Solutions to Modular Arithmetic

MAST

NQU

Contents

1 Unsourced

Find the inverse of $2 \pmod{p}$ for odd prime p in terms of p.

1.1 Solution

2 Unsourced

Find the remainder of 97! when divided by 101.

2.1 Solution

3 Unsourced

Find the remainder of $(p-2)!$ when divided by p, provided that p is prime.

3.1 Solution

4 AMC 12A 2003/18

Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

4.1 Solution

5 MAST Diagnostic 2020

How many integer values of $1 \le x \le 100$ makes $x^2 + 8x + 5$ divisible by 10?

5.1 Solution

6 1001 Problems in Number Theory

For which positive integers n is it true that $1 + 2 + \cdots + n \mid 1 \cdot 2 \cdot \cdots \cdot n$?

6.1 Solution

7 Unsourced

What is the residue of $\frac{1}{1\cdot 2} \cdot \frac{1}{2\cdot 3} \cdot \cdots \cdot \frac{1}{11\cdot 12} \pmod{13}$?

7.1 Solution

8 AMC 10A 2020/18

Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $0, 1, 2, 3$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

8.1 Solution

9 AMC 10B 2018/16

Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

9.1 Solution

10 PUMaC 2018

Find the number of positive integers $n < 2018$ such that $25^n + 9^n$ is divisible by 13.

10.1 Solution

11 Unsourced

Prove $\phi(n)$ is composite for $n \geq 7$.

11.1 Solution

12 AMC 10B 2019/14

The base-ten representation for 19! is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T + M + H$?

12.1 Solution

13 Unsourced

Find the remainder of $5^{31} + 5^{17} + 1$ when divided by 31.

13.1 Solution

14 OMO 15-16 Spring/9

Let $f(n) = 1 \times 3 \times 5 \times \cdots \times (2n-1)$. Compute the remainder when $f(1) + f(2) + f(3) + \cdots + f(2016)$ is divided by 100.

14.1 Solution

15 Unsourced

Prove that the equation $x^2 + y^2 + z^2 = x + y + z + 1$ has no solutions over the rationals.

15.1 Solution

16 MAST Diagnostic 2021

Find the remainder of $(1^3)(1^3+2^3)(1^3+2^3+3^3)\dots(1^3+2^3+3^3\cdots+99^3)$ when divided by 101.

16.1 Solution

17 Wolstenholme's Theorem

Prove that for all prime $p \geq 5$, we have $p^2 \mid (p-1)!\left(\sum_{i=1}^{p-1} \frac{1}{i}\right)$.

17.1 Solution

18 AIME 1989/9

One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer such that $133^5 + 110^5 + 84^5 + 27^5 = n^5$. Find the value of n.

18.1 Solution

19 USAMO 1979/1

Determine all non-negative integral solutions $(n_1, n_2, \dots, n_k)$, if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$

19.1 Solution

20 AIME II 2017/8

Find the number of positive integers n less than 2017 such that

$$1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \frac{n^4}{4!} + \frac{n^5}{5!} + \frac{n^6}{6!}$$

is an integer.

20.1 Solution

21 IMO 1970/4

Find all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets so that the product of the numbers in each subset is equal.

21.1 Solution

22 IMO 2005/4

Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

22.1 Solution

23 AIME I 2013/15

Let N be the number of ordered triples (A, B, C) of integers satisfying the conditions

- $0 \le A < B < C \le 99$,
- there exist integers a, b, and c, and prime p where $0 \le b < a < c < p$,
- p divides $A - a$, $B - b$, and $C - c$, and
- each ordered triple (A, B, C) and each ordered triple (b, a, c) form arithmetic sequences.

Find N.

23.1 Solution

24 USEMO 2019/4

Prove that for any prime p, there exists a positive integer n such that

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \equiv 2020 \pmod{p}.$$

24.1 Solution

25 Unsourced

The expansion of $\frac{1}{7}$ is $0.\overline{142857}$, which is a repeating decimal with a 6 digit long sequence. How many digits long is the expansion of $\frac{1}{13}$?

25.1 Solution

26 Unsourced

We define the cycle of a repeating fraction $\frac{m}{n}$ as the minimum number i such that $\frac{m}{n} = 0.\overline{a_1 a_2 a_3 \ldots a_i}$. Find the cycle of $\frac{1}{23}$.

26.1 Solution

27 AMC 10A 2019/18

For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323\ldots_k$. What is k?

27.1 Solution

28 e-dchen Mock MATHCOUNTS

What is the sum of all odd n such that $\frac{1}{n}$ expressed in base 8 is a repeating decimal with period 4?

28.1 Solution

29 AMC 12A 2014/23

The fraction

$$\frac{1}{99^2} = 0.\overline{b_{n-1}b_{n-2}\ldots b_2b_1b_0},$$

where n is the length of the period of the repeating decimal expansion. What is the sum $b_0 + b_1 + \cdots + b_{n-1}$?

29.1 Solution

30 AMC 12B 2016/22

For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. Find n.

30.1 Solution