Solutions to Fake Algebra

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AQU

Contents

§ 1 Unsourced

If $a < b < c < a + b$, order $\frac{b^2+c^2-a^2}{bc}, \frac{c^2+a^2-b^2}{ca}, \frac{a^2+b^2-c^2}{ab}$ in ascending order.

§ 1.1 Solution

Consider a triangle with side lengths $a < b < c$, and let $\angle A, \angle B, \angle C$ denote the angles opposite the sides a, b, c as usual. This is motivated by our somewhat-symmetric expressions, as well as the requirement that $c < a + b$. Observe that by Law of Cosines:

$$\begin{aligned} b^2 + c^2 - 2bc\cos\angle A &= a^2 \\ b^2 + c^2 - a^2 &= 2bc\cos\angle A \\ 2\cos\angle A &= \frac{b^2 + c^2 - a^2}{bc}. \end{aligned}$$

Doing this to the other two expressions, we want to sort $2\cos\angle A, 2\cos\angle B, 2\cos\angle C$ in ascending order. Since cos is decreasing on $[0, \pi]$—in degrees, this is $[0°, 180°]$—we want to sort $\angle A, \angle B, \angle C$ in descending order. Noting that, in a triangle, the shortest and longest sides are opposite the smallest and largest angles respectively, we have $\angle C > \angle B > \angle A$. Hence $2\cos\angle C < 2\cos\angle B < 2\cos\angle A$, so we conclude that $\boldsymbol{\frac{a^2+b^2-c^2}{ab}, \frac{c^2+a^2-b^2}{ca}, \frac{b^2+c^2-a^2}{bc}}$ is the correct ordering.

§ 2 Unsourced

Prove that the A and B angle bisectors of a triangle are equal in length if and only if $BC = CA$.

§ 2.1 Solution

§ 3 AIME 1986/2

Evaluate the product $(\sqrt{5}+\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})$.

§ 3.1 Solution

Consider a triangle with side lengths $2\sqrt{5}, 2\sqrt{6}, 2\sqrt{7}$. By Heron's formula, the area of this triangle is:

$$\sqrt{(\sqrt{5}+\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})}.$$

To be continued.

§ 4 Unsourced

Let x and y be real numbers such that $(x-5)^2+(y-5)^2=18$. Determine the maximum value of $\frac{y}{x}$.

§ 4.1 Solution 1

It is possible to solve this purely algebraically. Suppose $\frac{y}{x}=k \implies y=kx$. We wish to maximize k. Substituting $y=kx$ into our equation and treating it as a quadratic in x gives:

$$\begin{aligned}(x-5)^2+(kx-5)^2&=18\\ x^2-10x+25+k^2x^2-10kx+25&=18\\ (k^2+1)x^2-10(k+1)x+32&=0.\end{aligned}$$

For this to have real roots, the discriminant must be nonnegative, so:

$$\begin{aligned}(10(k+1))^2-4(k^2+1)(32)&\geq 0\\ 100k^2+200k+100-128k^2-128&\geq 0\\ -28k^2+200k-28&\geq 0\\ 7k^2-50k+7&\leq 0\\ (7k-1)(k-7)&\leq 0.\end{aligned}$$

Hence the maximum value of k is **7**, since for $k>7$ we have $(7k-1)(k-7)>0$.

§ 4.2 Solution 2

§ 5 Unsourced

Let a, b, c be positive reals. Prove that $\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$.

§ 5.1 Solution

§ 6 Unsourced

Minimze $\sqrt{x^2-3x+3}+\sqrt{y^2-3y+3}+\sqrt{x^2-\sqrt{3}xy+y^2}$ over the reals.

§ 6.1 Solution

§ 7 Unsourced

Prove that for reals $a, b \geq 1$,

$$\sqrt{a^2-1}+\sqrt{b^2-1} \leq ab.$$

§ 7.1 Solution

Consider a triangle with two sides of length $a, b \geq 1$ such that the altitude from the vertex formed from the sides of length a, b has length 1. Label the triangle and its altitudes as shown:
(There should be a diagram here, but things are broken)

We note that $AD = \sqrt{b^2-1}$ and $BD = \sqrt{a^2-1}$ by the pythagorean theorem. Hence by the $\frac{1}{2}bh$ area formula, we have $2[ABC] = \sqrt{a^2-1}+\sqrt{b^2-1}$. Now let $\angle ACB = \theta$. By the sine area formula, we have $2[ABC] = ab\sin\theta$. As $\sin\theta \leq 1$, it follows that $ab\sin\theta \leq ab$, so we have $\sqrt{a^2-1}+\sqrt{b^2-1} \leq ab$ as desired.

§ 8 Unsourced

What value of x maximizes $(21+x)(1+x)(x-1)(21-x)$, if x must be positive?

