Irreducibility of multilayer network dynamics: the case of the voter model

Marina Diakonova,[1] Vincenzo Nicosia,[2] Vito Latora,[2] and Maxi San Miguel[1]

[1]*Instituto de Física Interdisciplinar y Sistemas Complejos IFISC (CSIC-UIB), E07122 Palma de Mallorca, Spain*
[2]*School of Mathematical Sciences, Queen Mary University of London, E1 4NS Mile End Road, London, UK.*

We address the issue of the reducibility of the dynamics on a multilayer network to an equivalent process on an aggregated single-layer network. As a typical example of models for opinion formation in social networks, we implement the voter model on a two-layer multiplex network, and we study its dynamics as a function of two control parameters, namely the fraction of edges simultaneously existing in both layers of the network (edge overlap), and the fraction of nodes participating in both layers (interlayer connectivity or degree of multiplexity). We compute the asymptotic value of the number of active links (interface density) in the thermodynamic limit, and the time to reach an absorbing state for finite systems, and we compare the numerical results with the analytical predictions on equivalent single-layer networks obtained through various possible aggregation procedures. We find a large region of parameters where the interface density of large multiplexes gives systematic deviations from that of the aggregates. We show that neither of the standard aggregation procedures is able to capture the highly nonlinear increase in the lifetime of a finite size multiplex at small interlayer connectivity. These results indicate that multiplexity should be appropriately taken into account when studying voter model dynamics, and that, in general, single-layer approximations might be not accurate enough to properly understand processes occurring on multiplex networks, since they might flatten out relevant dynamical details.
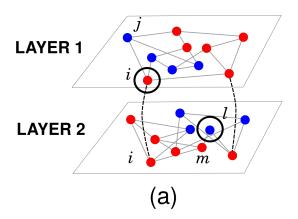
PACS numbers:

I. INTRODUCTION

Real-world interactions often happen at different levels and are therefore properly modelled by means of multi-layer networks. Such multilayer approaches [1–3] have been applied to fields ranging from energy infrastructure [4] and transport [5–7], to epidemiology [8]. The multi-layer set up can either describe interconnected networks with nodes of the same nature in each layer, but inter-acting with nodes of different nature in a different layer, or a multiplex structure with nodes of the same nature interacting via a different network in each layer. In any case, a central methodological question is that of multi-layer reducibility, that is, when the multilayer framework is really needed to explain new phenomena, or when the system description can be reduced to an appropriately aggregated or reformulated single-layer network. An interesting contribution in this direction has recently come from the study of the structural reducibility of multilayer networks, i.e. of the possibility of aggregating some of the layers of a multi-dimensional network while preserving its distinguishability from the corresponding single-layer aggregated graph [9]. Although some recent works have pointed out that multiplex dynamics can be intrinsically different from their equivalent single-layer counterparts [10–14], little attention has been devoted so far to the problem of reducing a process taking place on a multilayer network to a dynamically equivalent process on an appropriate single-layer network. In this paper we address this general question, by implementing the voter model on a multiplex networks and by studying the reducibility of its dynamics as a test case.

The voter model [15] is a nonequilibrium lattice model [16] which gives a standard framework for studying the influence of social imitation on the process of opinion formation [17]. A basic question considered in this context is when and how the system reaches an absorbing state with all the interacting nodes in the same state, or when an active dynamical situation of coexistence of different states prevails. The answer to these questions is known to depend crucially on the network structure and on the update rules employed [18–22]. The voter model has also been instrumental to understand fundamental phenomena in coevolution dynamics in which node states and network structure have coupled dynamical evolution with two different time scales [23–27]. In terms of comparison with real data, a metapopulation voter model has been recently shown to be able to account for voting patterns in the US general elections [28].

Quite frequently, social interactions happen on different concurrent contexts, so that any model of social imitation based on a single-layer representation of social relationships should probably be regarded only as a first-order approximation of a potentially more complex dynamics. In order to better mimic the multidimensional nature of social interactions, we consider here a multiplex voter model, where agents interact at two distinct layers. We assume also that the system exhibits a certain level of multiplexity, meaning that a fraction q of the agents is present on both layers [26]. As a consequence, any change in the state of those agents on either of the two layers is automatically transferred to the other layer, effectively coupling the voter model dynamics taking place on each of the two networks. By taking into account heterogeneity in the participation of agents to layers we aim at reproducing an interesting feature of real-world multilevel social systems [29]. In particular, it has been shown that the percentage of users of an online social
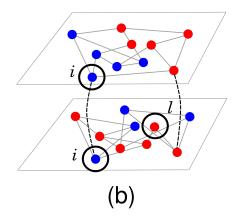
FIG. 1: (Color online) In the multiplex voter model each node i on layer α is associated to a state $s_i^{[\alpha]}(t) = \pm 1$ (the values $+1$ and -1 are respectively indicated in the figure by red and blue), which evolves according to one simple rule: select one of your neighbours x on layer α uniformly at random, and copy its state, i.e. set $s_i^{[\alpha]}(t+1) = s_x^{[\alpha]}(t)$. For instance, the state of node l on layer 2 changes from $s_l^{[2]} = -1$ (blue, panel a) to $s_l^{[2]} = +1$ (red, panel b) since l selected its neighbour m on layer 2 and copied its state on that layer. The presence of inter-layer edges (dashed lines) denotes identification between nodes at the two layers. An example is the inter-layer edge connecting the two replicas of node i. In this case, any change in the state of node i at one layer will enforce a change of its state on the other layer. In the figure, the state of node i flips from red (panel a) to blue (panel b) on both layers, as a consequence of the interaction of node i with its neighbour j on layer 1.

network (e.g., Twitter) which has also an account on another online social network (e.g., Facebook or LinkedIn) lies somewhere between 30% and 70%, as reported by the social media matrix periodically published by the Pew Research Center [30]. These empirical findings confirm the necessity to consider systems whose multiplexity is neither 0 nor 1, but somewhere in between. A second important feature of real-world social interactions is the fact that networks corresponding to different contexts share a number of common links, as can be found for instance in data from online games: analysis of the PARDUS online society in [31, 32] yields significant overlap between designation of other players as friends, correspondents, or trading partners. This property of multilayer networks can be quantified by the edge-overlap parameter [33].

