \le 15$.

Solution: Notice that $|x-p| = x-p$, $|x-15| = 15-x$, and $|x-p-15| = 15+p-x$.

Adding these together, we find that the sum is equal to $30-x$, which attains its minimum value (on the given interval $p \le x \le 15$) when $x = 15$, giving a minimum of $\boxed{015}$.

Example 7.1.2 — AIME II 2009/4
A group of children held a grape-eating contest. When the contest was over, the winner had eaten n grapes, and the child in k-th place had eaten $n+2-2k$ grapes. The total number of grapes eaten in the contest was 2009. Find the smallest possible value of n.

Solution: Let there be m children. The number of grapes eaten by the children collectively will be $(n)+(n-2)+(n-4)+\cdots+(n+2-2m) = \frac{1}{2}m(n+n+2-2m) = m(n+1-m)$. Note that $2009 = 7^2 \cdot 41$. Thus, there are only a limited number of values that m can take; testing each value, we conclude that the minimum n is achieved at $m = 41$ and $n = 89$.

Example 7.1.3 — AIME II 2003/12
The members of a distinguished committee were choosing a president, and each member gave one vote to one of the 27 candidates. For each candidate, the exact percentage of votes the candidate got was smaller by at least 1 than the number of votes for that candidate. What was the smallest possible number of members of the committee?

Solution: Let v_i be the number of votes that candidate i receives, and v the total number of votes. The given condition implies $100\frac{v_i}{v} + 1 \le v_i \implies v \ge \frac{100v_i}{v_i-1}$ for all $1 \le i \le 27$. Note that $v_i \ge 2$. If at least of v_i is 2, then $v \ge 200$. If at least one of v_i is 3, then $s \ge 150$. If at least one of v_i is 4, then $s \ge \frac{400}{3}$, and hence $s \ge 134$. If $s < 134$, then $v_i \ge 5$ for all i. However this implies $v = \sum_{i=1}^{27} v_i \ge 27 \cdot 5 = 135$, contradiction. Hence $v = 134$ is the minimum. Suppose all but one candidate received 5 votes, and the last received 4. Then $v = 134$, and the given criterion is satisfied. Thus 134 is the minimum number of voters in the committee.

7.2 Algebraic Inequalities

Very rarely we can finish such problems directly or indirectly with well known inequalities. Usually these problems have a more algebraic taste.

Theorem 7.2.1 — Trivial Inequality
For any real number x, $x^2 \ge 0$, with equality at $x = 0$.

Take a moment and convince yourself that this is true.

Theorem 7.2.2 — Arithmetic Mean-Geometric Mean Inequality
For any real numbers $x_1, x_2, \ldots, x_n \ge 0$,

$$\frac{\sum_{i=1}^n x_i}{n} \ge \sqrt[n]{\prod_{i=1}^n x_i}.$$

Equality holds for $x_1 = x_2 = \cdots = x_n$.

This is a relatively elegant and complicated proof, featuring a very special form of induction called Cauchy-Induction. You do not need to remember, or even read, this proof (at this level), though convince yourself you could understand it if you wanted to. Also note that this proof is moderately difficult to motivate. This inequality is often abbreviated as "AM-GM".

Proof: First, we prove that the inequality is true for the base case $n = 2$. Then, we show that if the inequality is true for $n = k$, it is too for $n = 2k$. Finally, if it is true for $n = k$, it is also true for $n = k - 1$.

For $n = 2$, the inequality rearranges to

$$\frac{a+b}{2} \ge \sqrt{ab} \implies a^2 + 2ab + b^2 \ge 4ab,$$

or $(a-b)^2 \ge 0$.

Now assume that this inequality is true for $n = k$; ie. $\frac{x_1+x_2+\cdots+x_k}{n} \ge \sqrt[n]{x_1x_2\cdots x_k}$. Then, we have

$$\begin{aligned}
\sqrt[2k]{x_1x_2\cdots x_{2k}} &= \sqrt{\sqrt[n]{x_1x_2\cdots x_k}\sqrt[k]{x_{k+1}x_{k+2}\cdots x_{2k}}} \\
&\le \frac{\sqrt[k]{x_1x_2\cdots x_k} + \sqrt[k]{x_1x_2\cdots x_{2k}}}{2} \\
&\le \frac{\frac{x_1+x_2+\cdots+x_k}{k} + \frac{x_{k+1}+x_{k+2}+\cdots+x_{2k}}{k}}{2} \\
&= \frac{x_1 + x_2 + \cdots + x_{2k}}{2k}.
\end{aligned}$$

Thus, the inequality is true for $n = 2k$. Again assume the inequality is true for $n = k$. Take $x_k = \frac{\sum\limits_{i=1}^{k-1} x_i}{k-1}$. Note that $\frac{1}{k}\sum\limits_{i=1}^{k} x_i = \frac{1}{k-1}\sum\limits_{i=1}^{k-1} x_i$.