§ 8.1 Solution 1

Note that this is the square of the area of a triangle with sides $20, 22, 2x$, by Heron's. From the sine area formula, we get that the area of the triangle is $220\sin\theta$, where θ is the measure of the angle between the sides of lengths 20 and 22. $\sin\theta$ attains its maximum value when $\theta = 90°$, where it is equal to 1. In this case, we get from the Pythagorean Theorem that $2x = \sqrt{20^2 = 22^2} \implies x = \boldsymbol{\sqrt{221}}$.

§ 8.2 Solution 2

Also possible to just use difference of squares and just do algebra.

§ 9 TrinMaC 2020/19

Compute

$$\sum_{n=0}^{\infty} \cos^{-1}\left(\frac{\sqrt{n(n+1)(n+2)(n+3)}+1}{(n+1)(n+2)}\right).$$

§ 9.1 Solution

§ 10 Unsourced

Let a, b, c, d be real numbers such that $a^2 - b^2 - c^2 + d^2 = ad + bc$ and $a^2 + b^2 - c^2 - d^2 = 0$. Determine the value of $\frac{ab+cd}{ad+bc}$.

§ 10.1 Solution

We note that the first condition rewrites as $a^2 + d^2 - 2ad\cos 60^\circ = b^2 + c^2 + 2bc\cos 120^\circ$, while the second rearranges as $a^2 + b^2 = c^2 + d^2$. So a, b, c, d are the side lengths of a cyclic quadrilateral with angles $60^\circ, 120^\circ$ inscribed in a circle. WLOG $AB = a, BC = b, CD = c, DA = d$. Now the Pythagorean inequality combined with $a^2 + b^2 = c^2 + d^2$ gives us $\angle ABC = \angle ADC = 90^\circ$. So $\triangle ABC, \triangle ADC$ are $30 - 60 - 90$. WLOG setting $b = c = 1$ then gives us $a = d = \sqrt{3}$, after which we can easily get the answer as $\boldsymbol{\frac{\sqrt{3}}{2}}$.

§ 11 AIME II 2006/15

Given that x, y, and z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

§ 11.1 Solution

The RHS looks suspiciously like the Pythagorean Theorem. After a bit of trial and error based on this observation, we realize that x, y, z are the side lengths of a triangle with altitudes $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ (the altitudes and the sides are ordered in the same way, so the altitude of length $\frac{1}{4}$ is perpendicular to the side of length x). Since the area is the same we have $\frac{x}{4} = \frac{y}{5} = \frac{z}{6}$. Let this quantity equal k, so $x = 4k, y = 5k, z = 6k$. Then the area is $\frac{k}{2}$. On the other hand, Heron's gives us the area as $\frac{15k^2\sqrt{7}}{4}$. Setting these equal gives us $k = \frac{2}{15\sqrt{7}}$. Since $x + y + z = 15k$ it follows that the desired quantity is $\frac{2}{\sqrt{7}} \implies$ **9**vv.

§ 12 Unsourced

Consider sequence a_n with $a_1 = \sqrt{2} + 1$ and $a_n a_{n-1}^2 + 2a_{n-1} - a_n = 0$ for $n \geq 2$. Find a_{1000}.

§ 12.1 Solution

§ 13 AIME 1991/15

For positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers whose sum is 17. There is a unique positive integer n for which S_n is also an integer. Find this n.

§ 13.1 Solution

§ 14 Unsourced

If x, y, z are positive numbers such that

$$x^2 + xy + \frac{1}{3}y^2 = 25$$

$$\frac{1}{3}y^2 + z^2 = 9$$

$$z^2 + zx + x^2 = 16,$$

find $xy + 2yz + 3zx$.

§ 14.1 Solution

We substitute $(a, b, c) = (x, \frac{y}{\sqrt{3}}, z)$. The equations rewrite as:

$$a^2 + ab\sqrt{3} + b^2 = 25$$

$$b^2 + c^2 = 9$$

$$a^2 + ac + c^2 = 16$$

We then use the implicit LoC trick to get that $\frac{1}{2}bc + \frac{1}{4}ab + \frac{\sqrt{3}}{4}ca = [ABC]$ where $\triangle ABC$ is a triangle with side lengths $3, 4, 5$. In this case, $[ABC]$ is simply 6, so

$$\frac{1}{2}bc + \frac{1}{4}ab + \frac{\sqrt{3}}{4}ca = 6.$$

Substituting into (x, y, z) gives us

$$\frac{1}{4\sqrt{3}}xy + \frac{1}{2\sqrt{3}}yz + \frac{\sqrt{3}}{4}zx = 6.$$

Multiplying by $4\sqrt{3}$ gives the desired quantity equal to $\mathbf{24\sqrt{3}}$.
Not completely sure this is right pls check!

§ 15 HMMT Feb. Algebra 2014/9

Given a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$
$$b^2 + bc + c^2 = -2$$
$$c^2 + ca + a^2 = 1,$$

compute $(ab + bc + ca)^2$. (Here, $i = \sqrt{-1}$.)

§ 15.1 Solution

The idea is to use LoC to show a more general statement for reals, which can be phrased as a polynomial identity and thus must hold in complex numbers as well! Will add more later.

