Transformations

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GQT

There are three types of transformations, and each of them is used to solve different types of problems. We jump into each type headfirst and provide some heuristics after.

§ 1 Reflection

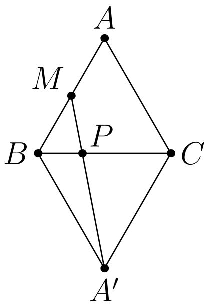
There's a couple of properties that make this the most important section. Some of these are exercises, such as the property of the tangent to an ellipse.

Theorem 1 (Running the River) Consider points A, B on the same side of line ℓ and point P on ℓ. Then $\min(AP + BP) = AB'$, where B' is the reflection of B about ℓ.

Proof: Note that by the definition of a reflection, $AP + BP = AP + PB'$. By the Triangle Inequality, $AP + PB' \geq AB'$, with equality when P is the intersection of AB' and ℓ. ■

Example 1 If $AB = BC = CA = 2$, M is the midpoint of AB, and P lies on BC, find the minimum value of $PA + PM$.

Solution: Reflect A about BC to get A'. Then $MP + PA = MP + PA' \leq MA' = \sqrt{(\frac{1}{2}^2 + \frac{3\sqrt{3}}{2})^2} = \sqrt{7}$.



Now for an example of a deceptively difficult problem.

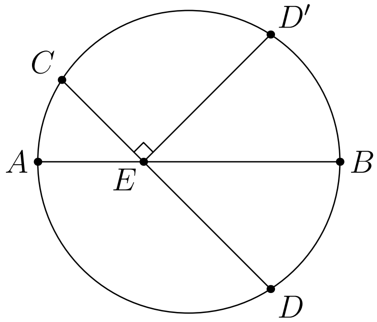
Example 2 (AMC 12B 2020/12) Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE = 2\sqrt{5}$ and $\angle AEC = 45°$. What is $CE^2 + DE^2$?

Solution: Reflect D about AB to get D'. Then note $\angle CED' = 90°$.

Now note $CE^2 + ED^2 = CE^2 + ED'^2 = CD'^2$.

Note $2\angle CEA = 90° = \widehat{CA} + \widehat{BD} = \widehat{CA} + \widehat{BD'}$, so $\widehat{CD'} = 90°$. Thus $CD'^2 = 2r^2 = 100$.

(Diagram taken from djmathman's post in the problem thread.)



§ 1.1 Heuristics

• Use this when you want to minimize the sum of some distances, with points moving on lines.

• This only works if you preserve all the lengths in some way and the endpoints of the line segment you're constructing are both constant (i.e. cannot move).

• More concretely, **only reflect the involved point(s) that are stationary.**

• There are some edge cases of problems where reflection helps and none of the above apply. But "don't reflect background points" (points which are not directly involved in the desired segments) is still a good rule of thumb.

§ 2 Rotation

We start with a generic example.

Example 3 (Autumn Mock AMC 10) Let $ABCD$ be a square and point P be placed in $ABCD$ such that $AP = 3$, $BP = 6$, and $CP = 9$. Find the side length of $ABCD$.

Solution: Rotate $\triangle APB$ about B such that A coincides with C. Then note $BP' = 6$ and $CP' = 3$. Since $\angle PBP' = 90°$, $PP' = 6\sqrt{2}$. Thus $\angle PP'C = 90°$ by the Pythagorean Theorem, so $\angle BP'C = 45° + 90° = 135°$. By Law of Cosines, $BC = \sqrt{6^2 + 3^2 - 2 \cdot 6 \cdot 3 \cdot \cos(135°)} = 3\sqrt{5 + 2\sqrt{2}}$.

These types of problems often have their difficulty overestimated. For example, see AMC 12A 2020/24.

§ 2.1 Heuristics

✦ When you have a regular polygon and some point P inside and you're given the distances from P to the vertices, rotate.

✦ You're probably going to be using Law of Cosines. Remember that $\cos(a + b) = \cos a \cos b - \sin a \sin b$.

✦ Preserve and create nice angles. These include but are not limited to $30°, 45°, 60°, 90°$.

