Computational Geometry Summit

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GRV

1 Lecture notes

In case the R-level geometry units were insufficient, here is a collection of incredibly difficult computational geometry problems of mixed styles, that many contestants would find challenging or tricky. For comedic value, the sections are labeled after the major musical periods: Baroque, Classical, Romantic, and Contemporary. These labels are just for fun and shouldn't be taken especially seriously. As with all[1] Summit units, in lieu of any walkthroughs or useful advice, here is a celebratory cake[2].

— 03/30/2017

[1]Hopefully people will make more!
[2]Thanks to Evan Chen for the asymptote code.

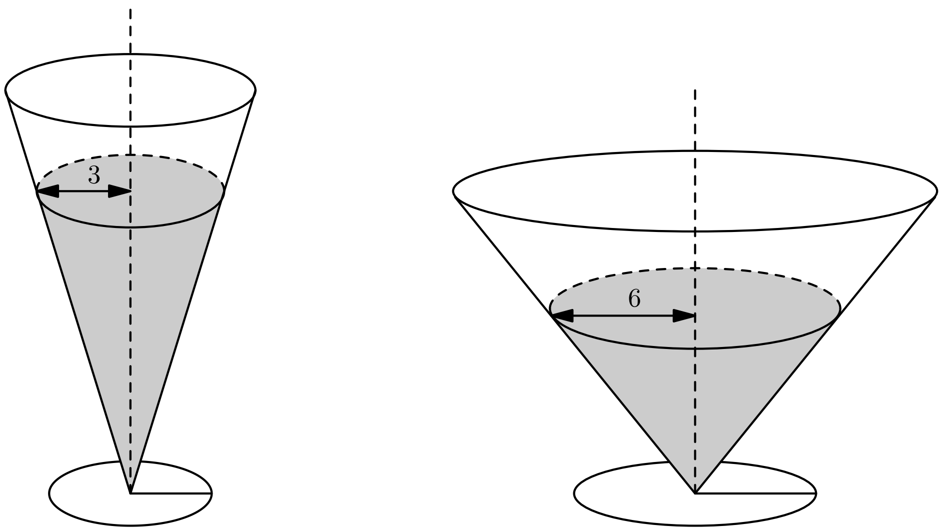
2 Problems

Minimum is [90 ♟]. Problems denoted with ♞ are required. (They still count towards the point total.)

> "Ishigami...you have friends...?"
>
> Miko Iino

2.1 Baroque-style geometry problems

[2 ♟] **Problem 1 (AMC 12A 2021/10)** Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?



[6 ♟] **Problem 2 (AIME II 2007/15)** Four circles ω, ω_A, ω_B, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are 13, 14, and 15, find the radius of ω.

[9 ♟] **Problem 3 (AIME I 2019/15)** Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[9 ♞] **Problem 4 (PuMaC 2020 G7)** Let $\triangle ABC$ be a triangle with sides $AB = 34$, $BC = 15$, $AC = 35$ and let Γ be the circle of smallest possible radius passing through A tangent to $\overline{BC}$. Let the second intersections of Γ and sides AB, AC be the points X, Y. Let the ray XY intersect the circumcircle of the triangle $\triangle ABC$ at Z. Find AZ.

[9 ♟] **Problem 5 (CMIMC 2017 G10)** Suppose $\triangle ABC$ is such that $AB = 13$, $AC = 15$, and $BC = 14$. It is given that there exists a unique point D on side $\overline{BC}$ such that the Euler lines of $\triangle ABD$ and $\triangle ACD$ are parallel. Determine the value of $\frac{BD}{CD}$. (The *Euler* line of a triangle ABC is the line connecting the centroid, circumcenter, and orthocenter of ABC.)