§ 16 Unsourced

Find all triples (x, y, z) such that $xy + yz + zx = 1$ and $5(x + \frac{1}{x}) = 12(y + \frac{1}{y}) = 13(z + \frac{1}{z})$.

§ 16.1 Solution

§ 17 rd123/tworigami Mock AIME 2020/13

If a, b, c, d are positive real numbers such that

$$\begin{aligned}ab + cd &= 90,\\ ad + bc &= 108,\\ ac + bd &= 120,\\ a^2 + b^2 &= c^2 + d^2,\end{aligned}$$

and $a + b + c + d = \sqrt{n}$ for some integer n, find n.

§ 17.1 Solution

Consider a quadrilateral $ABCD$ with $AB = a, BC = b, CD = c, DA = a$, and $\angle B = \angle D = 90^\circ$. Then from Pythagoras we have $a^2 + b^2 = c^2 + d^2 = AC$. Further since $\angle B + \angle D = 180^\circ$ this quadrilateral is cyclic, so inscribe it in a circle. This also means that $\angle C = 180^\circ - \angle A$. We know that

$$[ABCD] = [ABC] + [ADC] = \frac{ab + cd}{2}.$$

Since $ab + cd = 90$ is given, we get $[ABCD] = 45$. We can also write $[ABCD] = [ABD] + [CBD]$. Then by the sine area formula and using the fact that $\sin(180^\circ - \theta) = \sin\theta$, this is equal to

$$\frac{1}{2}\sin\angle A(ad + bc) = 54\sin\angle A.$$

But $[ABCD] = 45$ as well, so $\sin\angle A = \frac{5}{6}$. Finally, we note that by Ptolemy's we have:

$$ac + bd = AC \cdot BD \implies AC \cdot BD = 120.$$

Now, since the inscribed angle with measure θ of chord $\overline{BD}$ satisfies $\sin\theta = \frac{5}{6}$, it follows from LoS on either $\triangle BDA$ or $\triangle BDC$ that $BD = \frac{5}{6}AC$, since $\overline{AC}$ is a diameter and therefore $AC = 2R$. This gives us:

$$\frac{5}{6}AC^2 = 120 \implies AC^2 = 144 = a^2 + b^2 = c^2 + d^2.$$

To finish, we consider the identity:

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$$

Substituting $a^2 + b^2 = c^2 + d^2 = 144$ as well as the values given at the start of the problem, we get $(a + b + c + d)^2 = n = \mathbf{924}$.

§ 18 PUMaC Div. A Algebra 2018/6

Let a, b, c be nonzero reals such that $\frac{1}{abc} + \frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. The maximum possible value of

$$\frac{4}{a^2+1} + \frac{4}{b^2+1} + \frac{7}{c^2+1}$$

is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m+n$.

§ 18.1 Solution

§ 19 2018 Mock AIME, by TheUltimate123

Let a,b,c,d be positive real numbers such that

$$195 = a^2 + b^2 = c^2 + d^2 = \frac{13(ac + bd)^2}{13b^2 - 10bc + 13c^2} = \frac{5(ad + bc)^2}{5a^2 - 8ac + 5c^2}$$

Then $a + b + c + d$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

§ 19.1 Solution

§ 20 Mildort AIME 3/15

Let Ω denote the value of the sum

$$\sum_{k=1}^{40} \cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right).$$

The value of $\tan(\Omega)$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

§ 20.1 Solution

We note that $\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}} = \frac{k^2+k+1}{\sqrt{(k^2+k+1)^2+1}}$. Drawing out a right triangle quickly, it becomes clear that the summation is equivalent to:

$$\sum_{k=1}^{40} \arctan\left(\frac{1}{k^2+k+1}\right).$$

We would ideally like to make this sum telescope. Define a function f such that:

$$\arctan\left(\frac{1}{k^2+k+1}\right) = \arctan\left(\frac{1}{f(k)}\right) - \arctan\left(\frac{1}{f(k+1)}\right).$$

Then the summation telescopes to $\arctan\left(\frac{1}{f(1)}\right) - \arctan\left(\frac{1}{f(41)}\right)$ which is hopefully easier to evaluate. Using arctangent addition, we have $\arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{y}\right) = \frac{y-x}{1+xy}$, so we need:

$$\frac{1}{k^2+k+1} = \frac{f(k+1)-f(k)}{f(k)f(k+1)+1}.$$

After looking at this for a while it becomes clear that $f(k) = k$ works (verifiable with substitution). So we just have to evaluate $\arctan(1) - \arctan\left(\frac{1}{41}\right)$. Using the arctangent addition formula again, we get that this is equal to $\arctan\left(\frac{20}{21}\right)$, so $\tan(\Omega) = \frac{20}{21}$ which yields an answer of **41**.

§ 21 IMO 2001/6

Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

§ 21.1 Solution

Look at the problem for a few minutes and cry until you decide to give up and do another unit because Dennis made an IMO P6 required.