In this work we address a fundamental question, that is whether the coupling of several voter models into a multiplex dynamics, motivated by the multiplicity of contexts influencing real interactions among individuals, gives rise to qualitatively different phenomena or effects than those observed in the classical single-layer setup. In particular, we want to establish whether multiplexity makes any difference for a simple social dynamics like the voter model, or if instead the multiplex voter model can indeed be reduced to an equivalent voter model dynamics on an appropriately constructed single-layer network.

A preliminary result on the question of the multilayer reducibility exists for a bilayer of uncorrelated networks evolving according to the voter model with adaptive links [26], but in the limit of zero edge overlap. In this model each layer is associated with a network plas-

ticity parameter, that controls the rate at which relations among agents are rearranged, such that if the values of plasticity at the two layers are sufficiently different the system displays a network *shattered* fragmentation. On the other hand when the layers have the same plasticity one finds that the results coming from a pair approximation in the thermodynamic limit are equivalent to those of an appropriate aggregated single-layer network [26]. We build upon this, in time independent networks, examining both the asymptotic properties of the thermodynamic limit and the characteristic times to reach an absorbing state for finite systems. We also consider different schemes to obtain a possible equivalent aggregated single-layer network and highlight the importance of the degree of multiplexity and edge overlapping.

The paper is organized as follows. Sect II introduces our voter dynamics multilayer model. Sect. III describes our numerical findings. These are compared in Sect. IV with single-layer theoretical results for two natural methods of collapsing the multilayer into an aggregated single-layer network. Sect. V analyzes an optimum single-layer reduction method. Conclusions are discussed in Sect. VI.

II. THE MODEL

We consider here the case of a 2-layer undirected unweighted multiplex network, described by the pair of binary adjacency matrices $\{A^{[1]}, A^{[2]}\}$, where $A^{[\alpha]} \equiv \{a_{ij}^{[\alpha]}\}$, and $a_{ij}^{[\alpha]} = 1$ if and only if node i and node j are connected by a link at layer α, and zero otherwise. On each

layer we have N nodes. A parameter of interest in this study is the *average edge overlap* ω, that is the probability that an edge is present on both layers:

$$\omega = \frac{\sum_{i,j} a_{ij}^{[1]} a_{ij}^{[2]}}{2K} \qquad (1)$$

where

$$K = \sum_i \sum_{j>i} \frac{a_{ij}^{[1]} + a_{ij}^{[2]}}{1 + a_{ij}^{[1]} a_{ij}^{[2]}}$$

is the number of edges of the graph obtained by aggregating the two layers into a single one [33]. Notice that $\omega = 0$ only if each edge exists in exactly one of the two layers, but not in the other one, while we have $\omega = 1$ only if all the edges exist on both layers.

Each node i on layer α is associated to a binary state variable $s_i^{[\alpha]}(t)$, where $s_i^{[\alpha]}(t)$ can be either $+1$ or -1. Moreover, we assume that a fraction q of the N nodes participates in both layers, requiring that if i is one of these qN nodes then its state at the two layers will be identical at the end of every update. Such nodes participating in both layers are chosen randomly at initialization. We can think of the parameter q as the *interlayer connectivity* or the *degree of structural multiplexity* of the system. The model is illustrated in Fig. 1.

The multiplex voter dynamics consists of a sequence of time steps. During a time step we perform N updates, and each update consists of three elementary operations, as follows: *i)* a layer α is selected at random, with uniform probability; *ii)* one of the nodes i on layer α is chosen at random and its state $s_i^{[\alpha]}$ is updated according to the classical voter model dynamics, that is, $s_i^{[\alpha]}$ becomes the same as that of a randomly chosen neighbour of i on layer α; *iii)* if the updated node i participates in both layers, then the state of the corresponding node in the other layer β changes as well by setting $s_i^{[\beta]} = s_i^{[\alpha]}$. With this third operation, state changes can propagate across layers: indeed the presence of a fraction of nodes existing in both layers intertwines the voter dynamics on the two layers, so that in general the evolution of the overall multiplex dynamics might differ from the one we would observe on two independent networks of the same size. As a limiting case, the dynamics reduces to that of a classical voter model on a single-layer network only when $q = 1$.

It is well known that in connected finite-size single-layer networks the voter model dynamics always reaches an absorbing state, where all the nodes have exactly the same state, in a survival time that scales with the system size N [16–18, 20, 21]. In networks of high effective dimensionality (including random networks), when $N \to \infty$ the dynamics sustains an active disordered state in which nodes continue to change their state [19]. Such active state is the one observed asymptotically in large systems, before finite-size fluctuations pull the system

towards the absorbing state. The classical order parameter to measure the activity in the voter model is the so-called *interface density* $\rho(t)$, defined as the fraction of active edges of the network, i.e. of those edges whose endpoints have different states. In the multiplex model we can define:

$$\rho^{[\alpha]}(t) = \frac{1}{2K^{[\alpha]}} \sum_i \sum_{j<i} a_{ij}^{[\alpha]} |s_i^{[\alpha]}(t) - s_j^{[\alpha]}(t)|, \qquad (2)$$

for each layer α, $\alpha = 1, 2$, where we denote by $K^{[\alpha]} = \frac{1}{2}\sum_{i,j} a_{ij}^{[\alpha]}$ the total number of edges at layer α. Notice that $\rho = 0$ if and only if all the nodes have the same state, while larger values of ρ are associated to active configurations. A second quantity of interest for a finite system is the average time $\langle T \rangle$ to reach an absorbing state of consensus. Such a time can be defined as $\langle T \rangle = \int_0^\infty P_s(t)dt$ from the survival probability $P_s(t)$, i.e. the probability for a system to be active at time t [34].

