Logarithms

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§ 1 Problems

Minimum is [32 ✎]. Problems with the ♢ symbol are required.

> "I just long for a world in which ordinary things are done in an ordinary way."
>
> Psycho-Pass

[1 ✎] **Problem 1** (AIME II 2020/3) The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

[2 ✎] **Problem 2** (AIME 1986/8) Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1000000. What is the integer nearest to S?

[2 ✎] **Problem 3** (AIME I 2020/2) There is a unique positive real number x such that the three numbers $\log_8 2x$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

[2 ✎] **Problem 4** (AIME I 2007/7) Let $N = \sum\limits_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor)$.
Find the remainder when N is divided by 1000. ($\lfloor k \rfloor$ is the greatest integer less than or equal to k, and $\lceil k \rceil$ is the least integer greater than or equal to k.)

[3 ✎] **Problem 5** (SMT 2020) If a is the only real number that satisfies $\log_{2020} a = 202020 - a$ and b is the only real number that satisfies $2020^b = 202020 - b$, what is the value of $a+b$?

[3 ✎] **Problem 6** (AIME II 2013/2) Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a+b$.

[3 ♢] **Problem 7 (AIME II 2010/5)** Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

[3 ✎] **Problem 8** (AIME I 2006/9) The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

[4 ✎] **Problem 9** (HMMT 2020) Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

[4 ✎] **Problem 10** (AIME II 2007/12) The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that $\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57$, find $\log_3(x_{14})$.

[4 ✎] **Problem 11** (AIME I 2009/7) The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1} - a_n)} - 1 = \frac{1}{n + \frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

[4 ✎] **Problem 12** (AIME I 2010/14) For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$.
Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

[6 💎] **Problem 13 (AIME I 2005/8)** The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

[6 ✎] **Problem 14** (AIME I 2013/8) The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m+n$ is divided by 1000.

[7 ✎] **Problem 15** (hARMLess Mock ARML 2019/10) Compute the sum of all positive integers that can be expressed in the form

$$\log_b(404!) - \log_b(c),$$

where b and c are positive integers such that $b > 1$ and $b+c$ is odd.

[9 ✎] **Problem 16** (AIME I 2012/9) Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime positive integers. Find $p+q$.