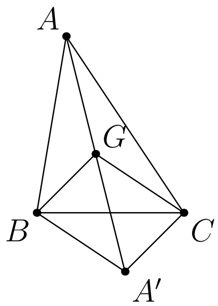
§ 3 Translation

This is probably one of the most beautiful classical geometry problems.

Example 4 (Area of Triangle with Lengths of Medians) Consider $\triangle ABC$ with medians AD, BE, CF. Then construct $\triangle XYZ$ such that $XY = AD, YZ = BE, ZX = CF$. Prove that $[XYZ] = \frac{3}{4}[ABC]$.

Solution: This is equivalent to proving that the triangle with side lengths AG, BG, CG has area $\frac{3}{4} \cdot (\frac{2}{3})^2 [ABC] = \frac{1}{3}[ABC]$. Then construct parallelogram $BGCA'$.

If M is the midpoint of BC, then $GA' = 2GM = AG$, by the properties of the centroid. So $\triangle BGA'$ has all of the necessary side lengths. But note that $[BGA'] = [BGM] + [BMA'] = [BGM] + [MGC] = [BGC] = \frac{2}{6}[ABC]$, as desired.



§ 3.1 Heuristics

- ✦ This is also known as "constructing a parallelogram."
- ✦ Geometry conditions that feel weird but don't fit into reflections or rotations.

§ 4 Problems

Minimum is [90✐]. Problems with the ♢ symbol are required.

> "When you come out of the storm you wont be the same person who walked in. Thats what this storms all about."
>
> Kafka on the Shore

[3✐] **Problem 1** (AMC 12A 2021/11) A laser is placed at the point $(3, 5)$. The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7, 5)$. What is the total distance the beam will travel along this path?

[3✐] **Problem 2** (AMC 10A 2021/21) Let $ABCDEF$ be an equiangular hexagon. The lines AB, CD, and EF determine a triangle with area $192\sqrt{3}$, and the lines BC, DE, and FA determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m = n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

[4✐] **Problem 3** Given a square $ABCD$ and a point P in its interior such that $AP = \sqrt{7}$, $BP = 1$, and $CP = 3$, find the side length of $ABCD$.

[4✐] **Problem 4** Find the area of a square $ABCD$ containing a point P such that $PA = 3$, $PB = 7$, and $PD = 5$.

[4✐] **Problem 5** (ART 2019/3) Consider $\triangle ABC$ with $AB = 5$, $BC = 7$, and $CA = 4\sqrt{2}$. Let H be the foot of the altitude from A to BC. If P is a point on AC, find the minimum value of $BP + HP$.

[4✐] **Problem 6** (ART 2020/2) Consider equilateral triangle $\triangle ABC$. Let the reflection of A about BC be D. Let the midpoint of AB be M. Then let MC intersect the circumcircle of $\triangle BCD$ at N. Then there is some point P on BC such that $MP + NP$ is minimized. Find $\frac{BP}{CP}$.

[4✐] **Problem 7** (MOP) Consider rectangle $ABCD$ with point M in its interior. If $\angle BMC + \angle AMD = 180°$, find $\angle BCM + \angle DAM$.

[4✐] **Problem 8** Isosceles $\triangle ABC$ has $40°$ base angles at B and C. Let M be the intersection of the angle bisector of C with $\overline{AB}$. Let G be on the extension of $\overrightarrow{CM}$ such that $AM = GM$. Calculate $\angle GBC$.

[4✐] **Problem 9** (Given equilateral $\triangle ABC$ with point O in its interior such that $\angle AOB = 115°$ and $\angle BOC = 125°$, find the angles of the triangle with side lengths OA, OB, OC.)

[4✐] **Problem 10** (Consider rectangle $ABCD$ with $AB = 20$ and $BC = 10$. If M is on AC and N is on AB, find the minimum value of $BM + MN$.)

[4✐] **Problem 11** (Consider $\triangle ABC$. Let $\angle A < 60°$, P lie on AB, and Q lie on AC. Construct a line segment such that its length is equal to the minimum value of $BQ + QP + PC$.)

