Solutions to Lengths and Areas in Triangles

Dennis Chen

GQU

Contents

§ 1 Unsourced

Find the inradius of the triangles with the following lengths:

- $3, 4, 5$
- $5, 12, 13$
- $13, 14, 15$
- $5, 7, 8$

(These are arranged by difficulty. All of these are good to know.)

§ 1.1 Solution

We find the area of each of the triangles and divide by the semiperimeter.

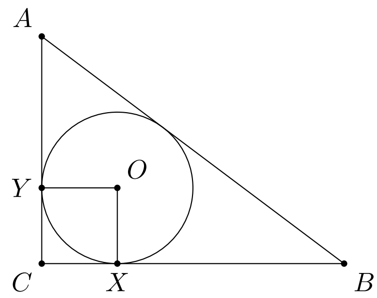
- This is a right triangle, so the area is $\frac{3 \cdot 4}{2} = 6$. Thus the inradius is $\frac{6}{\frac{3+4+5}{2}} = 1$.
- This is a right triangle, so the area is $\frac{5 \cdot 12}{2} = 30$. Thus the inradius is $\frac{30}{\frac{5+12+13}{2}} = 2$.
- By Heron's Formula, the area of this triangle is $\sqrt{21(21-13)(21-14)(21-15)} = 84$. Thus the inradius is $\frac{84}{\frac{13+14+15}{2}} = 4$.
- By Heron's Formula, the area of this triangle is $\sqrt{10(10-5)(10-7)(10-8)} = 10\sqrt{3}$. Thus the inradius is $\frac{10\sqrt{3}}{\frac{5+7+8}{2}} = \sqrt{3}$.

§ 2 Unsourced

Prove that in a right triangles with legs of length a, b, and hypotenuse with length c, $r = \frac{a+b-c}{2}$.

§ 2.1 Solution

Let the triangle have legs AC, BC and hypotenuse AB, let the incenter of $\triangle ABC$ be I, and let the incircle be tangent to AC, BC at X, Y. Then note that $CXOY$ is a square, since $\angle XCY = 90^\circ$ and $\angle OXC = \angle OYC = 90^\circ$ because AC, BC are tangents. So $r = s - c = \frac{a+b-c}{2}$.

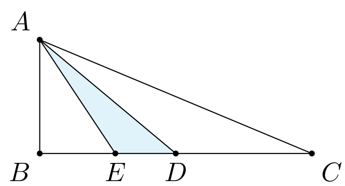


§ 3 Unsourced

In $\triangle ABC$, $AB = 5$, $BC = 12$, and $CA = 13$. Points D, E are on BC such that $BD = DC$ and $\angle BAE = \angle CAE$. Find $[ADE]$.

§ 3.1 Solution

Note that by the angle bisector theorem, $BE = \frac{5}{5+13} \cdot 12 = \frac{10}{3}$, so $ED = \frac{8}{3}$. Thus $[ADE] = \frac{1}{2} \cdot \frac{8}{3} \cdot 5 = \frac{20}{3}$.



§ 4 Unsourced

Find the maximum area of a triangle with two of its sides having lengths $10, 11$.

§ 4.1 Solution

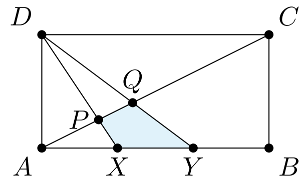
By $\frac{1}{2}ab\sin C$, the area is $\frac{1}{2} \cdot 10 \cdot 11 \cdot \sin\theta = 55\sin\theta$, where θ is the angle between the sides of lengths $10, 11$. Clearly $\sin\theta$ is maximized when $\theta = 90°$, so the maximum area is 55.

§ 5 e-dchen Mock MATHCOUNTS, Sprint Round, Problem 17

Consider rectangle $ABCD$ such that $AB = 2$ and $BC = 1$. Let X, Y trisect AB. Then let DX and DY intersect AC at P and Q, respectively. What is the area of quadrilateral $XYQP$?

§ 5.1 Solution

Without loss of generality, let $AX < AY$. Then note that we want to find $[AQY] - [APX]$. Since $\triangle AQY \sim \triangle CQD$, the height from Q to AY is $\frac{2}{5}$ and since $\triangle APX \sim \triangle CPD$, the height from P to AX is $\frac{1}{4}$. Thus $[AQY] = \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{2}{5} = \frac{4}{15}$ and $[APX] = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{12}$, so $[PQXY] = \frac{4}{15} - \frac{1}{12} = \frac{11}{60}$.



§ 6 Autumn Mock AMC 10, Problem 8

Equilateral triangle ABC has side length 6. Points D, E, F lie within the lines AB, BC and AC such that $BD = 2AD$, $BE = 2CE$, and $AF = 2CF$. Let N be the numerical value of the area of triangle DEF. Find N^2.

§ 6.1 Solution

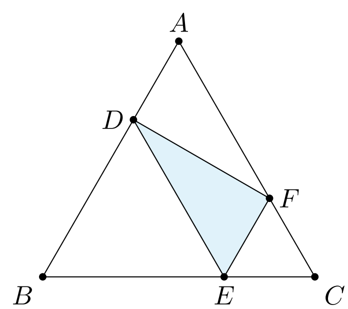
We know that $[ABC] = 9\sqrt{3}$. By $\frac{1}{2}ab\sin\theta$,

$$[AFD] = \frac{1}{2} \cdot 4 \cdot 2\sin 60^\circ = 2\sqrt{3}$$

$$[BDE] = \frac{1}{2} \cdot 4 \cdot 4\sin 60^\circ = 4\sqrt{3}$$

$$[CEF] = \frac{1}{2} \cdot 2 \cdot 2\sin 60^\circ = \sqrt{3}.$$

Thus $[ABC] = 9\sqrt{3} - 2\sqrt{3} - 4\sqrt{3} - \sqrt{3} = 2\sqrt{3}$, so the answer is 12.