[13 ♟] **Problem 6 (USMCA 2021/29)** Three circles $\Gamma_A, \Gamma_B, \Gamma_C$ are externally tangent. The circles are centered at A, B, C and have radii $4, 5, 6$ respectively. Circles Γ_B and Γ_C meet at D, circles Γ_C and Γ_A meet at E, and circles Γ_A and Γ_B meet at F. Let GH be a common external tangent of Γ_B and Γ_C on the opposite side of BC as EF, with G on Γ_B and H on Γ_C. Lines FG and EH meet at K. Point L is on Γ_A such that $\angle DLK = 90°$. Compute $\frac{LG}{LH}$.

2.2 Classical-style geometry problems

[4 ♟] **Problem 7 (AIME 1994/15)** Given a point P on a triangular piece of paper ABC, consider the creases that are formed in the paper when A, B, and C are folded onto P. Let us call P a fold point of $\triangle ABC$ if these creases, which number three unless P is one of the vertices, do not intersect. Suppose that $AB = 36$, $AC = 72$, and $\angle B = 90°$. Then the area of the set of all fold points of $\triangle ABC$ can be written in the form $q\pi - r\sqrt{s}$, where q, r, and s are positive integers and s is not divisible by the square of any prime. What is $q + r + s$?

[6 ♟] **Problem 8 (AIME I 2014/11)** In $\triangle RED$, $RD = 1$, $\angle DRE = 75°$ and $\angle RED = 45°$. Let M be the midpoint of segment $\overline{RD}$. Point C lies on side $\overline{ED}$ such that $\overline{RC} \perp \overline{EM}$. Extend segment $\overline{DE}$ through E to point A such that $CA = AR$. Then $AE = \frac{a-\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer. Find $a + b + c$.

[6 ♟] **Problem 9 (AIME II 2016/14)** Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside of the plane of $\triangle ABC$ and are on the opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a $120°$ dihedral angle (The angle between the two planes). There is a point O whose distance from each of A, B, C, P and Q is d. Find d.

[6 ♟] **Problem 10 (AIME II 2019/13)** Regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ is inscribed in a circle of area 1. Point P lies inside the circle so that the region bounded by $\overline{PA_1}$, $\overline{PA_2}$, and the minor arc $\widehat{A_1A_2}$ of the circle has area $\frac{1}{7}$, while the region bounded by $\overline{PA_3}$, $\overline{PA_4}$, and the minor arc $\widehat{A_3A_4}$ of the circle has area $\frac{1}{9}$. There is a positive integer n such that the area of the region bounded by $\overline{PA_6}$, $\overline{PA_7}$, and the minor arc $\widehat{A_6A_7}$ is equal to $\frac{1}{8} - \frac{\sqrt{2}}{n}$. Find n.

[9 ♞] **Problem 11 (AIME I 2019/13)** Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

[9 ♟] **Problem 12 (NICE Spring 2021/23)** Let ABC be a triangle with $\angle ABC = 90°$ and incircle ω of radius 4, which is tangent to $\overline{BC}$ and $\overline{AC}$ at E and F respectively. Suppose that $\overline{BF}$ is tangent to the circumcircle of $\triangle CEF$. Then BC can be written in the form $p + \sqrt{q}$ for positive integers p and q. Determine $1000p + q$.

[13 ♟] **Problem 13 (OMO Spring 2018/28)** In $\triangle ABC$, the incircle ω has center I and is tangent to $\overline{CA}$ and $\overline{AB}$ at E and F respectively. The circumcircle of $\triangle BIC$ meets ω at P and Q. Lines AI and BC meet at D, and the circumcircle of $\triangle PDQ$ meets $\overline{BC}$ again at X. Suppose that $EF = PQ = 16$ and $PX + QX = 17$. Find BC^2.