For single-layer uncorrelated networks at sufficiently large times $t > N$, $P_s(t)$ decreases exponentially $P_s(t) \sim e^{-2t/\tau}$ [35]. Hence, $\langle T \rangle \sim \tau/2$, so that the dependencies of $\langle T \rangle$ on system size and on the moments of the degree distribution are given precisely by the corresponding dependencies of the *characteristic time* τ. For uncorrelated networks one can find exact expressions for the value of the average interface density in the thermodynamic limit ρ^{single}, as well as for the characteristic time to reach the absorbing state τ^{single} [23, 35]. An important feature is that these analytical predictions of ρ^{single} and τ^{single} depend only on the values of the first two moments of the degree distribution $P(k)$ of the network, i.e. on $\mu_1 = \langle k \rangle = \sum_k kP(k)$ and $\mu_2 = \langle k^2 \rangle = \sum_k k^2 P(k)$, and not on any other microscopic property of the network, and read:

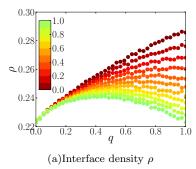$$\rho^{\text{single}} = \frac{\mu_1 - 2}{3(\mu_1 - 1)}, \qquad (3)$$

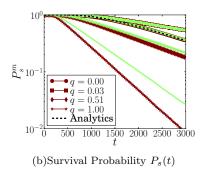where ρ^{single} is the average over surviving runs, and

$$\tau^{\text{single}} = \frac{(\mu_1 - 1)\mu_1^2 N}{(\mu_1 - 2)\mu_2}. \qquad (4)$$

In the following we will focus on the values $\rho(q, \omega)$ and $\tau(q, \omega)$ of interface density and characteristic time of the multiplex voter model as a function of the average edge overlap of the system ω and of the fraction q of nodes present in both layers.

III. NUMERICAL RESULTS

We studied the voter model on a duplex network made by two random regular graphs with N nodes, each node having a degree equal to μ. We therefore have $\mu_1 = \langle k^{[1]} \rangle = \langle k^{[2]} \rangle = \mu$, and $\mu_2 = \mu^2$. In our simulations we fixed $N = 1000$ and $\mu = 4$, and we studied the dynamics of the system by varying the value of the

(a)Interface density ρ (b)Survival Probability $P_s(t)$ (c)Rescaled characteristic time τ/N
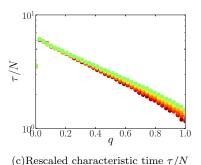
FIG. 2: (Color online) (a) Asymptotic value of the interface density averaged over the surviving runs; (b) Survival probability of the multiplex for different q values, for $\omega = 0$ (maroon, larger and darker markers), and $\omega = 1$ (green, smaller and lighter markers). The line denoted as *analytics* is $P_s^m(t, q = 0, \omega = 0)$ whose expression is given in the main text; (c) Logarithm of the average characteristic time τ of the approach to the absorbing state, rescaled by $(1/N)$ (legend same as in fig. 2(a)). Quantities are functions of multiplexity q, and edge overlap ω. We have considered an ensemble of 10^5 random initializations on a fixed duplex whose layers are random regular graphs, each with $N = 1000$ and $\langle k \rangle = 4$.

edge overlap ω and the fraction q of nodes participating in both layers. Since the system is finite, it will eventually converge to the absorbing frozen state corresponding to an interface density equal to zero. Consequently, for each time t, we will evaluate the average value of the interface density only over the surviving runs, i.e. on those realisations of the dynamics which are still active at time t. We associate the asymptotic value of this numerically-obtained quantity with the value at the thermodynamic limit, given by Eq. (3), and refer to it as interface density, while being clear whether we are referring to the numerical or analytical results. In the case of a network with two layers, we have of course two values of interface density, one for each layer. If $\langle k^{[1]} \rangle = \langle k^{[2]} \rangle$ the ensemble averages on the two layers will be equal, and hence the activity of an arbitrary layer is representative of the typical activity of the entire system. We therefore use the asymptotic value of the interface density of an arbitrary layer to reflect the activity of the multiplex, and dispense with the α superscript.

In Fig. 2(a) we report the values of interface density $\rho(q, \omega)$ as a function of q, and for different values of ω. Let us consider first the case $\omega = 0$ of no overlap between the structure of the two layers of the network. When $q = 0$, i.e. when inter-layer state passing is not allowed, the system effectively corresponds to two identical but independent single-layer voter dynamics, so that $\rho(q, \omega)$ is in accordance with the classical analytical predictions for single-layer networks (since $\mu_1 = 4$, Eq. (3) gives the activity $\rho(0, 0) = 0.22$). On the other hand, when $q = 1$, i.e. when all the nodes participate in both layers and each edge exists only in one layer, the system is in all respects identical to a single-layer network with $\mu_1 = 2\mu$ (for $\mu_1 = 8$, Eq. (3) gives the activity $\rho(1, 0) = 0.286$). For intermediate values of q the dynamics interpolates monotonically between the two extreme cases, i.e. two dynamically indistinguishable voter models on single-layer networks with $\mu_1 = \mu$ ($q = 0$), and

one voter model on a single-layer network with $\mu_1 = 2\mu$ ($q = 1$). The picture changes completely as soon as ω is large enough. In general, when the overlap is above some limit, then $\rho(q, \omega)$ is a non-monotonic concave function of q, with a maximum at a given value of q in $[0, 1]$ which depends on ω. Notice that, when $\omega = 1$ and $q = 1$, i.e. if the two layers are identical and all the nodes participate in both layers, the dynamics is identical to a voter model on a single-layer network with $\mu_1 = \mu$. In fact, since all the edges exist on both layers, a node participating in both layers will have only μ distinct neighbours, and will be connected to each of them on both layers. Hence, in a multiplex network with $\omega = 1$, the interface density ρ of the voter model takes the same value 0.22 at $q = 0$ and at $q = 1$, while for intermediate values of q the interface density is higher than that of a voter model on each of the two layers.

