Applications of Differentiation

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ART

Here, we will discuss the common uses of differentiation in contest problems that are not explicitly about Calculus.

1 Jensen's inequality and Tangent Line Trick

A part of why "Trvial by Jensen's" is often said as a meme regarding problems that may or may not involve inequalities is its true power.

2 Local maximas and minimas

Here is the first problem I have ever solved in any contest using differentiation, which is a prime example of how the roots of derivatives can give us critical information on a function.

Example (Stormersyle mock AMC 10/25). An ordered pair (a, b) is *spicy* if there exists real c such that the polynomial $f(x) = x^3 + ax^2 + bx + c$ has all real roots. For how many ordered pairs (a, b) of integers with $1 \le a, b \le 20$ is (a, b) spicy?

Solution: The key claim is the following: such a real c exists iff f has a local minima and maxima. Since nonreal roots of a real-coefficient polynomial come in complex conjugate pairs, f', which has degree 2, has either 2 zeroes, no zeroes or a double root. If it has no zeroes, then f strictly increases (as $3 > 0$), and it's obviously impossible to choose a c such that f has 3 roots. if it has a double root, then we can shift f so that the inflection point is a triple root. Otherwise, we can draw a horizontal line between the local minima and maxima; since the polynomial is continuous, the line will intersect f between the two critical points, once as $x \to -\infty$ and once as $x \to \infty$. Therefore, we just need to calculate the number of pairs (a, b) with $4a^2 - 12b \ge 0$, which can easily be computed to be **305**.
Here is a much more difficult example that still utilizes the properties of local minimas and maximas.

Example (2021 HMMT Feb. AlgNT/9). Find all monic cubic polynomials f following the following properties:

- f is odd, and
- over all reals c, $f(f(x)) - c$ has either $1, 5$ or 9 roots.

3 Problems

Minimum is [TBD ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

[2 ♟] **Problem 1 (SMT 2021)** Farley the frog starts at the first lily pad in an infinite row of lily pads. If she is currently on the nth lily pad, she has a $\frac{1}{n}$ probability of jumping to the $n+1$th lilypad. Find the expected number of lily pads that she will ever reach.