Exercise 1: σ-Algebra

a) Let $\Omega = \mathbb{N}$. Then the subset of even numbers $A_0 = \{2, 4, 6, \dots\}$ is not contained in $\mathcal{F}_\Omega$ as A_0 is not finite and it's complement A_1 is also not finite $A_1 = \{1, 3, 5, \dots\}$. So $\mathcal{F}_\Omega \neq 2^\Omega$. $\square$

b) No. the definition says:

$$\forall A \subseteq \Omega : ((|A| < \infty \Rightarrow A \in \mathcal{F}_\Omega) \wedge (A \in \mathcal{F}_\Omega \Rightarrow (\Omega \setminus A) \in \mathcal{F}_\Omega))$$

i.e. (i) means a finite subset is in $\mathcal{F}_\Omega$. (ii) means if the complement of a subset is finite, the subset is in $\mathcal{F}_\Omega$. Thus if neither the set is finite nor it's complement, none of them is a member of $\mathcal{F}_\Omega$. To summarize: Even if the largest class of subsets is considered the definition is non-trivial and the example from a) still holds i.e. it may still be that $\mathcal{F}_\Omega \neq 2^\Omega$. $\square$

c) (Proof or) disproof that $(\Omega, \mathcal{F}_\Omega)$ is a σ-Algebra.
Consider $A_i = \{2i\}$. $A_i \in \mathcal{F}_\Omega$ as it's finite, but $\cup_i A_i \notin \mathcal{F}_\Omega$ as it's not finite and it's complement is also not finite (see a)). So $\mathcal{F}_\Omega$ not a σ-Algebra $\square$

Exercise 2:

a. Given from slides:
$E[X] = \sum_{i=0}^{n} x_i \cdot Pr_X(X = x_i)$,
$Var[X] = E[X^2] - E[X]^2 = E[X(X-1)] + E[X] - E[X^2]$,
Given from Analysis I (Geometric series, first & second derivative):
$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$,
$\sum_{k=0}^{\infty} k \cdot q^{k-1} = \frac{1}{(1-q)^2}$.
$\sum_{k=0}^{\infty} k(k-1) \cdot q^{k-2} = \frac{-2}{(1-q)^3}$.

With X geometrically distributed and $q = 1 - p$
Expected Value:

$$E[X] = \sum_{k=1}^{\infty} k \cdot (q)^{k-1} \cdot p$$

$$\Leftrightarrow p \sum_{k=1}^{\infty} k \cdot q^{k-1}$$

$$\Leftrightarrow p \frac{1}{(1-q)^2} = \frac{p}{(1-1-p)(1-1-p)}$$

$$\Leftrightarrow \frac{p}{p^2} = \frac{1}{p}$$

$\square$

Variance:

$$Var[X] = E[X(X-1)] + E[X] - E[X^2] = \sum_{k=1}^{\infty} k(k-1) \cdot q^{k-1} \cdot p + \frac{1}{p} - \frac{1}{p^2}$$

$$\Leftrightarrow p \sum_{k=1}^{\infty} k(k-1) \cdot q^{k-1} + \frac{1}{p} - \frac{1}{p^2}$$

$$\Leftrightarrow p \sum_{k=1}^{\infty} k(k-1) \cdot q^{k-1} q^{-1} q^{1} + \frac{1}{p} - \frac{1}{p^2}$$

$$\Leftrightarrow pq \sum_{k=1}^{\infty} k(k-1) \cdot q^{k-2} + \frac{1}{p} - \frac{1}{p^2}$$

$$\Leftrightarrow pq \frac{-2}{(1-q)^3} + \frac{1}{p} - \frac{1}{p^2}$$

$$\Leftrightarrow \frac{-2p(1-p)}{(1-1-p)^3} + \frac{1}{p} - \frac{1}{p^2}$$

$$\Leftrightarrow \frac{-2(1-p)}{-p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$\Leftrightarrow \frac{2(1-p)}{p^2} - \frac{1}{p^2} + \frac{p}{p^2}$$

$$\Leftrightarrow \frac{2(1-p)+p-1}{p^2} = \frac{2-2p+p-1}{p^2}$$

$$\Leftrightarrow \frac{1-p}{p^2}$$

$\square$

b. Prove that $\Pr(X = k + m | X > m) = \Pr(X = k)$ for any $m, k \in \mathbb{N}$

$$\Pr(X = k + m | X > m) = \frac{\Pr(X = k + m \cap X > m)}{\Pr(X > m)}$$

$$\frac{\Pr(X = k + m | X > m)\Pr(X > m)}{\Pr(X > m)} = \frac{\Pr(X > m | X = k + m)\Pr(X = k + m)}{\Pr(X > m)}$$

$$\frac{1 \cdot (1-p)^{k+m-1} p}{(1-p)^m} = (1-p)^{k-1} p$$

Regarding $\Pr(X > m | X = k + m) = 1$ is trivial as $k, m \in \mathbb{N} \wedge k + m > m$ $\square$.

Exercise 3:

- Given:
  $\Pr(D) = 0.01$, $\Pr(T|D) = \Pr(Tc|Dc) = 0.95$ accuracy of T
- Wanted: $\Pr(D|T)$
- Ansatz: Conditional Probability, Bayes rule and true positives + false positives for the denominator.

$$\Pr(D|T) = \frac{\Pr(T|D)\Pr(D)}{\Pr(T)} = \frac{\Pr(T|D)\Pr(D)}{\Pr(T|D)\Pr(D) + \Pr(T|\overline{D})\Pr(\overline{D})}$$
$$= \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.05 \cdot 0.99} = \frac{0.0095}{0.0095 + 0.0495} = \frac{0.0095}{0.059}$$
$$= 0.161016949153 = 16.1016949153\%$$