We now consider the characteristic time τ of the multiplex dynamics, where by characteristic time we understand twice the inverse exponent of the multiplex survival probability $P_s^m(t)$. We consider the multiplex as active if at least one of the layers is active, and find that for all $q > 0$ the survival probability of the multiplex does decay exponentially with some exponent $\tau = \tau(q, \omega)$ (fig. 2(b)). The only exception is the $q = 0$ case of the fully-disconnected multiplex, where the survival probabilities of the layers are independent. In this case the survival probability of the multiplex is given by the probability that at least one of the layers is active, $P_s^m(t, q = 0, \omega = 0) \sim P_s(t)(2 - P_s(t))$, or $P_s^m(t, 0, 0) \sim 2e^{-2t/\tau^{\text{single}}} - e^{-4t/\tau^{\text{single}}}$. This means that for $q = 0$ the multiplex survival probability *does not scale exponentially*, and hence $\tau(q = 0)$ is not well-defined. Fig. 2(c) shows the characteristic times $\tau(q, \omega)$ obtained (the value at $q = 0$ is that of an approximate exponential fit). The trend shows a peak at small q, followed by an exponential decay with increasing number

of interlayer connections. The value of edge overlap ω controls the rate of decrease. We stress that the peak is *not* a consequence of the definition of τ for the multiplex, and the consequent rogue value of τ at $q = 0$: the limit is truly singular, with plots for the actual average time until absorption displaying the same features, as do the plots for the characteristic exponent layer by layer. We also note that the peak is robust with respect to system size. Therefore the slowest finite-size multiplexes are ones where the layers are interconnected by the smallest number of links. Multiplexes with more interlayer connections will stop being active faster, as will wholly disconnected systems. This insight can be understood by realizing that in a multiplex with few interlayer links, the layers will most of the time function as completely disconnected networks that half the time will try to settle into the absorbing states of consensus different from one another. Since this situation is now prohibited by the few interlayer links, the multiplex will freeze only when one of the layers switches over and both layers reach the same consensus. It is this switching behaviour that is responsible for the peak in τ for small q. We also notice that this non-linear effect does not depend on the edge overlap ω. Here, unlike with the behaviour of interface density, the overlap does not change qualitatively the behaviour of τ for increasing q. In fact, the more interconnected the multiplex, the higher the role of edge overlap: tuning up ω causes a decrease in the interface density and results in a longer-lived multiplex, and the effect becomes more pronounced with q.

IV. IRREDUCIBILITY OF THE DYNAMICS

It was shown in [26] that the interface density of the multiplex voter model dynamics as a function of q and in absence of edge overlap can be rewritten as the interface density of a voter model on a single-layer network having $\mu_1 = \langle k \rangle (1 + q)$ (where k is the average degree of each of the two original layers), under some appropriately rescaled time. Here it is important to note that the approximation of [26] treated the interlayer connections as inherently probabilistic: the q parameter was a probability that each node's state gets passed on to the other layer. It was shown that the analytics of such a system, in the particular case $\omega = 0$, are equivalent to those of the voter model on some properly aggregated single-layer network. However, as Ref. [26] only considered the thermodynamic limit, it is not obvious whether the aggregate displays a corresponding rescaling of characteristic time. Nevertheless, it should be possible to devise an aggregate that results from the flattening of the multiplex into a single-layer network, so that the resulting network will have a first moment of the degree distribution equal to the expression μ_1 given above.

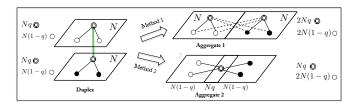The main question now is whether such a reducibility is possible in the most general case in which the multiplex has edge overlap $\omega \neq 0$. This is indeed the most interesting case, i.e. when in a multiplex there are correlations between the edges at the different layers [29, 33]. In the following our working hypothesis will be that, if the multiplex can indeed be reduced to a monoplex (in the variables of interest), then there exists an aggregate graph such that the behaviour of the voter model on the multiplex is completely described by Eq.(3) and Eq.(4) evaluated on the corresponding monoplex. Since those equations depend just on the number of nodes and on the first two moments of the degree distribution of the resulting single-layer network, we will now consider two standard aggregation procedures, and derive analytically the values of the first and second moments of the resulting degree distributions. Let us remind that, by definition, the first moment of the degree distribution of a graph is equal to $\mu_1 = \langle k \rangle = \sum_{k=1}^{\infty} k P(k)$, where $P(k)$ is the degree distribution, so that $P(k = \kappa)$ is the probability that the degree of a node sampled at random from the graph is equal to κ. The probability $P(k)$ can be also written as:

$$P(k) = \frac{N_k}{\sum_\ell N_\ell} \qquad (5)$$

where N_k is the number of nodes in the graph having degree equal to k and the normalization is just the total number of distinct nodes.

