Lengths and Areas in Triangles

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§ 1 Lengths

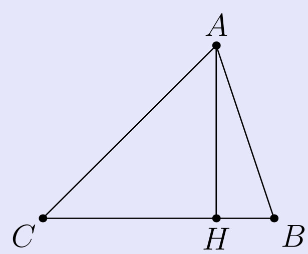
There are a couple of important lengths in a triangle. These are the lengths of cevians, the inradius/exradius, and the circumradius.

§ 1.1 Law of Cosines and Stewart's

We discuss how to find the third side of a triangle given two sides and an included angle, and use this to find a general formula for the length of a cevian.

Theorem 1 (Law of Cosines) Given $\triangle ABC$, $a^2 + b^2 - 2ab\cos C = c^2$.

Proof: Let the foot of the altitude from A to BC be H. Then note that $A = b\sin C$, $CH = b\cos C$, and $BH = |a - b\cos C|$. (The absolute value is because $\angle B$ can either be acute or obtuse.) Then note by the Pythagorean Theorem, $(b\sin C)^2 + (a - b\cos C)^2 = a^2 + b^2 - 2ab\cos C = c^2$.



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Theorem 2 (Stewart's Theorem) Consider $\triangle ABC$ with cevian AD, and denote $BD = m$, $CD = n$, and $AD = d$. Then $man + dad = bmb + cnc$.

Proof: We use the Law of Cosines. Note that

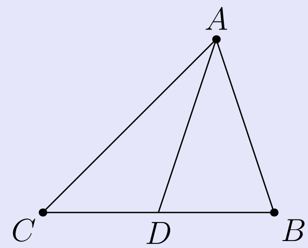
$$\cos\angle ADB = \frac{d^2+m^2-c^2}{2dm} = -\frac{d^2+n^2-b^2}{2dn} = -\cos\angle ADC.$$

Multiplying both sides by $2dmn$ yields

$$c^2n - d^2n - m^2n = -bm^2 + d^2m + mn^2$$

$$b^2m + c^2n = mn(m+n) + d^2(m+n)$$

$$bmb + cnc = man + dad.$$



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Here are two corollaries that will save you a lot of time in computational contests.

Fact 1 (Length of Angle Bisector) In $\triangle ABC$ with angle bisector AD, denote $BD = x$ and $CD = y$. Then

$$AD = \sqrt{bc - xy}.$$

Fact 2 (Length of Median) In $\triangle ABC$ with median AD,

$$AD = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2}.$$

§ 1.2 Law of Sines and the Circumradius

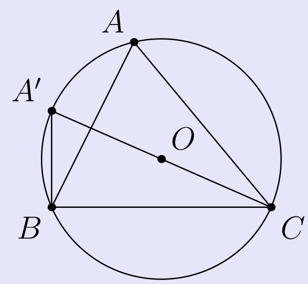
The Law of Sines is a good way to length chase with a lot of angles.

Theorem 3 (Law of Sines) In $\triangle ABC$ with circumradius R,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof: We only need to prove that $\frac{a}{\sin A} = 2R$, and the rest will follow.

Let the line through B perpendicular to BC intersect (ABC) again at A'. Then note that $A'C = 2R$ by Thale's. By the Inscribed Angle Theorem, $\sin\angle CA'B = \sin A$, so $\frac{a}{\sin A} = \frac{a}{\sin\angle CA'B} = \frac{a}{\frac{a}{2R}} = 2R$.



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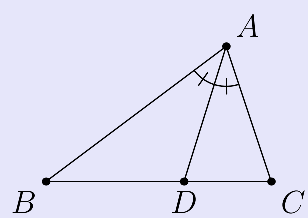
Other texts will call this the Extended Law of Sines. But the Extended Law of Sines has a better proof than the "normal" Law of Sines, and redundancy is bad.

The Law of Sines gives us the Angle Bisector Theorem.

Theorem 4 (Angle Bisector Theorem) Let D be the point on BC such that $\angle BAD = \angle DAC$. Then $\frac{AB}{BD} = \frac{AC}{CD}$.

