Exercise 1: Sigma Algebras (4 points)

Let Ω be a countably infinite set and define $\mathfrak{F}_\Omega$ as the smallest class of subsets of Ω such that for all $A \subseteq \Omega$

(i) if A is finite, then A $\in \mathfrak{F}_\Omega$, and

(ii) if A $\in \mathfrak{F}_\Omega$, then $A^c \in \mathfrak{F}_\Omega$ for $A^c := (\Omega \setminus A)$.

a) Show that the definition is non-trivial, i.e., in general $\mathfrak{F}_\Omega \neq 2^\Omega$.

(Hint: find a set Ω and a subset $A \subseteq \Omega$ which cannot be in $\mathfrak{F}_\Omega$ according to the above definition.)

b) Would this change if $\mathfrak{F}_\Omega$ is defined as the *largest* class of subsets defined as above (instead of the *smallest*)?

c) Prove or disprove that $\mathfrak{F}_\Omega$ is a $\sigma-$algebra as defined in the lecture for any countably infinite set Ω.

Exercise 2: Geometric Distribution (3+3 points)

Recall the definition of a geometric distribution as given in the lecture:

Definition 1. Let X be a discrete random variable, $k \in \mathbb{N}_{>0}$ and $0 < p \leq 1$. The mass function of a *geometric distribution* is given by:

$$\Pr\{X = k\} = (1-p)^{k-1} \cdot p$$

Let X now be a geometrically distributed with parameter p.

a) Show that $E[X]= \frac{1}{p}$ and $Var[X] = \frac{1-p}{p^2}$.

b) Prove that:

$$\Pr\{X = k + m | X > m\} = \Pr\{X = k\} \text{ for any } m, k \in \mathbb{N}_{>0}.$$

*Hint:*Use properties of probability measures and the geometric distribution as presented in the lecture.

Exercise 3: Conditional probability (5 points)

A patient named Fred is tested for a disease called *conditionitis*, a medical condition that affects 1% of the population. The test result is positive, i.e., the test claims that Fred has the disease. Let D be the event that Fred has the disease and T be the event that he tests positive. Suppose that the test is 95% accurate; there are different measures of the accuracy of a test, but in this problem it is assumed to mean that $Pr(T|D) = 0.95$ and $Pr(Tc|Dc) = 0.95$. The quantity $Pr(T|D)$ is known as the sensitivity or true positive rate of the test, and $Pr(Tc|Dc)$ is known as the specificity or true negative rate. Find the conditional probability that Fred has conditionitis, given the evidence provided by the test result.