Exercise 1: Probabilities in DTMCs

Let the transition matrix P:

$$P = \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

a. Compute $P^3(s_0, s_3)$
Using the Chapman-Kolmogorov equation:

$$p_{s_0,s_3}(3) = \sum_{s'} p_{s_0,s'}(l)p_{s',s_3}(3-l) = \sum_{s'} P(s_0,s')p_{s',s_3}(2) = \sum_{s'} P(s_0,s')(\sum_{s''} P(s',s'')P(s'',s_3))$$

$$\Leftrightarrow \sum_{s'} P(s_0,s')(P(s',s_1)P(s_1,s_3) + P(s',s_0)P(s_0,s_3) + P(s',s_3)P(s_3,s_3))$$

$$\Leftrightarrow P(s_0,s_3)P(s_3,s_1)P(s_1,s_3) + P(s_0,s_1)P(s_1,s_0)P(s_0,s_3) + P(s_0,s_3)P(s_3,s_3)P(s_3,s_3)$$
$$+P(s_0,s_1)P(s_1,s_3)P(s_3,s_3)$$

$$\Leftrightarrow \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{1}{3}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}$$

$$\Leftrightarrow \frac{1}{32} + \frac{3}{48} + \frac{1}{64} + \frac{3}{32} = \frac{2}{64} + \frac{4}{64} + \frac{1}{64} + \frac{6}{64}$$

$$\Leftrightarrow \frac{13}{64} = 0.203125$$

$$\Leftrightarrow \begin{pmatrix} 0.02083333 & 0.296875 & 0.47916667 & 0.203125 \\ 0.125 & 0.05208333 & 0.58333333 & 0.23958333 \\ 0 & 0 & 1 & 0 \\ 0.02083333 & 0.109375 & 0.77083333 & 0.09895833 \end{pmatrix}|_{(s_0,s_3)}$$

$$\Leftrightarrow \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}^3 |_{(s_0,s_3)} = P^3(s_0,s_3)$$

b. Compute the probability of being in state s3 after exactly 3 steps assuming a uniform initial distribution (over all states).

$$\Theta_3^{\mathcal{D}}(s_3) = \sum_{s\in S} l_{\text{init}} P^3(s,s_3) = \sum_{s\in S} \frac{1}{|S|} P^3(s,s_3)$$

With x_{ij} the element of P^3 in the i-th row and j-th column and

$$P^3 = \begin{pmatrix} 0.02083333 & 0.296875 & 0.47916667 & 0.203125 \\ 0.125 & 0.05208333 & 0.58333333 & 0.23958333 \\ 0 & 0 & 1 & 0 \\ 0.02083333 & 0.109375 & 0.77083333 & 0.09895833 \end{pmatrix}$$

$$\Leftrightarrow \frac{1}{4}\sum_i x_{i3} = \frac{0.203125 + 0.23958333 + 0 + 0.09895833}{4} = \frac{0.54166666}{4} = 0.135416\overline{6}$$

c. Compute the limiting probability of being in state s_3.

d. Compute the probability of going from s_0 to s_3 in at most 3 steps.

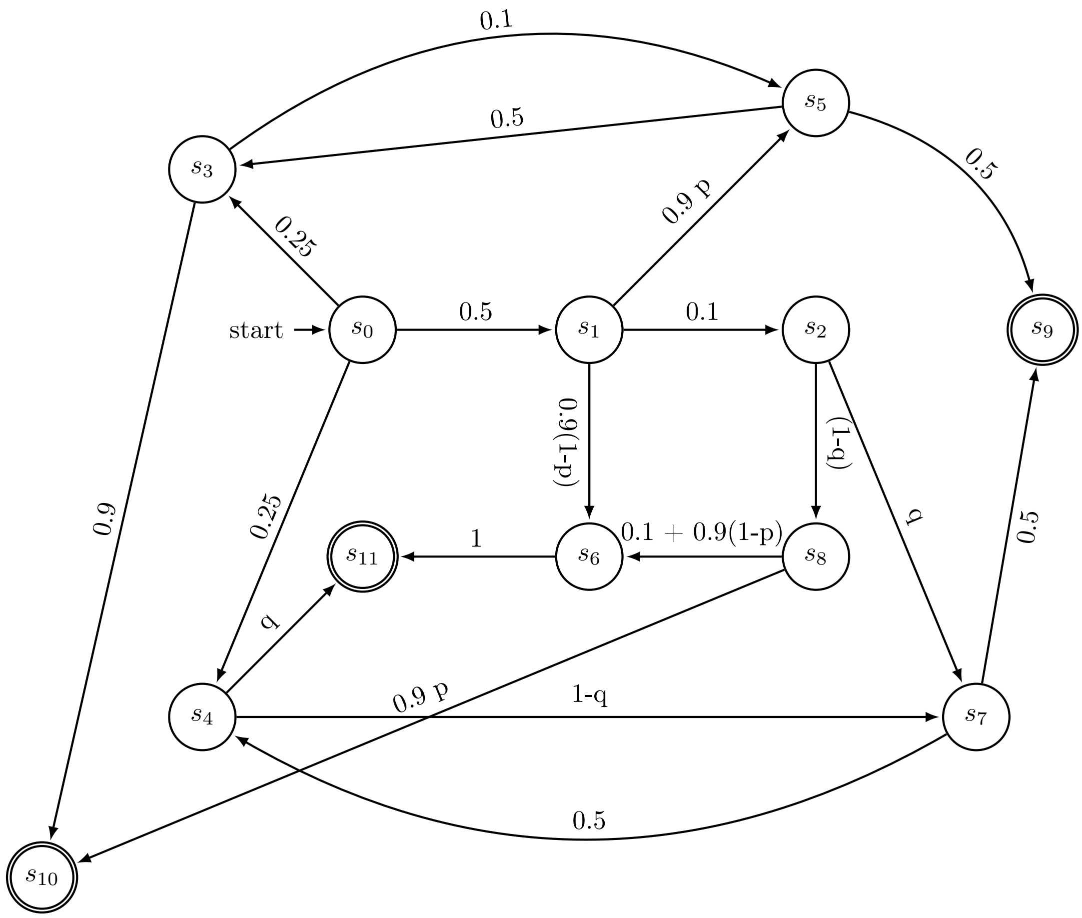
e. Compute the probability of reaching (without a bound on the number of steps) s_3 when starting in s_0.

Exercise 2: Duelling Cowboys

a.

b. Depict a DTMC for this process. Please indicate for each state (i) who is alive and (ii) whose turn it is.

state	turn	alive
s_0	Good	all
s_1	Bad	all
s_2	Ugly	all
s_3	Bad	Good, Bad
s_4	Ugly	Good, Ugly
s_5	Good	Good, Bad
s_6	Ugly	Bad, Ugly
s_7	Good	Good, Ugly
s_8	Bad	Bad, Ugly
s_9	Good	Good
s_{10}	Bad	Bad
s_{11}	Ugly	Ugly



Exercise 3: The Gruffalo Game

a.

b.