Given a duplex formed by two random regular graphs with identical degree μ, with edge overlap ω and where a fraction q of nodes participates in both layers, we can distinguish two classes of nodes. The nodes in the first class exist in only one layer, and we indicate as k_{single} the number of their neighbours, while nodes in the second class exist in both layers (these are the nodes having an inter-layer link) and we indicate their degree as k_{both}. The former class has degree:

$$k_{\text{single}} = \mu, \qquad (6)$$

whereas nodes present on both layers have degree $k_{\text{both}} = 2\mu$ when $\omega = 0$. However, if the edge overlap is not null, i.e. when $\omega > 0$, then the nodes being present in both layers have degree k_{both} equal to:

$$k_{\text{both}} = 2\mu - q\mu\omega = \mu(2 - q\omega). \qquad (7)$$

FIG. 3: Schematic illustration of two possible aggregation procedures that reduce a two-layer multiplex with N nodes on each to a single network. The multiplex contains qN interlayer links between associated nodes (pairwise links). These nodes are shown in double circles. These nodes will have twice as many links in the aggregates, independent of the flattening method. Method 1 keeps $2N$ nodes in the resultant aggregate, whereas method 2 has $N(2 - q)$.

(a)Interface density ρ

(b)Rescaled characteristic time τ/N
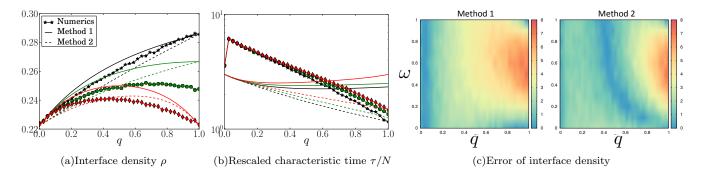
(c)Error of interface density

FIG. 4: (Color online) Comparison of the theoretical prediction for (a) the interface density and (b) the characteristic time based on the aggregation of the duplex. Numerical results are shown for $\omega = 0$ (stars, black), $\omega = 0.5$ (circles, green) and $\omega = 1$ (diamonds, red), where the other parameters are the same as in Fig.2. Analytics for the aggregates are computed for the respective values of ω for methods 1 and 2, and normalized by N. Panel (c): Absolute value of the relative error on the estimation of $\rho(q, \omega)$ using aggregation method 1 as a function of q and ω.

In fact, the degree of a node i present in both layers is equal to the sum of its degrees on the two layers (2μ) minus the expected number of its edges which are present on both layers. This number is equal to the probability that a neighbour j of i is also present on both layers (which is equal to q), times the probability that the edge (i, j) is present in both layers (which is equal to ω) multiplied by the number of neighbours of i (i.e., μ). Hence we get the correction $q\mu\omega$. In the particular case in which $\omega = 1$, we get $k_{\text{both}} = \mu(2-q)$, while for $\omega = 0$ we recover $k_{\text{both}} = 2\mu$.

Let us now consider the two following distinct aggregation procedures. They are illustrated in Fig. 3 and correspond to the two most standard ways to aggregate a duplex into a single-layer network. The two flattening procedures differ in the total number of nodes and also in the number of nodes of degree k_{single} and k_{both} that they produce. This in turn changes the effective system size of the aggregate network, and the first two moments of the degree distribution. In the following sections we compute these quantities for the two methods, and assess how well the aggregates fare in describing the behaviour of the true multiplex.

A. Aggregation method 1

We can obtain a single-layer representation of a multiplex by putting the layers side by side with no effective node overlap. Therefore $N_{\text{single}} = 2N(1 - q)$, $N_{\text{both}} = 2Nq$, and the total number of nodes in the aggregate is just $N_{\text{aggr}} = 2N$. The degrees of the nodes participating in just one or both layers are respectively equal to k_{single} and k_{both}. By using Eq. (5) we have:

$$P(k = k_{\text{single}}) = (1 - q), \tag{8}$$

and

$$P(k = k_{\text{both}}) = q. \tag{9}$$

The first moment μ_1^{aggr} of the aggregated graph is then equal to:

$$
\begin{aligned}
\mu_1^{\text{aggr}} &= \sum_{k=1}^{\infty} k P(k) \\
&= k_{\text{single}} P(k = k_{\text{single}}) + k_{\text{both}} P(k = k_{\text{both}}) \\
&= \mu_1 P(k = k_{\text{single}}) + \mu_1 (2 - q\omega) P(k = k_{\text{both}}) \\
&= (1 - q)\mu_1 + q(2\mu_1 - q\omega\mu_1) \\
&= \mu_1 (1 + q - \omega q^2),
\end{aligned}
\tag{10}
$$

and the second moment

$$
\begin{aligned}
\mu_2^{\text{aggr}} &= (1 - q)\mu_1^2 + q\mu_1^2 (2 - q\omega)^2 \\
&= \mu_1^2 \left[1 + 3q - 4\omega q^2 + \omega^2 q^3 \right].
\end{aligned}
\tag{11}
$$

Under no overlap these reduce to

$$
\begin{aligned}
\mu_1^{\text{aggr}} &= \mu_1 (1 + q), \\
\mu_2^{\text{aggr}} &= \mu_1^2 (1 + 3q).
\end{aligned}
\tag{12}
$$

Note that this method produces the same effective rescaling of the first moment μ_1 as given by the analytical estimation of the interface density of the multiplex in the thermodynamic limit (see Ref. [26]).