[6✐] **Problem 12** Let ellipse ω with focii A, B be tangent to line ℓ at P. Let α be the acute angle between AP and ℓ, and let β be the acute angle between BP and ℓ. Prove that $\alpha = \beta$.

[6✐] **Problem 13** Find the area of an equilateral triangle containing in its interior a point P, whose distances from the vertices of the triangle are $3, 4$, and 5.

[6✐] **Problem 14** (AMC 12A 2020/24) Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

[6] **Problem 15** (GGMT Speed 2020/20) There exists a point P inside regular hexagon $ABCDEF$ such that $AP = \sqrt{3}, BP = 2, CP = 3$. If the area of the hexagon can be expressed as $\frac{a\sqrt{b}}{c}$, where b is not divisible by the square of a prime, find $a + b + c$.

[6] **Problem 16** (China) Consider isosceles right triangle $\triangle ABC$ with $AB = AC = 2$. Let X be the midpoint of AC, and let Y and Z be points on AB and BC, respectively. Find the minimum perimeter of $\triangle XYZ$.

[6] **Problem 17** (AMC 12A 2014/20) In $\triangle BAC$, $\angle BAC = 40°$, $AB = 10$, and $AC = 6$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively. What is the minimum possible value of $BE + DE + CD$?

[6] **Problem 18** (AIME I 2012/13) Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as $a + \frac{b}{c}\sqrt{d}$, where a, b, c and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a + b + c + d$.

[6] **Problem 19** (HMMT 2019) Let $ABCD$ be an isosceles trapezoid with $AD = BC = 255$ and $AB = 128$. Let M be the midpoint of CD and let N be the foot of the perpendicular from A to CD. If $\angle MBC = 90°$, compute $\tan \angle NBM$.

[6] **Problem 20** If $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = n$, find the area of the triangle with side lengths AD, BE, CF.

[6] **Problem 21** (China) Consider $\triangle BAC$ such that $\angle A = 45°$. Let H be the foot of the A altitude. If $BH = 2$ and $CH = 3$, find $[ABC]$.

[6] **Problem 22** Consider isosceles triangle with $AC = BC$, $\angle ACB = 80°$, and point M in the interior of $\triangle ABC$ such that $\angle MAB = 10°$ and $\angle MBA = 30°$. Find $\angle AMC$.

[6] **Problem 23** Consider unit square $ABCD$ with P on AD and Q on AB such that the perimeter of $\triangle APQ$ is 2. Find $\angle PCQ$.

[9] **Problem 24** (CIME 2020) Let $ABCD$ be a cyclic quadrilateral with $AB = 6, AC = 8, BD = 5, CD = 2$. Let P be the point on $\overline{AD}$ such that $\angle APB = \angle CPD$. Then $\frac{BP}{CP}$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[9] **Problem 25** Consider the hexagon $A_1A_2A_3A_4A_5A_6$ with $A_1A_2 = A_2A_3$, $A_3A_4 = A_4A_5$, $A_5A_6 = A_6A_1$, and $\angle A_1 + \angle A_3 + \angle A_5 = \angle A_2 + \angle A_4 + \angle A_6$. Find $\frac{[A_2A_4A_6]}{[A_1A_2A_3A_4A_5A_6]}$ and $\frac{\angle A_6A_2A_4}{\angle A_2}$.

[9] **Problem 26** Consider $\triangle ABC$ with point O in its interior such that $\angle AOB = \angle BOC = \angle COA = 120°$. Then consider equilateral $\triangle XYZ$ with point P in its interior such that $XP = a$, $YP = b$, and $ZP = c$. Prove that the side length of $\triangle XYZ$ is equivalent to $AO + BO + CO$.

[13] **Problem 27** Consider unit square $ABCD$ and points P, Q in its interior. If $\angle PAQ = \angle PCQ = 45°$, find $[PAB] + [PCQ] + [AQD]$.

[13] **Problem 28** (IMO 1993/2) Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90° + \angle ACB$. Find the ratio

$$\frac{AB \cdot CD}{AC \cdot BD},$$

and prove that the circumcircles of the triangles ACD and BCD are orthogonal.