Applications of Calculus

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ART

Here, we will discuss the common uses of single-variable differentiaton and integration in contest problems that are not explicitly about Calculus.

1 Jensen's inequality and Tangent Line Trick

(This section is more useful for Olympiad math rather than computational, but its importance demands for it to be included.)

A part of why "Trivial by Jensen's" is often said as a meme regarding problems that may or may not involve inequalities is its true power.

Jensen's Inequality. In an interval, a function is **convex** if and only if its second derivative is nonnegative throughout the interval and **concave** if and only if its second derivative is nonpositive throughout the interval.

For real numbers $x_1, x_2 \ldots x_i$ in a convex function f's domain and positive real weights $w_1, w_2 \ldots w_n$, we have

$$f\left(\frac{\sum w_i x_i}{\sum w_i}\right) \leq \frac{\sum w_i f(x_i)}{\sum w_i}$$

When the function is concave, we have an analogous inequality

$$f\left(\frac{\sum w_i x_i}{\sum w_i}\right) \geq \frac{\sum w_i f(x_i)}{\sum w_i}$$

as $-f$ is convex.

It is useful to memorize the convexities of the most common functions, and below are some of them:

Exercise (List of functions to evaluate). $x^2, x, \frac{1}{x}, \sqrt{x}, \log(x)$ over the real numbers

(Answers: convex, convex and concave, convex, concave, concave)

The last function (log with respect to an arbitrary base), despite seemingly the least common, can be used to simplify a multitude of inequalities.

2 Local maximas and minimas

(This section can be thought of a follow-up to the graphing unit in a way, as it Here is the first problem I have ever solved in any contest using differentiation, which is a prime example of how the roots of derivatives can give us critical information on a function.

Example (Stormersyle mock AMC 10/25). An ordered pair (a,b) is *spicy* if there exists real c such that the polynomial $f(x) = x^3 + ax^2 + bx + c$ has all real roots. For how many ordered pairs (a,b) of integers with $1 \le a, b \le 20$ is (a,b) spicy?

Solution. The key claim is the following: such a real c exists iff f has a local minima and maxima.

Since nonreal roots of a real-coefficient polynomial come in complex conjugate pairs, f', which has degree 2, has either 2 distinct zeroes, no zeroes or a double root.

If it has two distinct roots, then we can draw a horizontal line between the local minima and maxima; since the polynomial is continuous, the line will intersect f between the two critical points, once as $x \to -\infty$ and once as $x \to \infty$.

if it has a double root, then we can shift f so that the inflection point is a triple root.

Otherwise, f strictly increases (as $3 > 0$), and it's obviously impossible to choose a c such that f has 3 roots.

Therefore, we just need to calculate the number of pairs (a,b) with $4a^2 - 12b \ge 0$, which can easily be computed to be **305**.

Here is a much more difficult example that still utilizes the properties of local minimas and maximas.

Example (2021 HMMT Feb. AlgNT/9). Find all monic cubic polynomials f that have the following properties:

- f is odd, and
- over all reals c, $f(f(x)) - c$ has either $1, 5$ or 9 roots.

Walkthrough:

1. Don't be scared by the problem number!
2. f is of the form $x^3 + ax$. Using simple reasoning, arrive at that $a < 0$.
3. Consider moving a horizontal line from a large y value (intersecting $f(f(x))$ once) downwards. What does it hopping from intersecting f once to five times tell us?
4. Using the chain rule, solve for the local maximas of $f(f(x))$. (It might be helpful to make the substitution $a = -3b^2$.)
5. Use the fact that the local maximas have equal y values to find b.

3 Estimating series

We can often estimate infinite sums with integrals, which solves many problems asking for the rough value (floor/ceil or rounded) of infinite sums.

Example. Find the floor of $\sum_{n=1}^{1000000} \frac{1}{\sqrt{n}}$.

Solution. Trying to look for smart telescopes or cancellations would be pointless, but fortunately, the sum is close to an easily evaluable integral.

The sum is a lower bound to $\int_{x=0}^{1000000} \frac{1}{\sqrt{x}} dx = 2000$ and an upper bound to $\int_{x=1}^{1000001} \frac{1}{\sqrt{x}} dx > 1998$. Therefore, the answer is either 1998 or 1999.

We can see that $\int_{x=0}^{1}(1 - \frac{1}{\sqrt{x}})dx = 1$, and therefore $\int_{x=0}^{1000000} \left\lceil \frac{1}{\sqrt{x}} \right\rceil - \frac{1}{\sqrt{x}} > 1$, making our answer **1998**.

4 Calculating area

5 Problems

Minimum is [TBD ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)
[2 ♟] **Problem 1 (SMT 2021)** Farley the frog starts at the first lily pad in an infinite row of lily pads. If she is currently on the nth lily pad, she has a $\frac{1}{n}$ probability of jumping to the $n+1$th lilypad. Find the expected number of lily pads that she will ever reach.