B. Aggregation method 2

This method reduces the number of effective nodes in the aggregate through treating nodes present in both layers as *one node*. So, while for the rest of the nodes we still have $N_{\text{single}} = 2N(1 - q)$, the number of 'multiplex' nodes is now half as much as in method 1, $N_{\text{both}} = Nq$. The total number of distinct nodes in the aggregate is equal to $N_{\text{aggr}} = 2N(1 - q) + Nq = N(2 - q)$. The degrees of the single and 'multiplex' nodes are as before equal to k_{single} and k_{both} respectively. Therefore

$$P(k = k_{\text{single}}) = \frac{2(1 - q)}{(2 - q)}, \tag{13}$$

(a)Effective degree distribution

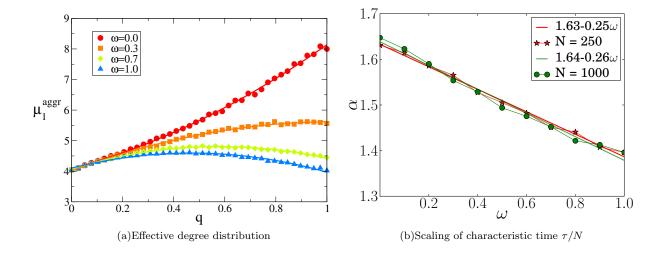(b)Scaling of characteristic time τ/N

FIG. 5: (Color online) (a) The values of μ_1^{aggr} resulting from the inversion of the measured interface density in the simulations as a function of q for different values of edge overlap (symbols) and the corresponding quadratic fits (lines). (b) Scaling of τ with q using the data shown in Fig. 2. For each ω, α corresponds to the slope of $-ln(\tau/N)$ with q computed for $q > 0$. Straight lines are the linear regression fits for each of the two N trends, with coefficients given in the legend.

and

$$P(k = k_{\mathrm{both}}) = \frac{q}{(2 - q)}. \tag{14}$$

Consequently, we have for the first moment of the degree distribution μ_1^{aggr}:

$$\mu_1^{\mathrm{aggr}} = \sum_{k=1}^{\infty} kP(k) = \frac{2N(1 - q) \times \mu_1 + Nq \times \mu_1(2 - q\omega)}{N(2 - q)}$$

$$= \mu_1 \frac{(2 - q^2\omega)}{(2 - q)} \tag{15}$$

and for the second moment μ_2^{aggr}:

$$\mu_2^{\mathrm{aggr}} = \mu_1^2 \frac{(q^3\omega^2 - 4q^2\omega + 2q + 2)}{(2 - q)}. \tag{16}$$

When the layers are uncorrelated and the overlap $\omega = 0$, we have

$$\mu_1^{\mathrm{aggr}} = \frac{2\mu_1}{(2 - q)},$$

$$\mu_2^{\mathrm{aggr}} = 2\mu_1^2 \frac{(1 + q)}{(2 - q)}$$

We notice that, when $q = 0$ (i.e., if we have two non-interacting layers), we get $\mu_1^{\mathrm{aggr}} = \mu_1$ and $\mu_2^{\mathrm{aggr}} = \mu_1^2 = \mu_2$, while for $q = 1$ we have $\mu_1^{\mathrm{aggr}} = 2\mu_1$ and $\mu_2^{\mathrm{aggr}} = 4\mu_1^2 = (2\mu_1)^2$.

C. Comparing aggregated to multiplex dynamics

We now compare the numerical results for the multiplex to the theoretical values given by the corresponding

equations for the aggregates. Eqns. (3) and (4) describe the behaviour of the monoplex. We contrast the two aggregates by substituting the respective effective value for the first and second moments and the total number of nodes. In other words, we take $N \to N_{\mathrm{aggr}}$, $\mu_1 \to \mu_1^{\mathrm{aggr}}$, and $\mu_2 \to \mu_2^{\mathrm{aggr}}$, where the effective values differ depending on the aggregation method.

The results are reported in Fig. 4. Consider first the behaviour in the thermodynamic limit, described by the interface density ρ. Both methods result in aggregates whose qualitative behaviour with ω and q corresponds to the trend observed in the multiplex (Fig. 4(a)): for small q the system becomes more active with increasing number of interlayer edges, whereas after a certain point activity may decrease, depending on the whether or not the edge overlap is significant enough. However, neither method gives correct quantitive understanding for a general ω and $q > 0$. This can be seen in Fig. 4(c), which compares the performance of the two methods through computing the absolute value of the relative error on ρ as heat map of q and ω. Method 1 in general does a better job than method 2 at being systematically consistent, albeit suggesting higher values (the semicircular drop in the error to zero, observed in method 2, comes about from the crossing of the analytical and the numerical trends). As we have seen before, the analytical results for Method 1 (at $\omega = 0$) correspond to a multiplex with probabilistic interlayer connections of intensity q. Therefore, at $\omega = 0$, comparing the numerical results for the multiplex, and the analytical trend for method 1 can be used to gain insight into the difference induced by an *alternative method of inteconnecting the layers* of the multiplex (in fact, when viewed in this way, the small magnitude of the differences becomes more surprising than their pres-

ence). The main quantitative differences arise for a wide range of intermediate overlap values, for medium-to-large interlayer connectivity - precisely the region of parameter space motivated by real-world systems. The conclusion we draw from this is that, as long as edge overlap is taken into account, standard aggregates can only inform on some qualitative features of large multiplex systems, but cannot capture quantitatively the behavior of real multiplex networks.

Consider now the characteristic time of the approach to absorbing states of finite-size systems (Fig. 4(b)). The difference at $q = 0$ is due to $\tau(q = 0, \omega)$ not being defined for the multiplex; yet the aggregate timescale is close. The monotonic decrease for $q > 0$ shown by the numerics is captured only by method 2, with coincidence of τ values for $q = 1$. In fact, method 2 does a better job at both qualitative and quantitative results, unlike method 1, which appears to work best in the thermodynamic limit (this aggregation method results in a much slower system for large q than the observed duplex). The main discrepancy, however, is that neither of the methods capture the jump of τ at small q. Therefore, although the two standard aggregation procedures can come close to describing the qualitative features of long-term duplex activity, they are not sophisticated enough to capture the long timescales associated with sparsely interconnected systems. In the following Section we will see whether and how it is possible to devise more complex aggregation procedures in order to reproduce quantitatively the dynamics of the multiplex voter model.