Proof: By the Law of Sines, $\frac{\sin\angle ADB}{\sin\angle BAD} = \frac{AB}{BD}$ and $\frac{\sin\angle ADC}{\sin\angle CAD} = \frac{AC}{CD}$. But note that $\angle BAD = \angle ADC$ and $\angle BAD + \angle CAD = 180°$, so $\frac{AB}{BD} = \frac{AC}{CD}$.



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In fact, the Angle Bisector Theorem can be generalized in what is known as the ratio lemma.

Theorem 5 (Ratio Lemma) Consider $\triangle ABC$ with point P on BC. Then $\frac{BP}{CP} = \frac{c\sin\angle BAP}{b\sin\angle CAP}$.

The proof is pretty much identical to the proof for Angle Bisector Theorem.

Proof: By the Law of Sines, $BP = \frac{c\sin\angle BAP}{\sin\angle APB}$ and $CP = \frac{b\sin\angle CAP}{\sin\angle APC}$. Since $\sin\angle APB = \sin\angle APC$,

$$\frac{BP}{CP} = \frac{c\sin\angle BAP}{b\sin\angle CAP}.$$

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Note that this remains true even if P is on the *extension* of BC.

Here's a classic example that cleverly utilizes the Law of Sines.

Example 1 Show that $\triangle ABC$ is similar to the triangle with side lengths $\sin A, \sin B, \sin C$.

Solution: Note that $\sin A = \frac{a}{2R}$, so the similarity factor is $2R$.

We'll utilize this concept further in the next example.

Example 2 Consider $\triangle ABC$ with side lengths $AB = 13$, $BC = 5$, and $CA = 12$. Find the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$.

Solution: Note that $[ABC] = 60$ and the triangle with lengths $\sin A$, $\sin B$, and $\sin C$ is similar to $\triangle ABC$ with a scale factor of 13. Thus the desired area is $\frac{60}{13^2} = \frac{60}{169}$.

It's possible to just directly use the values of $\sin A$, $\sin B$, and $\sin C$, but this will not work for general triangles.

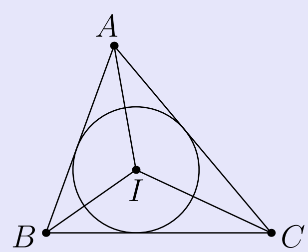
§ 1.3 The Incircle, Excircle, and Tangent Chasing

We provide formulas for the inradius, exradii, and take a look at some uses of the Two Tangent Theorem. Recall that the Two Tangent Theorem states that if the tangents from P to ω intersect ω at A, B, then $PA = PB$.

Theorem 6 (rs) In $\triangle ABC$ with inradius r,

$$[ABC] = rs.$$

Proof: Note that $[ABC] = r \cdot \frac{a+b+c}{2} = rs$.



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A useful fact of the incircle is that the length of the tangents from A is $s - a$. Similar results hold for the B, C tangents to the incircle.

Fact 3 (Tangents to Incircle) Let the incircle of $\triangle ABC$ be tangent to BC, CA, AB at D, E, F. Then

$$AE = AF = s - a$$

$$BF = BD = s - b$$

$$CD = CE = s - c.$$

Proof: Note that by the Two Tangent Theorem, $AE = AF = x$, $BF = BD = y$, and $CD = CE = z$. Also note that

$$BD + CD = y + z = a$$

$$CE + EA = z + x = b$$

$$AF + FB = x + y = c.$$

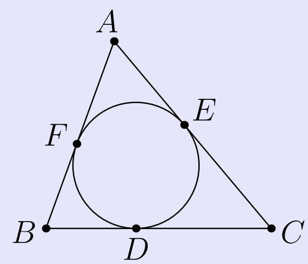
Adding these equations gives $2x + 2y + 2z = a + b + c = 2s$, implying $x + y + z = s$. Thus

$$x = AE = AF = s - a$$

$$y = BF = BD = s - b$$

$$z = CD = CE = s - c,$$

as desired.



