Mass Points

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1 Theory

Mass Points is a technique in computational Geometry that trivializes an entire class of Geometry problems featured prominently on MATHCOUNTS, AMC 8 and early-mid AMC 10/12, and can also help in solving some AIME problems, but **only when executed properly**. If used wrong, it will return nonsensical results and turn out to be a waste of time.

Mass point. A **mass point** is a point, P, that is assigned a positive real weight w. It is usually written in the form wP. (In this handout, we will often refer to a mass point more explicitly as "Point P with mass w" anyways.) A **system of mass points** is a set of mass points.

Its functionality will be revealed in the next two definitions:

Center of mass of 1 and 2 points.

1. The **center of mass** of any mass point oO is itself.
2. The center of mass of a pair of distinct mass points mM, nN is the unique point Q on segment MN such that $\frac{MQ}{QN} = \frac{n}{m}$. It has weight $m + n$.

To see why that is the specific location of the center of mass of two points, imagine them on a seesaw; the heavier point will be closer to the balancing point.

General center of mass. If aA is the center of mass of a system of mass points S_1 and bB is the center of mass of another system of mass points S_2 distinct from S_1, then the center of mass cC of aA and bB is also the center of mass of $S_1 \cup S_2$.

The center of mass of a system of mass points $w_1P_1, w_2P_2 \dots$ can be thought of as their weighted average: toss the mass points on the coordinate plane, then the x-coordinate of the center of mass is $\frac{w_1x_1+w_2x_2\dots}{w_1+w_2\dots}$ and similarly the y-coordinate is $\frac{w_1y_1+w_2y_2\dots}{w_1+w_2\dots}$.

Lastly, here is the single most important guiding principle (I personally refer to it as balancing) one should follow while solving problems with mass points.

Balancing Principle. Every mass point defined on a segment should be the gravity center of its two endpoints.

You shouldn't go wrong if you keep this in mind and define weights in a way that abides this principle. As a corollary, sometimes defining multiple systems of mass points is necessary; for example, in problems that demand you to solve for two points E, F inside a triangle, systems of mass points with E and F as unique gravity centers should both be set up.

2 Examples

Note: Contrary to usual practice in Geometry, a lower case letter in this section by default refers to the weight of its upper case counterpart rather than a side length.

2.1 Simpler problems

Mass points would be the most effective on problems that involve ratios, as they can be translated to weights. Having too many intersection points on a single line greatly complicate calculation by bringing in multiple systems to the picture, so it is not preferred.

First, here is how mass points can instantly nuke a theorem that otherwise would take some work to prove.

Example. The **centroid** G of a triangle is its center of mass and is the intersection of its three medians. Prove that it splits each median in the ratio $2 : 1$.

Solution: As usual, let the vertices of the triangle be A, B, C, and let M be the midpoint of segment BC. Assign each of the vertices weight 1; we can see that $2M$ is the gravity center of $1B$ and $1C$. The gravity center of $1A$ and $2M$ is the gravity center of the triangle, which immediately implies our desired result for the A-median. Proceed analogously for the other two medians.

Even though the method of mass points is not exactly obscure, there is a suprising amount of problems on official contests that are straightforward applications of it. We will present one such problem here.

Example (AMC 8 2019/24). In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?

Solution: Set A's weight as 2 and C's weight as 1. We can see that D has weight $2 + 1 = 3$, and since E is the midpoint of BD, E must have weight 6 and B has weight 3 by balancing. F must have weight $e - a = 4$, which means that $\frac{EF}{AE} = \frac{2}{4} = \frac{1}{2}$ and $\frac{BF}{FC} = \frac{1}{3}$. The area of triangle EBF is therefore $\frac{1}{3}$ of triangle ABF, which has $\frac{1}{4}$ the area of triangle ABC; our answer is $\frac{360}{3\cdot 4} = \mathbf{30}$.