V. NON-LINEAR MULTIPLEX EFFECTS

In principle, the results of the previous Section cannot absolutely rule out the possibility that there exist other aggregation methods reproducing the behaviour of $\rho(q, \omega)$ observed in the multiplex voter model. Our hypothesis is that the deviations from the theory found for $\omega > 0$ are due to the additional non-linearity induced by inter-layer state copying made possible by the presence of a fraction of inter-layer edges. In order to better investigate these non-linear effects, we inverted Eq. (3) to compute the effective value of the average degree $\mu_1^{\mathrm{aggr}}(q, \omega)$ of an ideal aggregated single-layer network able to reproduce the observed value of $\rho(q, \omega)$ for each value of ω and q. In formula:

$$\mu_1^{\mathrm{aggr}}(q, \omega) = \frac{3\rho(q, \omega) - 2}{3\rho(q, \omega) - 1} \qquad (17)$$

As made evident by Fig. 5, for each value of edge overlap, $\mu_1^{\mathrm{aggr}}(q, \omega)$ can be fitted very well by a quadratic polynomial in q. However, the actual values of the coefficients of the fit depend on ω in a non-trivial way. We can formally write:

$$\mu_1^{\mathrm{aggr}}(q, \omega) = F(\omega)q^2 + G(\omega)q + H(\omega). \qquad (18)$$

The problem now is to find an expression for $F(\omega)$, $G(\omega)$ and $H(\omega)$. We started by making an ansatz for the functional dependence of those coefficients on ω, and then fitting these functions, for each value of q, by using several realisations of $\mu_1^{\mathrm{aggr}}(q, \omega)$ corresponding to different values of ω.

We found that for all the values of ω the quadratic fit of $\mu_1^{\mathrm{aggr}}(q, \omega)$ yields $H(\omega) = \mu$, so we just focused on the other two coefficients. The values of $F(\omega)$ and $G(\omega)$ are reported in the panels of Fig. 6 as black circles. It is then reasonable to assume that $F(\omega)$, $G(\omega)$ are polynomial functions of ω. We found that in order to accurately reproduce the behaviour of $\mu_1^{\mathrm{aggr}}(q, \omega)$ in the whole range of ω, both $F(\omega)$ and $G(\omega)$ should be at least third-order polynomials in ω, as made evident by the plots reported in Fig. 6.

Notice that the predictions of μ_1^{aggr} provided by the two theoretical arguments reported above, based on the linear superpositions of the two layers, contained only linear terms in ω. However, the fit of $\mu_1^{\mathrm{aggr}}(q, \omega)$ confirms that, when $\omega > 0$, the behaviour of the multiplex voter model is the result of a highly non-linear combination of the two layers, suggesting that a *trivial* single-layer equivalent of the multiplex voter model dynamics does not exist, especially when the underlying multiplex network is characterised by a non-negligible overlap.

That simple aggregation procedures do not produce the observed scaling is also evident by examining the timescales on which finite systems approach absorbing states. For $q > 0$, the characteristic time $\tau(q, \omega)$ is an exponentially decreasing function of q, $\tau(q, \omega)/N \sim e^{-q\alpha(\omega, N)}$, $\alpha > 0$ (fig. 2(b)). Figure 5(b) shows that $\alpha(\omega, N) = -a\omega + b$, $a > 0$, and that the system-size dependency does not enter into it. Hence, $\tau(q, \omega)/N \sim e^{q(\omega a - b)}$. Thus increasing the edge overlap ω results in longer-lived systems, while adding more interlayer links produces the opposite effect. This was additionally confirmed by examining the behaviour of the rescaled time until absorption $\langle T \rangle$, which showed the same qualitative trend and almost identical a and b coefficients.

VI. DISCUSSION

Multilayer networks allow to extend the applicability of network theory to more realistic contexts in which nodes are connected through concurrent interaction patterns of different kinds. However, a fundamental open question to answer is whether the added complexity yielded by multilayer networks is really needed to model network phenomena, or if instead there exist simple ways of representing multiplex dynamics through appropriately constructed processes occurring on appropriately constructed single-layer networks. We have investigated here the problem of reducing the multiplex voter model to an equivalent single-layer dynamics. We have considered the predictions about the level of activity of the multiplex voter model in the thermodynamic limit, as measured
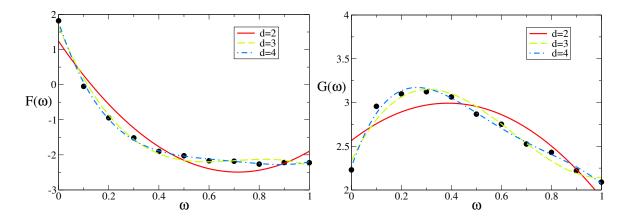
FIG. 6: The black dots in each panel represent the coefficients of the quadratic fit $F(\omega)q^2 + G(\omega)q + 1$ of $\mu_1^{\mathrm{aggr}}(q, \omega)$ for different values of ω. Notice that both $F(\omega)$ and $G(\omega)$ are non-linear functions of ω. The different lines correspond to fits of $F(\omega)$ and $G(\omega)$ using d-th order polynomials. Interestingly, a good fit is obtained only for $d \geq 3$, meaning that the presence of inter-layer state copying is introducing highly non-linear effects.

by the interface density, as well as the time to reach the absorbing state for finite systems. We have found that results for the interface density based on single-layer aggregated graphs are accurate only when there is little or no interaction between the layers (q sufficiently small), or when there is full connectivity ($q \sim 1$) and the edge overlap ω is either 0 or 1. For the complementary broad range of parameters, any standard aggregation procedure can only give some qualitative information about the interface density, but fails to reproduce the quantitative details. We showed that edge overlap and multiplexity have two opposite effects on the long-term dynamics of the multiplex voter model, and in particular that an increase in the value of edge overlap can counter the action of increasing the fraction of interlayer links, leading to an overall *decrease* in the interface density of the multiplex. In fact, we showed numerically that any equivalent single-layer representation of the multiplex voter model dynamics entails the construction of an aggregate network which is a highly non-linear combination of the original layers.