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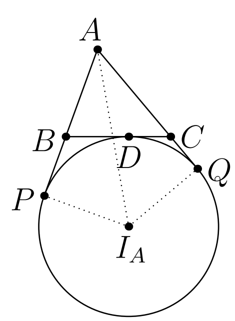
Theorem 7 ($r_a(s-a)$) In $\triangle ABC$ with A exradius r_a,

$$[ABC] = r_a(s-a).$$

Proof: Let AB, AC be tangent to the A excircle at P, Q, respectively, and let BC be tangent to the A excircle at D. Then note that by the Two Tangent Theorem, $PB = BD$ and $DC = CQ$. Thus $[ABC] = [API_A] + [AQI_A] - 2[BI_AC] = r_a \cdot \frac{s+s-2a}{2} = r_a(s-a)$. ■

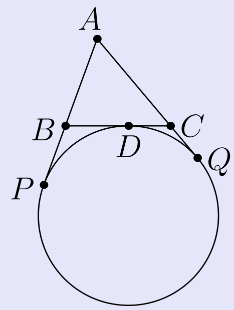


The proof also implies the following corollary.

Fact 4 (Tangents to Excircle) Let the A excircle of $\triangle ABC$ be tangent to BC at D. Then $BD = s - c$ and $CD = s - b$.

Analogous equations hold for the B and C excircles.

Proof: Let the A excircle be tangent to line AB at P and line AC at Q. Note that $AP = AB + BP = c + BD$ and $AQ = AC + CQ = b + CD$ by the Two Tangent Theorem. Applying the Two Tangent Theorem again gives $AP = AQ$, or $c + BD = b + CD$. Also note that $AP + AQ = b + c + BD + DC = 2s$, so $AP = AQ = s$ and $s = c + BD = b + CD$. Thus $BD = s - c$ and $CD = s - b$.



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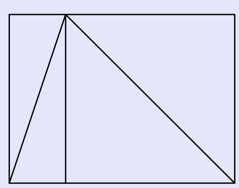
Keep these area and length conditions in mind when you see incircles and excircles.

§ 2 Areas

There are a variety of methods to find area. For harder problems, computing the area in two different ways can give useful information about the configuration.

Theorem 8 ($\frac{bh}{2}$) The area of a triangle is $\frac{bh}{2}$.

Proof: The area of each right triangle is half of the area of the rectangle it is in.



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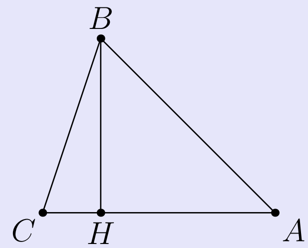
Theorem 9 (rs) The area of a triangle is rs, where r is the inradius and s is the semiperimeter.

We have already proved this in Length Chasing - but we mention this theorem again because it is useful for area too.

Theorem 10 ($\frac{1}{2}ab \sin C$) The area of a triangle is $\frac{1}{2}ab\sin C$, where a, b are side lengths and C is the included angle.

Proof: Drop an altitude from B to AC and let it have length h. Then note $\frac{1}{2} \cdot a \sin C \cdot b = \frac{1}{2} \cdot hb = \frac{bh}{2}$.



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We present a useful corollary of this theorem.

Fact 5 ($\frac{[PAB]}{[PXY]} = \frac{PA \cdot PB}{PX \cdot PY}$) Let P, A, X be on ℓ_1 and P, B, Y be on ℓ_2. Then $\frac{[PAB]}{[PXY]} = \frac{PA \cdot PB}{PX \cdot PY}$.

Proof: Note $\frac{[PAB]}{[PXY]} = \frac{\frac{1}{2} \cdot PA \cdot PB \cdot \sin\theta}{\frac{1}{2} \cdot PX \cdot PY \cdot \sin\theta} = \frac{PA \cdot PB}{PX \cdot PY}$, where $\theta = \angle APB$.
This works for all configurations since $\sin\theta = \sin(180 - \theta)$. ■

Here is a very difficult example done with the help of some tricky angle chasing, trig addition formulas, and the sine area formula.

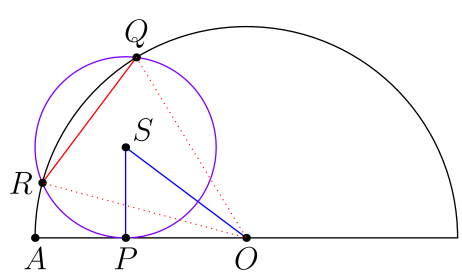
Example 3 (AMC 12A 2021/24) Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, then the area of $\triangle PQR$ is $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a + b + c$?