§ 7 AIME I 2019/3

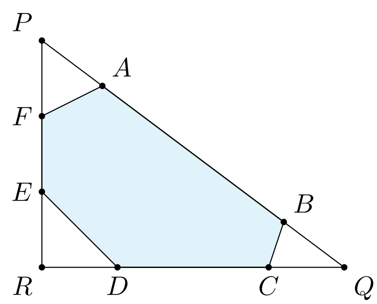
In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

§ 7.1 Solution

We know that $[PQR] = 150$. By $\frac{1}{2}ab\sin\theta$,

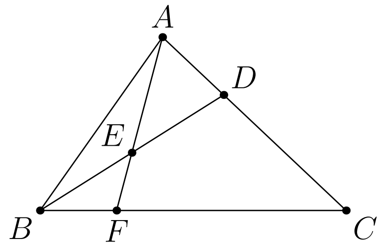
$$[PFA] = \frac{1}{2} \cdot 5 \cdot 5 \sin\angle APF = \frac{1}{2} \cdot 5 \cdot 5 \cdot \frac{4}{5} = 10$$
$$[QBC] = \frac{1}{2} \cdot 5 \cdot 5 \sin\angle BQC = \frac{1}{2} \cdot 5 \cdot 5 \cdot \frac{3}{5} = \frac{15}{2}$$
$$[RDE] = \frac{1}{2} \cdot 5 \cdot 5 \sin\angle DRE = \frac{1}{2} \cdot 5 \cdot 5 \cdot 1 = \frac{25}{2}.$$

Thus $[ABC] = 150 - 10 - \frac{15}{2} - \frac{25}{2} = 120$.



§ 8 AMC 8 2019/24

In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



§ 8.1 Solution

We use mass points. Note that we can assign $\diamond A = \diamond C = 1$, implying $\diamond D = 2$, which leads to $\diamond B = 2$ and $\diamond E = 4$. Thus $\diamond F = 3$.

Now note that $[EBF] = [ABC] \cdot \frac{EF}{AF} \cdot \frac{BF}{BC}$ by $\frac{bh}{2}$, and that

$$[ABC] \cdot \frac{EF}{AF} \cdot \frac{BF}{BC} = 360 \cdot \frac{1}{4} \cdot \frac{1}{3} = 30$$

by mass points.

§ 9 Unsourced

Consider $\triangle ABC$ such that $AB = 8$, $BC = 5$, and $CA = 7$. Let AB and CA be tangent to the incircle at T_C, T_B, respectively. Find $[AT_BT_C]$.

§ 9.1 Solution

The gist of the solution is that we use Heron's to find that $[ABC] = 10\sqrt{3}$, use the Two Tangent Theorem to find $AT_B = AT_C = 5$, and use $\frac{1}{2}ab\sin\theta$ to find $\frac{[AT_BT_C]}{[ABC]}$, which determines $[AT_BT_C]$.

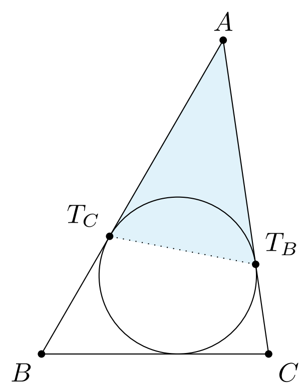
We first prove the following lemma, which we have implicitly used on the problems before this.

Lemma 1 (Area Ratio Given Collinear Points) Given collinear points P, A, X and P, B, Y,

$$\frac{[PXY]}{[PAB]} = \frac{PX \cdot PY}{PA \cdot PB}.$$

Proof: Note that $[PXY] = \frac{1}{2}PX \cdot PY \sin\angle XPY$ and $[PAB] = \frac{1}{2}PA \cdot PB \sin\angle APB$, and since P, A, X and P, B, Y are collinear, $\sin\angle XPY = \sin\angle APB$, which proves our desired result. ■

Now note that $\frac{[AT_BT_C]}{[ABC]} = \frac{5\cdot5}{7\cdot8}$, implying that $[AT_BT_C] = \frac{125\sqrt{3}}{28}$.



§ 10 Unsourced

Consider trapezoid $ABCD$ with bases AB and CD. If AC and BD intersect at P, prove the sum of the areas of $\triangle ABP$ and $\triangle CDP$ is at least half the area of trapezoid $ABCD$.

§ 10.1 Solution

Let $AB = b_1$, $CD = b_2$, and let the height from P to AB be h_1 and the height from P to CD be h_2. Note that we want to prove

$$\frac{b_1h_1 + b_2h_2}{2} \geq \frac{(b_1 + b_2)(h_1 + h_2)}{4},$$

which is equivalent to

$$2b_1h_1 + 2b_2h_2 \geq b_1h_1 + b_2h_2 + b_1h_2 + b_2h_1$$

$$b_1h_1 + b_2h_2 \geq b_1h_2 + b_2h_1.$$

Note that by similar triangles, $b_2 = b_1k$ and $h_2 = h_1k$ for some constant k. So this is equivalent to

$$b_1h_1(1 + k^2) \geq b_1h_1(2k)$$

$$k^2 + 1 \geq 2k$$

$$(k-1)^2 \geq 0,$$

and the last inequality is obviously true. Since all steps are reversible, we are done.