These results will be found at the same time surprising and interesting by all the researchers aiming at modelling social interaction in real-world scenarios. As a matter of fact, it has been recently shown that multilayer social networks are normally *truly multiplex*, meaning that they are characterised by non-negligible values of edge overlap [30] and by an intermediate level of multiplexity [29, 31, 32]. And as we have shown in this work, the single-layer voter model approximation is qualitatively

(when considering the characteristic time) and quantitatively (when considering both the characteristic time and the interface density) inaccurate when the edge overlap ω and the degree of multiplexing q are far from their extreme values 0 or 1. This is reflected in the average timescales of consensus for finite systems, as we found that a multiplex with very few interlayer connections takes the longest time to reach consensus, much more than would probabilistically be needed by two disconnected monoplexes. This nonlinear effect is not captured by either of the simple aggregation procedures proposed in the paper.

The case of dynamical irreducibility of a multiplex process presented in this work raises the important question of whether other unknown phenomena might be lurking in the multilayer structure of real-world systems. This question, together with the insights about the intrinsically multidimensional nature of the multiplex voter model, represents a stimulus to perform further research along these lines.

[1] M. Kivelä, A. Arenas, M. Barthelemy, J. P. Gleeson, Y. Moreno, and M. A. Porter, Journal of Complex Networks **2**, 203 (2014).

[2] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, and A. Arenas, Phys. Rev. X **3**, 041022 (2013).

[3] S. Boccaletti, G. Bianconi, R. Criado, C. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal,

Z. Wang, and M. Zanin, Physics Reports **544**, 1 (2014).

[4] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature **464**, 1025 (2010).

[5] R. G. Morris and M. Barthelemy, Phys. Rev. Lett. **109**, 128703 (2012).

[6] M. De Domenico, A. Solé-Ribalta, S. Gómez, and A. Arenas, Proceedings of the National Academy of Sciences **111**, 8351 (2014).

[7] M. Kurant and P. Thiran, Phys. Rev. Lett. **96**, 138701 (2006).

[8] C. Granell, S. Gómez, and A. Arenas, Phys. Rev. Lett. **111**, 128701 (2013).

[9] M. De Domenico, V. Nicosia, A. Arenas, and V. Latora, Nature Communications **6** (2015).

[10] J. Gómez-Gardeñes, I. Reinares, A. Arenas, and L. M. Floria, Sci. Rep. **2**, (2012).

[11] J. Sanz, C.-Y. Xia, S. Meloni, and Y. Moreno, Phys. Rev. X **4**, 041005 (2014).

[12] F. Battiston, V. Nicosia, and V. Latora, ArXiv e-prints (2015), 1505.01378.

[13] F. Battiston, A. Cairoli, V. Nicosia, A. Baule, and V. Latora, ArXiv e-prints (2015).

[14] A. Chmiel and K. Sznajd-Weron, ArXiv e-prints (2015), 1503.01400.

[15] R. A. Holley and T. M. Liggett, Ann. Prob. **3**, 643 (1975).

[16] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University press, 2010).

[17] C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).

[18] K. Suchecki, V. M. Eguíluz, and M. San Miguel, Europhys. Lett. **69**, 228 (2005).

[19] K. Suchecki, V. M. Eguíluz, and M. San Miguel, Phys. Rev. E **72**, 036132 (2005).

[20] V. Sood and S. Redner, Phys. Rev. Lett. **94**, 178701 (2005).

[21] V. Sood, T. Antal, and S. Redner, Phys. Rev. E **77**, 041121 (2008).

[22] J. Fernández-Gracia, V. M. Eguíluz, and M. San Miguel, Phys. Rev. E **84**, 015103 (2011).

[23] F. Vazquez, V. M. Eguíluz, and M. San Miguel, Phys. Rev. Lett. **100**, 108702 (2008).

[24] T. Gross and B. Blasius, Proc. R. Soc. Interface **5**, 259 (2008).

[25] H. Silk, G. Demirel, M. Homer, and T. Gross, New Journal of Physics **16**, 093051 (2014).

[26] M. Diakonova, M. San Miguel, and V. M. Eguíluz, Phys. Rev. E **89**, 062818 (2014).

[27] M. Diakonova, V. M. Eguíluz, and M. San Miguel, ArXiv e-prints (2014), 1411.5181.

[28] J. Fernández-Gracia, K. Suchecki, J. J. Ramasco, M. San Miguel, and V. M. Eguíluz, Phys. Rev. Lett. **112**, 158701 (2014).

[29] V. Nicosia and V. Latora, ArXiv e-prints (2014), 1403.1546.

[30] The Social Media Matrix from the Pew Research Center (2013), http://www.pewinternet.org/.

[31] M. Szell, R. Lambiotte, and S. Thurner, Proc. Natl. Acad. Sci. USA **107**, 13636 (2010).

[32] P. Klimek and S. Thurner, New Journal of Physics **15**, 063008 (2013).

[33] F. Battiston, V. Nicosia, and V. Latora, Phys. Rev. E **89**, 032804 (2014).

[34] J. Fernández-Gracia, V. M. Eguíluz, and M. San Miguel, *Timing Interactions in Social Simulations: The Voter Model* (Springer Berlin Heidelberg, 2013), pp. 331–352, Understanding Complex Systems.

[35] F. Vazquez and V. M. Eguíluz, New Journal of Physics **10**, 063011 (2008).