Solution: Let S be the center of Ω, and note that by the Law of Sines, the circumradius of $\triangle PQR$ is $\frac{QR}{2\sin RPQ} = \frac{3\sqrt{3}}{\sqrt{3}} = 3$. Also note that by the Pythagorean Theorem, the distance from O to RQ is $\sqrt{7^2 - (\frac{3\sqrt{3}}{2})^2} = \frac{13}{2}$, as $\triangle ORQ$ is isosceles. So $SO = \frac{13}{2} - \frac{3}{2} = 5$ and $SP = 4$ by the Pythagorean Theorem.

Let $\angle OSP = \theta$ and note that $\arccos\theta = \frac{3}{5}$. Now, by the Sine Area Formula,

$$\begin{aligned}
[PQR] = [PSR] + [PSQ] + [RSQ] &= \frac{9}{2}(\sin 60^\circ + \sin(150^\circ - \theta) + \sin(150^\circ + \theta)) \\
&= \frac{9}{2}(\frac{\sqrt{3}}{2} + 2\sin 150^\circ \cos\theta) \\
&= \frac{9}{2}(\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{5}) \\
&= \frac{99\sqrt{3}}{20}.
\end{aligned}$$

Thus the answer is $99 + 3 + 20 = 122$.



Theorem 11 ($\frac{abc}{4R}$) In $\triangle ABC$ with side lengths a, b, c and circumradius R,

$$[ABC] = \frac{abc}{4R}.$$

Proof: Note that $[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}ab \cdot \frac{c}{2R} = \frac{abc}{4R}$. ■

Heron's Formula can find the area of a triangle with *only* the side lengths.

Theorem 12 (Heron's Formula) In $\triangle ABC$ with sidelengths a, b, c such that $s = \frac{a+b+c}{2}$,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof: Since $\cos C = \frac{a^2+b^2-c^2}{2ab}$, the Pythagorean Identity gives us

$$\sin C = \sqrt{\frac{4a^2b^2 - (a^2+b^2-c^2)^2}{4a^2b^2}} = \sqrt{\frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2b^2}}.$$

So

$$\frac{1}{2}ab\sin C = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)} = \sqrt{s(s-a)(s-b)(s-c)}.$$

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Heron's Formula has a reputation for being notoriously tricky to prove, but the proof isn't too bad if you consider what we're actually doing.

1. Use the Law of Cosines to find $\cos C$.
2. Use the Pythagorean Identity to find $\sin C$.
3. Use $\frac{1}{2}ab\sin C$ to find $[ABC]$.
4. Clean the expression up.

Fact 6 (Heron's with Altitudes) If x, y, z are the lengths of the altitudes of $\triangle ABC$,

$$\frac{1}{[ABC]} = \sqrt{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(-\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{x}-\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}\right)}.$$

To prove this, substitute $x = \frac{[ABC]}{2a}$.

§ 3 Problems

Minimum is [55 ✎]. Problems with the ♢ symbol are required.

> "Do you know, Poole, that you and I are about to place ourselves in a situation of some peril?"
>
> Strange Case of Dr. Jekyll and Mr. Hyde

[1 ✎] **Problem 1** Find the inradius of the triangles with the following lengths:

- $3, 4, 5$
- $5, 12, 13$
- $13, 14, 15$
- $5, 7, 8$

(These are arranged by difficulty. All of these are good to know.)

[2 ✎] **Problem 2** Prove that in a right triangle with legs of length a, b and hypotenuse with length c, $r = \frac{a+b-c}{2}$.

[2 ✎] **Problem 3** In $\triangle ABC$, $AB = 5$, $BC = 12$, and $CA = 13$. Points D, E are on BC such that $BD = DC$ and $\angle BAE = \angle CAE$. Find $[ADE]$.

