30/10/2018
Probabilistic Modelling for Computer Scientists
Excercise lecture 1 notes by
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(1.) see photo   sorry, I did not know that I should do this

(sigma)
def2.1. σ-algebra $(\Omega, \mathcal{F})$ — complement
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
1. $\Omega \in \mathcal{F}$
3. $A_1, A_2, A_3 \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

powerset function
$2^\Omega = P(\Omega) = \{A \mid A \subseteq \Omega\}$

Ex.2.2 $\Omega = \{1,2,3\}$
$2^\Omega = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

(2.) ~~Homework~~ Excercise 1   (Ω countably infinite, F_Ω - smallest class of subsets of Ω, $F_\Omega \subseteq 2^\Omega$
(i) A is finite $\Rightarrow A \in F_\Omega$
(ii) $A \in F_\Omega \Rightarrow A^c \in F_\Omega$

a) Solution by student $\Omega = \{1,2,3,\dots\} = \mathbb{N}$
for $A = \Omega$
! does not relate with (i)

Example $\Omega = \mathbb{N} \Rightarrow$

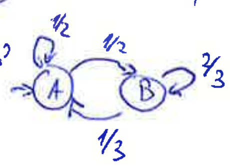
	(i)		(ii)
$\emptyset$	$\in F_\Omega$	$\Rightarrow$	$\Omega \in F_\Omega$
$\{1\}, \{2\}, \{3\}, \dots$	$\in F_\Omega$	complement $\Rightarrow$	$\in F_\Omega$
$\{1,2\}, \{1,3\}, \dots$	$\in F_\Omega$	$\Rightarrow$	$\in F_\Omega$
$\{1,2,3\} \dots$	$\in F_\Omega$	$\Rightarrow$	$\in F_\Omega$

! ~~SOLUTION~~ no $\Omega = \mathbb{N}$
$A = \{1,3,5,7,\dots\} \Rightarrow$ counterexample that does not need to be in F_Ω $\wedge$
since F_Ω should be smallest set it is not $\Rightarrow$ is not trivial. □

b) ~~yes~~ it is trivial $\wedge$ $F_\Omega = 2^\Omega$

c) no because it does not contain 1a counter example, but σ-algebra does
because F_Ω contains each odd number $\Rightarrow A = \bigcup\{\text{odd number}\} \Rightarrow$ (iii) does not hold for F_Ω □

(3) DTMC examples   transient

Ex.3.? 

def.3.1. $\mathcal{D} = (S, P, \mu_{init}, AP, L)$
$(\{A,B\}, \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}, (1,0), AP, L)$

def. 3.2 $Pr(X_1 = s' \mid X_0 = s) = P^{(1)}(s,s')$
$Pr(X_n = s' \mid X_0 = s) = P^{(n)}(s,s')$

conditional probability $P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}$
$\Rightarrow P(X \cap Y) = P(Y) \cdot P(X \mid Y)$
$= P(X) \cdot P(Y \mid X)$

def.3.3 Chapman - Kolmogorov relations
$Pr(X_n = s' \mid X_0 = s) = P^{(n)}(s,s') = \sum_{s'' \in S} P(X_n = s', X_i = s, X_0 = s')$
$= \sum P(X_i = s'', X_0 = s) \cdot P(X_n = s' \mid X_i = s'', \cancel{X_0 = s})$
↳ memoryless prop
$= \sum_{s'' \in S} P^{(i)}(s, s'') \cdot P^{(n-i)}(s'', s')$

def. 3.4
$\pi(n)$: a row vector denoting transient prob. distr. at time n
$\pi(0) = \mu_{init}$
$\pi(1) = \mu_{init} \cdot P \overset{\text{EXAMPLE}}{=} (1,0) \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} = (1/2 \;\; 1/2)$

$\pi(2) = \pi(1) \cdot P = \pi(0) \cdot P \cdot P = \pi(0) \cdot P^2$

def. 3.5 Stationary distribution $\tilde{\pi}$

$$\pi(n+1) = \pi(n) \cdot P = \pi(n) = \tilde{\pi} \quad \text{or} \quad \tilde{\pi} P = \tilde{\pi}$$

EXAMPLE $$[\tilde{\pi}_A \;\; \tilde{\pi}_B] \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [\pi_A \;\; \pi_B]$$

$$\tfrac{1}{2}\pi_A + \tfrac{1}{3}\pi_B = \pi_A \leadsto \pi_A = \tfrac{2}{3}\pi_B$$
$$\tfrac{1}{2}\pi_A + \tfrac{2}{3}\pi_B = \pi_B$$

def 3.6 Eigen value and Eigen vector

$$\underset{\in \mathbb{R}^{n \times n}}{M} \; \underset{\in \mathbb{R}^{n \times 1}}{v} = \underset{\in \mathbb{R}}{\lambda} \, v$$

Lemma 3.7 Each stochastic has a eigenvalue equal 1 and eigen vector of ones $= \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{n}$

$$\begin{bmatrix} P \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \land \lambda = 1$$

additional info Perron-T theorem

def. 3.8 P is stochastic matrix iff each row is a distribution

$$\begin{pmatrix} p_{11} & p_{1n} \\ & p_{nn} \end{pmatrix} \quad \hookrightarrow p_{ij} \qquad \Rightarrow \sum_{i \in \{1,\dots,n\}} p_{ij} = 1$$

Lemma 3.9 $P^{(n)}$ is stochastic matrix if P is stoch. matrix