$$\begin{aligned}
\frac{1}{k-1}\sum_{i=1}^{k-1} x_i &= \frac{1}{k}\sum_{i=1}^{k} x_i \\
&\geq \sqrt[k]{\prod_{i=1}^{k} x_i} \\
&= \sqrt[k]{\frac{1}{k-1}\left(\sum_{i=1}^{k-1} x_i\right)\left(\prod_{i=1}^{k=1} x_i\right)} \\
\implies \left(\frac{1}{k-1}\sum_{i=1}^{k-1} x_i\right)^k &\geq \frac{1}{k-1}\left(\sum_{i=1}^{k-1} x_i\right)\left(\prod_{i=1}^{k=1} x_i\right) \\
\implies \left(\frac{1}{k-1}\sum_{i=1}^{k-1} x_i\right)^{k-1} &\geq \prod_{i=1}^{k-1} x_i \\
\implies \frac{1}{k-1}\sum_{i=1}^{k-1} x_i &\geq \sqrt[n-1]{\prod_{i=1}^{k-1} x_i}.
\end{aligned}$$

This implies the inequality is true for $n = k - 1$. Our induction is complete. ■

Theorem 7.2.3 — Cauchy Schwarz Inequality
For any real numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, x_n$,

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2.$$

Equality holds for $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$.

This inequality is often abbreviated "C-S". We present two of the more elegant proofs. The first proof is by considering the discriminant of a cleverly defined function. Again, skip if you would like.

Proof: Let $f(x) = \sum_{i=1}^{n}(x_i x - y_i)^2 = \left(\sum_{i=1}^{n} x_i^2\right)x^2 - 2\left(\sum_{i=1}^{n} x_i y_i\right)x + \left(\sum_{i=1}^{n} y_i^2\right)$. By the trivial inequality, $f(x) \geq 0$. This means the discriminant of $f(x)$ is greater or equal to zero, which simplifies down to the desired inequality. Obviously equality holds if $f(x) = 0 \implies \frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$. ■

The second proof is using an identity called the Cauchy-Schwarz expansion that solves the inequality immediately.

Proof: Note that

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \sum_{1\leq i,j\leq n} (x_i y_i - x_j y_i)^2.$$

Since the expression is obviously greater than 0, we are done. ■

Theorem 7.2.4 — Triangle Inequality
Given a triangle with side lengths a, b, c, it is non-degenerate if and only if $a + b > c$, $b + c > a$, and $c + a > b$.

Take a moment and convince yourself that this is true.

7.2.1 Examples

Example 7.2.5 — AIME 1991/3
Expanding $(1+0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\binom{1000}{0}(0.2)^0+\binom{1000}{1}(0.2)^1+\cdots+\binom{1000}{1000}(0.2)^{1000}=A_0+A_1+A_2+\cdots+A_{1000},$$

where $A_k=\binom{1000}{k}(0.2)^k$ for $k=0,1,2,\ldots,1000$. For which k is A_k the largest?

Solution: We are looking for the smallest k such that

$$\frac{1}{5^k}\cdot\binom{1000}{k}>\frac{1}{5^{k+1}}\cdot\binom{1000}{k+1}\implies\binom{1000}{k}>\frac{1}{5}\cdot\binom{1000}{k+1}.$$

Expanding these binomial coefficients, we have

$$\frac{1000!}{(1000-k)!\cdot k!}>\frac{1}{5}\cdot\frac{1000!}{(1000-k-1)!\cdot(k+1)!}.$$

Simplifying, we must have $\frac{1}{1000-k}>\frac{1}{5k+5}\implies k>165.8$, so our desired answer is 166.

Example 7.2.6 — AIME 1983/9
Find the minimum value of $\frac{9x^2\sin^2 x+4}{x\sin x}$ for $0<x<\pi$.

Solution: By AM-GM, $9x\sin x+\frac{4}{x\sin x}\geq 2\cdot 6=12$. Note that equality is achieved at $x\sin x=\frac{2}{3}$, which is attainable in the range $0<x<\pi\implies 0<x\sin x<\frac{\pi}{2}$.

Example 7.2.7
For which real value of x is the function $f(x)=ax^2+bx+c$ minimized? Assume $a>0$ and the coefficients are real.

Solution: Completing the square, we have $f(x)=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$. By the trivial inequality, this expression is minimized when $a\left(x+\frac{b}{2a}\right)^2=0\implies x=-\frac{b}{2a}$.

Example 7.2.8 — WOOT
Let x be real. Find the maximum value of $f(x)=x^3(4-x)$.