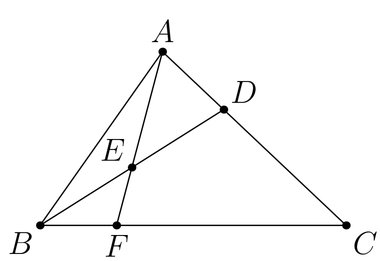
[2 ♢] **Problem 4** Find the maximum area of a triangle with two of its sides having lengths $10, 11$.

[2 ✎] **Problem 5** (e-dchen Mock MATHCOUNTS) Consider rectangle $ABCD$ such that $AB = 2$ and $BC = 1$. Let X, Y trisect AB. Then let DX and DY intersect AC at P and Q, respectively. What is the area of quadrilateral $XYQP$?

[2 ♢] **Problem 6 (Autumn Mock AMC 10)** Equilateral triangle ABC has side length 6. Points D, E, F lie within the lines AB, BC and AC such that $BD = 2AD$, $BE = 2CE$, and $AF = 2CF$. Let N be the numerical value of the area of triangle DEF. Find N^2.

[2 ✎] **Problem 7** (AIME I 2019/3) In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

[3 ✎] **Problem 8** (AMC 8 2019/24) In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



[3✎] **Problem 9** Consider $\triangle ABC$ such that $AB = 8$, $BC = 5$, and $CA = 7$. Let AB and CA be tangent to the incircle at T_C, T_B, respectively. Find $[AT_BT_C]$.

[3✎] **Problem 10** Consider trapezoid $ABCD$ with bases AB and CD. If AC and BD intersect at P, prove the sum of the areas of $\triangle ABP$ and $\triangle CDP$ is at least half the area of trapezoid $ABCD$.

[4✎] **Problem 11** (AIME I 2001/4) In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

[6✎] **Problem 12** (PUMaC 2016) Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10, AD = 24, XA = 17$, and $XB = 21$, then the area of $\triangle DEY$ can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

[6💎] **Problem 13 (CIME 2020)** An excircle of a triangle is a circle tangent to one of the sides of the triangle and the extensions of the other two sides. Let ABC be a triangle with $\angle ACB = 90^\circ$ and let r_A, r_B, r_C denote the radii of the excircles opposite to A, B, C, respectively. If $r_A = 9$ and $r_B = 11$, then r_C can be expressed in the form $m + \sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

[6✎] **Problem 14** (AIME II 2019/11) Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[6✎] **Problem 15** (AIME II 2005/14) In triangle $ABC, AB = 13, BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE = \angle CAD$. Given that $BE = \frac{p}{q}$ where p and q are relatively prime positive integers, find q.

[6✎] **Problem 16** (ART 2019/6) Consider unit circle O with diameter AB. Let T be on the circle such that $TA < TB$. Let the tangent line through T intersect AB at X and intersect the tangent line through B at Y. Let M be the midpoint of YB, and let XM intersect circle O at P and Q. If $XP = MQ$, find AT.

[9✎] **Problem 17** (AMC 12A 2017/24) Quadrilateral $ABCD$ is inscribed in circle O and has sides $AB = 3$, $BC = 2$, $CD = 6$, and $DA = 8$. Let X and Y be points on $\overline{BD}$ such that

$$\frac{DX}{BD} = \frac{1}{4} \quad \text{and} \quad \frac{BY}{BD} = \frac{11}{36}.$$

Let E be the intersection of intersection of line AX and the line through Y parallel to $\overline{AD}$. Let F be the intersection of line CX and the line through E parallel to $\overline{AC}$. Let G be the point on circle O other than C that lies on line CX. What is $XF \cdot XG$?

[9✎] **Problem 18** (AIME II 2016/10) Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[9✎] **Problem 19** (IMO 2003/4) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

[9✎] **Problem 20** (AIME I 2019/11) In $\triangle ABC$, the sides have integer lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An excircle of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

[9] **Problem 21** (AIME I 2020/13) Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

[13] **Problem 22** (CIME 2019) Let $\triangle ABC$ be a triangle with circumcenter O and incenter I such that the lengths of the three segments AB, BC and CA form an increasing arithmetic progression in this order. If $AO = 60$ and $AI = 58$, then the distance from A to BC can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[13] **Problem 23** (AIME I 2020/15) Let ABC be an acute triangle with circumcircle ω and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3$, $HX = 2$, $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.