2.3 Romantic-style geometry problems

[4 ♟] **Problem 14 (AIME II 2020/13)** Convex pentagon $ABCDE$ has side lengths $AB = 5$, $BC = CD = DE = 6$, and $EA = 7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

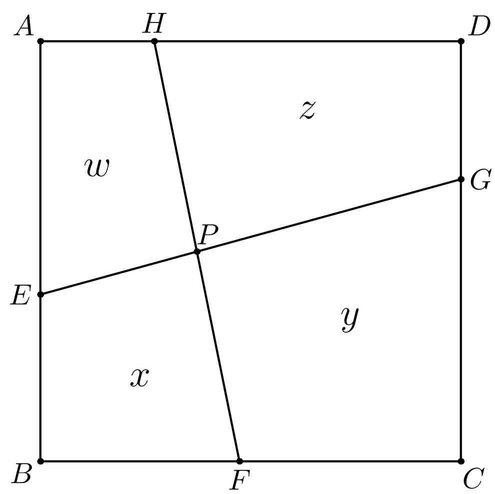
[4 ♟] **Problem 15 (MPfG 2014/17)** Let ABC be a triangle. Points D, E, and F are respectively on the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ of $\triangle ABC$. Suppose that

$$\frac{AE}{AC} = \frac{CD}{CB} = \frac{BF}{BA} = x$$

for some x with $\frac{1}{2} < x < 1$. Segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ cut the triangle into 7 nonoverlapping regions: 4 triangles and 3 quadrilaterals. The total area of the 4 triangles equals the total area of the 3 quadrilaterals. Compute the value of x.

[6 ♟] **Problem 16 (AIME I 2013/13)** In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

[6 ♞] **Problem 17 (AIME I 2014/13)** On square $ABCD$, points E, F, G, and H lie on sides $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$, respectively, so that $\overline{EG} \perp \overline{FH}$ and $EG = FH = 34$. Segments $\overline{EG}$ and $\overline{FH}$ intersect at a point P, and the areas of the quadrilaterals $AEPH, BFPE, CGPF$, and $DHPG$ are in the ratio 269 : 275 : 405 : 411. Find the area of square $ABCD$.



$w : x : y : z = 269 : 275 : 405 : 411$

[9 ♟] **Problem 18 (AIME I 2018/13)** Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

[13 ♟] **Problem 19 (AIME I 2011/13)** A cube with side length 10 is suspended above a plane. The vertex closest to the plane is labelled A. The three vertices adjacent to vertex A are at heights 10, 11, and 12 above the plane. The distance from vertex A to the plane can be expressed as $\frac{r-\sqrt{s}}{t}$, where r, s, and t are positive integers, and $r + s + t < 1000$. Find $r + s + t$.

2.4 Contemporary-style geometry problems

[3 ♟] **Problem 20 (AMC 12B 2018/23)** Amol is standing at point A near Pontianak, Indonesia, 0° latitude and 110° E longitude. Brian is standing at point B near Big Baldy Mountain, Idaho, USA, 45° N latitude and 115° W longitude. Assume that Earth is a perfect sphere with center C. What is the degree measure of $\angle ACB$?

[6 ♟] **Problem 21 (AMC 12A 2021/21)** The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6)=0$$

may be written in the form $x_k + y_k i$ for $1 \le k \le 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5). Find the eccentricity of $\mathcal{E}$. (Recall that the eccentricity of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of E and $2c$ is the is the distance between its two foci.)

[9 ♟] **Problem 22 (HMMT 2021 G8)** Two circles with radii 71 and 100 are externally tangent. Compute the largest possible area of a right triangle whose vertices are each on at least one of the circles.

[9 ♞] **Problem 23 (AIME I 2021/13)** Circles ω_1 and ω_2 with radii 961 and 625, respectively, intersect at distinct points A and B. A third circle ω is externally tangent to both ω_1 and ω_2. Suppose line AB intersects ω at two points P and Q such that the measure of minor arc $\widehat{PQ}$ is $120°$. Find the distance between the centers of ω_1 and ω_2.

[13 ♟] **Problem 24 (AMC 12B 2011/25)** Triangle ABC has $\angle BAC = 60°$, $\angle CBA \le 90°$, $BC = 1$, and $AC \ge AB$. Let H, I, and O be the orthocenter, incenter, and circumcenter of $\triangle ABC$, respectively. Assume that the area of the pentagon $BCOIH$ is the maximum possible. What is $\angle CBA$?

[13 ♟] **Problem 25 (AIME II 2017/15)** Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, define $f(X) = AX + BX + CX + DX$. The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.