Solution: By AM-GM,

$$f(x)=\frac{1}{3}[x\cdot x\cdot x\cdot(12-3x)]\leq\frac{1}{3}\cdot\left[\frac{x+x+x+(12-3x)}{4}\right]^4=27.$$

Equality holds if $x=x=x=12-3x$. Indeed, $f(3)=27$, so the maximum value is 27.

7.3 Problems

Minimum is [40 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

[2 ♞] **(WOOT)** In Example 7 we showed that the maximum value of $f(x) = x^3(4-x)$ is 27. However, we could have manipulated the inequality in many different ways. For instance, we could have written

$$f(x) = \frac{1}{36}[x(2x)(3x)(24-6x)] \le \frac{1}{36}\left[\frac{x+2x+3x+24-6x}{4}\right]^4 = 36.$$

So why is $\max(f(x)) = 27$ and not 36?

[2 ♟] Show that for any $n \ge 1$, the inequality $\sqrt{6+\cdots\sqrt{6+\sqrt{6}}} < 3$ holds, where there are n nested radicals.

[3 ♟] **(AIME II 2004/6)** Three clever monkeys divide a pile of bananas. The first monkey takes some bananas from the pile, keeps three-fourths of them, and divides the rest equally between the other two. The second monkey takes some bananas from the pile, keeps one-fourth of them, and divides the rest equally between the other two. The third monkey takes the remaining bananas from the pile, keeps one-twelfth of them, and divides the rest equally between the other two. Given that each monkey receives a whole number of bananas whenever the bananas are divided, and the numbers of bananas the first, second, and third monkeys have at the end of the process are in the ratio $3:2:1$,what is the least possible total for the number of bananas?

[3 ♟] **(AMC 10A 2019/19)** What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4)+2019$$

where x is a real number?

[3 ♟] **(AIME I 2010/5)** Positive integers a, b, c, and d satisfy $a > b > c > d$, $a+b+c+d = 2010$, and $a^2 - b^2 + c^2 - d^2 = 2010$. Find the number of possible values of a.

[4 ♟] **(AIME I 2004/5)** Alpha and Beta both took part in a two-day problem-solving competition. At the end of the second day, each had attempted questions worth a total of 500 points. Alpha scored 160 points out of 300 points attempted on the first day, and scored 140 points out of 200 points attempted on the second day. Beta who did not attempt 300 points on the first day, had a positive integer score on each of the two days, and Beta's daily success rate (points scored divided by points attempted) on each day was less than Alpha's on that day. Alpha's two-day success ratio was $300/500 = 3/5$. The largest possible two-day success ratio that Beta could achieve is m/n, where m and n are relatively prime positive integers. What is $m+n$?

[4 ♟] Let $\mathcal{S}$ be the set of all real numbers x such that $\lfloor x^2 \rfloor = \lfloor x \rfloor \lfloor x+1 \rfloor$. Let m be the minimum positive integer for which $m + 0.51$ is not in $\mathcal{S}$. Find m.

[4 ♟] **(Andreescu)** Let P be a polynomial with positive coefficients. Show that if

$$P\left(\frac{1}{x}\right) \ge \frac{1}{P(x)}$$

holds for $x = 1$, it holds for all $x > 0$.

[6 ♞] **(AIME 1990/11)** Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer n for which $n!$ can be expressed as the product of $n-3$ consecutive positive integers.

[6 ♟] **(AIME II 2011/9)** Let $x_1, x_2, \ldots, x_6$ be non-negative real numbers such that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, and $x_1x_3x_5 + x_2x_4x_6 \ge \frac{1}{540}$. Let p and q be positive relatively prime integers such that $\frac{p}{q}$ is the maximum possible value of $x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2$. Find $p+q$.

[6 ♞] **(AIME I 2008/12)** On a long straight stretch of one-way single-lane highway, cars all travel at the same speed and all obey the safety rule: the distance from the back of the car ahead to the front of the car behind is exactly one car length for each 15 kilometers per hour of speed or fraction thereof (Thus the front of a car traveling 52 kilometers per hour will be four car lengths behind the back of the car in front of it.) A photoelectric eye by the side of the road counts the number of cars that pass in one hour. Assuming that each car is 4 meters long and that the cars can travel at any speed, let M be the maximum whole number of cars that can pass the photoelectric eye in one hour. Find the quotient when M is divided by 10.

[9 ♟] **(AIME II 2009/11)** For certain pairs (m,n) of positive integers with $m \geq n$ there are exactly 50 distinct positive integers k such that $|\log m - \log k| < \log n$. Find the sum of all possible values of the product mn.

[9 ♟] A cubic polynomial has the property that all its roots are rational and its coefficients are positive prime integers. If $p(1) = 144$, compute the value of $p(2)$.

[9 ♟] **(AIME 1998/14)** An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where m, n, and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

[13 ♟] Triangle ABC has sides a, b, and c, and circumradius R. Prove that

$$b^2 + c^2 \geq a^2 - R^2.$$

When does equality occur?