Remark: It is not a fluke that $4F$, when seen as a point on the extension of AE, also happens to be the gravity center of $1B$ and $3C$; since E is the gravity center of the triangle in this case, E has weight $a + b + c$ and F has weight $(a + b + c) - a = b + c$. **If set up properly, mass points is guaranteed to work out.**

Exercise. On the triangle formed by the mass points aA, bB, cC, a point D is chosen on segment BC. Prove that AD goes through the gravity center of the triangle **if and only if** the mass point $(b + c)D$ is the gravity center of bB and cC. That is, we have to prove that segment AD goes through the gravity center if $(b + c)D$ is the gravity center of bB and cC, and that the segment would not go through the gravity center if $(b + c)D$ is not the gravity center of bB and cC.

(Keep in mind that mass points are still points, and can be referred to as points when only using their properties that are inherited from points.)

2.2 Splitting points and multiple systems

Sometimes, the problem is not as direct as the previous subsection, which would call for a few more tricks.

Example. In triangle ABC, we have E and F on sides AC, AB such that $\frac{AE}{EC} = \frac{5}{2}, \frac{AF}{FB} = \frac{3}{7}$. Let H be a point on EF with $\frac{EH}{HF} = \frac{20}{7}$; if line AH intersects BC again at D, find $\frac{AH}{HD}$.

This is a prime example of when splitting points is necessary. Its basic premise is to split A into two points: A_1 such that F is the gravity center of A_1 and B, A_2 such that E is the gravity center of A_2 and C, and treat A as having weight $a_1 + a_2$ when dealing with the cevian AD.

Solution: We split A into A_1 and A_2 as described.

Assign a weight of 27 to H. Then, E has weight 7, F has weight 20, A_1 has weight 14 and A_2 has weight 2. On the cevian AD, A has weight $14 + 2 = 16$ and D has weight $27 - 16 = 11$, therefore $\frac{AH}{HD} = \boldsymbol{\frac{11}{16}}$.

Here is a concrete example of attacking a problem with more than one systems of mass points:

Example. Points D, E, F are located on sides BC, AC, AB of triangle ABC, respectively, with $\frac{AF}{FB} = \frac{2}{5}, \frac{AE}{EC} = \frac{3}{2}, \frac{BD}{DC} = \frac{1}{6}$. Given that AD intersect BE, CF at distinct points R, S, find $\frac{AR}{AS}$.

Walkthrough:

1. Set R as the gravity center to get $\frac{AR}{AD}$, ignoring S.
2. Set S as the gravity center to get $\frac{AS}{AD}$, ignoring R.
3. Obtain the final answer with arithmetics.

2.3 Auxiliary devices

Deciding to use mass points does not mean that you can turn off your brain. Applying concepts from other branches of Geometry, such as similar triangles and area formulas, could often assist mass points to solve problems more efficiently.

Example (MATHCOUNTS 2012). Point M of rectangle $ABCD$ is the midpoint of side BC and point N lies on CD such that $\frac{DN}{NC} = \frac{1}{4}$. Segment BN intersects AM and AC at points R and S. If $NS : SR : RB = x : y : z$ for positive integers x, y, z with $\gcd(x, y, z) = 1$, find $x + y + z$.

In this example, mass points alone seems insufficient. However, upon noticing a critical pair of similar triangles, the problem is reduced to a standard exercise.

Walkthrough:

1. Find the pair of similar triangles. (Opposite angles are useful.) That immediately eliminates one term in the ratio.
2. Solve for the other two terms using the techniques that we have developed.
3. Finish the problem by taking the least common multiple of the denominators.

3 Problems

Generic mass points problems can be mass-produced, so I will only feature a small number of problems that I consider to be genuinely creative and/or nice applications of it.

Minimum is [TBD ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)
[2 ♞]Problem 1 (Review of fundamentals)

[1 ♟] Problem 2 (Ceva) On sides BC, AC, AB of triangle ABC, choose points D, E, F; prove that AD, BE and CF are concurrent (intersect at one single point) if and only if $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.